Calibrated subactions for Anosov maps and flows

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Summary

I. Position of the problem

- **Anosov maps and flows**
- Subactions for SFT
- Positive Livšic theorem
- Main theorem

I. Position of the problem: **Anosov maps and flows**

Definition (The discrete case) (M, f) is a C^1 map on a manifold of dimension $d_M \geq 2$, $\Lambda \subset M$ is a compact invariant set. We assume

1. Λ is *hyperbolic*: $\exists\, \lambda^s < 0 < \lambda^u$, $C_\Lambda \geq 1$ and a continuous equivariant splitting over Λ,

$$\forall\, x \in \Lambda,\ T_x M = E^u_\Lambda(x) \oplus E^s_\Lambda(x) \tag{1}$$

   such that

   a. $\forall\, x \in \Lambda,\ T_x f(E^u(x)) = E^u(f(x)),\ T_x f(E^s(x)) \subseteq E^s(f(x))$

   b. $\forall x \in \Lambda,\ \forall n \geq 0,\ \begin{cases} \forall v \in E^s_\Lambda(x),\ \|T_x f^n(v)\| \leq C_\Lambda\, e^{n\lambda^s} \|v\| \\ \forall v \in E^u_\Lambda(x),\ \|T_x f^n(v)\| \geq C_\Lambda^{-1}\, e^{n\lambda^u} \|v\| \end{cases}$

2. Λ is *locally maximal*: there exists an open neighborhood U of Λ of compact closure such that

$$\bigcap_{n \in \mathbb{Z}} f^n(\bar{U}) = \Lambda$$

I. Position of the problem: **Anosov maps and flows**

Definition (The continuous case) (M, V, f) is a C^1 flow, $\Lambda \subseteq M$ is a compact invariant set ($\forall\, t \in \mathbb{R},\ f^t(\Lambda) = \Lambda$).

1. Λ is *hyperbolic*: $\exists\, C_\Lambda \geq 1$, $\lambda^s < 0 < \lambda^u$ and a continuous equivariant splitting of Λ,

$$\forall\, x \in \Lambda,\ T_x M = E^u_\Lambda(x) \oplus E^0_\Lambda(x) \oplus E^s_\Lambda(x) \tag{1}$$

   such that,

   a. $\forall\, x \in \Lambda,\ T_x f^t(E^u(x)) = E^u(f^t(x)),\ T_x f^t(E^s(x)) = E^s(f^t(x))$

   b. $\forall x \in \Lambda,\ \forall t \geq 0,\ \begin{cases} \forall v \in E^s_\Lambda(x),\ \|T_x f^t(v)\| \leq C_\Lambda\, e^{t\lambda^s} \|v\| \\ E^0_\Lambda(x) = V(x)\mathbb{R}, \quad T_x f^t(V(x)) = V \circ f^t(x) \\ \forall v \in E^u_\Lambda(x),\ \|T_x f^t(v)\| \geq C_\Lambda^{-1}\, e^{t\lambda^u} \|v\| \end{cases}$

2. Λ is *locally maximal*: there exists an open neighborhood $U \supseteq \Lambda$ of compact closure such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} f^t(U)$$

I. Position of the problem

- Anosov maps and flows
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I. Position of the problem: **Subactions for SFT**

Definition We consider a Lipschitz observable

$$\phi : M \to \mathbb{R}$$

(M, f) is a one-sided topological dynamical system. $\Lambda \subseteq M$ a compact invariant set, $U \supseteq \Lambda$ an open neighborhood of Λ.

1. the *ergodic minimizing value*

$$\bar{\phi}_\Lambda := \lim_{n \to +\infty} \frac{1}{n} \inf_{x \in \Lambda} \sum_{k=0}^{n-1} \phi \circ f(x). \tag{1}$$

2. A *subaction on* (U, Λ) is a continuous function $u : U \to \mathbb{R}$

$$\forall\, x \in U,\ \phi(x) - \bar{\phi}_\Lambda \geq u \circ f(x) - u(x) \tag{2}$$

I. Position of the problem: **Subactions for SFT**

Definition (M, f) is a one-sided SFT

1. The *Lax-Oleinik non linear operator* acts on $C^0(M, \mathbb{R})$ by

$$T[u](y) = \min_{x \in M,\, f(x)=y} \big\{ u(x) + \phi(x) \big\} \qquad (1)$$

2. A *calibrated subaction* u is a fixed point of the Lax-Oleinik operator

$$T[u] = u + \bar{\phi}_M \qquad (2)$$

Remark

- Of course a calibrated subaction is a subaction
- A calibrated subaction is also a good numerical tool to construct the Mather set

I. Position of the problem

- Anosov maps and flows
- Subactions for SFT
- **Positive Livšic theorem**
- Main theorem

I. Position of the problem: **Positive Livšic theorem**

Theorem (Livšic) (M,f) is a C^1 Anosov diffeomorphism or (M,V,f) is a C^1 Anosov flow, $\phi : M \to \mathbb{R}$ is Lipschitz.
If

$$\frac{1}{\tau}\sum_{k=0}^{\tau-1}\phi\circ f^k(p) = 0 \tag{1}$$

or if

$$\frac{1}{\tau}\int_0^\tau \phi\circ f^s(p)\,ds = 0 \tag{2}$$

for every periodic orbit p of period τ,

then $\bar\phi_M = 0$ there exists a Lipschitz function $u : M \to \mathbb{R}$ such that

$$\phi - \bar\phi_M = u\circ f - u \tag{3}$$

or u is in addition differentiable along the flow and

$$\phi - \bar\phi_M = \mathcal{L}_V[u] =: \lim_{t\to 0}\frac{1}{t}\big(u\circ f^t - u\big) \tag{4}$$

I. Position of the problem: **Positive Livšic theorem**

Theorem (Positive Livšic)(A. Lopes, Ph. T or A. Lopes, V. Roasa, R. Ruggiero)
(M, V, f) is a C^1 Anosov flow, $\phi : M \to \mathbb{R}$ is Lipschitz.
If

$$\frac{1}{\tau}\int_0^\tau \phi \circ f^s(p)\,ds \geq 0 \tag{1}$$

for every periodic orbit p of period τ,

then there exists an Hölder $u : M \to \mathbb{R}$ differentiable along the flow such that

$$\phi - \bar{\phi}_M \geq \mathcal{L}_V[u], \qquad \bar{\phi} := \lim_{t\to+\infty} \inf_{x\in M} \frac{1}{t}\int_0^t \phi \circ f^s(x)\,ds \tag{2}$$

Remark It is important here to keep $\bar{\phi}_M$

I. Position of the problem

- Anosov maps and flows
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- **Main theorem**

I. Position of the problem: **Main theorem**

Theorem (X. Su, Ph. T) (The discrete case) (M, f) be a C^1 dynamical system, $\Lambda \subseteq M$ is a locally maximal hyperbolic compact set, $\phi : M \to \mathbb{R}$ is a Lipschitz continuous function, and $\bar\phi_\Lambda$ is the ergodic minimizing value of ϕ restricted to Λ.

Then there exists an open set Ω containing Λ and a Lipschitz continuous function $u : \Omega \to \mathbb{R}$ such that

$$\forall\, x \in \Omega, \quad \phi(x) - \bar\phi_\Lambda \geq u \circ f(x) - u(x). \tag{1}$$

Moreover,

$$\mathrm{Lip}(u) \leq K_\Lambda \mathrm{Lip}(\phi) \tag{2}$$

for some constant K_Λ depending only on the hyperbolicity of f on Λ.

I. Position of the problem: **Main theorem**

Theorem (X. Su, Ph. T (The continuous case) L (M, V, f) be a C^1 flow, $\Lambda \subseteq M$ is a locally maximal hyperbolic compact invariant set, and $\phi : M \to \mathbb{R}$ is a Lipschitz continuous function.

Then there exist an open neighborhood Ω of Λ and a Lipschitz continuous function $u : \Omega \to \mathbb{R}$ that satisfies

1. u is differentiable along the flow
2. $\mathcal{L}_V[u]$ is Lipschitz
3. $\forall\, x \in \Omega,\ \phi(x) - \bar{\phi}_\Lambda \geq \mathcal{L}_V[u](x)$
4. $\text{Lip}(u) \leq K_\Lambda \text{Lip}(\phi)$

for some constant K_Λ depending on the hyperbolicity of f on Λ

I. Position of the problem: **Main theorem**

Previous results

1. T. Bousch (2011) seems to state a similar theorem (Lipschitz regularity) in the discrete case
2. Wen Huang, Zeng Lian, Xiao Ma, Leiye Xu, and Yiwei Zhang (2019) have obtained an integrated formula:

   let $T > 0$, then there exists a Lipschitz continuous $u_T : \Omega \to \mathbb{R}$ such that

   $$\forall\, x \in \Omega,\ \int_0^T \phi \circ f^s(x)\, ds \geq u_T \circ f^T(x) - u_T(x) + T\bar{\phi}_\Lambda \qquad (1)$$

   There is no reason that u_T is independent of T

Summary

II. The main tools

- **Discrete Livšic positive criteria**
- Discrete Lax-Oleinik operator
- Continuous Livšic criteria
- Continuous Lax-Oleinik semigroup

II. The main tools: **Discrete Livšic positive criteria**

Definition (M, f, Λ) is a locally maximal hyperbolic compact invariant set, U is a neighborhood of Λ, and $\phi : U :\to \mathbb{R}$ is Lipschitz continuous.

We say that ϕ satisfies the *discrete positive Livšic criteria on* (U, Λ) *with distortion constant* C if

$$\inf_{n \geq 1} \inf_{(x_0, x_1, \ldots, x_n) \in U^{n+1}} \sum_{i=0}^{n-1} \big(\phi(x_i) - \bar{\phi}_\Lambda + C d(f(x_i), x_{i+1}) \big) > -\infty$$

where

$$\bar{\phi}_\Lambda := \lim_{n \to +\infty} \inf_{x \in \Lambda} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) \qquad (1)$$

Remark

1. if $\bar{\phi}_\Lambda$ is replaced by a constant $\beta > \bar{\phi}_\Lambda$, the infimum is $-\infty$
2. if $x_i = f^i(x)$ is a true orbit, the infimum may be $-\infty$ for non hyperbolic systems and smooth observable

II. The main tools: **Discrete Livšic positive criteria**

It seems that the positive Livšic criteria is very restrictive.

Lemma If ϕ admits a Lipschitz subaction $u : U \to \mathbb{R}$, that is

$$\forall\, x \in U,\ \phi(x) - \bar{\phi}_\Lambda \geq u \circ f(x) - u(x) \tag{1}$$

and $\text{Lip}(u) \leq C$, then for every sequence $(x_i)_{i=0}^n$

$$\sum_{k=0}^{n-1} \big(\phi(x_k) - \bar{\phi}_\Lambda + Cd(x_{k+1}, f(x_k))\big) \geq -2\|u\|_\infty \tag{2}$$

Proof

$$\sum_{k=0}^{n-1} \big(\phi(x_k) - \bar{\phi}_\Lambda + Cd(x_{k+1}, f(x_k))\big)$$

$$\geq \sum_{k=0}^{n-1} \big(u \circ f(x_k) - u(x_k) + Cd(x_{k+1}, f(x_k))\big)$$

$$\geq \sum_{k=0}^{n-1} \big(u \circ f(x_k) - u(x_{k+1}) + Cd(x_{k+1}, f(x_k))\big) + u(x_n) - u(x_0)$$

II. The main tools: **Discrete Livšic positive criteria**

Theorem The positive Livšic criteria is satisfied for every locally maximal hyperbolic invariant set Λ and Lipschitz continuous observable ϕ. More precisely,

there exist an open neighborhood $\Omega \supseteq \Lambda$, two constants $K_\Lambda \geq 0$ and $\delta_\Lambda \geq 0$, depending only on the hyperbolicity of Λ, such that for every Lipschitz $\phi : \Omega \to \mathbb{R}$, for every sequence $(x_i)_{i=0}^n$ of Ω

$$\sum_{k=0}^{n-1} \big(\phi(x_k) - \bar{\phi}_\Lambda + K_\Lambda \mathrm{Lip}(\phi) d(x_{k+1}, f(x_k))\big) \geq -\delta_\Lambda \mathrm{Lip}(\phi) \qquad (1)$$

II. The main tools

- Discrete Livšic positive criteria
- **Discrete Lax-Oleinik operator**
- Continuous Livšic criteria
- Continuous Lax-Oleinik semigroup

II. The main tools: **Discrete Lax-Oleinik operator**

Definition (M, f) be a topological dynamical system, $\Lambda \subseteq M$ is a compact f-invariant subset, $\Omega \supset \Lambda$ is an open neighborhood of Λ, $\phi : \Omega \to \mathbb{R}$ is a bounded function, and $C \geq 0$ is a constant,

1. The *Discrete Lax-Oleinik operator* is the nonlinear operator T acting on the space of bounded functions $u : \Omega \to \mathbb{R}$ defined by

$$\forall x' \in \Omega,\ T[u](x') := \inf_{x \in \Omega} \big\{ u(x) + \phi(x) - \bar{\phi}_\Lambda + C d(f(x), x') \big\}.$$

2. A *calibrated subaction of the Lax-Oleinik operator* is a bounded function $u : \Omega \to \mathbb{R}$ solution of the equation

$$T[u] = u. \tag{1}$$

II. The main tools: **Discrete Lax-Oleinik operator**

Theorem (M, f, Λ, Ω) is as in the previous definition, $\phi : \Omega \to \mathbb{R}$ is a bounded function. Assume ϕ satisfies the positive Livšic criteria on (Ω, Λ)

$$\inf_{n \geq 1} \inf_{(x_0, x_1, \ldots, x_n) \in U^{n+1}} \sum_{i=0}^{n-1} \big(\phi(x_i) - \bar{\phi}_\Lambda + Cd(f(x_i), x_{i+1})\big) > -\infty$$

Then there exists a C-Lipschitz calibrated subaction u.

Remark A calibrated subaction is a subaction

$$\forall\, x, x' \in \Omega,\ u(x') = T[u](x') \leq u(x) + \phi(x) - \bar{\phi}_\Lambda + Cd(f(x), x')$$

in particular $x' = f(x)$ and then

$$\forall\, x \in \Omega,\ u \circ f(x) \leq u(x) + \phi(x) - \bar{\phi}_\Lambda$$

Conclusion On a locally maximal hyperbolic compact set Λ, a Lispchitz observable ϕ satisfies the positive Livšic criteria and therefore admits a Lipschitz subaction u

$$\mathrm{Lip}(u) \leq K_\Lambda \mathrm{Lip}(\phi)$$

II. The main tools

- Discrete Livšic positive criteria
- Discrete Lax-Oleinik operator
- **Continuous Livšic criteria**
- Continuous Lax-Oleinik semigroup

II. The main tools: **Continuous Livšic criteria**

Definition (M, V, f) is a C^1 flow, $\Lambda \subseteq M$ is a hyperbolic locally maximal compact invariant set, U is an open neighborhood of Λ, and $\phi : U \to \mathbb{R}$ is Lipschitz continuous. We set

$$\bar{\phi}_\Lambda := \lim_{T \to +\infty} \inf_{x \in \Lambda} \frac{1}{T} \int_0^T \phi \circ f^s(x) \, ds \qquad (1)$$

The *weighted action of* ϕ *of weight* $C \geq 0$ over a piecewise C^1 continuous path $z : [0, T] \to M$ is the real number given by

$$\mathcal{A}_{\phi,C}(z) := \int_0^T \left[(\phi - \bar{\phi}_\Lambda) \circ z(s) + C \| V \circ z(s) - z'(s) \| \right] ds \qquad (2)$$

Recall In the discrete case

$$\mathcal{A}_{\phi,C}(x_i)_{i=0}^n := \sum_{i=0}^{n-1} \big(\phi(x_i) - \bar{\phi}_\Lambda + C d(f(x_i), x_{i+1}) \big)$$

II. The main tools: **Continuous Livšic criteria**

Definition Same notations (M, V, f) as before.

$$\mathcal{A}_{\phi,C}(z) := \int_0^T \left[(\phi - \bar{\phi}_\Lambda) \circ z(s) + C \| V \circ z(s) - z'(s) \| \right] ds \qquad (1)$$

We say that ϕ satisfies the *positive Livšic criteria on* (U, Λ) *with distortion constant* C if

$$\inf_{z:[0,T] \to U} \mathcal{A}_{\phi,C}(z) > -\infty, \qquad (2)$$

where the infimum is realized over the set of piecewise C^1 continuous path $z : [0, T] \to U$

Remark As in the discrete case, if ϕ admits a smooth subaction, then ϕ satisfies the positive Livšic criteria

II. The main tools: **Continuous Livšic criteria**

Lemma Assume there exists a $C^1(M)$ function $u : M \to \mathbb{R}$ such that

$$\phi(x) - \bar{\phi}_\Lambda \geq \mathcal{L}_V[u](x) = \frac{d}{dt}\Big|_{t=0} u \circ f^t(x) \tag{1}$$

Then for every piecewise C^1 continuous path $z : [0, T] \to M$

$$\mathcal{A}_{\phi,C}(z) \geq -2 \inf_{c \in \mathbb{R}} \|u - c\|_\infty. \tag{2}$$

where $C = \|du\|_\infty$

II. The main tools: **Continuous Livšic criteria**

Proof

$$\mathcal{A}_{\phi,C}(z) = \int_0^T \Big(\phi \circ z(s) - \bar{\phi}_\Lambda + \|du\|_\infty \|V \circ z - z'\|\Big) ds \tag{1}$$

$$\geq \int_0^T \Big(du \circ z \cdot V \circ z + \|du\|_\infty \|V \circ z - z'\|\Big)\, ds \tag{2}$$

$$= \int_0^T \Big(du \circ z \cdot V \circ z - du \circ z \cdot z') + \|du\|_\infty \|V \circ z - z'\|\Big) ds \tag{3}$$

$$+ \int_0^T du \circ z \cdot z'\, ds \tag{4}$$

$$\geq \int_0^T du \circ z \cdot z'\, ds = u \circ z(T) - u \circ z(0) \geq -2\|u\|_\infty. \tag{5}$$

II. The main tools: **Continuous Livšic criteria**

Theorem The positive Livšic criteria is satisfied for every locally maximal hyperbloic set Λ of a C^1 flow. More precisely

There exists a neighborhood Ω of Λ and constants $C_\Lambda \geq 0$, $\delta_\Lambda \geq 0$ such that for every piecewise C^1 continuous path $z : [0, T] \to \Omega$

$$\int_0^T \big[(\phi - \bar{\phi}_\Lambda) \circ z(s) + C_\Lambda \mathrm{Lip}(\phi) \| V \circ z(s) - z'(s) \| \big] \, ds \geq -\delta_\Lambda \mathrm{Lip}(\phi)$$

II. The main tools

- Discrete Livšic positive criteria
- Discrete Lax-Oleinik operator
- Continuous Livšic criteria
- **Continuous Lax-Oleinik semigroup**

II. The main tools: **Continuous Lax-Oleinik semigroup**

Definition $\phi : U \to \mathbb{R}$ is a C^0 bounded function and $C \geq 0$ is a constant. Assume ϕ satisfies the positive Livšic criteria on (U, Λ) with distortion constant C

1. The *continuous Lax-Oleinik semigroup on* (U, Λ) *of generator* ϕ is a nonlinear operator acting on bounded functions $u : U \to \mathbb{R}$ defined for every $t > 0$ by, for every $q \in U$

$$T^t[u](q) := \inf_{\substack{z:[-t,0]\to U \\ z(0)=q}} \Big\{ u \circ z(-t) + \int_{-t}^{0} \big[\phi \circ z - \bar{\phi}_\Lambda + C\|V \circ z - z'\|\big]\, ds \Big\}$$

   where the infimum is taken over the set of piecewise C^1 continuous paths z ending at q.

2. A *calibrated subaction* of the Lax-Oleinik semigroup is a bounded function $u : U \to \mathbb{R}$ solution of the equation

$$\forall\, t > 0,\ T^t[u] = u.$$

II. The main tools: **Continuous Lax-Oleinik semigroup**

Theorem (M, V, f) be a C^1 flow, Λ is a compact invariant set, $U \supseteq \Lambda$ is a connected open set of compact closer, $\phi : U \to \mathbb{R}$ be a bounded Lipschitz continuous function and $C \geq 0$ be a constant.

Assume ϕ satisfies the positive Livšic criteria on (U, Λ) with distortion constant C. Then there exists a C-Lipschitz calibrated subaction $u : U \to \mathbb{R}$: for every $t > 0$, for every $q \in U$

$$u(q) := \inf_{\substack{z:[-t,0]\to U \\ z(0)=q}} \Big\{ u \circ z(-t) + \int_{-t}^{0} \big[\phi \circ z - \bar{\phi}_\Lambda + C\|V \circ z - z'\|\big]\, ds \Big\}$$

Remark

1. In the discrete Aubry-Mather theory, $\big[\cdots\big]$ is replaced by a Lagrangian $L(z, z')$. A calibrated subaction is called weak KAM solution
2. The Lipschitz regularity C is the same as the distortion constant C

Summary

III. Ideas of the proof

- **Calibrated subaction**
- Livšic criteria

III. Ideas of the proof: **Calibrated subaction**

Definition $\phi : U \to \mathbb{R}$ is a bounded continuous function and $C \geq 0$ is a constant. The *weighted action of* ϕ *between two points* $p, q \in U$ for a weight C and a time laps $t > 0$ is the quantity

$$\mathcal{A}^t_{\phi,C}(p,q) := \inf_{\substack{z:[0,t]\to U \\ z(0)=p,\, z(t)=q}} \int_0^t \big[\phi \circ z - \bar{\phi}_\Lambda + C\|V \circ z - z'\|\big]\, ds,$$

where the infimum is realized over the set of piecewise C^1 continuous paths.

Corollary The Lax-Oleinik semigroup has a different expression

$$T^t[u](q) = \inf_{p\in U} \big\{u(p) + \mathcal{A}^t_{\phi,C}(p,q)\big\}.$$

III. Ideas of the proof: **Existence of a calibrated subaction**

Definition The *distance function between two points* $p, q \in U$ is the real number

$$d_U(p,q) = \inf \Big\{ \int_0^1 \|z'(s)\|\, ds : z : [0,1] \to U \text{ is piecewise } C^1, \text{ continuous, and } z(0)=p,\ z(1)=q \Big\}.$$

Lemma $\forall\, p, \tilde{p}, q, \tilde{q} \in U$

1. $\big|\mathcal{A}^t_{\phi,C}(p,q) - \mathcal{A}^t_{\phi,C}(p,\tilde{q})\big| \le C d_U(q,\tilde{q})$,
2. $\big|\mathcal{A}^t_{\phi,C}(p,q) - \mathcal{A}^t_{\phi,C}(\tilde{p},q)\big| \le C d_U(p,\tilde{p})$,

III. Ideas of the proof: **Calibrated subaction**

Proof of the existence of a calibrated subaction

1. If u is bounded then $T^t[u]$ is C-Lipschitz for every $t > 0$
2. Define $v = \inf_{s>0} T^s[0]$. Then $s \mapsto T^s[v]$ is increasing
3. Define $u = \sup_{s>0} T^s[v]$. Then $T^t[u] = u$

$$u(q) := \inf_{\substack{z:[-t,0]\to U \\ z(0)=q}} \Big\{ u \circ z(-t) + \int_{-t}^{0} \big[\phi \circ z - \bar{\phi}_\Lambda + C\|V \circ z - z'\|\big]\, ds \Big\}$$

4. by taking orbits of the flow $z(s) = f^s(p)$ one obtains

$$u \circ f^t(p) \le u(p) + \int_0^t \big(\phi - \bar{\phi}_\Lambda\big) \circ f^s(p)\, ds$$

5. by modifying u with a partition of unity

$$\frac{d}{dt}\Big|_{t=0} u \circ f^t(p) \le (\phi - \bar{\phi}_\Lambda)(p)$$

III. Ideas of the proof

- Calibrated subaction
- **Livšic criteria**

III. Ideas of the proof: **Livšic criteria**

Lemma (Shadowing) (M, f) is a C^1 dynamical system and $\Lambda \subseteq M$ is a compact hyperbolic set. Then there exist constants $\epsilon_\Lambda > 0$, $K_\Lambda \geq 1$, such that for every $n \geq 1$, for every ϵ_Λ-pseudo orbit $(x_i)_{0 \leq i \leq n}$ of the neighborhood $\Omega_\Lambda = \{x \in M : d(x, \Lambda) < \epsilon_\Lambda\}$, there exists a point $y \in M$ such that

$$\max_{0 \leq i \leq n} d(x_i, f^i(y)) \leq K_\Lambda \max_{1 \leq k \leq n} d(f(x_{k-1}), x_k). \tag{1}$$

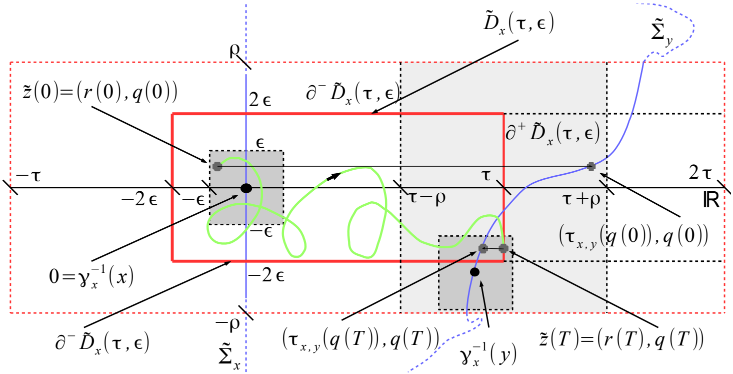
Lemma (Improved shadowing)

$$\sum_{i=0}^{n} d(x_i, f^i(y)) \leq K_\Lambda \sum_{k=1}^{n} d(f(x_{k-1}), x_k), \tag{2}$$

Corollary The improved shadowing lemma is also true for iteration of Poincaré maps

III. Ideas of the proof: **Livšic criteria**

Definition Flow boxes $\gamma_x : (-\tau, 2\tau) \times B_x(\rho) \to M$



$$\partial^+ \tilde{D}_x(\tau, \epsilon) := \{\tau\} \times B_x(2\epsilon),$$

$$\partial^- \tilde{D}_x(\tau, \epsilon) := \{-2\epsilon\} \times \overline{B_x(2\epsilon)} \cup [-2\epsilon,$$

III. Ideas of the proof: **Livšic criteria**

Definition We discuss three cases:

1. *The pseudo orbit case*: the path z exits at the forward boundary

$$z(T) \in \partial^+ D_x(\tau, \epsilon).$$

2. *The escaped orbit case*: the path exits at the backward boundary

$$z(T) \in \partial^- D_x(\tau, \epsilon).$$

3. *the trapped orbit case*: the path stays inside $D_x(\tau, \epsilon)$.

III. Ideas of the proof: **Livšic criteria**

Lemma Given $\mathcal{A}_* \geq 0$, for $C = C(\mathcal{A}_*)$ sufficiently large

1. The pseudo orbit case:

$$\mathcal{A}_{\phi,C}(z) \geq \Phi_{x,y} + \Psi_y - \Psi_x + C\|f_{x,y}(\tilde{q}_x) - \tilde{q}_y\|_y$$

$$\Phi_{x,y} := \int_0^{\tau_{x,y}(\tilde{q}_x)} (\phi - \bar{\phi}_\Lambda) \circ f^{s-\tilde{r}_x} \circ z(0)\, ds$$

2. The escaped-orbit case:

$$\mathcal{A}_{\phi,C}(z) \geq \mathcal{A}_*$$

3. The trapped orbit case:

$$\mathcal{A}_{\phi,C}(z) \geq -\mathcal{A}_*$$

Bibliographie I

- T. Bousch, Le lemme de Mañé-Conze-Guivarc'h pour les systèmes amphidynamiques rectifiables, Annales de la Faculté des Sciences de Toulouse, Vol. 20, No. 1 (2011), 1–14.
- A. Fathi, Weak KAM theorem in Lagrangian dynamics, Cambridge University Press (2020).
- Wen Huang, Zeng Lian, Xiao Ma, Leiye Xu, and Yiwei Zhang. Ergodic optimization theory for axiom A flows. Preprint 2019.
- A.O. Lopes, V. Roasa, R. Ruggiero. Cohomology and subcohomology for expansive geodesic flows. Discrete Contin. Dyn. Syst. 17(2007), 403–422.
- A.O. Lopes, Ph. Thieullen, Sub-actions for Anosov diffeomorphisms. Geometric Methods in Dynamics (II). Astérisque, Vol. 287 (2003), 135–146.
- A.O. Lopes, Ph. Thieullen, Sub-actions for Anosov flows. Ergodic Theory and Dynamical Systems, Vol. 25, No. 2 (2005), 605–628.
- Xifeng Su, Ph. Thieullen, Wenzhe Yu. Lipschitz sub-actions for locally maximal hyperbolic sets of a C^1 maps. Preprint (2021).
- Xifeng Su, Ph. Thieullen. Lipschitz sub-actions for locally maximal hyperbolic sets of a C^1 flow. Preprint (2021).