# Tropical methods for ergodic control and zero-sum games

## Minilecture, Part III

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Dynamical Optimization in PDE and Geometry
Applications to Hamilton-Jacobi
Ergodic Optimization, Weak KAM
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# Today

Spectral theory

Algorithmic aspects

# The max-plus spectral problem

Given $A = (A_{ij}) \in (\mathbb{R} \cup \{-\infty\})^{n \times n}$, find $v \in \mathbb{R} \cup \{-\infty\}^n$, $v \not\equiv -\infty$, $\lambda \in \mathbb{R}$, such that

$$\max_j A_{ij} + v_j = \lambda + v_i$$

$$\text{“}Av = \lambda v\text{”}$$

Among the oldest max-plus results.

Goes back to Cuninghame-Green 61, Vorobyev, Romanovski, Gondran and Minoux 77, Cohen, Dubois, Quadrat 83, ... Some references in Akian, SG, Bapat: Handbook of linear algebra (finite dim) and Max-plus Martin boundary / discrete spectral theory (infinite dim).

# Interpretation: dynamic programming, one player

Set of nodes $[d] := \{1, \dots, d\}$, arc $(i, j)$ with weight $A_{ij}$

$$A^k_{ij} = \sum_{m_1,\dots,m_{k-1}\in[d]} A_{im_1} A_{m_1 m_2} \cdots A_{m_{k-1} j}$$

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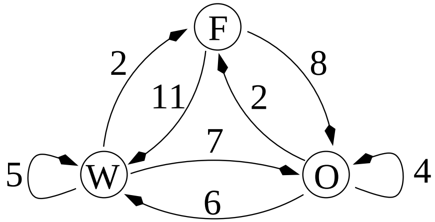
$$= \text{max weight path } i \to j \text{ length } k$$

# Crop rotation




$A_{ij}$ = reward of the year if crop $j$ follows crop $i$
F=fallow (no crop), W=wheat, O=oat,

$$(A^k v)_i = \sum_{j \in [d]} A^k_{ij} v_j$$

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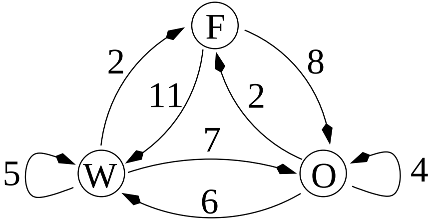


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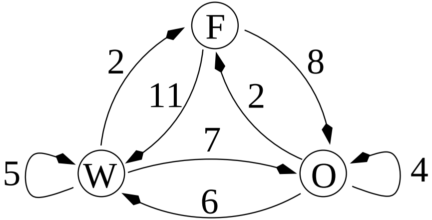


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$$(A^k v)_i = \max_{j \in [d]} A^k_{ij} + v_j$$

= reward in $k$ years, init. crop $i$; $v_j$ term. reward

# Eigenvector

Find $v \in \mathbb{R}^d_{\max}$, $v \not\equiv 0$, $\lambda \in \mathbb{R}_{\max}$, such that

$$Av = \lambda v$$

$$A^k v = \lambda^k v$$

# Eigenvector

Find $v \in \mathbb{R}_{\max}^d$, $v \not\equiv -\infty$, $\lambda \in \mathbb{R}_{\max}$, such that

$$\max_{j \in [d]} A_{ij} + v_j = \lambda + v_i$$

$$A^k v = k\lambda + v$$

Theorem (Max-plus spectral theorem, Cuninghame-Green, 61, Gondran & Minoux 77, Cohen et al. 83)

*Assume $G(A)$ is strongly connected. Then*

- *the eigenvalue is unique:*

$$\rho_{\max}(A) := \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

- *Assume WLOG $\rho_{\max}(A) = 0$, then, $\exists \alpha_i \in \mathbb{R} \cup \{-\infty\}$,*

$$u = \max_{j \in \textit{maximizing circuits}} \alpha_j + A^*_{\cdot,j}$$

$A^*_{ij} :=$ *max weight path arbitrary length $i \to j$.*

- "$A^{N+c} = \rho_{\max}(A)^c A^N$", $\exists N, c$

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- $A^{N+c} = c\rho_{\max}(A) + A^N$, $\exists N, c$

The dual linear problem of

$$\min \lambda,\ A_{ij} + v_j \leq \lambda + u_i \qquad \forall i, j$$

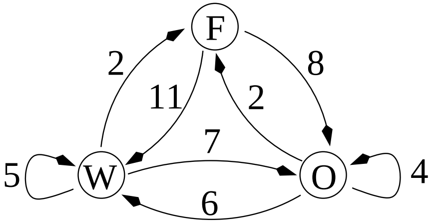
is

$$\rho(A) = \max_x \sum_{ij} A_{ij} x_{ij},\ x_{ij} \geq 0, \sum_j x_{ij} = \sum_j x_{ji}, \sum_{ij} x_{ij} = 1$$

The extreme points of the polytope of circulations are uniform measures supported by elementary circuits.

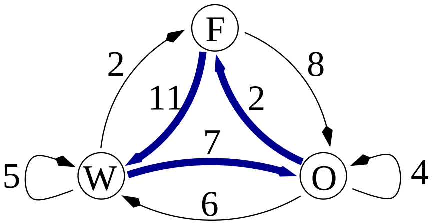
Complementary slackness shows that $v, \lambda, x$ optimal iff $x_{ij}(\lambda + u_i - A_{ij} - v_j)$

Discrete version of maximizing measures.




F=fallow (no crop), W=wheat, O=oat, $\rho_{\max}(A) = 20/3$

N. Bacaer, C.R. Acad. d'Agriculture de France, 03

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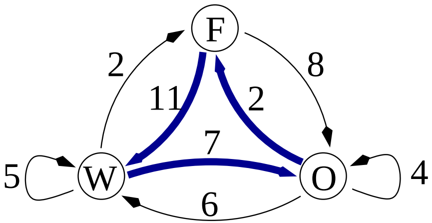



F=fallow (no crop), W=wheat, O=oat, $\rho_{\max}(A) = 20/3$

Actually, Bacaer showed that a memory of two years is needed to recover the different historical rotations

The critical graph $G^c(A)$ is the union of the maximizing circuits (analogue of Mather and Aubry sets - no difference between them in this discrete case).

### Lemma

*If $i, j$ are in the same strongly connected component of the critical graph, then $A^*_{\cdot i}$ and $A^*_{\cdot j}$ are tropically proportional.*

$$\text{“}A^* A^* = A^*\text{”}$$

$$\max_k A^*_{ik} + A^*_{kj} = A^*_{ij}$$

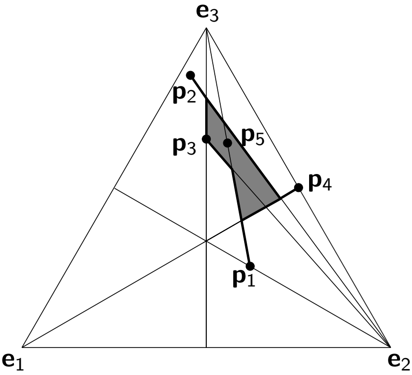
$i, j$ in the same component means $A^*_{ij} + A^*_{ji} = 0$.

$$A^*_{kj} \geq A^*_{ki} + A^*_{ij} \geq A^*_{kj} + A^*_{ji} + A^*_{ij} = A^*_{kj}$$

A vector $u \in C$ is extreme if $u = \sup(v, w)$, $v, w \in C$ implies $u = v$ or $u = w$. I.e.,
$u \in [v, w], v, w \in C \implies u = v$ or $u = w$.

Theorem (Tropical Minkowski-Carathéodory, SG, Katz LAA07; Butkovič, Sergeev, Schneider LAA07; infinite dim Choquet Poncet thesis 11)

*Every element of a closed tropical convex set of $\mathbb{R}^n_{\max}$ is the tropical convex combination of at most $n$ extreme points.*




Proof.

$$S_i(u) = \{x \in C \mid x \leq u \mid x_i = u_i\}$$

$$\text{Extr}\, C = \cup_i \,\text{Min}\, S_i$$

## Proposition

*Every* $A^*_{\cdot j}$, $j \in G^c(A)$ *is* *extreme* *in the tropical cone* $\{v \mid Av = \lambda v\}$.

# Cyclicity

WLOG: $\rho(A) = 0$.
The smallest $c$ such that $A^{N+c} = A^N$ for some $N$ (cylicity) is

$$c = \mathrm{lcm}(\mathrm{cyc}(K_1), \ldots, \mathrm{cyc}(K_s))$$

where $K_1, \ldots, K_s$ are the strongly connected components of the critical graph, and the cyclicity of a strongly connected component is the gcd of the lengths of its circuits.

Cohen, Dubois, Quadrat, Viot 83, Nussbaum 88

Give example at the blackboard.

- If $T$ is a nonexpansive mapping $\mathbb{R}^n \to \mathbb{R}^n$ with respect to a polyhedral norm, and if $T$ has bounded orbits, then, any orbit converges to a periodic orbit of length bounded by a function of $n$ and of the number of facets of the ball.
Weller, Sine, Nussbaum, Verdyun-Lunel, Scheutzow, Lemmens, ...
- If $T$ is a Shapley operator (order preserving, additively homogeneous) and convex (=1 player), possible orbits lengths are the orders of permutations Akian, SG 03.
- If $T$ is a Shapley operator (2-player), the optimal bound on the length is $\binom{n}{\lfloor n/2 \rfloor}$, the size of a maximal antichain in $\{0,1\}^n$: Lemmens and Scheutzow, Ergodic Th. and Dyn. Sys.
- If $T$ is sup-norm nonexpansive, Nussbaum conjectured the optimal length to be $2^n$.

# Spectral projector

WLOG $\rho(A) = 1$, $c = 1$.

$$A^N = A^{N+1} = \cdots = P, \qquad P = P^2, \qquad AP = PA$$

$$P_{ij} = \sup_k A^*_{ik} + A^*_{kj}$$

$=$ Turnpike theorem (every long path goes through a maximizing circuit).

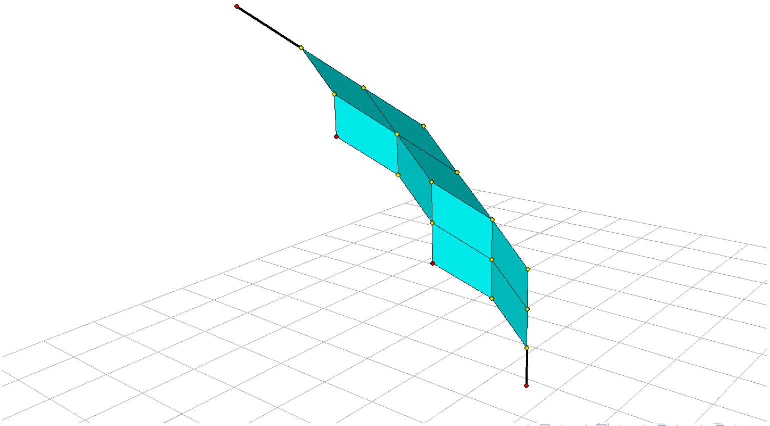
Let $K$ denote the set of critical nodes, $E = \{u \mid Au = u\}$. The restriction $u \mapsto (u_i)_{i\in K}$ (trace on the projected Aubry set) is an isomorphism, with image

$$\{v \in \mathbb{R}^K \mid v_i - v_j \geq A^*_{ij}, \qquad \forall i,j \in K\}$$

$=$ Space of Lipschitz functions for the metric $-A^*$

Note all the tropical convex sets are images of linear projectors. The images of linear projectors arise precisely in this way.

For a Shapley operator (2 player), the tropical convex set $\{u \mid u \leq T(u)\}$ is a polyhedral complex Develin, Sturmfels Doc. Math. 04. Every cell of this complex corresponds to a strategy, and is the image of a linear projector.



The representation of the eigenspace carries over to the infinite dimensional setting.

- Generalizations to kernels appeared in works of Nussbaum and Mallet-Parret, under quasi-compactness conditions (essential spectral radius)

the existence of a continuous eigenvector is in general a difficult problem.

- Lax-Oleinik semigroups treated in book by Maslov and Kolokoltsov, Kluwer 97 (typically when the projected Aubry set is finite). Spectral projector written in this context. WKB asymptotics.

Here: abstract boundary theory

# Martin boundary, discrete case (Dynkin)

Given $P_{xy}$ Markov kernel, over a discrete infinite set $E$, find all nonnegative harmonic functions: $u = Pu$.

1) Define the Green kernel: $G = P^0 + P + P^2 + \cdots$

2) The Martin kernel is:

$$K_{xy} = \frac{G_{xy}}{G_{by}}$$

where $b \in E$ is a basepoint.

3) Let $\mathcal{K} := \{K_{\cdot y} \mid y \in E\}$

4) The Martin space $\mathcal{M}$ is the closure of $\mathcal{K}$ in the product topology.

5) The Martin boundary is $\mathcal{B} := \mathcal{M} \setminus \mathcal{K}$.

## Theorem (classical Martin representation)

*Every harmonic function u can be written as a positive linear combination of functions from the boundary:*

$$u = \int_{\mathcal{B}} w\mu(dw) \enspace .$$

$\mu$ can be choosen to be supported by a subset of $\mathcal{B}$, the minimal Martin boundary. (We recognise Choquet's theorem!).

Computing the probabilistic Martin boundary is difficult, eg. Ney and Spitzer 65, boundary of random walk in $\mathbb{Z}^2$ is the circle, computing the tropical analogue is much easier!

# The max-plus Martin boundary

Akian, SG, Walsh, CDC06, Doc. Math. 09 (Semigroup version), Ishii, Mitake 07 (PDE version).

Consider the eigenproblem over an arbitrary state space $S$

$$u_x = \sup_{y \in S} A_{xy} + u_y, \qquad \forall x \in S$$

The Martin kernel reads: $K_{xy} = A^*_{xy} - A^*_{by}$.
The Martin space $\mathcal{M}$ is the closure of $\mathcal{K} := \{K_{\cdot,y} \mid y \in S\}$ in the product topology (compact, Tychonoff). Martin boundary (set of horofunctions) is $\mathcal{B} = \mathcal{M} \setminus \mathcal{K}$.

When $A^*_{x,y} = -d(x,y)$ is the opposite of a metric, recover the construction of the horoboundary by Gromov.

# The detour metric

$$A^* = \text{“}I + A^+\text{”}, \qquad A^+ = \text{“}A + A^2 + A^3 + \ldots\text{”}$$

$$A^+_{xy} = \sup(A_{xy}, A^2_{xy}, A^3_{xy}, \ldots)$$

$$H^\flat_{xy} = A^+_{bx} + A^+_{xy} - A^+_{by} \quad \text{detour penalty}$$

Extend $H^\flat$ to the whole Martin space

$$H^\flat(u, v) = \limsup_{x_d \to u} \liminf_{y_e \to v} H^\flat_{x_d, y_e}$$

where the limsup,inf are taken along nets $x_d$ and $y_e$ converging to $u$ and $v$ in the topology of the Martin space.

The Minimal Martin space is
$\mathcal{M}^m := \{w \in \mathcal{M} \mid H^\flat(w,w) = 0\}$.

Theorem (Max-plus Martin representation Akian, SG, Walsh, CDC06, Doc. Math. 09)

*$\mathcal{M}^m$ is the set of extreme elements of $\{u \mid Au = u\}$. Any such $u$ can be written as*

$$u = \sup_{w\in\mathcal{M}_m} w + \mu(w), \qquad \mu : \mathcal{M}_m \to \mathbb{R}\cup\{-\infty\} \quad scs$$

$$\mu_u(w) := \limsup_{x_d\to w} A^*_{bx_d} + u(x_d)$$

Analogous to max-plus integral representations by Fathi, Siconolfi, Contreras, Ishii, Mitake, in different settings.

If the Martin space is metrisable, then $\mathcal{M}_m$ is precisely the set of Busemann points = limits of quasi-geodesics, i.e. of sequences $x_1, x_2, \ldots$ such that there exists $\alpha \in \mathbb{R}$

$$A^*_{bx_k} \leq A^*_{bx_1} + A_{x_1 x_2} + \cdots + A_{x_{k-1} x_k} + \alpha, \qquad \forall k$$

Quasi geodesics correspond to almost-sure trajectories of the renormalized H-process of Dynkin.

Lax-Oleinik (continuous time) version in CDC06.

# Linear quadratic control - nonquadratic solutions

Hamilton–Jacobi equation

$$\lambda = -|\mathbf{x}|^2 + \frac{1}{4}|\nabla w|^2$$

Maximise reward:

$$-\int_0^T (|\gamma(t)|^2 + |\dot{\gamma}(t)|^2 + \lambda)\, dt,$$

If $\lambda > 0$, solutions are

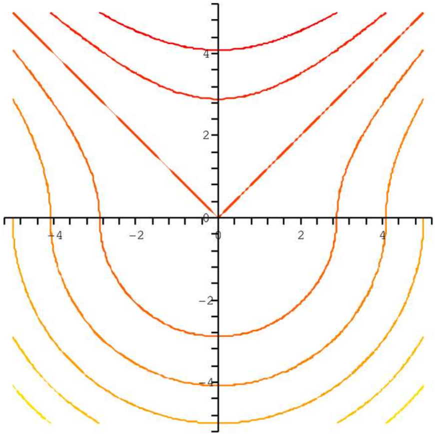
$$w(\mathbf{x}) = \sup_{\mathbf{n}} (\nu(\mathbf{n}) + h_{\mathbf{n}}(\mathbf{x})),$$

where $\nu$ is an upper semi–continuous map from the unit vectors to $\mathbb{R} \cup \{-\infty\}$.

When $\lambda = 0$, there is a horofunction for each direction $\mathbf{n}$:

$$h_{\mathbf{n}}(\mathbf{x}) = \begin{cases} -|\mathbf{x}|^2 + 2(\mathbf{x} \cdot \mathbf{n})^2, & \text{if } \mathbf{x} \cdot \mathbf{n} > 0, \\ -|\mathbf{x}|^2, & \text{otherwise.} \end{cases}$$

The function $-|\mathbf{x}|^2$ is also a horofunction.

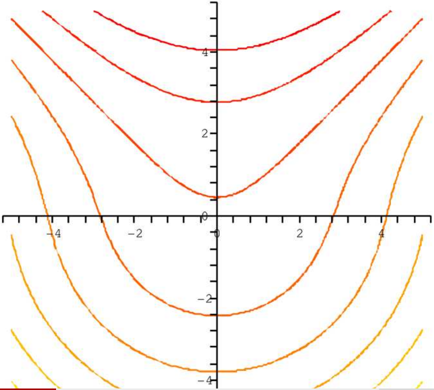



Horospheres of $h_n$ with $n = (0, 1)$.

When $\lambda > 0$: for each direction $\mathbf{n}$,

$$h_{\mathbf{n}}(\mathbf{x}) = -\lambda \frac{|\mathbf{x}|^2}{R^2} + \mathbf{x} \cdot \mathbf{n} \frac{\lambda + 2|\mathbf{x}|^2}{R} - \lambda \log \frac{R}{\sqrt{\lambda}},$$

where $R := \sqrt{(\mathbf{x} \cdot \mathbf{n})^2 + \lambda} - \mathbf{x} \cdot \mathbf{n}$.



Back to finite dimension.

The max-plus spectral problem as a limit of the Perron-Frobenius problem

# Deformation of the Perron root

Chain of spins (Ising)

$$Z = \sum_{\sigma_1,\dots,\sigma_n \in \Sigma^N} \exp(-\sum_{i=1}^{N} E(\sigma_i, \sigma_{i+1})/T), \qquad \sigma_{N+1} := \sigma_1$$

$-E(\sigma,\sigma') = H\sigma + J\sigma\sigma'$, $\sigma,\sigma' \in \{\pm 1\}$ (Ising)

$$Z_N = \operatorname{tr} M_T^N, \qquad (M_T)_{\sigma\sigma'} = \exp(-E(\sigma,\sigma')/T)$$

$F_N = N^{-1} T \log Z_N \sim T \log \rho(M_T)$ free energy per site,

$T \to 0$, ground state

$$\epsilon := \exp(-1/T), \qquad (M_T)_{\sigma,\sigma'} = \epsilon^{E(\sigma,\sigma')}$$

Similar to perturbation problems, but now, the "Puiseux series" have real exponents (Dirichlet series).

Kingman 61:

$$\log \circ \rho \circ \exp \qquad \text{convex [entrywise exp]}$$

Let $A, B \geq 0$, and $C = A^{(s)} \circ B^{(t)}$, with $s + t = 1, s, t \geq 0$ [entrywise product and exponent] then

$$\rho(C) \leq \rho(A)^s \rho(B)^t \quad .$$

Indeed, $\log \rho(C) = \lim_m \log \|C^m\|/m$ is a pointwise limit of convex functions of $(\log C_{ij})$, for any monotone norm. □

So

$$\rho(A \circ B) \leq \rho(A^{(p)})^{1/p} \rho(B^{(q)})^{1/q} \qquad 1/p + 1/q = 1$$

$$\rho(B^{(q)})^{1/q} \to \max_{i_1,\dots,i_m} (B_{i_1 i_2} \cdots B_{i_{m-1} i_m})^{1/m} =: \rho_\infty(B)$$

Theorem (Friedland 86)

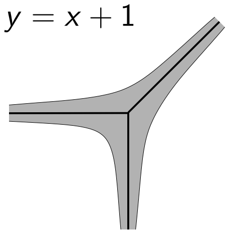
*For all $A \in \mathbb{C}^{n\times n}$,*

$$\rho(A) \leq \rho(pattern(A))\rho_\infty(|A|) \leq n\rho_\infty(|A|)$$

*and*

$$\rho(A) \geq \rho_\infty(A) \qquad \text{if } A_{ij} \geq 0 \; .$$

Explanation: approximation of an amoeba by its skeletton $V \subset (\mathbb{C}^*)^n$, $A(V) = \{(\log|z_1|, \dots, \log|z_n|) \mid x \in V\}$.




Cf. Gelfand, Kapranov, Zelevinsky; Passare, Rüllgaard; Purbhoo; Yger.

Limit of the Perron eigenvector. Consider $A^{(p)} = (A_{ij}^p)$, and let $U(p)$ denote the normalized Perron eigenvector of $A^{(p)}$.

Taking $p^{-1}\log$ / passing in the limit in

$$\lambda(p) U_i^p(p) = \sum_j A_{ij}^p U_j^p$$

we get that

$$\lambda + u_i = \max_j \log A_{ij} + u_j$$

where $\lambda$ and $u_j$ are accumulation points of $p^{-1}\log\lambda(p)$, $\log U_j(p)$, resp.

Which tropical eigenvector is selected?

WLOG $\lambda = \log \rho_\infty(A) = 0$.

**Theorem** (Akian, Bapat, SG CRAS 1998)

*If there is only one SCC of the critical graph with maximal Perron root, then $u_i = (\log A)^*_{ij}$, for any $j$ in this class.*

Related work by Lopes, Mohr, Souza, Thieullen.

Give example at the blackboard.

Proof idea. Make diagonal scaling

$$B(p) = diag(\exp(-pu))A^p \,\mathrm{diag}(\exp(pu)) \quad .$$

The matrix $B(p)$ has a limit in $[0,1]^{n\times n}$ as $p \to \infty$.
We want $B(\infty)$ to have a positive eigenvector. A nonnegative matrix has a positive eigenvector iff the basic classes are exactly the final classes.
For the choice of eigenvector $u = (\log A)^*_{\cdot j}$, this is the case, because the saturation graph

$$\{(k,l) \mid \log A_{kl} + u_l = u_k\}$$

is a river network with sea SCC$(j)$. Make drawing.

# An application: perturbation of eigenvalues

Exercise.

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

# An application: perturbation of eigenvalues

Exercise.

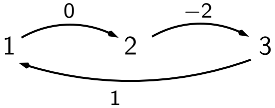
$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} \quad ,$$

Show without computation that the eigenvalues have the following asymptotics as $\epsilon \to 0$

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} , \qquad A = \begin{bmatrix} 1 & 0 & 4 \\ \infty & 1 & -2 \\ 1 & 2 & \infty \end{bmatrix} .$$

We have $\gamma_1 = -1/3$, corresponding to the critical circuit:




Eigenvalues:

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

Assume that the entries of $\mathcal{A}_\varepsilon$ have Puiseux series expansions in $\epsilon$, or even that $\mathcal{A}_\varepsilon = a + \epsilon b$, $a, b \in \mathbb{C}^{n\times n}$.

$\mathcal{L}_1, \dots, \mathcal{L}_n$ eigenvalues of $\mathcal{A}_\varepsilon$.

$v(s)$: opposite of the smallest exponent of a Puiseux series $s$.

$\gamma_1 \geq \cdots \geq \gamma_n$: tropical eigenvalues of $v(A_\epsilon)$.

Theorem (Akian, Bapat, SG CRAS04, arXiv:0402090)

$$v(\mathcal{L}_1) + \cdots + v(\mathcal{L}_n) \leq \gamma_1 + \cdots + \gamma_n$$

*and equality holds under generic (Lidski-type) conditions.*

The maximal tropical eigenvalue $\gamma_1$ coincides with the ergodic constant of the one-player game

$$\lambda + u_i = \max_{1 \leq j \leq n} \big( \mathrm{val}(A_\epsilon)_{ij} + u_j \big), \forall i$$

$\lambda$ is the maximal circuit mean.

In general, tropical eigenvalues are non-differentiability points of a parametric optimal assignment problem = Legendre transform a the generic Newton polygon

The (algebraic) tropical eigenvalues of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ are the roots of

$$\text{“}\operatorname{per}(A + xI)\text{”}$$

where

$$\text{“}\operatorname{per}(M)\text{”} := \text{“}\sum_{\sigma\in S_n} \prod_{i\in[n]} M_{i\sigma(i)}\text{”}$$

All geom. eigenvalues $\lambda$ (“$Au = \lambda u$”) are algebraic eigenvalues, but the converse does not hold. $\rho_{|max}(A)$ is the max algebraic eigenvalue.

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- Trop. eigs. can be computed in $O(n)$ calls to an optimal assignment solver (Butkovič and Burkard) (not known whether the formal characteristic polynomial can be computed in polynomial time).

## Theorem (Kapranov)

*If $f(z) = \sum_k f_k z^k \in \mathbb{C}\{\{\epsilon\}\}[z_1, \ldots, z_n]$, the closure of the image of $f = 0$ by $v$ is the set of points $x \in \mathbb{R}^n$ at which the maximum*

$$\max_k v(f_k) + \langle k, x \rangle$$

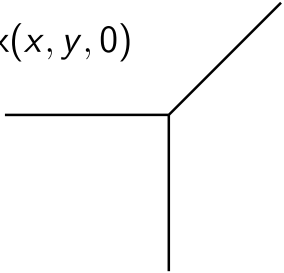
*is attained at least twice.*

Follows from Puiseux theorem when $n = 1$. Inclusion $\subset$ obvious. Converse: reduction to Puiseux.

When $n = 1$: the set of tropical roots is a zero-dimensional amoeba

Example. $y = x + 1$, $K = \mathbb{C}\{\{\epsilon\}\}$

$\max(x, y, 0)$



# Algorithms for games

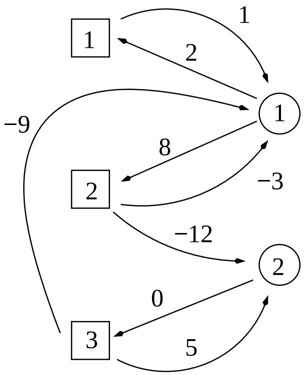
$$H_i : \max_{1\le j\le n} a_{ij} + x_j \le \max_{1\le k\le n} b_{ik} + x_k$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\le k\le n} b_{ik} + x_k \quad .$$

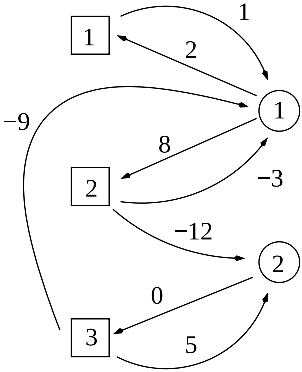
## Interpretation of the game

- State of MIN: variable $x_j$, $j \in \{1, \ldots, n\}$
- State of MAX: half-space $H_i$, $i \in I$
- In state $x_j$, Player MIN chooses a tropical half-space $H_i$ with $x_j$ in the LHS
- In state $H_i$, player MAX chooses a variable $x_k$ at the RHS of $H_i$
- Payment $-a_{ij} + b_{ik}$.

$$A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}$$



$$
\begin{aligned}
2 + x_1 &\leq 1 + x_1 \\
8 + x_1 &\leq \max(-3 + x_1, -12 + x_2) \\
x_2 &\leq \max(-9 + x_1, 5 + x_2)
\end{aligned}
$$




$$\chi(T) = \lim_k v^k/k = (-1, 5)$$

## Proposition

*If $T$ is nonexpansive and piecewise affine $\mathbb{R}^n \to \mathbb{R}^n$, the discounted value $v_\alpha = T(\alpha v_\alpha)$ has a Laurent series expansion*

$$v_\alpha = \frac{a_{-1}}{1-\alpha} + a_0 + (1-\alpha)a_1 + \dots, a_i \in \mathbb{R}^n$$

This is the case for a stochastic game with perfect information and finite action spaces.

Then

$$\chi(T) = \lim_k T^k(0)/k = a_{-1} \ .$$

- Strategy of MAX $\sigma : \{H_1, \dots, H_m\} \to \{x_1, \dots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$

Duality theorem (coro of Kohlberg) If finite action spaces, then

$$\chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T_{\pi}) \enspace .$$

- Strategy of MAX $\sigma : \{H_1, \dots, H_m\} \to \{x_1, \dots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$
- Strategy of MIN $\pi : \{1, \dots, n\} \to \{1, \dots, m\}$, in state $x_j$ choose hyperplane $H_{\pi(j)}$

Duality theorem (coro of Kohlberg) If finite action spaces, then

$$\chi(T) = \max_{\sigma} \chi(T^{\sigma}) = \min_{\pi} \chi(T_{\pi}) \ .$$

- Strategy of MAX $\sigma : \{H_1, \dots, H_m\} \to \{x_1, \dots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$
- Strategy of MIN $\pi : \{1, \dots, n\} \to \{1, \dots, m\}$, in state $x_j$ choose hyperplane $H_{\pi(j)}$
- One player Shapley operators

$$[T^\sigma(x)]_j = \inf_{1 \le i \le m} -a_{ij} + b_{i\sigma(i)} + x_{\sigma(i)} \quad .$$

$$[T_\pi(x)]_j = -a_{\pi(j)j} + \max_{1 \le k \le n} b_{\pi(j)k} + x_k \quad .$$

Duality theorem (coro of Kohlberg) If finite action spaces, then

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi) \quad .$$

- Strategy of MAX $\sigma : \{H_1, \ldots, H_m\} \to \{x_1, \ldots, x_n\}$, in state $H_i$ choose coordinate $x_{\sigma(i)}$
- Strategy of MIN $\pi : \{1, \ldots, n\} \to \{1, \ldots, m\}$, in state $x_j$ choose hyperplane $H_{\pi(j)}$
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$$[T_\pi(x)]_j = -a_{\pi(j)j} + \max_{1 \le k \le n} b_{\pi(j)k} + x_k \enspace .$$

Duality theorem (coro of Kohlberg) If finite action spaces, then

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi) \enspace .$$

Every $\chi(T^\sigma)$ and $\chi(T_\pi)$ can be computed in polynomial time.

# Proof: Blackwell optimality

For all $x \in \mathbb{R}^n$, we have a selection

$$\exists \sigma, \pi, \; T(x) = T^\sigma(x) = T_\pi(x) \; .$$

So for all $0 < \alpha < 1$, the discounted value $v_\alpha = T(\alpha v_\alpha)$ satsifies

$$v_\alpha(T) = \max_\sigma v_\alpha(T^\sigma) = \min_\pi v_\alpha(T_\pi) \; .$$

Since $\chi$ is the first coefficient of the Laurent series

$$\chi(T) = \max_\sigma \chi(T^\sigma) = \min_\pi \chi(T_\pi) \; .$$

$\sigma, \pi$ are Blackwell optimal if optimal for all $\alpha \in (\bar{\alpha}, 1)$ (exist because the zeros of a Laurent series cant accumulate at $1^-$).

**Corollary (Condon 92, Zwick and Paterson, TCS 96)**

*Mean payoff games are in NP $\cap$ co-NP.*

- I can convince you that $\chi_i(T) \geq 0$ (initial state $i$ is winning) by giving you a strategy $\sigma$ of MAX such that $\chi_i(T^\sigma) \geq 0$. You can check that in polynomial time by solving a one player game.
- I can convince you that the opposite is true by giving you a strategy $\pi$ of MIN such that $\chi_i(T_\pi) < 0$. You can also check this in polynomial time.

The class NP $\cap$ co-NP captures the good characterizations of Edmonds. Evidence that the problem is not NP-complete.

- "$Ax \leq Bx$" unfeasible iff $\exists \pi$, $\overline{\chi}(T_\pi) < 0$.

- "$Ax \leq Bx$" unfeasible iff $\exists \pi$, $\overline{\chi}(T_\pi) < 0$.
- "$Ax \leq Bx$" feasible iff $\exists \sigma$, $\overline{\chi}(T^\sigma) \geq 0$.

- “$Ax \leq Bx$” unfeasible iff $\exists \pi$, $\overline{\chi}(T_\pi) < 0$.
- “$Ax \leq Bx$” feasible iff $\exists \sigma$, $\overline{\chi}(T^\sigma) \geq 0$.
- $\exists x \in \mathbb{R}^n_{\max}$, $Ax \leq Bx$? is in NP $\cap$ co-NP

### Corollary

*Feasibiliby in tropical linear programming, i.e.,*

$$\exists ? u \in (\mathbb{R} \cup \{-\infty\})^n, \ \max_j a_{ij} + u_j \leq \max_j b_{ij} + u_j, \ 1 \leq i \leq p$$

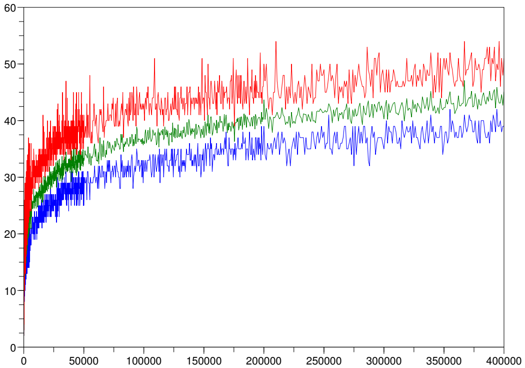
*is polynomial-time equivalent to mean payoff games.*

are in NP $\cap$ coNP: Zwick, Paterson 96.

Tropical convex sets are log-limits of classical convex sets: polynomial time solvability of mean payoff games might follow from a strongly polynomial-time algorithm in linear programming (Schewe).

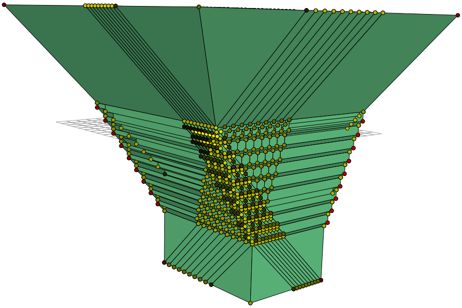
Several pseudo-polynomial algorithms exist for (deterministic) mean payoff games: Zwick, Paterson TCS96. No pseudo-polynomial algorithm seems to be known for stochastic mean payoff game. However, Policy iteration works (Cochet,SG 06), - based on a tropical idea = spectral projectors - ; alternative algorithm by Boros, Gurvich, Elbassioni, Makino, . . .

Policy iteration for games scales well in practice. ♯ iterations / ♯ nodes




However, Friedmann LICS 10 showed that policy iteration for games can be exponential.

Intersection of 10 affine tropical hyperplanes in dimension 3, only 24 vertices, but 1215 pseudo-vertices.



Tropical double description Allamigeon, SG, Goubault.
Efficient implementation in TPLib/caml by Allamigeon.

# Concluding remarks

- Tropical algebra $\sim$ discrete version of Weak KAM
- Tropical convex cones arises when considering spaces of weak KAM solutions (1-player), or sub/super solutions.
- Combinatorial properties in the discrete case (lenghts of periodic orbits)
- Thinking tropical brings "complex" perspective on Lax-Oleinik semigroups (not just one eigenvalue)
- Relation between ergodic problem and optimal assignment appears in the discrete case (the eigenvalues are nondifferentiability points of an optimal assignment problem), is there a PDE analogue (relation with mass transport problem)?
- Tropical algebra is fun!

Thank you