Weak asymptotics of the Spectral Shift Function in strong constant magnetic field

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We consider the three-dimensional Schrödinger operator with constant magnetic field of strength $b > 0$, and with smooth electric potential. The weak asymptotics of the Spectral Shift Function with respect to $b \nearrow +\infty$ is studied. First, we fix the distance to the Landau levels, then the distance to Landau levels tends to infinity as $b \nearrow +\infty$. In particular we give explicitly the leading terms in the asymptotics and in some case we obtain full asymptotics expansions.

1 Introduction

The three-dimensional Schr¨odinger operator with electric and constant magnetic fi elds can be written as:

$$H_1(b) = (D_x + \frac{b}{2}y)^2 + (D_y - \frac{b}{2}x)^2 + D_z^2 + V(x,y,z), \quad D_\nu = -i\frac{\partial}{\partial \nu}$$
$$= H_0(b) + V(x,y,z),$$

where b represents the strength of the magnetic fi elds. We assume that $V \in C^\infty(\mathbf{R}^3;\mathbf{R})$ and there exists $p > 2$, $s > 1$ such that

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma V(x,y,z)| \leq C_{\alpha,\beta,\gamma} \langle x,y\rangle^{-p} \langle z\rangle^{-s}. \tag{1.1}$$

Here $\langle X\rangle = (1+|X|^2)^{1/2}$.

It is well known [1], [15] that the spectrum of $H_0(b)$ is absolutely continuous, equals to $[b,+\infty[$, and has an infi nite set of thresholds $b\Lambda_q = b(2q-1)$, $q \geq 1$ (called Landau levels). By the Weyl criterion the essential spectrum of $H_1(b)$ and $H_0(b)$ are the same.

There are many papers dealing with different aspects of the spectral theory of $H_1(b)$. In particular, the asymptotics of the counting function of the number of eigenvalues of $H_1(b)$ in the gap $]-\infty, b[$ have been studied by many authors in different aspects. We refer to [1],[18], [19],[16], [15], [26], [10], [20] and the references given there.

The assumption (1.1) implies that the spectral shift function (SSF for short) $\xi(\lambda, b)$ related to $H_1(b)$ and $H_0(b)$ is well defi ned in the sense of distribution :

$$\langle \xi'(\cdot,b), f(\cdot)\rangle := \mathrm{tr}\Big(f(H_1(b)) - f(H_0(b))\Big), \ \ f \in C_0^\infty(\mathbf{R}). \tag{1.2}$$

The SSF may be considered as a generalization of the eigenvalues counting function. Under suitable assumptions it can be identifi ed with the scattering phase. For more details, we refer to [27] or to the survey paper [22].

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For continuous properties of ξ away from Landau levels, we refer to [7]. Let us remark that according to Theorem 5.1 of [1], the operator H could have embedded eigenvalues and then the derivative of the SSF could be locally a Dirac distribution.

Recently a substantial progress has been given in the analysis of the spectral shift function and the works around trace formula. Many results on the upper and lower bounds of resonances can be obtained easily by proving an asymptotic expansion of the right hand side of (1.2) and combining it with a representation of the derivative of the SSF related to the resonances [5], [23], [24], [6], [12], [4], [25].... For this reason, it is natural to study the asymptotic behavior of the right hand side of (1.2) as $b \to +\infty$.

Since the distance between two Landau levels grows to infinity as $b \to \infty$ and since the external potential is uniformly bounded with respect to b, the effect caused by the potential V will be located near the Landau levels.

To simplify the notation and the exposition of this paper we only consider the first Landau level $b\Lambda_1 = b$ and we refer to Remark 1.4 for the other Landau levels $b\Lambda_q$, $q \geq 2$. We should say that throughout this article we have opted for ease of exposition over generality.

Our first result is the following :

Theorem 1.1 *Fix δ in $]0, +\infty[$, and let $b \mapsto \kappa(b)$ be a non-negative bounded function. We assume that either, $\kappa(b)b \to \infty$ when b tends to infinity or $\kappa(b) = \frac{\delta}{2b}$. For $f \in C_0^\infty(] - \delta, +\delta[; \mathbf{R})$, there exists $b_0 > 0$ such that for all $b > b_0$, we have*

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(f(\kappa(b)(H_1(b)-b)) - f(\kappa(b)(H_0(b)-b))\Big) = \tag{1.3}$$

$$\mathrm{tr}_{L^2(\mathbf{R}^2_{y,z})}\Big(f(\kappa(b)Q(b^{-1})) - f(\kappa(b)D_z^2)\Big) + \mathcal{O}_f(b^{-1}\kappa(b)^{\frac{1}{2}})$$

where

$$Q(b^{-1}) := D_z^2 + V^w(b^{-1}D_y, y, z) + \frac{b^{-1}}{4}(\Delta_{x,y}V)^w(b^{-1}D_y, y, z). \tag{1.4}$$

Here and in the following, tr_E denote the trace on $\mathcal{L}(E)$. For a symbol a, a^w denote the Weyl quantization (see Section 2 and Appendix).

Using Theorem 1.1 and the b^{-1}-pseudodifferential calculus, we prove in section 3 the following asymptotics :

Theorem 1.2 *(Asymptotics near $b\Lambda_1$). Let $f \in C_0^\infty(\mathbf{R})$. We assume that the support of f is independent of b. Under the assumption (1.1), the following asymptotic holds :*

$$\langle \xi'(\cdot + b, b), f(\cdot)\rangle = \mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(f(H_1(b) - b) - f(H_0(b) - b)\Big) \tag{1.5}$$

$$= b\gamma_1(f) + \gamma_0(f) + \mathcal{O}(b^{-1}), \quad b \to +\infty,$$

with

$$\gamma_1(f) = \frac{1}{2\pi}\int\int_{\mathbf{R}^2} \mathrm{tr}_{L^2(\mathbf{R}_z)}\Big(f(D_z^2 + V(x,y,z)) - f(D_z^2)\Big)dxdy,$$

$$\gamma_0(f) = \frac{1}{8\pi}\int\int_{\mathbf{R}^2} \mathrm{tr}_{L^2(\mathbf{R}_z)}\Big(\Delta_{x,y}V(x,y,z)f'(D_z^2 + V(x,y,z))\Big)dxdy +$$

$$\frac{1}{2\pi}\int\int_{\mathbf{R}^2} \mathrm{tr}_{L^2(\mathbf{R}_z)}\Big(T(\partial_x V, \partial_y V) - T(\partial_y V, \partial_x V)\Big)(x,y)dxdy,$$

where T is the operator valued function of two operators defined by (3.5).

Theorem 1.3 *(Asymptotics away from $b\Lambda_1$). Let f and $\kappa(b)$ be as in Theorem 1.1 and Theorem 1.2. In addition, we assume that $\kappa(b) \to 0$ as b tends to infinity. We have the following asymptotics expansion: for all $M, N \in \mathbf{N}$*

$$\frac{1}{\kappa(b)}\langle \xi'(\frac{\cdot}{\kappa(b)} + b, b), f(\cdot)\rangle = \mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(f(\kappa(b)(H_1(b)-b)) - f(\kappa(b)(H_0(b)-b))\Big) =$$

$$b\sum_{k=0}^{N}\kappa(b)^{\frac{1}{2}+k}\beta_k(f)+\sum_{k=0}^{M}\kappa(b)^{\frac{1}{2}+k}\alpha_k(f)+\mathcal{O}(\min(b^{-1}\kappa(b)^{\frac{1}{2}},b\kappa(b)^{N+1},\kappa(b)^{M+1})),\quad b\to+\infty, \tag{1.6}$$

with

$$\beta_0(f)=\frac{2}{(2\pi)^2}\int\int\int_{\mathbf{R}^3}V(x,y,z)dxdydz\ \times\ \int_0^\infty f'(r^2)dr.$$

Notice that, in the case where $b^{-1}=\mathcal{O}(\kappa(b)^\infty)$, Theorem 1.3 gives a full asymptotic expansion in powers of $\kappa(b)$ given by the first sum in (1.6).

Let us introduce the (x,y)-dependent spectral shift function $\xi_{x,y}(\lambda)$ corresponding to the pair $(-\Delta_z+V(x,y,z),-\Delta_z)$. Then $\gamma_1(f)$ can be expressed in term of $\xi'_{x,y}(\lambda)$:

$$\gamma_1(f)=(2\pi)^{-1}\int\int\langle\xi'_{x,y}(.),f\rangle\,dxdy.$$

Let f be in $C_0^\infty(]0,\delta[)$. By a change of variables, we have:

$$\beta_0(f)=\frac{1}{2(2\pi)^2}\int_{\mathbf{R}}f(t)t^{-\frac{3}{2}}dt\ \int\int\int_{\mathbf{R}^3}V(x,y,z)dxdydz.$$

The asymptotics (1.5) and (1.6) tell us that the following asymptotics hold in $\mathcal{D}'$ as $b\to\infty$:

$$(2\pi b^{-1})\xi'(t+b,b)\to\int\int\xi'_{x,y}(t)\,dxdy,$$

$$\frac{1}{b}(\frac{t}{\kappa(b)})^{3/2}\xi'(\frac{t}{\kappa(b)}+b,b)\to(2(2\pi)^2)^{-1}\int\int\int_{\mathbf{R}^3}V(x,y,z)dxdydz.$$

To our best knowledge, there is only two results concerning the spectral shift function corresponding to $(H_1(b),H_0(b))$ for large b. The first one is due to [7], and the second is the recent paper of L. Michel [17]. In [7], the first term of the asymptotic (in the strong sense) of

$$\Delta_1\ni\lambda\to\xi(\lambda+b\Lambda_q+\mathcal{E}b,b)$$

is given for $\mathcal{E}=0$ (*near the Landau levels*) or $\mathcal{E}>0$ (*far from the Landau levels*) without remainder estimate (under weaker assumption on V). Here Δ_1 is some compact (b-independent) interval in $(0,+\infty)$. In [17], the Schr¨odinger operator with constant magnetic field is considered in all dimension. For energy far from Landau levels and under weak assumptions on V, L. Michel [17] obtains the first term of the asymptotic of the scattering amplitude and complete asymptotic expansion of the scattering phase (which can be identified with the SSF).

For b fixed, the behavior of the SSF is also studied into two directions: In [13], the behavior of $\xi(\lambda+b\Lambda_q)$ is considered as $\lambda\to 0$ and in [8] the high energy asymptotics is discussed.

In the 2D dimensional case, a Weyl type asymptotic with optimal remainder estimate for the counting function of eigenvalues was obtained in [10]. In our case the situation is more complicated, since our operator are $\mathcal{L}(L^2(\mathbf{R}_z))$-valued symbol. This question will be treated elsewhere. On the other hand, for the weak asymptotics, our proofs are very simple. We don't need the construction of Grushin problem as in [10].

Remark 1.4 Similar techniques could be applied near other Landau levels. In this case we prove the same asymptotics expansions and the leading terms of the weak asymptotics *near* $b\Lambda_q$ and *away from* $b\Lambda_q$ are independent of q.

The paper is organized as follows: In the next section, we introduce some notations and prove Theorem 1.1. In section 3, we develop some b^{-1}-pseudodifferential calculus and prove Theorem 1.2 and Theorem 1.3. In an appendix, we recall some results on h-pseudodifferential calculus with operator valued symbol.

2 The strong field reduction

Throughout this section we assume that the assumptions of Theorem 1.1 are satisfied and we prove Theorem 1.1.

In this work we will use the standard notations for symbols and pseudodifferential operators (see [21], [11], [15]).

Let $m:\ \mathbf{R}^{2d}\to[0,\infty[$ be an order function (see Definition 7.5 in [11]), we define the class of semi-classical symbols on $T^*\mathbf{R}^d=\mathbf{R}^{2d}$:

$$S(R^{2d},m)=\{a(x,\xi;h)\in C^\infty(\mathbf{R}^{2d}\times]0,1]);\ \forall\alpha,\beta\in\mathbf{N}^d,\ \exists\ C_{\alpha,\beta},$$
$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi;h)|\le C_{\alpha,\beta}\,m(x,\xi)\}. \tag{2.1}$$

In the special case when $m=1$ we will write $S(\mathbf{R}^{2d})$ instead of $S(\mathbf{R}^{2d};1)$.

If $a=a(x,\xi;\lambda,h)$ depends also on some parameter $\lambda\in\Omega$, we say that $a\in S(\mathbf{R}^{2d},m)$, if the constant $C_{\alpha,\beta}$ in (2.1) is independent of $\lambda\in\Omega$.

When the symbol a is an operator in $\mathcal{L}(K,L)$, (K, L being subspaces of a Hilbert space), $|.|$ denote the norm in $\mathcal{L}(K,L)$ and the associated class of symbols will be denoted by $S(\mathbf{R}^{2d},m;\mathcal{L}(K,L))$.

Let $a(x,\xi;h)\in S(\mathbf{R}^{2d},m)$. We say that $a(x,\xi;h)$ has an asymptotic expansion in powers of h in $S(\mathbf{R}^{2d},m)$, and we write

$$a(x,\xi;h)\sim\sum_{j=0}^{\infty}a_j(x,\xi)h^j \text{ in } S(\mathbf{R}^{2d},m),$$

if for every $N\in\mathbf{N}$, $h^{-(N+1)}(a-\sum_{j=0}^N a_jh^j)\in S(\mathbf{R}^{2d},m)$.

We will use the standard Weyl quantization of symbols. More precisely, if $P(x,\xi)$ is a function on $T^*\mathbf{R}^d$ satisfying suitable estimates, $P^w(y,D_y)$ is the operator defined by

$$P^w(y,D_y)u(y)=(2\pi)^{-n}\int\int_{\mathbf{R}^{2d}}e^{i(y-y')\cdot\eta}P(\frac{y+y'}{2},\eta)u(y')dy'd\eta,$$

for $u\in\mathcal{S}(\mathbf{R}^d)$, the class of rapidly decreasing functions. Sometimes we will quantize a function $P(x,y,\xi,\eta)$ only with respect to the variable (y,η): in this case we will denote by $P^w(x,y,\xi,D_y)$ the operator obtained as above by considering (x,ξ) as a parameter.

Finally, when $P(x,\xi)$ is a function on $T^*\mathbf{R}^d$ (possibly operator valued), we denote by $P^w(x,hD_x)$ the semi-classical quantization obtained as above by quantizing $P(x,h\xi)$. In an appendix, we recall some results on h-pseudodifferential calculus.

By using a symplectic change of variables (see [14], [15]), we have :

Proposition 2.1 *There exits a unitary operator $\mathcal{U}\in\mathcal{L}(L^2(\mathbf{R}^3))$ such that:*

$$\mathcal{U}H_0\mathcal{U}^*=\tilde H_0+b,\qquad \mathcal{U}H_1\mathcal{U}^*=\tilde H_1+b$$

where

$$\tilde H_0=b(D_x^2+x^2)\otimes I_{yz}+I_{xy}\otimes D_z^2-bI_{xyz}, \tag{2.2}$$
$$\tilde H_1=\tilde H_0+V^w(b^{-\frac12}D_x+b^{-1}D_y,y-b^{-\frac12}x,z), \tag{2.3}$$

are the self-adjoint operators with domain $\mathcal{D}:=B^2(\mathbf{R}_x)\otimes L^2(\mathbf{R}_y)\otimes H^2(\mathbf{R}_z)$, $B^2(\mathbf{R}_x)$ being the domain of the Harmonic Oscillator $(D_x^2+x^2)$.

For the simplicity of the notation we will note V^w instead of $V^w(b^{-\frac12}D_x+b^{-1}D_y,y-b^{-\frac12}x,z)$. Let f_n, $n\in\mathbf{N}^*$ be the n-th normalized Hermite function:

$$(D_x^2+x^2)f_n=\Lambda_nf_n,\quad \Lambda_n=2n-1\quad \|f_n\|_{L^2}=1.$$

We then introduce the following operators :

$$\Pi\ \ :L^2(\mathbf{R}^3)\to L^2(\mathbf{R}^3),v\mapsto\langle v(.,y,z),f_1\rangle f_1(x),$$

where $\langle .,. \rangle$ denotes the scalar product on $L^2(\mathbf{R}_x)$.

Obviously,

$$\Pi \circ \Pi = \Pi.$$

From now on, we denote :

$$\widehat{\Pi} := 1 - \Pi, \quad \widehat{H}_j = \widehat{\Pi}\tilde{H}_j\widehat{\Pi}, \qquad j = 0, 1.$$

The following lemma is a simple consequence of the spectral theorem, the fact that V^w is uniformly bounded with respect to b as well as the fact that $\sigma\left(\widehat{H}_0\right) = [2b, +\infty[$.

Lemma 2.2 *Fix δ in $]0, +\infty[$, and let $b \mapsto \kappa(b) =: \kappa$ be a non-negative bounded function. We assume that either, $\kappa b \to \infty$ when b tends to infinity or $\kappa = \frac{\delta}{2b}$. There exists $b_0 > 0$, such that for all $b > b_0$, the operators*

$$E_j(\lambda, \kappa) := \left(\lambda - \kappa\widehat{H}_j\right)^{-1}\widehat{\Pi}, \qquad j = 0, 1, \tag{2.4}$$

are well defined, holomorphic for $\lambda \in D(0, \delta) := \{z \in \mathbf{C};\ |z| < \delta\}$, and

$$\|E_j(\lambda, \kappa)\| = \mathcal{O}((\kappa b)^{-1}), \tag{2.5}$$

uniformly with respect to $\lambda \in D(0, \delta - \eta)$ and $b \in [b_0, +\infty[$. Here η is some constant in $]0, \delta[$.

Lemma 2.3 *Under the assumptions of Lemma 2.2, there exists $c_0 > 0$ and $b_0 > 0$, such that for all $b > b_0$,*

$$\|[\Pi, V^w]\langle z \rangle^s\| = \mathcal{O}(b^{-1/2}), \tag{2.6}$$

and

$$(c_0 b + D_z^2)E_j(\lambda, \kappa) = \mathcal{O}(\kappa^{-1}), \tag{2.7}$$

uniformly with respect to $\lambda \in D(0, \delta - \eta)$.
Moreover, the operator $[\Pi, V^w]E_j(\lambda, \kappa)$ is trace class one, and

$$\|[\Pi, \kappa V^w]E_j(\lambda, \kappa)\|_{\mathrm{tr}} = \mathcal{O}(1), \tag{2.8}$$

uniformly with respect to $\lambda \in D(0, \delta - \eta)$. Here $[.,.]$ denotes the commutator: $[A, B] := AB - BA$.

Proof. First, we claim that

$$b^{1/2}\,[\Pi, V^w] \in \mathrm{Op}^w\Big(S(\mathbf{R}^6; \langle (b^{-1}\eta, y)\rangle^{-p}\langle x\rangle^{-\infty}\langle \xi\rangle^{-\infty}\langle z\rangle^{-s})\Big).$$

Here, by $a(x) = \langle x\rangle^{-\infty}$ we mean $a(x) = \langle x\rangle^{-q}$ for all $q \in \mathbf{N}$.

To see this, choose a function $f \in C_0^\infty(]0, 3[; [0, 1])$ such that $f(x) = 1$ near $x = 1$. Obviously, $f(D_x^2 + x^2) \otimes I_{yz} = \Pi$, and according to Theorem 8 in [11], we have

$$\Pi \in \mathrm{Op}^w\Big(S(\mathbf{R}^6; \langle x\rangle^{-\infty}\langle \xi\rangle^{-\infty})\Big).$$

On the other hand, from (1.1) and the expression of V^w, we have

$$V^w \in \mathrm{Op}^w\Big(S(\mathbf{R}^6; \langle (b^{-\frac{1}{2}}\xi + b^{-1}\eta, y - b^{-\frac{1}{2}}x)\rangle^{-p}\langle z\rangle^{-s})\Big).$$

By observing that

$$\langle (b^{-\frac{1}{2}}\xi + b^{-1}\eta, y - b^{-\frac{1}{2}}x)\rangle^{-p} \le C\langle (b^{-1}\eta, y)\rangle^{-p}\langle b^{-\frac{1}{2}}x\rangle^{p}\langle b^{-\frac{1}{2}}\xi\rangle^{p} \le C\langle (b^{-1}\eta, y)\rangle^{-p}\langle x\rangle^{p}\langle \xi\rangle^{p},$$

the claim follows from the composition formula of pseudodifferential operators (see Theorem A1) and the assumption (1.1). Notice that, the term $b^{1/2}$ comes from the expression of V^w and the fact that we are working with a commutator operator.

Now applying Theorem A2 to the claim we obtain (2.6).

Since $\sigma(\widehat{H}_0) \subset [2b, +\infty[$, it follows from the assumption on $\kappa(b)$ that : there exists $c_0 > 0$ and $b_0 >> 1$ such that for $b > b_0$,

$$\Re\langle\big(\kappa D_z^2 + b\kappa(D_x^2 + x^2) - \kappa b - \Re\lambda\big)\widehat{\Pi}u, \widehat{\Pi}u\rangle \geq \kappa\,\langle\big(D_z^2 + c_0 b\big)\widehat{\Pi}u, \widehat{\Pi}u\rangle.$$

From this we deduce (2.7) for $j = 0$. The case $j = 1$ follows by using resolvent equation, estimate (2.5) and that $\|V^w\|$ is uniformly bounded with respect to b.

Notice that, for $s > 1$ the operator

$$\langle z\rangle^{-s}(c_0 b + D_z^2)^{-1} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$$

is trace class one and $\|\langle z\rangle^{-s}(c_0 b + D_z^2)^{-1}\|_{\mathrm{tr}} = \mathcal{O}(b^{-1/2})$. Combining this with the claim and using (2.6), (2.7) as well as Theorem A3 in the Appendix we get (2.8). □

Lemma 2.4 *Assume that V satisfies (1.1). For all $N \in \mathbf{N}$ there exists a b^{-1}-pseudodifferential operator (independent of x),*

$$Q_N(y, b^{-1}D_y, z) = \sum_{i=0}^{N} q_i^w(y, b^{-1}D_y, z)b^{-i},$$

such that

$$\Pi V^w \Pi = \Pi Q_N(y, b^{-1}D_y, z)\Pi + b^{-(N+1)}R_N(b),$$

with

$$\sup_{b\in[1,+\infty[}\|R_N(b)\| < +\infty, \qquad \sup_{b\in[1,+\infty[}\|R_N(b)(\kappa D_z^2 + 1)^{-1}\|_{\mathrm{tr}} = \mathcal{O}(b\kappa^{-\frac{1}{2}}).$$

Moreover,

$$q_0(y, \eta, z) = V(\eta, y, z), \quad q_1(y, \eta, z) = \frac{1}{4}(\Delta_{x,y}V)(\eta, y, z).$$

P r o o f. Notice that, we can view V^w as a b^{-1}-pseudodifferential operator on (y, z) with $\mathcal{L}(L^2(\mathbf{R}_x))$-operator valued symbol. Its symbol is given by :

$$\widehat{V}(y, \eta) := V^w(\eta + b^{-1/2}D_x, y - b^{-1/2}x, z).$$

On the other hand, by Taylor's formula (see also the proof of Proposition 2.5 in [10]), we can write

$$\widehat{V}(y, \eta) = V(\eta, y, z) + b^{-1/2}\partial_\eta V(\eta, y, z) \cdot D_x - b^{-1/2}\partial_y V(\eta, y, z) \cdot x + ...$$

Combining this with the fact that

$$\langle x^l \partial_x^m f_j, f_j\rangle_{L^2(\mathbf{R}_x)} = 0, \quad \text{for } \ l + m + 1 \in 2\mathbf{N},$$

$$\langle D_x^2 f_j, f_j\rangle_{L^2(\mathbf{R})} = \langle x^2 f_j, f_j\rangle_{L^2(\mathbf{R})} = \frac{2j-1}{2},$$

as well as the fact that $\partial_x^m f_j(x) = \mathcal{O}(e^{-x^2/3})$ for any $m \in \mathbf{N}$, we get the lemma. The trace class estimate is obtained repeating the arguments of the proof of (2.8). □

Lemma 2.5 *The following estimate holds uniformly for $\lambda \in D(0, \delta)$*

$$\|\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\widehat{\Pi} - E_1(\lambda, \kappa)\|_{\mathrm{tr}} = \mathcal{O}(b^{-3/2}|\Im\lambda|^{-4}). \tag{2.9}$$

P r o o f. Making use of the fact that $\widehat{\Pi}\Pi = 0$, $\Pi^2 = \Pi$, $[\widehat{\Pi}, \tilde{H}_0] = 0$ and $\widehat{\Pi}^2 = \widehat{\Pi}$, we obtain

$$\Pi V^w \widehat{\Pi} = \Pi[\Pi, V^w]\widehat{\Pi}, \tag{2.10}$$

$$\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\Pi = -\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}[\Pi, \kappa V^w](\lambda - \kappa\tilde{H}_1)^{-1}\Pi, \tag{2.11}$$

$$(\lambda - \kappa\tilde{H}_1)E_1(\lambda,\kappa) = \widehat{\Pi} - \Pi[\Pi, \kappa V^w]E_1(\lambda,\kappa). \tag{2.12}$$

The last equality implies that

$$\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\widehat{\Pi} = E_1(\lambda,\kappa) + \widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\Pi[\Pi, \kappa V^w]E_1(\lambda,\kappa). \tag{2.13}$$

Next, the left hand side of (2.11) can be written as

$$\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\Pi = \widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\widehat{\Pi}[\Pi, \kappa V^w](\lambda - \kappa\tilde{H}_1)^{-1}\Pi \tag{2.14}$$

$$+\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\Pi[\Pi, \kappa V^w](\lambda - \kappa\tilde{H}_1)^{-1}\Pi = (1) + (2).$$

From (2.5), (2.6) and (2.13) we deduce

$$\|(1)\| = \mathcal{O}(b^{-3/2}|\Im\lambda|^{-2}). \tag{2.15}$$

Substituting $\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\Pi$ in (2) by the right hand side of (2.14), and using the rough estimate

$$\|\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\Pi\| = \mathcal{O}(b^{-1/2}|\Im\lambda|^{-2})$$

(which follows from (2.6) and (2.11)), we obtain

$$\|(2)\| = \mathcal{O}(b^{-3/2}|\Im\lambda|^{-4}). \tag{2.16}$$

Putting together (2.15), (2.16), (2.8) we get (2.9). □

Now, let $f \in C_0^\infty(]-\delta,\delta[)$ and let $\tilde{f} \in C_0^\infty(D(0,\delta))$ be an almost analytic extension of f such that

$$\overline{\partial}_\lambda \tilde{f}(\lambda) = \mathcal{O}(|\Im\lambda|^\infty). \tag{2.17}$$

By Helffer-Sj¨ostrand formula (see for instance [11]), we have :

$$f(\kappa\tilde{H}_1) - f(\kappa\tilde{H}_0) = -\frac{1}{\pi}\int \overline{\partial}\tilde{f}(\lambda)\Big((\lambda - \kappa\tilde{H}_1)^{-1} - (\lambda - \kappa\tilde{H}_0)^{-1}\Big)(Ld\lambda),$$

where $L(d\lambda)$ denotes the Lebesgue measure on $\mathbf{C}$.

Recall that $\widehat{\Pi}$ commutes with $\tilde{H}_0$. Then $\widehat{\Pi}(\lambda - \kappa\tilde{H}_0)^{-1}\widehat{\Pi} = \widehat{\Pi}(\lambda - \kappa\widehat{\Pi}\tilde{H}_0\widehat{\Pi})^{-1}\widehat{\Pi} = E_0(\lambda,\kappa)$ is holomorphic for $\lambda \in D(0,\delta)$. Combining this with Lemma 2.5, Lemma 2.2 and using the cyclicity of the trace and the fact that $\overline{\partial}_\lambda \tilde{f}(\lambda) = \mathcal{O}(|\Im\lambda|^\infty)$ we obtain :

Proposition 2.6 *Let δ and $b \to \kappa(b)$ be as in Lemma 2.2. For all $f \in C_0^\infty(]-\delta,\delta[)$, we have :*

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\widehat{\Pi}(f(\kappa\tilde{H}_1) - f(\kappa\tilde{H}_0))\widehat{\Pi}\Big) = \mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\widehat{\Pi}f(\kappa\tilde{H}_1)\widehat{\Pi}\Big) = \mathcal{O}(b^{-3/2}). \tag{2.18}$$

Set

$$\overline{H}_1 = \tilde{H}_0 + \Pi V^w \Pi.$$

Proposition 2.7 *Under the assumption of Proposition 2.6, we have*

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\Pi(f(\kappa\tilde{H}_1) - f(\kappa\overline{H}_1))\Pi\Big) = \mathcal{O}(b^{-1}\kappa^{1/2}). \tag{2.19}$$

P r o o f. The ideas of the proof is quite similar to the one in Lemma 2.5 and that is why we omit some details. Making use of the resolvent equation and taking into account (2.10), we obtain

$$\Pi\Big(\lambda - \kappa\tilde{H}_1)^{-1} - (\lambda - \kappa\overline{H}_1)^{-1}\Big)\Pi = \Pi(\lambda - \kappa\tilde{H}_1)^{-1}\kappa\widehat{\Pi}V^w\Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi \tag{2.20}$$

$$= \Pi(\lambda - \kappa\overline{H}_1)^{-1}\kappa\widehat{\Pi}V^w\Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi - \Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi\kappa V^w\widehat{\Pi}(\lambda - \tilde{H}_1)^{-1}\widehat{\Pi}\kappa V^w\Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi$$

$$= -\Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi[\Pi, \kappa V^w]\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\widehat{\Pi}[\kappa V^w, \Pi]\Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi,$$

where we used that $\Pi(\lambda - \kappa\overline{H}_1)^{-1}\widehat{\Pi} = \widehat{\Pi}(\lambda - \kappa\overline{H}_1)^{-1}\Pi = 0$.

Next, by observing that $\Pi(\lambda - \kappa\tilde{H}_0)^{-1}\Pi = \Pi(\lambda - \kappa D_z^2)^{-1}\Pi$, and repeating the arguments in the proof of (2.8), we get

$$\|\Pi(\lambda - \kappa\overline{H}_1)^{-1}\Pi[\Pi, \kappa V^w]\|_{\mathrm{tr}} = \mathcal{O}(|\Im\lambda|^{-2}(b\kappa)^{1/2}). \tag{2.21}$$

On the other hand, (2.5), (2.6) and (2.13) yield

$$\|\widehat{\Pi}(\lambda - \kappa\tilde{H}_1)^{-1}\widehat{\Pi}[\Pi, \kappa V^w]\| = \mathcal{O}(|\Im\lambda|^{-3}b^{-3/2}),$$

which together with (2.20) and (2.21) gives

$$\|\Pi\Big(\lambda - \kappa\tilde{H}_1)^{-1} - (\lambda - \kappa\overline{H}_1)^{-1}\Big)\Pi\|_{\mathrm{tr}} = \mathcal{O}(b^{-1}\kappa^{1/2}|\Im\lambda|^{-6}).$$

Now, applying the Helffer-Sjöstrand formula and using the above estimate as well as the fact that $\overline{\partial}_\lambda\tilde{f}(\lambda) = \mathcal{O}(|\Im\lambda|^\infty)$, we obtain (2.19). □

Let us now give the proof of Theorem 1.1.

P r o o f. Using the cyclicity of the trace, Proposition 2.1, as well as the fact that $\Pi\widehat{\Pi} = 0$, we obtain

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big((f(\kappa(H_1(b) - b)) - f(\kappa(H_0(b) - b))\Big) = \mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(f(\kappa\tilde{H}_1) - f(\kappa\tilde{H}_0)\Big) =$$

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\widehat{\Pi}(f(\kappa\tilde{H}_1) - f(\kappa\tilde{H}_0))\widehat{\Pi}\Big) + \mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\Pi(f(\kappa\tilde{H}_1) - f(\kappa\tilde{H}_0))\Pi\Big),$$

which together with Proposition 2.6, Proposition 2.7 leads to:

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big((f(\kappa(H_1(b) - b)) - f(\kappa(H_0(b) - b))\Big) = \mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\Pi(f(\kappa\overline{H}_1) - f(\kappa\tilde{H}_0))\Pi\Big) + \mathcal{O}(b^{-1}\kappa^{1/2}).$$

Moreover by Lemma 2.4 and Theorem A3 we have

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\Pi(f(\kappa\overline{H}_1) - f(\kappa\tilde{H}_0))\Pi\Big) =$$

$$\mathrm{tr}_{L^2(\mathbf{R}^3)}\Big(\Pi(f(\kappa\tilde{H}_0 + \kappa\Pi q_0^w\Pi + \kappa b^{-1}\Pi q_1^w\Pi) - f(\kappa\tilde{H}_0))\Pi\Big) + \mathcal{O}(b^{-1}\kappa^{1/2}).$$

Since Π commutes with $\tilde{H}_0$, q_0 and q_1 (we recall that q_0 and q_1 are independent of (x, ξ)), we have $\Pi f(\kappa(\tilde{H}_0 - b))\Pi = \Pi f(\kappa\Pi(\tilde{H}_0 - b)\Pi))\Pi = f(\kappa D_z^2)\Pi$ and

$$\Pi f(\kappa\tilde{H}_0 + \kappa\Pi q_0^w\Pi + \kappa b^{-1}\Pi q_1^w\Pi)\Pi = f(\kappa D_z^2 + \kappa q_0^w + \kappa b^{-1}q_1^w)\Pi.$$

Summing up we have proved Theorem 1.1. □

3 Proofs of Theorem 1.2 and Theorem 1.3

From now on, we denote $h = b^{-1}$, and we assume (1.1). We recall that

$$Q(h) := D_z^2 + V^w(hD_y, y, z) + \frac{h}{4}(\Delta_{x,y}V)^w(hD_y, y, z).$$

Theorem 3.1 *For every $f \in C_0^\infty(\mathbf{R})$, the following full asymptotic expansion holds as $h \searrow 0$:*

$$\mathrm{tr}(f(Q(h)) - f(D_z^2)) \sim \sum_{j=0}^{\infty} c_j(f)\, h^{j-1}. \tag{3.1}$$

In particular,

$$c_0(f) = \frac{1}{2\pi} \int \int_{\mathbf{R}^2} \mathrm{tr}_{L^2(\mathbf{R}_z)} \Big(f(D_z^2 + V(x,y,z)) - f(D_z^2) \Big) dxdy, \tag{3.2}$$

$$\begin{aligned} c_1(f) &= \frac{1}{8\pi} \int \int_{\mathbf{R}^2} \mathrm{tr}_{L^2(\mathbf{R}_z)} \Big(\Delta_{x,y} V(x,y,z) f'(D_z^2 + V(x,y,z)) \Big) dxdy && (3.3)\\ &+ \frac{1}{2\pi} \int \int_{\mathbf{R}^2} \mathrm{tr}_{L^2(\mathbf{R}_z)} \Big(T(\partial_x V, \partial_y V) - T(\partial_y V, \partial_x V) \Big)(x,y) dxdy, && (3.4) \end{aligned}$$

where $(A,B) \to T(A,B)$ is defined by

$$T(A,B) = \lim_{\epsilon \to 0} \Big[\frac{1}{2i} \int f(t) \Big(G(t+i\epsilon) - G(t-i\epsilon) \Big)\, dt \Big] \tag{3.5}$$

with

$$G(t \pm i\epsilon) = (t \pm i\epsilon - D_z^2 - V)^{-1}\, A\, (t \pm i\epsilon - D_z^2 - V)^{-1}\, B\, (t \pm i\epsilon - D_z^2 - V)^{-1}.$$

P r o o f. The proof of Theorem is quite standard, and uses the h-pseudodifferential calculus of operator with operator valued symbol. We only mimic the proof in [10] (see also [9], [11]).

Let $\tilde{f} \in C_0^\infty(\mathbf{C})$ be an almost analytic extension of f with

$$\bar{\partial}_\lambda \tilde{f}(\lambda) = \mathcal{O}(|\Im \lambda|^\infty). \tag{3.6}$$

By Helffer-Sj¨ostrand formula, we have

$$f(Q(h)) - f(D_z^2) = -\frac{1}{\pi} \int \bar{\partial}_\lambda \tilde{f}(\lambda) \Big[(\lambda - Q(h))^{-1} - (\lambda - D_z^2)^{-1} \Big]\, L(d\lambda). \tag{3.7}$$

From the resolvent equation, we have

$$(\lambda - Q(h))^{-1} - (\lambda - D_z^2)^{-1} = (\lambda - Q(h))^{-1} W(h) (\lambda - D_z^2)^{-1},$$

where

$$W(h) = V^w(hD_y, y, z) + \frac{h}{4} (\Delta_{x,y} V)^w (hD_y, y, z).$$

Under assumption (1.1) it follows from Theorem A3 that $W(h)(\lambda - D_z^2)^{-1}$ is a trace class operator with

$$\| W(h)(\lambda - D_z^2)^{-1} \|_{\mathrm{tr}} = \mathcal{O}(h^{-1} |\Im \lambda|^{-1}). \tag{3.8}$$

Recalling that, we can view $Q(h)$ as an h-pseudodifferential operator with $\mathcal{L}(H^2, L^2)$-valued symbol. Hence, for every $N \in \mathbf{N}$, we can construct a symbol (see [9], [11], [2])

$$c(y, \eta, \lambda, h) = E_0(y, \eta, \lambda) + hE_1(y, \eta, \lambda) + ... + h^N E_N(y, \eta, \lambda)$$

such that

$$(\lambda - Q(h))^{-1} = c^w(y, hD_y, \lambda, h) + \mathcal{O}(h^{N(1-2\delta)}), \tag{3.9}$$

uniformly for $\{\lambda \in \mathbf{C}; |\Im \lambda| \geq h^\delta\}$, where δ is some fi xed constant in $]0, 1/2[$. The symbol $E_j(y, \eta, \lambda)$ is a fi nite sum of terms of the form

$$(\lambda - D_z^2 - V(\eta, y, z))^{-1} b_1(y, \eta, z) (\lambda - D_z^2 - V(\eta, y, z))^{-1} .... b_k(y, \eta, z) (\lambda - D_z^2 - V(\eta, y, z))^{-1},$$

$1 \leq k \leq 2j+1$ and b_k depends on V and their derivatives.

On the other hand, the composition formula of h-pseudodifferential calculus shows that

$$(\lambda - Q(h))^{-1} W(h)(\lambda - D_z^2)^{-1} = \tag{3.10}$$

$$\tilde{E}_0^w(y, hD_y, \lambda) + h\tilde{E}_1^w(y, hD_y, \lambda) + ... + h^N \tilde{E}_N^w(y, hD_y, \lambda) + \mathcal{O}(h^{N(1-2\delta)}),$$

where $\tilde{E}_j$ has the same properties as E_j. In particular,

$$\tilde{E}_0(y,\eta,\lambda) = (\lambda - D_z^2 - V(\eta,y,z))^{-1} V(\eta,y,z)(\lambda - D_z^2)^{-1} = (\lambda - D_z^2 - V(\eta,y,z))^{-1} - (\lambda - D_z^2)^{-1},$$

$$\tilde{E}_1(y,\eta,\lambda) = \frac{1}{4}(\lambda - D_z^2 - V(\eta,y,z))^{-1} \Delta_{x,y} V(\eta,y,z)(\lambda - D_z^2 - V(\eta,y,z))^{-1}$$

$$-\frac{i}{2}\Big(\tilde{T}(\partial_x V, \partial_y V) - \tilde{T}(\partial_y V, \partial_x V)\Big)(\eta,y,z),$$

$$\tilde{T}(A,B) := (\lambda - D_z^2 - V(\eta,y,z))^{-1} A (\lambda - D_z^2 - V(\eta,y,z))^{-1} B (\lambda - D_z^2 - V(\eta,y,z))^{-1}. \tag{3.11}$$

Using Theorem A3 we check easily that trace norm of the remainder is also $\mathcal{O}(h^{N(1-2\delta)-1})$.

Next, fi x δ in $]0,1/2[$. We decompose the right hand side of (3.7) as a sum of two terms

$$f(Q(h)) - f(D_z^2) = -\frac{1}{\pi}\int_{\{|\Im\lambda| \leq h^\delta\}} \bar{\partial}_\lambda \tilde{f}(\lambda)\Big[(\lambda - Q(h))^{-1} - (\lambda - D_z^2)^{-1}\Big]\, L(d\lambda) \tag{3.12}$$

$$-\frac{1}{\pi}\int_{\{|\Im\lambda| \geq h^\delta\}} \bar{\partial}_\lambda \tilde{f}(\lambda)\Big[(\lambda - Q(h))^{-1} - (\lambda - D_z^2)^{-1}\Big]\, L(d\lambda) = I_1 + I_2.$$

If follows from (3.6) and (3.8) that $I_1 = \mathcal{O}(h^\infty)$. Inserting the right hand side of (3.10) in I_2 and using Theorem A3 we get (3.1).

Formula (3.2) (resp. (3.3)-(3.4)) follows from the expression of $\tilde{E}_0(y,\eta,\lambda)$ (resp. $\tilde{E}_1(y,\eta,\lambda)$), the cyclicity of the trace and the Cauchy formula (see [9]). □

Set $Q(h,\kappa) = \kappa Q(h)$.

Theorem 3.2 *Assume that $\frac{h}{\kappa} \leq C$ and $\kappa \to 0$ as h tends to 0. For $f \in C_0^\infty(\mathbf{R})$, the following full asymptotic expansion holds as $h \searrow 0$:*

$$\mathrm{tr}(f(Q(h,\kappa)) - f(\kappa D_z^2)) = \sum_{k=0}^\infty \sum_{j=0}^\infty c_{k,j}(f)\, \kappa^{\frac{1}{2}+j} h^{k-1}. \tag{3.13}$$

In particular,

$$c_{0,0} = \frac{2}{(2\pi)^2}\int_0^\infty f'(r^2)dr \int\int\int_{\mathbf{R}^3} V(x,y,z)dxdydz. \tag{3.14}$$

P r o o f. Helffer-Sj¨ostrand formula yields

$$f(Q(h,\kappa)) - f(\kappa D_z^2) = -\frac{1}{\pi}\int \bar{\partial}_\lambda \tilde{f}(\lambda)\Big[(\lambda - Q(h,\kappa))^{-1} - (\lambda - \kappa D_z^2)^{-1}\Big]\, L(d\lambda). \tag{3.15}$$

From the resolvent equation, we obtain :

$$(\lambda - Q(h,\kappa))^{-1} = \sum_{j=0}^N \kappa^j \Big[(\lambda - \kappa D_z^2)^{-1} W(h)\Big]^j (\lambda - \kappa D_z^2)^{-1}$$

$$+\kappa^{N+1}(\lambda - Q(h,\kappa))^{-1} W(h)\Big[(\lambda - \kappa D_z^2)^{-1} W(h)\Big]^N (\lambda - \kappa D_z^2)^{-1}.$$

Consequently,

$$(\lambda - Q(h,\kappa))^{-1} - (\lambda - \kappa D_z^2)^{-1} = \tag{3.16}$$

$$\sum_{j=1}^{N} \kappa^j \Big[(\lambda - \kappa D_z^2)^{-1} W(h)\Big]^j (\lambda - \kappa D_z^2)^{-1}$$

$$+\kappa^{N+1}(\lambda - Q(h,\kappa))^{-1} W(h) \Big[(\lambda - \kappa D_z^2)^{-1} W(h)\Big]^N (\lambda - \kappa D_z^2)^{-1}$$

$$= \sum_{j=1}^{N} A_j(\lambda) + B(\lambda).$$

Clearly, $\|B(\lambda)\|_{\rm tr} = \mathcal{O}(\kappa^N |\Im\lambda|^{-(N+2)} h^{-1})$, which together with (3.6) implies

$$\| \int \bar\partial_\lambda \tilde f(\lambda) B(\lambda)\, L(d\lambda)\|_{\rm tr} = \mathcal{O}(\kappa^N h^{-1}).$$

For $j = 0, 1, ...$, set

$$I_j := -\frac{1}{\pi} \int \bar\partial_z \tilde f(\lambda)\, {\rm tr}(A_j(\lambda))\, L(d\lambda).$$

In view of (3.15), (3.16) and the above estimate, it suffices to show that I_j has an asymptotic expansion like the right hand side of (3.13).

Fix δ in $]0, 1/2[$. From (3.6) and (3.8), we obtain

$$\| \int_{|\Im\lambda| \le \kappa^{\frac{\delta}{2}}} \bar\partial_\lambda \tilde f(\lambda) A_j(\lambda)\, L(d\lambda)\|_{\rm tr} = \mathcal{O}(\kappa^\infty h^{-1}).$$

Fix N in $\mathbf{N}$. By the h-pseudodifferential calculus (see [11] chapters 7,8), there exists $A(y,\eta,z,k,\lambda,\kappa,h) \in S^4\Big(\mathbf{R}^4; \langle y,\eta\rangle^{-p}\langle z\rangle^{-s}\langle k\rangle^{-2}\Big)$ such that

$$A_j(\lambda) = A^w(y, hD_y, z, \kappa^{\frac12} D_z, \lambda, \kappa, h) + \mathcal{O}(\kappa^{\frac{N}{2}(1-2\delta)} h^{-1})$$

uniformly for $\lambda \in \Omega_\delta := \{\lambda \in \mathbf{C};\ |\Im\lambda| > \kappa^{\frac{\delta}{2}}\}$ in the trace norm class. Moreover

$$A(y,\eta,z,k,\lambda,\kappa,h) \sim \sum_{i,l=0}^{\infty} A_{i,l}(y,\eta,z,k)\kappa^{\frac{i}{2}+1} h^l, \quad \text{in} \quad S^4\Big(\mathbf{R}^4; \langle y,\eta\rangle^{-p}\langle z\rangle^{-s}\langle k\rangle^{-2}\Big)$$

where $A_{i,l}(y,\eta,z,k)$ is a finite sum of terms of the form

$$a_i(k)(\lambda - k^2)^{-m-i-1} g_i(y,\eta,z),$$

with $a_i(k)$ is a homogeneous polynomial of degree i and g_i are functions depending on V, W and their derivatives. In particular,

$$A_{0,0}(y,\eta,z,k) = (\lambda - k^2)^{-2} V(\eta, y, z). \tag{3.17}$$

Since $a_i(-k) = -a_i(k)$ for i odd, it follows from the above discussion that $\kappa^{\frac12} I_j$ has an asymptotic expansion in powers of κ. It remains to prove (3.14).

Applying the following formula

$$-\frac{1}{\pi} \int \bar\partial_z \tilde f(z)(z-\mu)^{-2}\, L(dz) = f'(\mu), \quad \text{for all } \mu \in \mathbf{R},$$

to $\mu = A_{0,0}(y,\eta,z,k)$ and using Theorem A3 we get (3.14). □

At last, combining Theorem 1.1 with Theorem 3.1 (resp. Theorem 3.2) we deduce Theorem 1.2 (resp. Theorem 1.3).

4 Appendix: Operator valued pseudo-differential operators

Here, we recall some results about operator valued pseudodifferential operators. Concerning the proofs, we refer to [2] and [11]. Let K, L be two subspaces of a Hilbert space H. Let $m:\ \mathbf{R}^{2d}\to[0,\infty[$ be an order function (see Definition 7.5 in [11]). We introduce the class of symbols $S(\mathbf{R}^{2d},m;\mathcal{L}(K,L))$ defined by the set of operator valued functions $a\in C^\infty(\mathbf{R}^{2d},\mathcal{L}(K,L))$ such that for any $(\alpha,\beta)\in\mathbf{N}^{2d}$, we have:

$$\|\partial_x^\alpha\partial_y^\beta a(x,\xi)\|_{\mathcal{L}(K,L)}=\mathcal{O}_{\alpha,\beta}\Big(m(x,\xi)\Big).$$

In the special cases when $m=1$ we will write $S(\mathbf{R}^{2d};\mathcal{L}(K,L))$ instead of $S(\mathbf{R}^{2d},1;\mathcal{L}(K,L))$, and when $K=L=H$ we will write $S(\mathbf{R}^{2d},m)$ instead of $S(\mathbf{R}^{2d},m;\mathcal{L}(H))$.

For $a\in S(\mathbf{R}^{2d},m;\mathcal{L}(K,L))$ we define the Weyl quantization, $a^w(x,hD_x):=Op_h^w(a)$ by:

$$Op_h^w(a)u(x)=\frac{1}{(2\pi h)^d}\int\int_{\mathbf{R}^{2d}}e^{ih^{-1}(x-y).\xi}a(\frac{x+y}{2},\xi)u(y)dyd\xi,$$

when $u\in\mathcal{S}(\mathbf{R}^d,K)$ the class of rapidly decreasing functions.

As in the case of scalar symbols, we have the following composition Theorem.

Theorem A 1 [2] *Let $a\in S(\mathbf{R}^{2d},m_1;\mathcal{L}(K,L))$ and $b\in S(\mathbf{R}^{2d},m_2;\mathcal{L}(L,H))$. Then there exists $c(h)\in S(\mathbf{R}^{2d},m_1m_2;\mathcal{L}(K,H))$ such that*

$$Op_h^w(a)oOp_h^w(b)=Op_h^w(c(h)).$$

The symbol $c(h)$ is given by:

$$c(x,\xi,h)=\Big(\exp(\frac{ih}{2}(D_\xi\ D_y-D_\eta\ D_x))a(x,\xi)b(y,\eta)\Big)|_{(y,\eta)=(x,\xi)}.$$

The L^2-boundedness can be established exactly as in the scalar case:

Theorem A 2 [2] *For $a\in S(\mathbf{R}^{2d},1;\mathcal{L}(K,L))$ the operator $a(x,hD_x)$ can be extended to a bounded operator from $L^2(\mathbf{R}^d,K)$ to $L^2(\mathbf{R}^d,L)$ and there exist $C_d>0$, $P_d\in\mathbf{N}$ such that*

$$\|a(x,hD_x)\|_{\mathcal{L}(L^2(\mathbf{R}^d,K),L^2(\mathbf{R}^d,L))}\le C_d\sup_{\substack{(x,\xi)\in\mathbf{R}^{2d}\\|\alpha|,|\beta|\le P_d}}\|\partial_x^\alpha\partial_y^\beta a(x,\xi)\|_{\mathcal{L}(K,L)}.$$

We have also trace class properties for such operators:

Theorem A 3 [2] *Suppose that the injection $K\hookrightarrow H$ is of the Schatten class $\sigma_{1-\nu}$, $0<\nu<1$ (in particular this injection is of trace class). For $a\in S(\mathbf{R}^{2d},m;\mathcal{L}(K,L))$, with $m\in L^1(\mathbf{R}^{2d})$, the bounded operator $a(x,hD_x)\in\mathcal{L}(L^2(\mathbf{R}^d,H))$ is of trace class and there exists $C_d>0$ such that for any integer $P>\frac{d}{\nu}$, we have*

$$\|a(x,hD_x)\|_{\mathrm{tr}}\le C_dh^{-d}\sum_{|\alpha|+|\beta|\le 2P}\int\int_{\mathbf{R}^{2d}}\|\partial_x^\alpha\partial_y^\beta a(x,\xi)\|_{\mathrm{tr}}dxd\xi,$$

$$\mathrm{tr}\Big(a(x,hD_x)\Big)=(2\pi h)^{-d}\int\int_{\mathbf{R}^{2d}}\mathrm{tr}(a(x,\xi))dxd\xi$$

where the trace in the LHS is in $\mathcal{L}(L^2(\mathbf{R}^d,H))$ and the trace in the RHS is in $\mathcal{L}(H)$.

At last, we give a characterization of pseudo-differential operators due to Beals [3] (see also section 8 of [11]).

Theorem A 4 *Let $A=A_h$ be an operator from $\mathcal{S}(\mathbf{R}^d,K)$ to $\mathcal{S}'(\mathbf{R}^d,H)$, $0<h<h_0$. The following two statements are equivalent:*

(1) $A_h=Op_h^w(a(h))$ for some $a(h)\in S(\mathbf{R}^{2d},1;\mathcal{L}(K,H))$.

(2) For every $N\in\mathbf{N}$ and for every sequence $l_1(x,\xi),\cdots,l_N(x,\xi)$ of linear forms on $\mathbf{R}^{2d}$, the operators

$$[l_1(x,hD_x)\,[l_2(x,hD_x)\,[\cdots[l_N(x,hD_x),A_h]\cdots]]]\,,$$

belongs to $\mathcal{L}(L^2(\mathbf{R}^d,K),L^2(\mathbf{R}^d,H))$ and is of norm $\mathcal{O}(h^N)$ in that space. Here $[.,.]$ denote the commutator: $[A,B]:=AB-BA$.

The proof of this result for operator valued symbols follows the proof of the scalar case (see section 8 of [11]). The main difference is that the numerical functions $\Phi(x),\Psi(x)\in\mathcal{S}(\mathbf{R}^d)$ are replaced by $\Phi(x)f\in\mathcal{S}(\mathbf{R}^d,K)$, $\Psi(x)g\in\mathcal{S}(\mathbf{R}^d,H)$ (for $f\in K$, $g\in H$) and the product $\overline{\Phi}(x)\hat{\Psi}(\xi)a(x,\xi)$ becomes $\overline{\Phi}(x)\hat{\Psi}(\xi)\langle a(x,\xi)f,g\rangle_{H\times H}$.

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