EIGENVALUES FOR MAXWELL'S EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

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Abstract. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup generated by Maxwell's equations in an exterior domain $\Omega \subset \mathbb{R}^3$ with dissipative boundary condition $E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0, \gamma(x) > 0, \forall x \in \Gamma = \partial\Omega$. We prove that if $\gamma(x)$ is nowhere equal to 1, then for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ the eigenvalues of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where $\Lambda_\epsilon = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2}+\epsilon} + 1),\ \operatorname{Re} z < 0\}$, $\mathcal{R}_N = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N},\ \operatorname{Re} z < 0\}$.

1. Introduction

Suppose that $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ is an open connected domain and $\Omega := \mathbb{R}^3 \setminus \bar{K}$ is an open connected domain with C^∞ smooth boundary Γ. Consider the boundary problem

$$\begin{aligned} &\partial_t E = \operatorname{curl} B, \qquad \partial_t B = -\operatorname{curl} E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega, \\ &E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma, \\ &E(0, x) = e_0(x), \qquad B(0, x) = b_0(x). \end{aligned} \tag{1.1}$$

with initial data $f = (e_0, b_0) \in (L^2(\Omega))^6 = \mathcal{H}$. Here $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at $x \in \Gamma$ pointing into Ω, $\langle , \rangle$ denotes the scalar product in $\mathbb{C}^3$, $u_{tan} := u - \langle u, \nu\rangle\nu$, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. The solution of the problem (1.1) is given by a contraction semigroup $(E, B) = V(t)f = e^{tG_b}f$, $t \geq 0$, where the generator G_b has domain $D(G_b)$ that is the closure in the graph norm of functions $u = (v, w) \in (C^\infty_{(0)}(\mathbb{R}^3))^3 \times (C^\infty_{(0)}(\mathbb{R}^3))^3$ satisfying the boundary condition $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$ on Γ.

In an earlier paper [2] we proved that the spectrum of G_b in $\operatorname{Re} z < 0$ consists of isolated eigenvalues with finite multiplicity. If $G_b f = \lambda f$ with $\operatorname{Re}\lambda < 0$, the solution $u(t, x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are invisible for inverse scattering problems. It was proved [2] that if there is at least one eigenvalue λ of G_b with $\operatorname{Re}\lambda < 0$, then the wave operators $W_\pm$ are not complete, that is $\operatorname{Ran} W_- \neq \operatorname{Ran} W_+$. Hence we cannot define the scattering operator S related to the Cauchy problem for the Maxwell system and (1.1) by the product $W_+^{-1}W_-$. For the perfect conductor boundary conditions for Maxwell's equations, the energy is conserved in time and the unperturbed and perturbed problems are associated to unitary groups. The corresponding scattering operator $S(z) : (L^2(\mathbb{S}^2))^2 \to (L^2(\mathbb{S}^2))^2$ satisfies the identity

$$S^{-1}(z) = S^*(\bar{z}), \quad z \in \mathbb{C} \tag{1.2}$$

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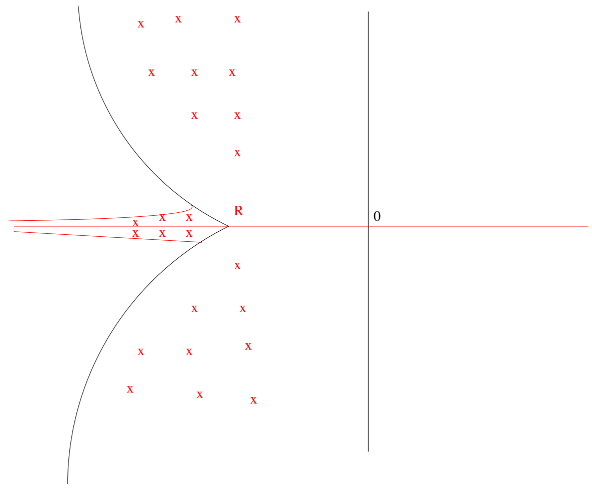

FIGURE 1. Eigenvalues of G_b

if $S(z)$ is invertible at z. The scattering operator $S(z)$ defined in [4] is such that $S(z)$ and $S^*(z)$ are analytic in the "physical" half plane $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and the above relation for conservative boundary conditions implies that $S(z)$ is invertible for $\operatorname{Im} z > 0$. For dissipative boundary conditions the relation (1.2) in general is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. Lax and Phillips [4] proved that this implies that $\mathbf{i}z_0$ is an eigenvalue of G_b. The analysis of the location of the eigenvalues of G_b is important for the location of the points where the kernel of $S(z)$ is not trivial.

The main result of this paper is the following

Theorem 1.1. *Assume that for all $x \in \Gamma$, $\gamma(x) \neq 1$. Then for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ there are constants $C_\epsilon > 0$ and $C_N > 0$ such that the eigenvalues of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where*

$$\Lambda_\epsilon = \{z \in \mathbb{C} : \ |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{1/2+\epsilon} + 1), \ \operatorname{Re} z < 0\},$$

$$\mathcal{R}_N = \{z \in \mathbb{C} : \ |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \ \operatorname{Re} z < 0\}.$$

Example 1.2. *For Maxwell's equations, $K = \{|x| \leq 1\}$, and $0 < \gamma =$ constant, the generator G_b has real eigenvalues (see [1]) and these eigenvalues are stable under small perturbations of the boundary and the boundary conditions (see [2]).*

If $\operatorname{Re} \lambda < 0$ and $G_b(E, B) = \lambda(E, B) \neq 0$, then

$$\begin{aligned} &\lambda E = \operatorname{curl} B && \text{on} \quad \Omega, \\ &\lambda B = -\operatorname{curl} E && \text{on} \quad \Omega, \\ &\operatorname{div} E = \operatorname{div} B = 0, && \text{on} \quad \Omega, \\ &E_{tan} - \gamma(\nu \wedge B_{tan}) = 0 && \text{on} \quad \Gamma. \end{aligned} \tag{1.3}$$

This implies that $u := (E, B)$ satisfies

$$\Delta u - \lambda^2 u = 0, \qquad \text{on} \quad \Omega.$$

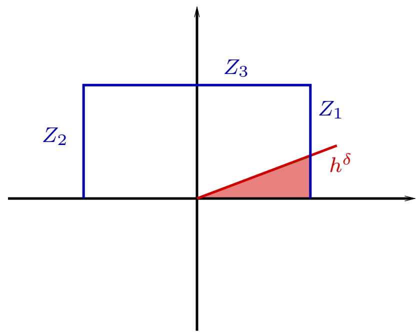

FIGURE 2. Contours $Z_1, Z_2, Z_3, \delta = 1/2 - \epsilon$

The eigenvalues of G_b are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is nonnegative. The mapping $z \mapsto z^2$ maps the positive quadrant $\{z \in \mathbb{C} : \operatorname{Re} z > 0\,, \operatorname{Im} z > 0\}$ bijectively to the upper half space. Denote by $\sqrt{z}$ the inverse map. The part of the spectral domain $\{\lambda \in \mathbb{C} :\ \operatorname{Re}\lambda < 0\,, \operatorname{Im}\lambda > 0\}$ is mapped by $\lambda = \mathbf{i}\sqrt{z}$ to the upper half plane $\{z \in \mathbb{C} :\ \operatorname{Im} z > 0\}$. In $\{z \in \mathbb{C} :\ \operatorname{Im} z \geq 0\}$ introduce the sets

$$\begin{aligned} Z_1 &:= \{z \in \mathbb{C} :\ \operatorname{Re} z = 1,\ \ h^\delta \leq \operatorname{Im} z \leq 1\}, \quad 0 < h \ll 1, \quad 0 < \delta < 1/2, \\ Z_2 &:= \{z \in \mathbb{C} : \operatorname{Re} z = -1,\ \ 0 \leq \operatorname{Im} z \leq 1\}, \\ Z_3 &:= \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1,\ \ \operatorname{Im} z = 1\}. \end{aligned}$$

Set $\lambda = \mathbf{i}\sqrt{z}/h$, $z \in Z_1 \cup Z_2 \cup Z_3$. To study the eigenvalues λ, $|\lambda| > R_0$, it is sufficient to consider $0 < h \ll 1$. As z runs over the blue rectangle in Figure 2, with $0 < h \ll 1$, λ sweeps out the large values in the intersection of left and upper half planes. The values of $z \in Z_2$ near the lower left hand corner, $z = -1$, of the blue rectangle go the spectral values near the negative real axis. The spectral analysis near these values in Z_2 for dissipative Maxwell's equations does not have clear analogue with the spectral problems for the wave equation with dissipative boundary conditions. In fact, for the wave equation if $0 < \gamma(x) < 1$, $\forall x \in \Gamma$, the eigenvalues of the generator of the corresponding semigroup are located in the domain Λ_ϵ (see Section 3, [6] and [5]). For Maxwell's equations the eigenvalues of G_b lie in the domain $\Lambda_\epsilon \cup \mathcal{R}_N$ and for $0 < \gamma(x) < 1$ and $\gamma(x) > 1$ we have the same location.

Equation (1.3) implies that on Ω each eigenfunction $u = (E, B)$ of G_b satisfies

$$\sqrt{z}\, E \;=\; \frac{h}{\mathbf{i}} \operatorname{curl} B\,, \qquad \sqrt{z}\, B \;=\; -\frac{h}{\mathbf{i}} \operatorname{curl} E\,, \tag{1.4}$$

and therefore $(-h^2\Delta - z)E = (-h^2\Delta - z)B = 0$. For eigenfunctions $(E, B) \neq 0$, we derive a pseudodifferential system on the boundary involving $E_{tan} = E - \langle E, \nu\rangle\nu$ and $E_{nor} = \langle E, \nu\rangle$. A semi-classical analysis shows that for $z \in Z_1 \cup Z_3$ this system implies that for h small enough we have $E|_\Gamma = 0$ which yields $E = B = 0$. By scaling one concludes that the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{M}$, where

$$\mathcal{M} = \{z \in \mathbb{C} :\ |\arg z - \pi| \leq \pi/4,\ |z| \geq R_0 > 0,\ \operatorname{Re} z < 0\}.$$

The strategy for the analysis of the case $z \in Z_1 \cup Z_3$ is similar to that exploited in [7] and [6]. In these papers the semi-classical Dirichlet-to-Neumann map $\mathcal{N}(z,h)$ plays a crucial role and the problem is reduced to the proof that some $h-$pseudodifferantial operators is elliptic in a suitable class. For the Maxwell system the pseudodifferential equation on the boundary is more complicated. Using the equation $\operatorname{div} E = 0$, yields a pseudodifferential system for E_{tan} and E_{nor}. We show that if $(E,B) \neq 0$ is an eigenfunction of G_b, then $\|E_{nor}\|_{H_h^1(\Gamma)}$ is bounded by $Ch\|E_{tan}\|_{H_h^1(\Gamma)}$. The term involving E_{nor} then plays the role of a negligible perturbation in the pseudodifferentrial system on the boundary and this reduces the analysis to one involving only E_{tan}. The system concerning E_{tan} has a diagonal leading term and we may apply the same arguments as those of [6] to conclude that $E_{tan} = 0$ and hence $E_{nor} = 0$.

The analysis of the case $z \in Z_2$ is more difficult since the principal symbol g of the pseudodifferential system for E_{tan} need not be elliptic at some points (see Section 3). Even where g is elliptic, if $|\operatorname{Im} z| \leq h^{1/2}$ it is difficult to estimate the norm of the difference $Op_h(g)Op_h(g^{-1}) - I$. To show that the eigenvalues of G_b lying in $\mathcal{M}$ are in fact confined to the region $\mathcal{R}_N$ for every $N \in \mathbb{N}$, we analyze the real part of the following scalar product in $L^2(\Gamma)$

$$Q(E_0) \ := \ \operatorname{Re}\langle(\mathcal{N}(z,h) - \sqrt{z}\gamma)E_0, E_0\rangle_{L^2(\Gamma)}, \qquad E_0 \ := \ E|_\Gamma.$$

We follow the approach in [7], [6] based on a Taylor expansion of $Q(E_0)$ at $z = -1$ and the fact that for $z = -1$ we have $Q(E_0) = \mathcal{O}(h^N)$, $\forall N \in \mathbb{N}$.

2. Pseudodifferential equation on the boundary

Introduce geodesic normal coordinates $(y_1, y') \in \mathbb{R}^3$ on a neighborhood of a point $x_0 \in \Gamma$ as follows. For a point x, $y'(x)$ is the closest point in Γ and $y_1 = \operatorname{dist}(x, \Gamma)$. Define $\nu(x)$ to be the unit normal in the direction of increasing y_1 to the surface $y_1 =$ constant through x. Thus $\nu(x)$ is an extension of the unit normal vector to a unit vector field. The boundary Γ is mapped to $y_1 = 0$ and

$$x \ = \ \alpha(y_1, y') \ = \ \beta(y') \ + \ y_1\nu(y').$$

We have

$$\frac{\partial}{\partial x_k} \ = \ \nu_k(y')\frac{\partial}{\partial y_1} + \sum_{j=2}^{3} \frac{\partial y_j}{\partial x_k}\frac{\partial}{\partial y_j}, \qquad k = 1,2,3.$$

Moreover,

$$\sum_{k=1}^{3} \nu_k(y')\frac{\partial y_j}{\partial x_k}(y_1, y') = \langle \nu, \frac{\partial y_i}{\partial x}\rangle = 0, \qquad j = 1,2,3, \quad \text{and}$$

$$\sum_{k=1}^{3} \nu_k(x)\partial_{x_k} f(x) = \partial_{y_1}(f(\alpha(y_1, y')).$$

Since $\|\nu(x)\| = 1$, $\langle \nu, \partial_{x_j}\nu\rangle = 0$, $j = 1,2,3$.

A straight forward computation yields

$$\begin{aligned}\nu(x) \wedge \frac{h}{\mathbf{i}}\operatorname{curl} u(x) &= \mathbf{i}h\partial_\nu u_{tan} + \Big(\langle D_{x_1}u, \nu\rangle, \langle D_{x_2}u, \nu\rangle, \langle D_{x_3}u, \nu\rangle\Big)\Big|_{tan} \\ &= \mathbf{i}h\partial_\nu u_{tan} + \Big(\operatorname{grad}_h\langle u, \nu\rangle\Big)\Big|_{tan} - \mathbf{i}h\langle g_0(u_{tan}), \qquad x \in \Gamma,\end{aligned}$$

where

$$D_{x_j} = -\mathbf{i}h\partial_{x_j}, \qquad j = 1,2,3, \quad \mathrm{grad}_{\,h} f = \{D_{x_j} f\}_{j=1,2,3},$$
$$g_0(u_{tan}) = \{\langle u_{tan}, \partial_{x_j}\nu\rangle\}_{j=1,2,3}.$$

Setting $E_{nor} = \langle E, \nu\rangle$, from (1.3) one deduces

$$\nu \wedge B = -\frac{1}{\sqrt{z}}\nu \wedge \frac{h}{\mathbf{i}}\mathrm{curl}\, E = \frac{1}{\sqrt{z}} D_\nu E_{tan} - \frac{1}{\sqrt{z}}\Big[\Big(\mathrm{grad}_{\,h} E_{nor}\Big)\Big|_{tan} - \mathbf{i}hg_0(E_{tan})\Big],$$

where $D_\nu = -\mathbf{i}h\partial_\nu$ and the boundary condition in (1.3) becomes

$$\Big(D_\nu - \frac{1}{\gamma}\sqrt{z}\Big)E_{tan} - \Big(\mathrm{grad}_{\,h} E_{nor}\Big)\Big|_{tan} + \mathbf{i}hg_0(E_{tan}) = 0, \qquad x \in \Gamma. \tag{2.1}$$

Next

$$\mathrm{grad}_{\,h} f(x)|_{tan} \;=\; \Big\{\sum_{j=2}^{3} \frac{\partial y_j}{\partial_{x_k}} D_{y_j} f(\alpha(y_1, y'))\Big\}_{k=1,2,3}$$

and for $u = (u_1, u_2, u_3) \in \mathbb{C}^3$,

$$\frac{h}{\mathbf{i}}\mathrm{div}\, u(\alpha(y_1, y')) = \langle D_{y_1} u(\alpha(y_1, y')), \nu(y')\rangle + \sum_{k=1}^{3}\sum_{j=2}^{3} \frac{\partial y_j}{\partial_{x_k}} D_{y_j} u_k(\alpha(y_1, y'))$$

$$= D_{y_1}\Big(u_{nor}(t, y')\Big) + \sum_{j=2}^{3} D_{y_j}\Big\langle u_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x}\Big\rangle + h\langle u_{tan}, Z\rangle,$$

where $\langle u(\alpha(y_1, y')), \nu(y')\rangle := u_{nor}(y_1, y')$ and Z depends on the second derivatives of $y_j,\ j = 2.3$. Apply the operator $D_{y_1} - \frac{\sqrt{z}}{\gamma(y')}$ to $\mathrm{div}\, E(\alpha(y_1, y')) = 0$ to find

$$(D_{y_1}^2 - \frac{\sqrt{z}}{\gamma(y')} D_{y_1})E_{nor}(y_1, y') + \sum_{j=2}^{3} D_{y_j}\Big\langle (D_{y_1} - \frac{\sqrt{z}}{\gamma(y')})E_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x}\Big\rangle$$

$$= h\langle (D_{y_1} - \frac{\sqrt{z}}{\gamma})E_{tan}, Z\rangle + h\langle E_{tan}, Z_1\rangle,$$

where $\gamma(y') := \gamma(\beta(y'))$.

Taking the trace $y_1 = 0$ and applying the boundary condition (2.1), yields

$$\Big(D_{y_1}^2 + \sum_{j,\mu=2}^{3}\sum_{k=1}^{3} \frac{\partial y_j}{\partial x_k}\frac{\partial y_\mu}{\partial x_k} D^2_{y_j, y_\mu}\Big)E_{nor}(0, y') - \frac{\sqrt{z}}{\gamma(y')} D_{y_1} E_{nor}(0, y')$$
$$= h\Big\langle \Big(\mathrm{grad}_{\,h} E_{nor}\Big)\Big|_{tan}(0, y'), Z\Big\rangle + hQ_1(E_{tan}(0, y')), \tag{2.2}$$

with

$$\|Q_1(E_{tan}(0, y'))\|_{L^2(\mathbb{R}^2)} \leq C_2\|E_{tan}(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Here $H_h^s(\Gamma), s \in \mathbb{R}$, denotes the semi-classical Sobolev spaces with norm $\|\langle h\partial_x\rangle^s u\|_{L^2(\Gamma)}$, $\langle h\partial_x\rangle = (1 + \|h\partial_x\|^2)^{1/2}$. In the exposition below we use the spaces $(L^2(\Gamma))^3$ and $(H_h^s(\Gamma))^3$ of vector-valued functions but we will omit this in the notations writing simply $L^2(\Gamma)$ and $H_h^s(\Gamma)$.

The operator $-h^2\Delta_x - z$ in the coordinates (y_1, y') has the form

$$\mathcal{P}(z, h) = D_{y_1}^2 + r(y, D_{y'}) + hq_1(y, D_y) + h^2\tilde{q} - z$$

with $r(y, \eta') = \langle R(y)\eta', \eta' \rangle$, $q_1(y, \eta) = \langle q_1(y), \eta) \rangle$. Here

$$R(y) = \Big\{ \sum_{k=1}^{3} \frac{\partial y_j}{\partial x_k} \frac{\partial y_\mu}{\partial x_k} \Big\}_{j,\mu=2}^{3} = \Big\{ \Big\langle \frac{\partial y_j}{\partial x}, \frac{\partial y_\mu}{\partial x} \Big\rangle \Big\}_{j,\mu=2}^{3}$$

is a symmetric (2×2) matrix and $r(0, y', \eta') = r_0(y', \eta')$, where $r_0(y', \eta')$ is the principal symbol of the Laplace-Beltrami operator $-h^2 \Delta_\Gamma$ on Γ equipped with the Riemannian metric induced by the Euclidean one in $\mathbb{R}^3$. We have

$$\Big(\mathcal{P}(z, h) E_{nor}\Big)(0, y') \;=\; \langle \mathcal{P}(z, h) E, \nu \rangle (0, y') \;+\; h Q_2(E(0, y')),$$

where

$$\|Q_2(E(0, y'))\|_{L^2(\mathbb{R}^2)} \;\leq\; C_2 \, \|E(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Since $\mathcal{P}(z, h)E = 0$, this lets us replace the terms with all second derivatives of E_{nor} in (2.4) by $zE_{nor}(0, y')$ modulo terms having a factor h and containing first order derivatives of E_{nor}. This follows from the form of the matrix $R(y)$ given above. After a multiplication by $-\frac{\gamma(y')}{\sqrt{z}}$ the equation (2.2) yields

$$(D_{y_1} - \gamma(y')\sqrt{z}) E_{nor}(0, y') = h Q_3(E(0, y')), \tag{2.3}$$

where $Q_3(E(0, y'))$ has the same properties as $Q_2(E(0, y'))$.

Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support in small neighborhood of $x_0 \in \Gamma$. Replace E, B by $E_\psi = E\psi$, $B_\psi = B\psi$. The above analysis works for E_ψ and B_ψ with lower order terms depending on ψ. We obtain

$$\langle (D_\nu - \gamma(x)\sqrt{z}) E|_\Gamma \psi(x), \nu(x) \rangle \;=\; h \, Q_{3,\psi}(E|_\Gamma).$$

Taking a partition of unity in a neighborhood of Γ, yields

$$\langle (D_\nu - \gamma(x)\sqrt{z}) E|_\Gamma, \nu \rangle = h Q_4(E|_\Gamma), \qquad \|Q_4(E|_\Gamma)\|_{L^2(\Gamma)} \leq C \|E|_\Gamma\|_{H_h^1(\Gamma)}. \tag{2.4}$$

For $z \in Z_1 \cup Z_2 \cup Z_3$ let $\rho(x', \xi', z) = \sqrt{z - r_0(x', \xi')} \in C^\infty(T^*\Gamma)$ be the root of the equation

$$\rho^2 + r_0(x', \xi') - z = 0$$

with $\operatorname{Im} \rho(x', \xi', z) > 0$. For large $|\xi'|$,

$$\rho(x', \xi', z) \;\sim\; |\xi'|, \qquad \operatorname{Im} \rho(x', \xi', z) \;\sim\; |\xi'|,$$

while for bounded $|\xi'|$,

$$\operatorname{Im} \rho(x', \xi', z) \;\geq\; \frac{h^\delta}{C}.$$

We recall some basic facts about h-pseudodifferential operators that the reader can find in [3]. Let X be a C^∞ smooth compact manifold without boundary with dimension $d \geq 2$. Let (x, ξ) be the coordinates in $T^*(X)$ and let $a(x, \xi, h) \in C^\infty(T^*(X))$. Given $m \in \mathbb{R}$, $l \in \mathbb{R}, \delta > 0$ and a function $c(h) > 0$, one denotes by $S_\delta^{l,m}(c(h))$ the set of symbols so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha,\beta} (c(h))^{l - \delta(|\alpha| + |\beta|)} (1 + |\xi|)^{m - |\beta|}, \; \forall \alpha, \forall \beta, \quad (x, \xi) \in T^*(X).$$

If $c(h) = 1$, we denote $S_\delta^{l,m}(c(h))$ simply by $S_\delta^{l,m}$. Symbols restricted to a domain where $|\xi| \leq C$ will be denoted by $a \in S_\delta^l(c(h))$. The $h-$pseudodifferential operator with symbol $a(x, \xi, h)$ acts by

$$(Op_h(a)f)(x) \ := \ (2\pi h)^{-d+1} \int_{T^*X} e^{-\mathbf{i}\langle x-y, \xi\rangle/h} a(x,\xi,h) f(y) dy d\xi.$$

For matrix valued symbols we use the same definition. This means that every element of a matrix symbol is in the class $S_\delta^{l,m}(c(h))$.
Now suppose that $a(x, \xi, h)$ satisfies the estimates

$$|\partial_x^\alpha a(x,\xi,h)| \leq c_0(h) h^{-|\alpha|/2}, \qquad (x,\xi) \in T^*(X) \tag{2.5}$$

for $|\alpha| \leq d-1$, where $c_0(h) > 0$ is a parameter. Then there exists a constant $C > 0$ independent of h such that

$$\|Op_h(a)\|_{L^2(X)\to L^2(X)} \ \leq \ C\, c_0(h). \tag{2.6}$$

For $0 \leq \delta < 1/2$ products of h-pseudodifferential operators are well behaved. If $a \in S_\delta^{l_1,m_1}$, $b \in S_\delta^{l_2,m_2}$ and $s \in \mathbb{R}$, then

$$\|Op_h(a)Op_h(b) - Op_h(ab)\|_{H^s(X)\to H^{s-m_1-m_2+1}(X)} \leq Ch^{-l_1-l_2-2\delta+1}. \tag{2.7}$$

Let $u \in \mathbb{C}^3$ be the solution of the Dirichlet problem

$$(-h^2\Delta - z)u = 0 \quad \text{on} \quad \Omega, \qquad u = F \quad \text{on} \quad \Gamma. \tag{2.8}$$

Introduce the semi-classical Dirichlet-to-Neumann map

$$\mathcal{N}(z,h) : H_h^s(\Gamma) \ni F \longrightarrow D_\nu u|_\Gamma \in H_h^{s-1}(\Gamma).$$

G. Vodev [7] established for bounded domains $K \subset \mathbb{R}^d$, $d \geq 2$, with C^∞ boundary the following approximation of the interior Dirichlet-to-Neumann map $\mathcal{N}_{int}(z,h)$ related to (2.8), where the equation $(-h^2\Delta - z)u = 0$ is satisfied in K.

Theorem 2.1 ([7])**.** *For every $0 < \epsilon \ll 1$ there exists $0 < h_0(\epsilon) \ll 1$ such that for $z \in Z_{1,\epsilon} := \{z \in Z_1,\ |\operatorname{Im} z| \geq h^{\frac{1}{2}-\epsilon}\}$ and $0 < h \leq h_0(\epsilon)$ we have*

$$\|\mathcal{N}_{int}(z,h)(F) - Op_h(\rho + hb)F\|_{H_h^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|F\|_{L^2(\Gamma)}, \tag{2.9}$$

where $b \in S_{0,1}^0(\Gamma)$ does not depend on h and z. Moreover, (2.9) *holds for $z \in Z_2 \cup Z_3$ with $|\operatorname{Im} z|$ replaced by* 1.

With small modifications (2.9) holds for the Dirichlet-to-Neumann map $\mathcal{N}(z,h)$ related to (2.8) (see [6]). Applying (2.9) with $\mathcal{N}(z,h)$ and $F = E_0 = E|_\Gamma$, we obtain

$$\Big\| \langle \mathcal{N}(z,h)E_0, \nu\rangle - \langle Op_h(\rho)E_0, \nu\rangle \Big\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)}. \tag{2.10}$$

Therefore (2.4) yields

$$\Big\| \langle Op_h(\rho) - \gamma\sqrt{z})E_0, \nu\rangle - hQ_4(E_0) \Big\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)}. \tag{2.11}$$

The commutator $[Op_h(\rho), \nu(x)]$ is a pseudodifferential operator with symbol in $hS_\delta^{0,0}$ and so

$$\|[Op_h(\rho), \nu_k(x)]E_{nor}\|_{H_h^j(\Gamma)} \ \leq \ C_2 h^{1-\delta} \|E_{nor}\|_{H_h^j(\Gamma)}, \quad k = 1,2,3, \quad j = 0,1.$$

The last estimate combined with (2.11) implies

$$\Big\| (Op_h(\rho) - \gamma\sqrt{z})E_{nor} - hQ_4(E_0) \Big\|_{L^2(\Gamma)} \leq C_3\Big(\frac{h}{|\operatorname{Im} z|} + h^{1-\delta}\Big) \|E_0\|_{L^2(\Gamma)}. \tag{2.12}$$

3. Eigenvalues-free regions

For $z \in Z_{1,\epsilon}$ we have $\rho \in S^{0,1}_{\delta}$ with $0 < \delta = 1/2 - \epsilon < 1/2$, while for $z \in Z_2 \cup Z_3$ we have $\rho \in S^{0,1}_0$ (see [7]). Since Γ is connected one has either $\gamma(x) > 1$ or $0 < \gamma(z) < 1$. We present the analysis in the case where $0 < \gamma(x) < 1$, $\forall x \in \Gamma$. The case $1 < \gamma(x)$ is reduced to this case at the end of the section. Clearly, there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0, \qquad \forall x \in \Gamma.$$

Combing (2.4) and (2.9), yields

$$\|\langle (Op_h(\rho) - \gamma(x)\sqrt{z})E_0, \nu(x)\rangle\|_{H^1_h(\Gamma)} \leq C\frac{h}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)} + C_1 h\|E_0\|_{H^1_h(\Gamma)},$$

where for $z \in Z_2 \cup Z_3$ we can replace $|\operatorname{Im} z|$ by 1. This estimate for E_0 and the estimate for the commutator $[Op_h(\rho), \nu_k(x)]$ imply

$$\|(Op_h(\rho) - \gamma(x)\sqrt{z})E_{nor}\|_{H^1_h(\Gamma)} \leq \frac{C_3\, h}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)} + C_4 h^{1-\delta}\|E_0\|_{H^1_h(\Gamma)}. \quad (3.1)$$

Let (x', ξ') be coordinates on $T^*(\Gamma)$. Consider the symbol

$$c(x', \xi', z) := \ \rho(x', \xi', z) \ - \ \gamma(x')\sqrt{z}, \qquad x' \in \Gamma.$$

Following the analysis in Section 3, [6], we know that c is elliptic in the case $0 < \gamma(x') < 1$ and if $z \in Z_1$ we have $c \in S^{0,1}_{\delta}$, $|\operatorname{Im} z| c^{-1} \in S^{0,-1}_{\delta}$, while if $z \in Z_2 \cup Z_3$ one gets $c \in S^{0,1}_0$, $c^{-1} \in S^{0,-1}_0$. This implies

$$\|Op_h(c^{-1})Op_h(c)E_{nor}\|_{H^1_h(\Gamma)} \ \leq \ \frac{C}{|\operatorname{Im} z|}\|Op_h(c)E_{nor}\|_{L^2(\Gamma)}.$$

On the other hand, according to Section 7 in [3], the symbol of the operator $Op_h(c^{-1})Op_h(c) - I$ is given by

$$\sum_{j=1}^{N}\frac{(ih)^j}{j!}\sum_{|\alpha|=j} D^{\alpha}_{\xi'}(c^{-1})(x', \xi')D^{\alpha}_{y'}c(y', \eta')\big|_{x'=y', \xi'=\eta'} + \tilde{b}_N(x', \xi')$$

$$:= b_N(x', \xi') + \tilde{b}_N(x', \xi'),$$

where

$$|\partial^{\alpha}_{x'}\tilde{b}_N(x', \xi')| \ \leq \ C_{\alpha} h^{N(1-2\delta) - s_d - |\alpha|/2}.$$

Taking into account the estimates for c^{-1} and c, and applying (2.5), and (2.6) yields

$$\Big\|\Big(Op_h(c^{-1})Op_h(c) - I\Big)E_{nor}\Big\|_{H^j_h(\Gamma)} \leq C_5\frac{h}{|\operatorname{Im} z|^2}\|E_{nor}\|_{H^j_h(\Gamma)}, \quad j = 0, 1.$$

Repeating the argument in Section 3 in [6] concerning the case $0 < \gamma(x') < 1$, for $z \in Z_1$ and $0 < \delta < 1/2$, one finds

$$\begin{aligned}\|E_{nor}\|_{H^1_h(\Gamma)} &\leq \Big\|\Big(Op_h(c^{-1})Op_h(c) - I\Big)E_{nor}\Big\|_{H^1_h(\Gamma)} \\ &\leq C_6 h^{1-2\delta}\|E_0\|_{L^2(\Gamma)} + C_5 h^{1-2\delta}\|E_{nor}\|_{H^1_h(\Gamma)} + C_7 h^{1-\delta}\|E_0\|_{H^1_h(\Gamma)}.\end{aligned} \quad (3.2)$$

Clearly,

$$\|E_0\|_{H^k_h(\Gamma)} \leq \|E_{tan}\|_{H^k_h(\Gamma)} + B_k\|E_{nor}\|_{H^k_h(\Gamma)}, \qquad k \in \mathbb{N}$$

with B_k independent of h. Hence we can absorb the terms involving the norms of E_{nor} in the right hand side of (3.2) choosing h small enough, and we get

$$\|E_{nor}\|_{H_h^1(\Gamma)} \leq Ch^{1-2\delta}\|E_{tan}\|_{H_h^1(\Gamma)}. \tag{3.3}$$

The analysis of the case $z \in Z_2 \cup Z_3$ is simpler since in the estimates above we have no coefficient $|\operatorname{Im} z|^{-1}$ and we obtain the same result with a factor h on the right hand side of (3.3).

With a similar argument it is easy to show that

$$\|E_{nor}\|_{L^2(\Gamma)} \ \leq \ C'h^{1-2\delta}\|E_{tan}\|_{L^2(\Gamma)}. \tag{3.4}$$

In fact from (2.11) one obtains

$$\Big\|Op_h(c^{-1})\Big[(\,Op_h(\rho) - \gamma\sqrt{z})E_{nor} - hQ_4(E_0)\Big]\Big\|_{L^2(\Gamma)} \leq C_8\Big(\frac{h}{\sqrt{|\operatorname{Im} z|}} + h^{1-\delta}\Big)\|E_0\|_{L^2(\Gamma)}$$

and

$$\|Op_h(c^{-1})Q_4(E_0)\|_{L^2(\Gamma)} \leq C_9\|E_0\|_{L^2(\Gamma)}.$$

Combining these estimates with the estimate of $\|Op_h(c^{-1})Op_h(c) - I\|_{L^2(\Gamma)\to L^2(\Gamma)}$ yields (3.4).

Going back to the equation (2.1), we have

$$\Big(D_\nu - \frac{1}{\gamma}\sqrt{z}\Big)E = \Big(D_\nu - \gamma\sqrt{z}\Big)E_{nor}\nu - \Big(\frac{1}{\gamma} - \gamma\Big)\sqrt{z}E_{nor}\nu$$
$$+\mathbf{i}hg_0(E_{tan}) + \Big(\operatorname{grad}_h(E_{nor})\Big)\Big|_{\tan}, \qquad x \in \Gamma. \tag{3.5}$$

Notice that for the first term on the right hand side of (3.5) we can apply the equality (2.4), while for E_{nor} and $\Big(\operatorname{grad}_h(E_{nor})\Big)\Big|_{\tan}$ we have a control by the estimate (3.3). Consequently, setting $E_0 = E|_\Gamma$, the right hand side of (3.5) is bounded by $Ch\|E_0\|_{H_h^1(\Gamma)}$. Next

$$1 \ < \ \frac{1}{1-\epsilon_0} \ \leq \ \frac{1}{\gamma(x)} \ \leq \ \frac{1}{\epsilon_0}, \qquad \forall x \in \Gamma.$$

This corresponds to the case (B) examined in Section 4 of [6]. The approximation of the operator $\mathcal{N}(z,h)$ given by (2.9) yields the estimate

$$\|(Op_h(\rho) - \frac{1}{\gamma}\sqrt{z})E_0\|_{L^2(\Gamma)} \ \leq \ C\frac{h}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)}. \tag{3.6}$$

For $z \in Z_1 \cup Z_3$ the symbol

$$d(x', \xi', z) := \rho(x', \xi', z) - \frac{1}{\gamma(x')}\sqrt{z}$$

is elliptic (see Section 4, [6]) and from (3.6) we obtain $E_0 = 0$ for h small enough. This implies $E = B = 0$.

Now recall that we have

$$\operatorname{Re}\lambda = -\frac{\operatorname{Im}\sqrt{z}}{h},\ \operatorname{Im}\lambda = \frac{\operatorname{Re}\sqrt{z}}{h}.$$

Suppose that $z \in Z_1$. Then

$$|\operatorname{Re}\lambda| \geq C(h^{-1})^{1-\delta},\ |\operatorname{Im}\lambda| \leq C_1 h^{-1} \leq C_2 |\operatorname{Re}\lambda|^{\frac{1}{1-\delta}}.$$

So if

$$|\operatorname{Re}\lambda| \geq C_3 |\operatorname{Im}\lambda|^{1-\delta},\ \operatorname{Re}\lambda \leq -C_4 < 0,$$

there are no eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b. In the same way we handle the case $z \in Z_3$ and we conclude that if $z \in Z_1 \cup Z_3$ for every $\epsilon > 0$ the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the domain $\Lambda_\epsilon \cup \mathcal{M}$, where

$$\mathcal{M} = \{z \in \mathbb{C} :\ |\arg z - \pi| \leq \pi/4,\ |z| \geq R_0 > 0,\ \operatorname{Re} z < 0\},$$

Λ_ϵ being the domain introduced in Theorem 1.1. Of course, if we consider the domain

$$Z_{3,\delta_0} = \{z \in \mathbb{C} :\ |\operatorname{Re} z| \leq 1,\quad \operatorname{Im} z = \delta_0 > 0\},$$

instead of Z_3, we obtain an eigenvalue-free region with $\mathcal{M}$ replaced by

$$\mathcal{M}_{\delta_0} = \{z \in \mathbb{C} :\ |\arg z - \pi| \leq \operatorname{a}rctg\,\delta_0,\ |z| \geq R_0(\delta_0) > 0,\ \operatorname{Re} z < 0\}.$$

The investigation of the case $z \in Z_2$ is more complicated since the symbol d may vanish for $\operatorname{Im} z = 0$ and $(x_0', \xi_0') \in T^*(\Gamma)$ satisfying the equation

$$\sqrt{1 + r_0(x_0', \xi_0')} - \frac{1}{\gamma(x_0')} \ = \ 0.$$

To cover this case and to prove that the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ with $z \in Z_2$ are confined in the domain $\mathcal{R}_N$, $\forall N \in \mathbb{N}$, we follow the arguments in [7] and [6]. For $z \in Z_2$ we introduce an operator $T(z,h)$ that yields a better approximation of $\mathcal{N}(z,h)$. In fact, $T(z,h)$ is defined by the construction of the semi-classical parametrix in Section 3, [7] for the problem (2.8) with $F = E_0$. We refer to [7] for the precise definition of $T(z,h)$ and more details. For our exposition we need the next proposition. Since $(\Delta - z)E = 0$, as in [7], we obtain

Proposition 3.1. *For $z \in Z_2$ and every $N \in \mathbb{N}$ we have the estimate*

$$\|\mathcal{N}(z,h)E_0 - T(z,h)E_0\|_{H_h^1(\Gamma)} \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)} \tag{3.7}$$

with constants $C_N, s_0 > 0$, independent of E_0, h and z, and s_0 independent of N.

Consider the system

$$\begin{cases} \Big(D_\nu - \frac{1}{\gamma}\sqrt{z}\Big)E_{tan} - \Big(\operatorname{grad}_h E_{nor}\Big)\Big|_{tan} + \mathbf{i}hg_0(E_{tan}) = 0, \quad x \in \Gamma, \\ \operatorname{div}_h E_{tan} + \operatorname{div}_h\Big(E_{nor}\nu\Big) = 0, \quad x \in \Gamma, \end{cases} \tag{3.8}$$

where $\operatorname{div}_h F = \sum_{k=1}^3 D_{x_k} F_k$.

Take the scalar product $\langle , \rangle_{L^2(\Gamma)}$ in $L^2(\Gamma)$ of the first equation of (3.8) and E_{tan}. Applying Green formula, it easy to see that

$$-\operatorname{Re}\langle \operatorname{grad}_h E_{nor}\Big|_{tan}, E_{tan}\rangle_{L^2(\Gamma)} = -\operatorname{Re}\langle \operatorname{div}_h E_{tan}, E_{nor}\rangle_{L^2(\Gamma)}. \tag{3.9}$$

We claim that

$$\operatorname{Im}\langle g_0(E_{tan}), E_{tan}\rangle_{L^2(\Gamma)} = 0. \tag{3.10}$$

Let $E_{tan} = (w_1, w_2, w_3)$. Then

$$\langle g_0(E_{tan}), E_{tan}\rangle_{\mathbb{C}^3} = \sum_{k,j=1}^3 w_k \frac{\partial \nu_k}{\partial_{x_j}} \overline{w_j} = \frac{1}{q} \sum_{k,j=1}^3 w_k \frac{\partial V_k}{\partial_{x_j}} \overline{w_j} = \frac{1}{q}\langle Sw, w\rangle_{\mathbb{C}^3},$$

where $S := \{\frac{\partial V_k}{\partial_{x_j}}\}_{k,j=1}^3$ with $V(x) = q(x)\nu(x)$, $q(x) > 0$ because $\sum_{k=1}^3 (\partial_{x_j} q) w_k \nu_k = 0$. Thus if the boundary is given locally by $x_3 = G(x_1, x_2)$, we choose $V(x) = (-\partial_{x_1} G, -\partial_{x_2} G, 1)$ and it is obvious that S is symmetric. Therefore $\operatorname{Im}\langle Sw, w\rangle_{\mathbb{C}^3} = 0$ and this proves the claim. Hence (3.10) implies

$$\operatorname{Re}[\mathbf{i}h\langle g_0(E_{tan}), E_{tan}\rangle_{L^2(\Gamma)}] = 0. \tag{3.11}$$

From the $L^2(\Gamma)$ scalar product of the second equation in (3.8) with E_{nor}, we obtain

$$\operatorname{Re}\langle \operatorname{div}_h E_{tan}, E_{nor}\rangle_{L^2(\Gamma)} + \operatorname{Re}\langle D_\nu E_{nor}, E_{nor}\rangle_{L^2(\Gamma)} = 0. \tag{3.12}$$

In fact,

$$\operatorname{div}_h(E_{nor}\nu) = D_\nu E_{nor} - \mathbf{i}hE_{nor}\operatorname{div}\nu$$

and $\operatorname{Im}\Big(\operatorname{div}\nu |E_{nor}|^2\Big) = 0$.

Taking together (3.9), (3.11) and (3.12), we conclude that

$$\operatorname{Re}\Big[\langle (D_\nu - \frac{\sqrt{z}}{\gamma})E_{tan}, E_{tan}\rangle_{L^2(\Gamma)} + \langle D_\nu E_{nor}\nu, E_{nor}\nu\rangle_{L^2(\Gamma)}\Big]$$

$$= \operatorname{Re}\Big\langle D_\nu E, E\Big\rangle_{L^2(\Gamma)} - \operatorname{Re}\langle \frac{\sqrt{z}}{\gamma}E_{tan}, E_{tan}\rangle_{L^2(\Gamma)} = 0.$$

Here we have used the fact that

$$\langle D_\nu E_{tan}, E_{nor}\nu\rangle_{\mathbb{C}^3} = D_\nu\Big(\langle E_{tan}, E_{nor}\nu\rangle_{\mathbb{C}^3}\Big) = 0.$$

Applying Proposition 3.1 with $E|_\Gamma = E_0$, yields

$$\Big|\operatorname{Re}\Big\langle T(z,h)E_0, E_0\Big\rangle_{L^2(\Gamma)} - \operatorname{Re}\Big\langle \frac{\sqrt{z}}{\gamma}E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big| \le C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)}. \tag{3.13}$$

For $z = -1$, as in Lemma 3.9 in [7] and Lemma 4.1 in [6], we have

$$|\operatorname{Re}\langle T(-1,h)E_0, E_0\rangle_{L^2(\Gamma)}| \le C_N h^{-s_0+N}\|E_0\|^2_{L^2(\Gamma)} = 0.$$

Consequently, by using Taylor formula for the real-valued function

$$\operatorname{Re}\Big[\Big\langle T(z,h)E_0, E_0\Big\rangle_{L^2(\Gamma)} - \Big\langle \frac{\sqrt{z}}{\gamma}E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big],$$

we get for every $N \in \mathbb{N}$ the estimate

$$\Big|\operatorname{Im}\Big[\Big\langle (\frac{\partial T}{\partial z}(z_t,h))E_0, E_0\Big\rangle_{L^2(\Gamma)} - \Big\langle \frac{\gamma_1}{2\sqrt{z_t}}E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big]\Big|$$
$$\le C_N \frac{h^{-s_0+N}}{|\operatorname{Im} z|}\|E_0\|^2_{L^2(\Gamma)}, \tag{3.14}$$

where $z_t = -1 + \mathbf{i}t\operatorname{Im} z$, $0 < t < 1$.

According to Lemma 3.9 in [7], in (3.14) we can replace $\frac{\partial T}{\partial z}(z_t, h)$ by $Op_h(\frac{\partial \rho}{\partial z}(z_t))$ and this yields an error term bounded by $Ch\|E_0\|^2_{H_h^{-1}(\Gamma)}$. On the other hand,

$$\Big|\Big\langle Op_h(\frac{\partial\rho}{\partial z}(z_t))E_{tan}, E_{nor}\nu\Big\rangle_{L^2(\Gamma)} + \Big\langle Op_h(\frac{\partial\rho}{\partial z}(z_t))E_{nor}, E_{tan}\nu\Big\rangle_{L^2(\Gamma)}\Big|$$
$$\le Ch\|E_0\|^2_{L^2(\Gamma)}$$

since the estimate (3.4) holds for $z \in Z_2$ and $\frac{\partial\rho}{\partial z}(z_t) \in S_0^{0,-1}$.

Thus the problem is reduced to a lower bound of

$$J := \Big|\mathrm{Im}\Big[\langle\Big(Op_h(\frac{\partial\rho}{\partial z}(z_t)) - \frac{\gamma_1}{2\sqrt{z}}\Big)E_{tan}, E_{tan}\rangle_{L^2(\Gamma)} + \langle Op_h(\frac{\partial\rho}{\partial z}(z_t))E_{nor}\nu, E_{nor}\nu\rangle_{L^2(\Gamma)}\Big]\Big|$$

$$\geq \Big|\mathrm{Im}\langle\Big(Op_h(\frac{\partial\rho}{\partial z}(z_t)) - \frac{\gamma_1}{2\sqrt{z}}\Big)E_{tan}, E_{tan}\rangle_{L^2(\Gamma)}\Big| - C_1\|E_{nor}\|^2_{L^2(\Gamma)}.$$

Since $\gamma_1(x) > 1,\ \forall x \in \Gamma$, applying the analysis of Section 4 in [6] for the scalar product involving E_{tan}, one deduces

$$\Big|\mathrm{Im}\Big\langle\Big(Op_h(\frac{\partial\rho}{\partial z}(z_t)) - \frac{\gamma_1}{2\sqrt{z}}\Big)E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big| \geq \eta_1\|E_{tan}\|^2_{L^2(\Gamma)}, \quad \eta_1 > 0.$$

By using once more the estimate (3.4), for h small enough we obtain

$$J \geq \eta_1\Big(\|E_{tan}\|^2_{L^2(\Gamma)} + \|E_{nor}\|^2_{L^2(\Gamma)}\Big) - B_0h\|E_{tan}\|^2_{L^2(\Gamma)} \geq \eta_2\|E_0\|^2_{L^2(\Gamma)},\ \ 0 < \eta_2 < \eta_1.$$

Consequently, (3.14) yields

$$(\eta_2 - B_1h)\|E_0\|^2_{L^2(\Gamma)} \leq C_N\frac{h^{-s_0+N}}{|\mathrm{Im}\, z|}\|E_0\|^2_{L^2(\Gamma)}$$

and for small h we conclude that for $z \in Z_2$ the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the region $\mathcal{R}_N$. This completes the analysis of the case $0 < \gamma(x) < 1,\ \forall x \in \Gamma$.

To study the case $\gamma(x) > 1,\ \forall x \in \Gamma$, we write the boundary condition in (1.1) as

$$\frac{1}{\gamma(x)}(\nu \wedge E_{tan}) - (\nu \wedge (\nu \wedge B_{tan})) = \frac{1}{\gamma(x)}(\nu \wedge E_{tan}) + B_{\mathrm{tan}} = 0.$$

Next

$$\nu \wedge E = \frac{1}{\sqrt{z}}\nu \wedge \frac{h}{\mathbf{i}}\mathrm{curl}\, B = -\frac{1}{\sqrt{z}}D_\nu B_{tan} + \frac{1}{\sqrt{z}}\Big[\Big(\mathrm{grad}_{\,h} B_{nor}\Big)\Big|_{tan} - \mathbf{i}hg_0(B_{tan})\Big]$$

and one obtains

$$\Big(D_\nu - \gamma(x)\sqrt{z}\Big)B_{tan} - \Big(\mathrm{grad}_{\,h}B_{nor}\Big)\Big|_{tan} + \mathbf{i}hg_0(B_{tan}) = 0,\ x \in \Gamma \tag{3.15}$$

which is the same as (2.1) with E_{tan}, E_{nor} replaced respectively by B_{tan}, B_{nor} and $\frac{1}{\gamma(x)}$ replaced by $\gamma(x) > 1$. We apply the operator $D_{y_1} - \gamma\sqrt{z}$ to the equation $\mathrm{div}\, B = 0$ and repeat without any change the above analysis concerning E_{tan}, E_{nor}. Thus the proof of Theorem 1.1 is complete.

Remark 3.2. The result of Theorem 1.1 holds for obstacles $K = \cup_{j=1}^J K_j$, where $K_j,\ j = 1, ..., J$ are open connected domains with C^∞ boundary and $K_i \cap K_j = \emptyset,\ i \neq j$. Let $\Gamma_j = \partial K_j,\ j = 1, ..., J$. In this case we may have $\gamma(x) < 1$ for some obstacles Γ_j and $\gamma(x) > 1$ for other ones. The proof extends with only minor modifications. The construction of the semi-classical parametrix in [7] is local and for the Dirichlet-to-Neumann map $\mathcal{N}_j(z, h)$ related to Γ_j we get the estimate

$$\|\mathcal{N}_j(z, h)(F) - Op_h(\rho + hb)F\|_{H^1_h(\Gamma_j)} \leq \frac{Ch}{\sqrt{|\mathrm{Im}\, z|}}\|F\|_{L^2(\Gamma_j)}.$$

The boundary condition in (1.1) is local and we can reduce the analysis to a fixed obstacle K_j. If $(E, B) \neq 0$ is an eigenfunction of G_b, our argument implies $E_{tan} = 0$ for $x \in \Gamma_j$ if $0 < \gamma(x) < 1$ on Γ_j and $B_{tan} = 0$ for $x \in \Gamma_j$ in the case $\gamma(x) > 1$ on Γ_j. By the boundary condition we get $E_{tan} = 0$ on Γ and this yields $E = B = 0$

since the Maxwell system with boundary condition $E_{tan} = 0$ has no eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

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