

# UPPER BOUND FOR THE COUNTING FUNCTION OF INTERIOR TRANSMISSION EIGENVALUES

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ABSTRACT. For the complex interior transmission eigenvalues (ITE) we study for small  $\theta > 0$  the counting function

$$N(\theta, r) = \#\{\lambda \in \mathbb{C} : \lambda \text{ is (ITE), } |\lambda| \leq r, |\arg \lambda| \leq \theta\}.$$

We obtain for fixed  $\theta > 0$  an upper bound  $N(\theta, r) \leq Cr^{n/2}$ ,  $r \geq r(\theta)$ .

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  be a bounded connected open domain with smooth boundary  $\partial\Omega$ . A complex number  $\lambda \neq 0$  is called an interior transmission eigenvalue (ITE) if and only if there exist  $u \neq 0$ ,  $v \neq 0$ ,  $u, v \in H^2(\Omega)$ , so that

$$\begin{cases} -\Delta u - \lambda u = 0, & x \in \Omega, \\ -\Delta v - \lambda(1 + m(x))v = 0, & x \in \Omega, \\ \gamma_0(u - v) = 0, & \gamma_0 \partial_\nu(u - v) = 0. \end{cases} \quad (1.1)$$

Here  $\nu$  is the exterior unit normal to  $\partial\Omega$  and  $\gamma_0$  is the operator of restriction on  $\partial\Omega$ . In this work we suppose that  $m(x) \in C^\infty(\bar{\Omega})$ , is a real-valued function satisfying the conditions

$$1 + m(x) \geq \delta_0 > 0, \forall x \in \bar{\Omega}, \quad m(x) \neq 0, \forall x \in \partial\Omega. \quad (1.2)$$

Consider in the space  $\mathcal{H} = L^2(\Omega) \oplus L^2(\Omega)$  the operator

$$\mathcal{P} = \begin{pmatrix} -\Delta & 0 \\ 0 & -\frac{1}{1+m} \Delta \end{pmatrix}$$

with domain

$$D(\mathcal{P}) = \{(u, v) \in \mathcal{H} : \Delta u \in L^2(\Omega), \Delta v \in L^2(\Omega), (u - v)|_{\partial\Omega} = 0, \partial_\nu(u - v)|_{\partial\Omega} = 0\}.$$

This operator is closed (see Lemma 1) and the (ITE) are the eigenvalues of  $\mathcal{P}$  in  $\mathbb{C} \setminus \{0\}$ .

The (ITE) are related to the scattering theory. In particular, if  $\lambda = k^2 \in \mathbb{R}$  is not a real (ITE), then the far-field operator  $F(\lambda) : L^2(\mathbb{S}^{n-1}) \rightarrow L^2(\mathbb{S}^{n-1})$  with kernel the scattering amplitude  $a(\lambda, \theta, \omega)$  is injective and has a dense range. The (ITE) play an important role in the inverse scattering methods and there is an increasing interest on the problems of the existence and distribution of (ITE). The reader may consult the survey [6] for a comprehensive review of recent results and

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references.

There are several works proving that under some conditions the (ITE) form a discrete set in  $\mathbb{C}$  (see [5], [9], [10], [6], [23]). In the case when  $m(x)$  satisfies (1.2) the results in [23], [17] guarantee that (ITE) are discrete with finite dimensional generalized eigenspaces. There are many works on the distribution of the (ITE) (see [2], [12], [13], [14], [19], [11], [17] and the references cited there). The distribution of the positive (ITE) has been studied in [13], [14], [19] and a lower bound for the number of the positive (ITE) has been established. The analysis of the complex (ITE) is more difficult than that of the positive ones. Under the assumption (1.2), it has been proved in [25] that for every  $\epsilon > 0$  outside a parabolic neighborhood of  $\mathbb{R}^+$  having the form

$$\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq C_\epsilon (|\operatorname{Re} \lambda| + 1)^{3/4+\epsilon}, \operatorname{Re} \lambda > 0\}$$

there are at most a finite number of (ITE) (see also [11] for an weaker result). From this it follows that the (ITE) can have accumulation point only at infinity and for every fixed  $0 < \theta < \pi/2$  outside the domain

$$D(\theta) = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \theta\}$$

we have at most finite number (ITE) (see also [15] for another proof of this result). The multiplicity of an (ITE)  $\lambda$  is given by the dimension of the eigenspace  $E_\lambda$  generated by the eigenfunctions and the generalized eigenfunctions corresponding to  $\lambda$ . In this direction we mention the interesting work [16], where the case of a constant function  $m(x) = m$  and the ball  $\{x \in \mathbb{R}^n : |x| \leq 1\}$  has been studied. This work shows that even in the case  $n = 1$  if  $m = p/q$  is rational and  $p/q$  is irreducible, the geometric and algebraic multiplicities of an (ITE) may be different (see Theorem 2.4 and 2.5 in [16]). The reader should consult also [24] for the analysis of the case when  $\Omega = \{|x| \leq 1\}$  and  $n = 1$ . In the following in the definition of the counting function  $N(r) = \#\{\lambda \in \mathbb{C} : \lambda \text{ is (ITE), } |\lambda| \leq r\}$  every eigenvalue  $\lambda$  will be counted with its *algebraic* multiplicity. Moreover, it follows from Theorem 3, [17], that under the assumptions (1.2) the set of the eigenvalues of  $\mathcal{P}$  is infinite and it is natural to study the asymptotics of  $N(r)$  as  $r \rightarrow \infty$ .

In [12] a Weyl type asymptotics has been obtained in the *anisotropic case*, when in (1.1) for  $v$  we have the equation  $-\nabla A(x) \nabla v - \lambda(1 + m(x))v = 0$  with a positive definite matrix  $A(x)$  different from the identity. Under some conditions on  $A(x)$ , which guarantee that the boundary problem is parameter-elliptic (see [1]), it is possible to apply the results in [3]. More precisely, in [12] it is shown that the counting function  $N(r)$  has the asymptotics

$$N(r) \sim \frac{\omega_n}{(2\pi)^n} \int_{\Omega} \left(1 + \frac{(1 + m(x))^{n/2}}{(\det A(x))^{1/2}}\right) dx r^{n/2}, \quad r \rightarrow \infty, \quad (1.3)$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . It is important to note that in the case treated in [12] the estimates for the solution in [1] imply that the domain of the corresponding operator is included in  $H^2(\Omega) \times H^2(\Omega)$  and this is one of the conditions for the application of the results in [3].

The *isotropic case* when  $A(x) = I$  leads to several difficulties. First, the boundary problem (1.1) is not parameter-elliptic and the matrix operator related to (1.1) is not self-adjoint. Second, the domain  $D(\mathcal{P})$  of  $\mathcal{P}$  is not included in  $H^2(\Omega) \times H^2(\Omega)$  (see Section 2) and the results in [3], [4] cannot be applied directly. In [17] Robbiano proved that in the isotropic case we have

$$N(r) \leq Cr^{n/2+2}, \quad r \rightarrow +\infty,$$

covering also the case of complex-valued  $m(x)$ .

In this paper we study for small  $\theta > 0$  the upper bounds of the counting function

$$N(\theta, r) = \#\{\lambda \in \mathbb{C} : \lambda \text{ is (ITE), } |\lambda| \leq r, |\arg \lambda| \leq \theta\}.$$

Our main result is the following

**Theorem 1.** *Let  $0 < \theta < \pi/2$ . Then for small  $\theta$  there exist  $0 < \epsilon(\theta) \rightarrow 0$  as  $\theta \rightarrow 0$  and  $C_\theta > 0$ ,  $r(\theta) > 0$  so that*

$$N(\theta, r) \leq 3\sqrt{3}(1 + \epsilon(\theta)) \frac{\omega_n}{(2\pi)^n} \int_{\Omega} \left(1 + (1 + m(x))^{n/2}\right) dx r^{n/2} + C_\theta, \quad (1.4)$$

provided  $r \geq r(\theta)$ .

Since outside the angle  $\{z \in \mathbb{C} : |\arg z| \leq \theta\}$  we have only finite number of eigenvalues (see [25], [15]), from our result we obtain

**Corollary 1.** *For the counting function  $N(r)$  we have the upper bound*

$$N(r) \leq 3\sqrt{3}(1 + o(1)) \frac{\omega_n}{(2\pi)^n} \int_{\Omega} \left(1 + (1 + m(x))^{n/2}\right) dx r^{n/2} \quad (1.5)$$

with  $o(1) \rightarrow 0$  as  $r \rightarrow \infty$ .

The constant  $3\sqrt{3}(1 + \epsilon(\theta))$  is not optimal and the factor  $3\sqrt{3}$  comes from the choice of the cut-off function  $f$  in Section 5 and the Weyl inequality for non-compact operators. By a more complicated argument in Section 5 we can slightly improve this constant. On the other hand, for  $n = 1$  and  $\Omega = \{|x| \leq 1\}$ ,  $m(x) = \text{const}$ , a precise Weyl formula has been obtained in [16]. After the submission of this paper we have been informed that Robbiano [18] obtained the leading term in (1.5) without the factor  $3\sqrt{3}(1 + o(1))$ . In the special case  $m(x) > 0$ ,  $\forall x \in \bar{\Omega}$ , a such asymptotics has been proved recently by Faierman [8].

The problem of sharp Weyl asymptotic of the counting function  $N(r)$  with a remainder  $\mathcal{O}(r^{n/2-1/2})$  is open. In this direction the construction of a semi-classical parametrix in Section 3 and our analysis in Section 5 of the counting function of the self-adjoint operator  $Q$  defined below could be useful and we hope to discuss in a further work the problem of the remainder in the asymptotic of  $N(r)$ .

We expect that the (ITE) are mainly concentrated in a neighborhood of the positive real axis and in this direction it is natural to make the conjecture that for every fixed  $a > 0$  there exists  $\eta = \eta(a) > 0$  such that

$$\#\{\lambda \in \mathbb{C} : \lambda \text{ is (ITE), } |\lambda| \leq r, |\operatorname{Im} \lambda| \geq a\} \sim \mathcal{O}(r^{n/2-\eta}), \quad r \rightarrow +\infty. \quad (1.6)$$

The proof of Theorem 1 is based on the semi-classical techniques developed in [20], [21], [22] for the distribution of the scattering resonances. First we write the spectral problem as a problem for a matrix operator  $\mathcal{P}$  which is convenient for our argument. Next our goal is to study the eigenvalues of the operator  $h^2\mathcal{P} - \omega_0$ ,  $0 < h \ll 1$ , where  $\omega_0 \in \mathbb{C}$  with  $\operatorname{Re} \omega_0 = 1$  and small  $0 < \operatorname{Im} \omega_0 < 1$  is fixed. We apply the strategy of [22] and investigate the eigenvalues of the self-adjoint operator

$$Q = (h^2\mathcal{P} - \omega_0)^*(h^2\mathcal{P} - \omega_0), \quad 0 < h \ll 1$$

with suitably boundary conditions on  $\partial\Omega$ . We show that the operator  $Q - z$  is elliptic with parameter for  $z \in \mathbb{C} \setminus [(\operatorname{Im} \omega_0)^2, +\infty[$ . The next step is to examine the trace class operator  $f(Q)$  for  $f \in$

$C_0^\infty([-\infty, (\operatorname{Im} \omega_0 + 3\delta)^2]; [0, 1])$  which is equal to 1 on  $[0, (\operatorname{Im} \omega_0 + 2\delta)^2]$ . To calculate the trace of  $f(Q)$ , we need a representation of the resolvent  $(Q - z)^{-1}$  and for this purpose we construct a semi-classical parametrix for  $Q - z$  in the region  $|\operatorname{Im} z| \geq h^\epsilon$ ,  $0 < \epsilon < 1/2$  and a microlocal parametrix in the region  $|\xi| \geq C_0 > 0$  with a large constant  $C_0$ . We calculate the leading term of  $\operatorname{tr} f(Q)$  and this yields an estimate for the number of the eigenvalues  $\mu_1, \dots, \mu_N$  of the operator  $|Q|^{1/2}$  in the interval  $[\operatorname{Im} \omega_0, \operatorname{Im} \omega_0 + 2\delta]$ . Next to obtain a bound for the eigenvalues of  $h^2 \mathcal{P}$  in the disk  $\{z \in \mathbb{C} : |z - \omega_0| \leq \operatorname{Im} \omega_0 + \delta\}$  we apply the Weyl inequality for non-compact operators proved in Appendix a. in [20]. This leads to an estimate of the eigenvalues of  $\mathcal{P}$  included in the rectangle  $h^{-2}S_0$ , where

$$S_0 = \{(x, y) \in \mathbb{R}^2 : 1 - \delta_2 \leq x \leq 1 + \delta_2, -X \leq y \leq \operatorname{Im} \omega_0\}$$

with  $\delta_2 = \sqrt{1 - \delta^2}(2\delta \operatorname{Im} \omega_0 + \delta^2)^{1/2}$ ,  $\tan \theta = \frac{X}{1 - \delta_2}$  and  $X \searrow 0$  as  $\delta \searrow 0$ . Summing over the union of rectangles, we obtain the result of Theorem 1.

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## 2. ELLIPTIC BOUNDARY PROBLEM FOR $Q - z$

Consider in  $\mathcal{H} = L^2(\Omega) \oplus L^2(\Omega)$  the operator

$$\mathcal{P} = \begin{pmatrix} -\Delta & 0 \\ 0 & -\frac{1}{1+m}\Delta \end{pmatrix}$$

with domain  $D(\mathcal{P})$  introduced in Section 1. We have the following

**Lemma 1.** *The operator  $\mathcal{P}$  with domain  $D(\mathcal{P})$  is closed in  $\mathcal{H}$ .*

**Proof.** Let  $D(\mathcal{P}) \ni (u_j, v_j) \rightarrow (u, v)$  in  $\mathcal{H}$  and let  $\mathcal{P}(u_j, v_j) \rightarrow (f, g)$  in  $\mathcal{H}$  as  $j \rightarrow \infty$ . Since  $(\Delta u_j, \Delta v_j) \rightarrow (\Delta u, \Delta v)$  in the distribution sense, we deduce  $\mathcal{P}(u, v) = (f, g)$ . Next,  $\Delta(u_j - v_j) \rightarrow f - (1+m)g \in L^2(\Omega)$  and  $(u_j - v_j)|_{\partial\Omega} = \partial_\nu(u_j - v_j)|_{\partial\Omega} = 0$ . The Dirichlet and Neumann problems for  $\Delta$  in  $\Omega$  are related to closed operators in  $L^2(\Omega)$  and this implies

$$(u - v)|_{\partial\Omega} = \partial_\nu(u - v)|_{\partial\Omega} = 0.$$

□

It is clear that the (ITE) are the eigenvalues  $\lambda \neq 0$  of the operator  $\mathcal{P}$ . On the other hand, the domain  $D(\mathcal{P})$  is not included in  $H^2(\Omega) \oplus H^2(\Omega)$ . This leads to difficulties if we want to apply the results in [3], [4] for the spectral asymptotics of non-self-adjoint operators with domains included in  $H^2(\Omega) \oplus H^2(\Omega)$ .

To establish the discreteness of the spectrum of  $\mathcal{P}$  in  $\mathbb{C} \setminus \{0\}$ , it is more convenient to pass to the problem studied by Sylvester [23] and Robbiano [17]. For  $\lambda \neq 0$  we reduce (1.1) to the problem

$$\begin{cases} (\Delta + \lambda(1+m))U + mV = 0, & (\Delta + \lambda)V = 0 \text{ in } \Omega, \\ u = 0, & \partial_\nu u = 0 \text{ on } \partial\Omega, \end{cases} \quad (2.1)$$

where  $U = v - u$ ,  $V = \lambda u$ . Let

$$B = \begin{pmatrix} \Delta_{00} & m \\ 0 & \Delta_{--} \end{pmatrix}, \quad I_m = \begin{pmatrix} (1+m) & 0 \\ 0 & 1 \end{pmatrix}.$$

Here  $\Delta_{00}$  is the Laplacian with domain  $H_0^2(\Omega)$  and  $\Delta_{--}$  is the Laplacian with domain  $\{v \in L^2(\Omega) : \Delta v \in L^2(\Omega)\}$ . As it is mentioned in [23],  $B$  is a closed non-self-adjoint operator in  $\mathcal{H}$  with domain

$$D(B) = H_0^2(\Omega) \oplus \{v \in L^2(\Omega) : \Delta v \in L^2(\Omega)\}.$$

The problem (2.1) can be written as  $(B + \lambda I_m)(u, v) = (0, 0)$ . In [23] it was proved that the resolvent  $(I_m^{-1}B + \lambda_0)^{-1}$  exists for real  $\lambda_0 \ll 1$  and this resolvent is upper triangular compact (UTC) operator (see Section 3 in [23] for the definition of (UTC)). Following [23] (see also [17] for another proof in a more general setup),  $(I_m B + \lambda)^{-1}$  has a meromorphic extension in  $\mathbb{C}$ . Moreover, in the Laurent expansion of  $(I_m^{-1}B + \lambda)^{-1}$  in a neighborhood of a pole the coefficients are finite dimensional operators. A simple calculus for  $\lambda \neq 0$  yields

$$(\mathcal{P} - \lambda) = - \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+m} \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & \lambda^{-1} \end{pmatrix} (B + \lambda I_m) \begin{pmatrix} -1 & 1 \\ \lambda & 0 \end{pmatrix}, \quad (2.2)$$

$$(\mathcal{P} - \lambda)^* = - \begin{pmatrix} -1 & \bar{\lambda} \\ 1 & 0 \end{pmatrix} (B + \lambda I_m)^* \begin{pmatrix} 0 & 1 \\ \bar{\lambda}^{-1} & \bar{\lambda}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+m} \end{pmatrix}, \quad (2.3)$$

$$(\mathcal{P} - \lambda)^{-1} = - \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & \lambda^{-1} \end{pmatrix} (B + \lambda I_m)^{-1} \begin{pmatrix} -1 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+m \end{pmatrix}. \quad (2.4)$$

Obviously, we have  $\begin{pmatrix} 0 & \lambda^{-1} \\ 1 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ \lambda & 0 \end{pmatrix} = I$ . Consequently,  $(\mathcal{P} - \lambda)^{-1}$  is meromorphic in  $\mathbb{C} \setminus \{0\}$  and  $\mathcal{P}$  has a discrete set of eigenvalues in  $\mathbb{C} \setminus \{0\}$  with finite algebraic multiplicity. Moreover, the algebraic multiplicity of the eigenvalues  $\lambda_0$  of  $\mathcal{P}$  coincide with the algebraic multiplicity of the eigenvalue  $-\lambda_0$  of  $I_m^{-1}B$ .

**Remark 1.** *If  $(u, v) \in D(\mathcal{P})$  is an eigenfunction of  $\mathcal{P}$  with eigenvalue  $\lambda \neq 0$ , then the problem (1.1) has a weak solution. It is proved in Theorem 5.1, [15] that this weak solution is always in  $H^2(\Omega) \oplus H^2(\Omega)$ . This agrees with the definition of eigenfunction in Section 1.*

Let  $\omega_0 \in \mathbb{C}$  be fixed with  $\operatorname{Re} \omega_0 > 0$ ,  $\operatorname{Im} \omega_0 > 0$ . Consider the operator  $h^2 \mathcal{P} - \omega_0$ ,  $0 < h \ll 1$ . Throughout our paper we use the notation  $D_{x_j} = \frac{1}{i} \partial_{x_j}$ ,  $j = 1, \dots, n$ . It is important to show that the boundary problem for  $h^2 \mathcal{P} - \omega_0$  is elliptic in the semi-classical sense. To study the problem close to the boundary, we pass to local coordinates. Given a point  $x_0 \in \partial\Omega$ , introduce boundary **normal coordinates**  $x = \alpha(y', y_n)$  in a neighborhood of 0 in  $\mathbb{R}^n$  by

$$x = \alpha(y) = \alpha(y', 0) + y_n N(y', 0), \quad y = (y', y_n),$$

where  $N(y', 0)$  is the unit normal at  $(y', 0)$  and  $y' = (y_1, \dots, y_{n-1})$ . For simplicity we will use below the notation  $x = (x', x_n)$ ,  $x' = (x_1, \dots, x_{n-1})$  for the new coordinates. Locally near  $x_0$ , the boundary  $\partial\Omega$  is defined by  $x_n = 0$  and we have  $x_n > 0$  in the domain  $\Omega$ . The principal symbol of  $-h^2 \Delta - \omega_0$  in the new coordinates becomes

$$p_0(x, \xi) = \xi_n^2 + s(x, \xi') - \omega_0,$$

**Lemma 2.** *The boundary problem*

$$\begin{cases} (h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u = 0, & x_n > 0, \\ (h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0)v = 0, & x_n > 0, \\ u(0) = v(0), & (hD_{x_n}u)(0) = (hD_{x_n}v)(0), \end{cases} \quad (2.5)$$

*has only a trivial bounded solution  $u = v = 0$ .*

**Proof.** Let  $\xi_n = \tilde{\lambda}_\pm(x', \xi')$ ,  $j = 1, 2$ , be the roots of  $\xi_n^2 + s(x', 0, \xi') - \omega_0 = 0$  with respect to  $\xi_n$  with  $\pm \operatorname{Im} \tilde{\lambda}_\pm(x', \xi') > 0$  and let  $\xi_n = \tilde{\mu}_\pm(x', \xi')$  be the roots of  $\xi_n^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0 = 0$  with respect to  $\xi_n$  with  $\pm \operatorname{Im} \tilde{\mu}_\pm(x', \xi') > 0$ . Then the problem

$$(h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u = 0, (h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x))\omega_0)v = 0, x_n > 0$$

with boundary conditions  $u(0) = v(0)$ ,  $hD_{x_n}u(0) = hD_{x_n}v(0)$  has no bounded non-trivial solutions. In fact, we have the representation

$$u(x_n) = \tilde{C}_1(x', \xi')e^{ix_n\tilde{\lambda}_+/h}, v(x_n) = \tilde{C}_2(x', \xi')e^{ix_n\tilde{\mu}_+/h}$$

and we get  $\tilde{C}_1(x', \xi') = \tilde{C}_2(x', \xi')$ ,  $(\tilde{\lambda}_+ - \tilde{\mu}_+)\tilde{C}_1(x', \xi') = 0$ . It is easy to see that  $m(x', 0) \neq 0$  implies  $\tilde{\lambda}_+ \neq \tilde{\mu}_+$  and we obtain  $\tilde{C}_1 = \tilde{C}_2 = 0$ .  $\square$

Let  $(h^2\mathcal{P} - \omega_0)^*$  be the adjoint operator to  $h^2\mathcal{P} - \omega_0$ . The adjoint operator  $B^*$  has the representation

$$B^* = \begin{pmatrix} \Delta_{--} & 0 \\ m & \Delta_{00} \end{pmatrix},$$

hence the domain of  $B^*$  is given by

$$D(B^*) = \{(u, v) \in L^2(\Omega) \oplus H_0^2(\Omega) : \Delta u \in L^2(\Omega)\}.$$

Therefore we obtain easily from (2.3) that the operator  $(h^2\mathcal{P} - \omega_0)^*$  has domain

$$D((h^2\mathcal{P} - \omega_0)^*) = \{(u, v) \in L^2(\Omega) \oplus L^2(\Omega) : \Delta u \in L^2(\Omega), \Delta(\frac{1}{1+m}v) \in L^2(\Omega),$$

$$(u + \frac{1}{1+m}v)|_{\partial\Omega} = 0, \partial_\nu(u + \frac{1}{1+m}v)|_\Omega = 0\}.$$

Next we introduce the operator

$$Q = (h^2\mathcal{P} - \omega_0)^*(h^2\mathcal{P} - \omega_0), 0 < h \ll 1$$

with boundary conditions

$$\gamma_0(u - v), \gamma_0(\partial_\nu(u - v)) = 0, \quad (2.6)$$

$$\gamma_0(h^2\Delta u + \omega_0 u) = \gamma_0\left(\frac{1}{1+m}\left(-\frac{h^2}{1+m}\Delta v - \omega_0 v\right)\right), \quad (2.7)$$

$$\gamma_0(\partial_\nu(h^2\Delta u + \omega_0 u)) = \gamma_0\left(\partial_\nu\left(\frac{1}{1+m}\left[-\frac{h^2}{1+m}\Delta v - \omega_0 v\right]\right)\right). \quad (2.8)$$

Clearly, the boundary conditions are determined from the inclusion

$$(h^2\mathcal{P} - \omega_0)(u, v) \subset D(h^2\mathcal{P} - \omega_0)^*.$$

This implies immediately the following

**Proposition 1.** *The operator  $Q$  with domain*

$$D = \{(u, v) \in L^2(\Omega) \oplus L^2(\Omega) : (u, v) \in D(\mathcal{P}), (h^2\mathcal{P} - \omega_0)(u, v) \in D(h^2\mathcal{P} - \omega_0)^*\}$$

*is self-adjoint.*

The determinant of the principal symbol of  $Q$  is

$$q(x, \xi) = (|\xi|^2 - \bar{\omega}_0)(|\xi|^2 - \omega_0) \left( \frac{1}{1+m} |\xi|^2 - \bar{\omega}_0 \right) \left( \frac{1}{1+m} |\xi|^2 - \omega_0 \right)$$

and the symbol  $q(x, \xi)$  is clearly elliptic since

$$|q(x, \xi)| \geq (\text{Im } \omega_0)^4.$$

Now we pass to the ellipticity of the boundary problems for  $Q$  and  $Q - z$ . First we study the boundary problem for  $Q$  and establish the following

**Proposition 2.** *The boundary problem*

$$\begin{cases} (h^2 D_{x_n}^2 + s(x', 0, \xi') - \bar{\omega}_0)(h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u = 0, & x_n > 0, \\ (h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\bar{\omega}_0)(h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0)v = 0, & x_n > 0, \\ u(0) = v(0), & (hD_{x_n}u)(0) = (hD_{x_n}v)(0), \\ -\left((h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u\right)(0) = \frac{1}{(1+m(x', 0))^2} \left( \left[ h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0 \right] v \right)(0), \\ -hD_{x_n} \left( (h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u \right)(0) = \frac{1}{(1+m(x', 0))^2} hD_{x_n} \left( \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi')) - (1 + m(x', 0))\omega_0 \right] v \right)(0). \end{cases} \quad (2.9)$$

has only a trivial bounded solution  $u = v = 0$ .

**Proof.** Multiply the second equation by  $\frac{1}{(1+m(x', 0))^2}$ . Then

$$w_1 = -(h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u$$

and

$$w_2 = \frac{1}{(1 + m(x', 0))^2} \left[ h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0 \right] v$$

are solutions of the equations

$$(h^2 D_{x_n}^2 + s(x', 0, \xi') - \bar{\omega}_0)w_1 = 0, \quad x_n > 0,$$

$$(h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\bar{\omega}_0)w_2 = 0, \quad x_n > 0.$$

Exploiting the boundary conditions for  $w_1$  and  $w_2$  and the above argument with  $\omega_0$  replaced by  $\bar{\omega}_0$ , we conclude that  $w_1 = w_2 = 0$ . Therefore

$$(h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0)u = (h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0)v = 0$$

Repeating the argument of Lemma 2, we get  $u = v = 0$  and this completes the proof.  $\square$

Next we choose  $z \in \mathbb{C} \setminus [(\text{Im } \omega_0)^2, \infty[$  and consider the operator  $Q - z$ . We will show that  $Q - z$  with boundary conditions (2.6)-(2.8) is elliptic in the semi-classical sense. Set  $p(x, \xi', \xi_n) = \xi_n^2 + s(x, \xi')$  and let

$$(p(x, \xi', \xi_n) - \bar{\omega}_0)(p(x, \xi', \xi_n) - \omega_0) - z = 0 \quad (2.10)$$

have roots  $\lambda_{j,+}(x, \xi', z)$ ,  $j = 1, 2$  in the upper half plane and roots  $\lambda_{j,-}(x, \xi', z)$ ,  $j = 1, 2$  in the lower half plane. Also let

$$(p(x, \xi', \xi_n) - (1 + m(x))\bar{\omega}_0)(p(x, \xi', \xi_n) - (1 + m(x))\omega_0) - (1 + m(x))^2 z = 0 \quad (2.11)$$

have roots  $\mu_{j,+}(x, \xi', z)$ ,  $j = 1, 2$  in the upper half plane and roots  $\mu_{j,-}(x, \xi', z)$ ,  $j = 1, 2$  in the lower half plane.

**Lemma 3.** *If  $z \in \mathbb{C} \setminus [(\operatorname{Im} \omega_0 + \delta)^2, \infty[$  and  $\delta > 0$  is small enough, we have  $\lambda_{j,+}(x', 0, \xi', z) \neq \mu_{k,+}(x', 0, \xi', z)$ ,  $j = 1, 2$ ,  $k = 1, 2$ .*

**Proof.** For simplicity we write  $m$  instead of  $m(x', 0)$ . If we have  $\lambda_{j,+} = \mu_{k,+}$ , then

$$(p(x', 0, \xi', \lambda_{j,+}) - (1 + m)\bar{\omega}_0)(p(x', 0, \xi', \lambda_{j,+}) - (1 + m)\omega_0) - (1 + m)^2 z = 0$$

yields

$$-\omega_0(p(x', 0, \xi', \lambda_{j,+}) - \bar{\omega}_0) - \bar{\omega}_0(p(x', 0, \xi', \lambda_{j,+}) - (1 + m)\omega_0) = (2 + m)z.$$

Thus we get

$$\frac{(m + 2)(|\omega_0|^2 - z)}{2 \operatorname{Re} \omega_0} = p(x', 0, \xi', \lambda_{j,+}).$$

On the other hand, the equality  $p^2 - 2 \operatorname{Re} \omega_0 p + |\omega_0|^2 - z = 0$  with  $\xi_n = \lambda_{j,+}$  leads to

$$\frac{(2 + m)^2(|\omega_0|^2 - z)^2}{4(\operatorname{Re} \omega_0)^2} - (1 + m)(|\omega_0|^2 - z) = 0.$$

For small  $\delta > 0$  it is clear that  $|\omega_0|^2 - z \neq 0$ . In fact, if  $z$  is real, we have  $(\operatorname{Im} \omega_0)^2 - \operatorname{Re} z > -o(\delta)$ . Next, dividing by  $|\omega_0|^2 - z$ , we obtain

$$(\operatorname{Im} \omega_0)^2 + (\operatorname{Re} \omega_0)^2 - z = \frac{4(1 + m)}{(2 + m)^2} (\operatorname{Re} \omega_0)^2,$$

hence

$$(\operatorname{Im} \omega_0)^2 - z = -\frac{m^2}{(2 + m)^2} (\operatorname{Re} \omega_0)^2$$

which is impossible for small  $\delta$ . □

Now we will examine the boundary problem for  $Q - z$  and prove the following

**Proposition 3.** *Let  $z \in \mathbb{C} \setminus [(\operatorname{Im} \omega_0)^2, +\infty[$ . Then the following problem*

$$\left\{ \begin{array}{l} \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi') - \bar{\omega}_0)(h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0) - z \right] u = 0, \quad x_n > 0, \\ \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\bar{\omega}_0)(h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0) \right. \\ \left. - (1 + m(x', 0))^2 z \right] v = 0, \quad x_n > 0, \\ u(0) = v(0), \quad (h D_{x_n} u)(0) = (h D_{x_n} v)(0), \\ -\left( (h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0) u \right)(0) = \frac{1}{(1 + m(x', 0))^2} \left( \left[ h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0 \right] v \right)(0), \\ -h D_{x_n} \left( (h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0) u \right)(0) = \frac{1}{(1 + m(x', 0))^2} h D_{x_n} \left( \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi')) - (1 + m(x', 0))\omega_0 \right] v \right)(0). \end{array} \right. \quad (2.12)$$

has only the trivial solution  $u = v = 0$ .

**Proof.** For simplicity of the notations we will write  $m, s$  instead of  $m(x', 0), s(x', 0, \xi')$ . We can simplify the last two boundary conditions in (2.12). In fact, since  $u(0) = v(0)$ , the third boundary condition can be written as follows

$$-(1 + m)^2 (h^2 D_{x_n}^2 u)(0) - (2 + 2m + m^2)(s - \omega_0)u(0) + m\omega_0 u(0) - (h^2 D_{x_n}^2 v)(0) = 0.$$

A similar modification can be done for the fourth boundary condition by using  $(D_{x_n} u)(0) = (D_{x_n} v)(0)$ . Introduce the semi-classical symbol

$$Y(x', \xi') := (2 + 2m(x', 0) + m^2(x', 0))(\omega_0 - s(x', 0, \xi')) + m(x', 0)\omega_0.$$



Then the last two boundary conditions in (2.12) have the form

$$\begin{aligned} -(1+m)^2 h^2 D_{x_n}^2 u(0) - h^2 D_{x_n}^2 v(0) + Y u(0) &= 0, \\ -(1+m)^2 h^3 D_{x_n}^3 u(0) - h^3 D_{x_n}^3 v(0) + Y h D_{x_n} u(0) &= 0. \end{aligned}$$

First we consider the case when  $z$  is real and  $z < (\operatorname{Im} \omega_0)^2$ . We will apply Proposition 2 with suitable choice of  $\omega_0$ . The equation (2.10) has the form

$$\begin{aligned} &(p(x', 0, \xi', \xi_n) - \bar{\omega}_0)(p(x', 0, \xi', \xi_n) - \omega_0) - z \\ &= p(x', 0, \xi', \xi_n)^2 - 2(\operatorname{Re} \omega_0)p(x', 0, \xi', \xi_n) + (\operatorname{Re} \omega_0)^2 + \left((\operatorname{Im} \omega_0)^2 - \operatorname{Re} z\right) = 0. \end{aligned}$$

Choose  $\omega_1 \in \mathbb{C}$  so that  $\operatorname{Re} \omega_1 = \operatorname{Re} \omega_0$ ,  $\operatorname{Im} \omega_1 > 0$  with  $(\operatorname{Im} \omega_1)^2 = (\operatorname{Im} \omega_0)^2 - z > 0$ . This makes possible to write (2.10) as follows

$$(p(x', 0, \xi', \xi_n) - \bar{\omega}_1)(p(x', 0, \xi', \xi_n) - \omega_1) = 0.$$

The equation (2.11) is transformed in the same way and we get

$$(p(x', 0, \xi', \xi_n) - (1 + m(x', 0))\bar{\omega}_1)(p(x', 0, \xi', \xi_n) - (1 + m(x', 0))\omega_1) = 0.$$

Now we apply Proposition 2 with  $\omega_1$  and obtain the result.

Passing to the case  $\operatorname{Im} z \neq 0$ , notice that every bounded solution of (2.12) satisfies

$$D_{x_n}^k u \rightarrow 0, D_{x_n}^k v \rightarrow 0, k = 0, 1, 2, 3, \text{ for } x_n \rightarrow +\infty. \quad (2.13)$$

We will study the existence of  $L^2(\mathbb{R}^+)$ -solutions depending on  $(x', \xi')$  of the homogeneous problem (2.12) following the approach in [22]. Let

$$U = (u(x', x_n, \xi', z), v(x', x_n, \xi', z)) \in H^4([0, +\infty[: \mathbb{C}^2),$$

where we consider  $x', \xi'$  and  $z$  as parameters. Assume that  $U \in H^4([0, +\infty[: \mathbb{C}^2)$  is a solution of (2.12) such that (2.13) holds. Let

$$\mathbf{p}(x', 0, \xi', hD_{x_n}) := \begin{pmatrix} h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0 & 0 \\ 0 & \frac{1}{(1+m(x', 0))^2} \left( h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0 \right) \end{pmatrix}.$$

By using the boundary conditions in (2.12), we obtain

$$((\mathbf{p}^* \mathbf{p} - z)U, U) = -z\|U\|^2 + \|\mathbf{p}U\|^2,$$

where  $(., .)$  is the scalar product in  $L^2(\mathbb{R}^+ : \mathbb{C}^2)$  and  $\|.\|$  is the  $L^2(\mathbb{R}^+ : \mathbb{C}^2)$ - norm. Therefore

$$\|U\| \leq \frac{1}{|\operatorname{Im} z|} \|(Q|_{x_n=0, hD_{x'}=\xi'} - z)U\|. \quad (2.14)$$

This proves that for fixed  $(x', \xi')$  we have do not non-trivial solutions of (2.12).  $\square$

The estimate (2.14) holds for every  $U \in C_0^\infty([0, +\infty[: \mathbb{C}^2)$  satisfying the boundary conditions in (2.12). Since we have an elliptic boundary problem, we deduce that for  $U = (u, v) \in H^4([0, +\infty[: \mathbb{C}^2)$  satisfying the boundary conditions in (2.12) we have

$$\|U\|_4 \leq \frac{C}{|\operatorname{Im} z|} \|(Q|_{x_n=0, hD_{x'}=\xi'} - z)U\|, \quad (2.15)$$

where the constant  $C$  depends of  $(x', \xi')$ . For  $|x'| \leq \rho, |\xi'| \leq C_0$  we may take  $C$  uniformly with respect to  $(x', \xi')$ .

Applying (2.15), it is easy to obtain an estimate for the solution of the non-homogeneous problem

$$\begin{cases} \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi') - \bar{\omega}_0)(h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0) - z \right] u = F_1, & x_n > 0, \\ \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\bar{\omega}_0)(h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0) \right. \\ \left. - (1 + m(x', 0))^2 z \right] v = F_2, & x_n > 0, \\ u(0) - v(0) = f_0, & (D_{x_n} u)(0) - (D_{x_n} v)(0) = f_1, \\ - \left( (h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0) u \right)(0) - \frac{1}{(1+m(x', 0))^2} \left( \left[ h^2 D_{x_n}^2 + s(x', 0, \xi') - (1 + m(x', 0))\omega_0 \right] v \right)(0) = f_2, \\ - h D_{x_n} \left( (h^2 D_{x_n}^2 + s(x', 0, \xi') - \omega_0) u \right)(0) \\ - \frac{1}{(1+m(x', 0))^2} h D_{x_n} \left( \left[ (h^2 D_{x_n}^2 + s(x', 0, \xi')) - (1 + m(x', 0))\omega_0 \right] v \right)(0) = f_3. \end{cases} \quad (2.16)$$

Let  $F = (F_1, F_2)$ . Choose functions  $H_j(x_n; h) \in C_0^\infty([0, +\infty[)$ ,  $j = 0, 1, 2, 3$ , so that

$$D_{x_n}^k H_j(0; h) = \delta_{k,j}, \quad k, j = 0, 1, \quad h^k D_{x_n}^k H_j(0; h) = \delta_{k,j}, \quad k = 0, 1, 2, 3, \quad j = 2, 3.$$

It is easy to see that we can choose  $H_j(x_n; h)$ ,  $j = 2, 3$ , so that their  $H^4([0, +\infty[)$  norms are bounded uniformly with respect to  $h$ . Consider

$$\tilde{u} = u - f_0 H_0, \quad \tilde{v} = v - f_1 H_1.$$

It is clear that

$$\tilde{u}(0) - \tilde{v}(0) = 0, \quad D_{x_n} \tilde{u}(0) - D_{x_n} \tilde{v}(0) = 0.$$

Thus changing  $F, f_2$  and  $f_3$ , we may write (2.12) as a boundary problem for  $(\tilde{u}, \tilde{v})$  with two homogeneous boundary conditions. Now let  $Y = Y(x', \xi')$  be the symbol introduced in the proof of Proposition 3. The last two boundary conditions for the problem for  $\tilde{u}, \tilde{v}$  can be written in the form

$$\begin{aligned} -(1+m)^2 h^2 D_{x_n}^2 \tilde{u}(0) - h^2 D_{x_n}^2 \tilde{v}(0) + Y(x', \xi') \tilde{u}(0) &= f_4, \\ -(1+m)^2 h^3 D_{x_n}^3 \tilde{u}(0) - h^3 D_{x_n}^3 \tilde{v}(0) + Y D_{x_n} \tilde{u}(0) &= f_5, \end{aligned}$$

where  $f_4, f_5$  may be expressed by  $f_0, f_1, f_2, f_3, H_0, H_1$ . Introduce

$$u_1 = \tilde{u} + f_4(1+m)^{-2} H_2 + f_5(1+m)^{-2} H_3, \quad v_1 = \tilde{v}.$$

Then

$$-(1+m)^2 h^2 D_{x_n}^2 u_1(0) - h^2 D_{x_n}^2 v_1(0) + Y u_1(0) = f_4 - f_4 = 0$$

and the last boundary condition for  $(u_1, v_1)$  is also homogeneous. Thus  $(u_1, v_1)$  satisfies a boundary problem with homogeneous boundary conditions. On the other hand,

$$(Q|_{x_n=0, hD_{x'}=\xi'} - z)(u_1, v_1) = F + M = \tilde{F},$$

where  $M$  depends on  $F, f_0, f_1, f_2, f_3, H_0, H_1, H_2, H_3$ . We obtain

$$\|\tilde{F}\| \leq \|F\| + C_1 \sum_{j=0}^3 |f_j|$$

and, applying the estimate (2.15), we deduce that the solution  $U = (u, v)$  of the problem (2.16) satisfies

$$\|U\|_4 \leq \frac{C_2}{|\operatorname{Im} z|} \left( \|F\| + \sum_{j=0}^3 |f_j| \right). \quad (2.17)$$

In the following we assume that  $z \in \mathbb{C} \setminus [(\operatorname{Im} \omega_0 + \delta)^2, +\infty[$  with small  $\delta > 0$ ,  $|z| \leq a_0$ . It is important to note that the problem (2.12) has no bounded solutions only for  $z \in \mathbb{C} \setminus [(\operatorname{Im} \omega_0)^2, +\infty[$ . However, we will show that for  $|\xi'| \geq C_0$  with a large fixed constant  $C_0 > 0$  and for small  $\delta > 0$  our analysis works even for  $(\operatorname{Im} \omega_0)^2 \leq z \leq (\operatorname{Im} \omega_0 + \delta)^2$ . This argument plays a crucial role in the construction of a semi-classical parametrix in the region (A) in Section 3 as well as in the calculus of the trace of  $f(Q)$  in Section 4.

We start with the case  $r = \operatorname{Re} z - (\operatorname{Im} \omega_0)^2 \neq 0$ . It is clear that  $\lambda_{1,+} \neq \lambda_{2,+}$  and  $\mu_{1,+} \neq \mu_{2,+}$ . Then a bounded solution of (2.12) has the form

$$\begin{cases} u(x', x_n, \xi'; z) = C_1(x', \xi') e^{ix_n \lambda_{1,+}/h} + C_2(x', \xi') e^{ix_n \lambda_{2,+}/h}, \\ v(x', x_n, \xi'; z) = C_3(x', \xi') e^{ix_n \mu_{1,+}/h} + C_4(x', \xi') e^{ix_n \mu_{2,+}/h}. \end{cases} \quad (2.18)$$

For the coefficients  $C_j(x', \xi')$ ,  $j = 1, \dots, 4$ , in the representation of  $u(x', x_n, \xi'; z)$ ,  $v(x', x_n, \xi'; z)$  we obtain a linear system with matrix

$$\Lambda(x', \xi', z) = \begin{pmatrix} 1 & 1 & -1 & -1 \\ \lambda_1 & \lambda_2 & -\mu_1 & -\mu_2 \\ -(1+m)^2 \lambda_1^2 + Y & -(1+m)^2 \lambda_2^2 + Y & -\mu_1^2 & -\mu_2^2 \\ -(1+m)^2 \lambda_1^3 + \lambda_1 Y & -(1+m)^2 \lambda_2^3 + \lambda_2 Y & -\mu_1^3 & -\mu_2^3 \end{pmatrix},$$

where for simplicity we write  $\lambda_j, \mu_k$  instead of  $\lambda_{j,+}, \mu_{k,+}$ . Clearly,

$$\frac{\det \Lambda}{(\lambda_2 - \lambda_1)} = \det \begin{pmatrix} 1 & \lambda_2 - \mu_1 & \mu_1 - \mu_2 \\ -(1+m)^2(\lambda_1 + \lambda_2) & -(1+m)^2 \lambda_2^2 - \mu_1^2 + Y & \mu_1^2 - \mu_2^2 \\ -(1+m)^2(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) + Y & -(1+m)^2 \lambda_2^3 - \mu_1^3 + \lambda_2 Y & \mu_1^3 - \mu_2^3 \end{pmatrix}.$$

Multiplying the first column by  $\lambda_2$  and taking the difference with the second one, we get

$$\det \Lambda(x', \xi', z) = -(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) \det \Lambda_1(x', \xi', z),$$

where

$$\Lambda_1 := \begin{pmatrix} 1 & -\mu_1 & 1 \\ -(1+m)^2(\lambda_1 + \lambda_2) & (1+m)^2 \lambda_1 \lambda_2 - \mu_1^2 + Y & \mu_1 + \mu_2 \\ -(1+m)^2(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) + Y & (1+m)^2 \lambda_1 \lambda_2 (\lambda_1 + \lambda_2) - \mu_1^3 & \mu_1^2 + \mu_1 \mu_2 + \mu_2^2 \end{pmatrix}.$$

Next multiplying the third column by  $\mu_1$  and adding it to the second one, we get

$$\begin{aligned} & \det \Lambda_1(x', \xi', z) \\ &= \det \begin{pmatrix} 1 & 0 & 1 \\ -(1+m)^2(\lambda_1 + \lambda_2) & (1+m)^2 \lambda_1 \lambda_2 + \mu_1 \mu_2 + Y & \mu_1 + \mu_2 \\ -(1+m)^2(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) + Y & (1+m)^2 \lambda_1 \lambda_2 (\lambda_1 + \lambda_2) + \mu_1 \mu_2 (\mu_1 + \mu_2) & \mu_1^2 + \mu_1 \mu_2 + \mu_2^2 \end{pmatrix} \\ &= (1+m)^2 \left( \mu_1 \mu_2 (\lambda_1 + \lambda_2)^2 - (\lambda_1 \lambda_2 + \mu_1 \mu_2) (\mu_1 + \mu_2) (\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 (\mu_1 + \mu_2)^2 \right) \end{aligned}$$

$$-\left((1+m)^2\lambda_1\lambda_2 + \mu_1\mu_2\right)^2 + \left((1+m)^2(\lambda_1^2 + \lambda_2^2) + (\mu_1^2 + \mu_2^2)\right)Y - Y^2.$$

We will prove the following

**Proposition 4.** *Assume*

$$(2 + 3m + m^2) \operatorname{Im} \omega_0 \leq \eta m(m+1) \operatorname{Re} \omega_0, \quad 0 < \eta < 1. \quad (2.19)$$

Then for  $z \in \mathbb{C} \setminus \left( [(\operatorname{Im} \omega_0 + \delta)^2, +\infty[ \cup \{(\operatorname{Im} \omega_0)^2\} \right)$ ,  $|z| \leq a_0$  and  $\delta > 0$  small enough there exists a constant  $C_0 > 0$  such that for  $|\xi'| \geq C_0$  we have the estimate

$$|\det \Lambda_1(x', \xi', z)| \geq C_2 \quad (2.20)$$

with  $C_2 > 0$  independent of  $(x', \xi')$  and  $z$ . In particular, for these values of  $z$  and  $\xi'$  the problem (2.12) has only a trivial bounded solution  $u = v = 0$ .

**Proof.** We start with a representation of  $\lambda_{j,+}$  and  $\mu_{k,+}$  for  $|\xi'| \geq C_0$ . If we have

$$p(x', 0, \xi', \xi_n)^2 - 2(\operatorname{Re} \omega_0)p(x', 0, \xi', \xi_n) + |\omega_0|^2 - z = 0,$$

then  $p(x', 0, \xi', \xi_n) = \operatorname{Re} \omega_0 \pm \sqrt{\operatorname{Re} z - (\operatorname{Im} \omega_0)^2 + i \operatorname{Im} z}$ . Set

$$r = \operatorname{Re} z - (\operatorname{Im} \omega_0)^2 \in \mathbb{R}, \quad \operatorname{Im} z = w.$$

We choose the branch  $0 \leq \operatorname{Arg} z \leq 2\pi$ . Then for  $|\xi'| \geq C_0$  and large  $C_0 > 0$  we have

$$\begin{aligned} \lambda_{1,\pm} &= \pm \sqrt{\operatorname{Re} \omega_0 - s + \sqrt{r + iw}}, \quad \lambda_{2,\pm} = \pm \sqrt{\operatorname{Re} \omega_0 - s - \sqrt{r + iw}}, \\ \mu_{1,\pm} &= \pm \sqrt{(m+1) \operatorname{Re} \omega_0 - s + (1+m)\sqrt{r + iw}}, \\ \mu_{2,\pm} &= \pm \sqrt{(m+1) \operatorname{Re} \omega_0 - s - (1+m)\sqrt{r + iw}}. \end{aligned}$$

It is clear that the equations (2.10), (2.11) imply

$$\lambda_1^2 + \lambda_2^2 = 2(\operatorname{Re} \omega_0 - s), \quad \mu_1^2 + \mu_2^2 = 2(1+m) \operatorname{Re} \omega_0 - 2s.$$

Therefore,

$$\begin{aligned} & (1+m)^2(\lambda_1^2 + \lambda_2^2) + (\mu_1^2 + \mu_2^2) \\ &= 2 \left[ ((1+m)^2 + (1+m)) \operatorname{Re} \omega_0 - ((1+m)^2 + 1)s \right] = 2 \operatorname{Re} Y \end{aligned}$$

and we obtain

$$[2 \operatorname{Re} Y - Y]Y = (\operatorname{Re} Y - i \operatorname{Im} Y)Y = |Y|^2.$$

Set  $\zeta = \sqrt{r + iw}$ . Then for  $|\xi'| \rightarrow +\infty$ , we have

$$\begin{aligned} \lambda_{1,2} &= i\sqrt{s} - i \frac{\operatorname{Re} \omega_0 \pm \zeta}{2\sqrt{s}} - i \frac{(\operatorname{Re} \omega_0 \pm \zeta)^2}{8s^{3/2}} + \mathcal{O}\left(\frac{1}{s^{5/2}}\right), \\ \mu_{1,2} &= i\sqrt{s} - i \frac{(m+1)(\operatorname{Re} \omega_0) \pm \zeta}{2\sqrt{s}} - i \frac{((1+m) \operatorname{Re} \omega_0 \pm \zeta)^2}{8s^{3/2}} + \mathcal{O}\left(\frac{1}{s^{5/2}}\right), \end{aligned}$$

where we have the sign (+) for  $\lambda_1, \mu_1$  and the sign (−) for  $\lambda_2, \mu_2$ . This implies

$$\lambda_1 \lambda_2 = -s + \operatorname{Re} \omega_0 + \frac{r + iw}{2s} + \mathcal{O}\left(\frac{1}{s^2}\right)$$

and

$$\mu_1 \mu_2 = -s + (m+1) \operatorname{Re} \omega_0 + \frac{(m+1)^2(r + iw)}{2s} + \mathcal{O}\left(\frac{1}{s^2}\right).$$

Also notice that we have

$$\begin{aligned}\lambda_1 + \lambda_2 &= 2i\sqrt{s} - \frac{i \operatorname{Re} \omega_0}{\sqrt{s}} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \\ \mu_1 + \mu_2 &= 2i\sqrt{s} - \frac{i(m+1) \operatorname{Re} \omega_0}{\sqrt{s}} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}(1+m)^2 \lambda_1 \lambda_2 + \mu_1 \mu_2 &= -(2+2m+m^2)s + (2+3m+m^2) \operatorname{Re} \omega_0 + \frac{(m+1)^2(r+iw)}{s} + \mathcal{O}\left(\frac{1}{s^2}\right) \\ &= \operatorname{Re} Y + \frac{(m+1)^2(r+iw)}{s} + \mathcal{O}\left(\frac{1}{s^2}\right)\end{aligned}$$

and we deduce

$$\left((1+m)^2 \lambda_1 \lambda_2 + \mu_1 \mu_2\right)^2 = |\operatorname{Re} Y|^2 - 2(2+2m+m^2)(m+1)^2(r+iw) + \mathcal{O}\left(\frac{1}{s}\right).$$

We multiply by (-1) the above expression, sum it with  $|Y|^2$  and obtain

$$\begin{aligned}-\left((1+m)^2 \lambda_1 \lambda_2 + \mu_1 \mu_2\right)^2 + |Y|^2 &= |\operatorname{Im} Y|^2 + 2(2+2m+m^2)(m+1)^2(r+iw) + \mathcal{O}\left(\frac{1}{s}\right) \\ &= (2+3m+m^2)^2(\operatorname{Im} \omega_0)^2 + 2(2+2m+m^2)(m+1)^2(r+iw) + \mathcal{O}\left(\frac{1}{s}\right).\end{aligned}\quad (2.21)$$

It remains to examine the behavior of the term

$$\begin{aligned}Z &:= \mu_1 \mu_2 (\lambda_1 + \lambda_2)^2 - (\lambda_1 \lambda_2 + \mu_1 \mu_2)(\mu_1 + \mu_2)(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 (\mu_1 + \mu_2)^2 \\ &= \left((\lambda_1 + \lambda_2) - (\mu_1 + \mu_2)\right) \left(\mu_1 \mu_2 (\lambda_1 + \lambda_2) - \lambda_1 \lambda_2 (\mu_1 + \mu_2)\right).\end{aligned}$$

We have

$$\begin{aligned}(\lambda_1 + \lambda_2) - (\mu_1 + \mu_2) &= \frac{im \operatorname{Re} \omega_0}{\sqrt{s}} + \mathcal{O}\left(\frac{1}{s^{3/2}}\right), \\ \mu_1 \mu_2 (\lambda_1 + \lambda_2) - \lambda_1 \lambda_2 (\mu_1 + \mu_2) &= im \operatorname{Re} \omega_0 \sqrt{s} + \mathcal{O}\left(\frac{1}{\sqrt{s}}\right),\end{aligned}$$

hence

$$(1+m)^2 Z = -m^2(1+m)^2(\operatorname{Re} \omega_0)^2 + \mathcal{O}\left(\frac{1}{s}\right).\quad (2.22)$$

Taking account of (2.21), (2.22), for  $|\operatorname{Im} z| \geq \eta_0 > 0$  we obtain an uniform lower bound for  $|\det \Lambda_1|$  since  $\operatorname{Im}(\det \Lambda_1) = 2(2+2m+m^2)(m+1)^2 \operatorname{Im} z + \mathcal{O}\left(\frac{1}{s}\right)$ . To treat the case  $|\operatorname{Im} z| < \eta_0$ , we exploit the condition (2.19) which yields

$$(2+3m+m^2)^2(\operatorname{Im} \omega_0)^2 - \eta^2 m^2(m+1)^2(\operatorname{Re} \omega_0)^2 \leq 0.$$

Since  $r \leq C\delta$ , the leading term of the real part of  $\det \Lambda_1$  is less than

$$-(1-\eta^2)m^2(m+1)^2(\operatorname{Re} \omega_0)^2 + 2(2+2m+m^2)(m+1)^2 C\delta.$$

For small  $\delta > 0$  depending on  $\operatorname{Re} \omega_0$  and  $m$  we obtain an uniform lower bound and this completes the proof.  $\square$

**Remark 2.** For  $\text{Im } z \neq 0$  the lower bound of  $|\det \Lambda_1|$  has the form  $C(m)|\text{Im } z|$  and this agrees with the estimates (2.14), (2.17). For  $\text{Im } z = 0$  the situation is more delicate. For  $\text{Re } z < (\text{Im } \omega_0)^2$  we obtain a lower bound depending only of  $\omega_0$  and  $m$ . If the assumption (2.19) is not satisfied, a more fine analysis of the lower order terms is necessary. For our purpose Proposition 4 is sufficient since in Section 5 we will choose  $\text{Re } \omega_0 = 1$  and  $0 < \text{Im } \omega_0 < 1$  small.

Obviously, for  $z = (\text{Im } \omega_0)^2$  and  $|\xi'| \geq C_0$  we have double roots

$$\lambda_+ := \sqrt{\text{Re } \omega_0 - s}, \quad \mu_+ := \sqrt{(1+m)\text{Re } \omega_0 - s}.$$

In this case a bounded solution of (2.12) has the form

$$\begin{cases} u(x', x_n, \xi'; z) = C_1(x', \xi')e^{ix_n\lambda_+/h} + C_2(x', \xi')\frac{ix_n}{h}e^{ix_n\lambda_+/h}, \\ v(x', x_n, \xi'; z) = C_3(x', \xi')e^{ix_n\mu_+/h} + C_4(x', \xi')\frac{ix_n}{h}e^{ix_n\mu_+/h}. \end{cases}$$

For the coefficients  $C_j(x', \xi')$  we obtain a linear system with the matrix

$$\Lambda_2(x', \xi', z) := \begin{pmatrix} 1 & 0 & -1 & 0 \\ \lambda & 1 & -\mu & -1 \\ -(1+m)^2\lambda^2 + Y & -2(1+m)^2\lambda & -\mu^2 & -2\mu \\ -(1+m)^2\lambda^3 + Y\lambda & -3(1+m)^2\lambda^2 + Y & -\mu^3 & -3\mu^2 \end{pmatrix},$$

where we write simply  $\lambda, \mu$  for  $\lambda_+, \mu_+$ . A calculus of  $\det \Lambda_2$  implies

$$\begin{aligned} \det \Lambda_2 &= \mu^4 + (1+m)^4\lambda^4 + (1+m)^2(4\lambda\mu^3 + 4\lambda^3\mu - 6\lambda^2\mu^2) - 2((1+m)^2\lambda^2 + \mu^2)Y + Y^2 \\ &= ((1+m)^2\lambda^2 + \mu^2)^2 + 4(1+m)^2\lambda\mu(\lambda - \mu)^2 - 2((1+m)^2\lambda^2 + \mu^2)Y + Y^2. \\ &= [(1+m)^2\lambda^2 + \mu^2 - Y]^2 + 4(1+m)^2\lambda\mu(\lambda - \mu)^2. \end{aligned}$$

We have

$$\begin{aligned} &(1+m)^2\lambda^2 + \mu^2 - Y \\ &= (1+m)^2(\text{Re } \omega_0 - s) + (1+m)\text{Re } \omega_0 - s + (2+2m+m^2)s - (2+3m+m^2)\omega_0 \\ &= -(2+3m+m^2)i\text{Im } \omega_0. \end{aligned}$$

The term  $4\lambda\mu(\lambda - \mu)^2$  is the analog of the term  $-Z$  in the proof of Proposition 4. By using the continuity of the roots  $\lambda_{j,+}, \mu_{j,+}$  as  $z \rightarrow (\text{Im } \omega_0)^2$ , we get

$$4(m+1)^2\lambda\mu(\lambda - \mu)^2 = m^2(m+1)^2(\text{Re } \omega_0)^2 + \mathcal{O}\left(\frac{1}{s}\right).$$

Thus we obtain the following

**Proposition 5.** Assume the condition (2.19) fulfilled. Then for  $z = (\text{Im } \omega_0)^2$  there exists a constant  $C_0 > 0$  such that for  $|\xi'| \geq C_0$  we have the estimate

$$|\det \Lambda_2(x', \xi', z)| \geq C_3 \quad (2.23)$$

with  $C_3 > 0$  independent of  $(x', \xi')$ . In particular, for these values of  $\xi'$  the problem (2.12) has only a trivial bounded solution  $u = v = 0$ .

**Remark 3.** We can obtain  $\Lambda_2$  from  $\frac{\Lambda}{(\lambda_2 - \lambda_1)(\mu_2 - \mu_1)} = -\Lambda_1$  by taking the limit  $z \rightarrow (\text{Im } \omega_0)^2$ . This explains the appearance of the sign  $(-)$  in the expression of  $\Lambda_2$ .

## 3. SEMI-CLASSICAL PARAMETRIX

In this section we assume that  $z$  belongs to a compact set  $Z_0$  in  $\mathbb{C} \setminus [(\operatorname{Im} \omega_0) + \delta)^2, \infty[$  which is independent on  $h$ . We assume also that  $\delta > 0$  is small enough and  $\omega_0$  is such that (2.19) holds. This guarantees that the estimates (2.20) and (2.23) are satisfied for  $|\xi'| \geq C_0$ .

Since the boundary problem (2.12) is not elliptic for all  $z \in Z_0$  and all  $\xi' \in \mathbb{R}^{n-1}$ , we will construct a semi-classical parametrix in two regions:

$$(A) : z \in Z_0, |\xi| \geq C_0,$$

$$(B) : z \in Z_0, |\operatorname{Im} z| \geq h^\epsilon.$$

Here  $C_0 > 0$  is the constant from Propositions 4, 5 and  $0 \leq \epsilon < 1/2$  is a small fixed constant. In the region (A) we construct a microlocal parametrix which will be defined precisely below. The semi-classical parametrix will be a sum of an operator with matrix symbol  $R(x, \xi; z, h)$  defined for all  $(x, \xi) \in \mathbb{R}^{2n}$  and a matrix-valued operator corresponding to the boundary conditions.

We extend  $m(x)$  to  $\mathbb{R}^n$  so that  $m(x) \in C_0^\infty(\mathbb{R}^n)$  with  $m(x) = 0$  for  $|x| \geq \rho > 0$ . The operator  $Q$  with coefficient  $m(x)$  extended for all  $x \in \mathbb{R}$  will be denoted also by  $Q$ . First we consider the region (B) and the modifications for the region (A) will be explained later. We start with the construction of a parametrix for  $Q - z$  in  $\mathbb{R}^n$ . We assume  $|\operatorname{Im} z| \geq h^\epsilon$  and we search a matrix symbol  $R(x, \xi; z, h)$  determined for all  $x \in \mathbb{R}^n$  and all  $\xi \in \mathbb{R}^n$  so that

$$(Q - z) \#_h R \sim I.$$

Set

$$Q_1(x, hD_x) = (-h^2 \Delta - \bar{\omega}_0)(-h^2 \Delta - \omega_0),$$

$$Q_2(x, hD_x) = (-h^2 \Delta \frac{1}{1+m} - \bar{\omega}_0)(-h^2 \Delta \frac{1}{1+m} - \omega_0).$$

We denote by  $q_k(x, \xi)$  the principal symbols of the semi-classical operators  $Q_k(x, hD_x)$ ,  $k = 1, 2$ . Clearly, the symbol  $q_1$  is independent on  $x$  but in the following we use the notation  $q_1(x, \xi)$  which is convenient when we treat the boundary conditions. Since the operator  $Q - z$  has a diagonal form, the construction follows a standard argument and we find  $R$  as a matrix-valued operator

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

so that

$$(Q_1 - z) \circ R_1 = I + K_1, (Q_2 - z) \circ R_2 = I + K_2.$$

Moreover, for the distribution kernels  $K_j(x, y; z, h)$  of the operators  $K_j$ ,  $j = 1, 2$ , we have

$$|\partial_x^\alpha \partial_y^\beta K_j(x, y; z, h)| \leq C_{\alpha, \beta, N} h^N, \quad \forall \alpha, \forall \beta, \forall N.$$

Below we use the spaces of symbols  $S_\epsilon^{m, k} := S_\epsilon^{m, k}(\mathbb{R}^{2n}; \mathcal{M}_2(\mathbb{C}))$  (see [22] and [11] for the properties of the operators with symbols in these classes.) For reader convenience recall that  $a(x, \xi; h) \in S_\epsilon^{m, k}(\mathbb{R}^{2n}; \mathcal{M}_2(\mathbb{C}))$  if  $a$  is square matrix-valued symbol smooth with respect to  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$  and such that

(i) for  $\xi$  outside some  $h$ -independent compact set every component  $d$  of  $a$  satisfies  $d \in S^k(\mathbb{R}^{2n})$ , where  $S^k(\mathbb{R}^{2n})$  denotes the Hörmander space, i.e.,  $|\partial_x^\alpha \partial_\xi^\beta d(x, \xi; h)| \leq C_{\alpha, \beta} (1 + |\xi|)^{k - |\beta|}$  uniformly

with respect to  $h$ ,

(ii) For every compact set  $\Xi$  in  $\mathbb{R}^n$  we have the estimates

$$\|\partial_x^\alpha \partial_\xi^\beta d(x, \xi; h)\| \leq C_{\alpha, \beta, \Xi} h^{-m} h^{-\epsilon(|\alpha| + |\beta|)}, \quad \forall (x, \xi) \in \mathbb{R}^n \times \Xi, \quad \forall \alpha, \forall \beta \in \mathbb{N}^n.$$

It is clear that

$$\left| \left( (|\xi|^2 - \bar{\omega}_0)(|\xi|^2 - \omega_0) - z \right) \left( \left( \frac{1}{1+m} |\xi|^2 - \bar{\omega}_0 \right) \left( \frac{1}{1+m} |\xi|^2 - \omega_0 \right) - z \right) \right| \geq |\operatorname{Im} z|^2.$$

Using the construction of a semi-classical parametrix depending on a parameter in the region (B) (see for example, [7], [22], [11]), we find  $\mathcal{R}_N = \sum_{j=0}^N h^j c_j$  with  $c_j \in S_\epsilon^{\epsilon+j, -4-j}(\mathbb{R}^{2n}; \mathcal{M}_2(\mathbb{C}))$  so that

$$(Q - z)Op_h \mathcal{R}_N = I + h^{N+1}Op_h(c_{N+1}).$$

The symbol of the operator  $R$  is defined as  $R \sim \sum_{j=0}^\infty h^j c_j$ . Since the construction is a repetition of that in [11], [22], we omit the details.

Now we pass to the construction close to the boundary. Our argument is similar to that in Section 3 in [22]. We introduce local **geodesic coordinates** in a neighborhood of a point  $x_0 \in \partial\Omega$  as in the previous section. For simplicity we will use the notations  $(x', x_n, \xi', \xi_n)$ .

Let  $\lambda_{j,\pm}(x, \xi', z)$  and  $\mu_{j,\pm}(x, \xi', z)$ ,  $j = 1, 2$ , be the roots defined in the previous section. We can choose  $R \in S_\epsilon^{\epsilon, -4}$  with a holomorphic extension in  $\xi_n$ -variable in the domain

$$\Omega_1(x, \xi', z) = \{\xi_n \in \mathbb{C}, |\xi_n| \leq \eta^{-1} \langle \xi' \rangle, |\xi_n - \lambda_{j,\pm}| > \eta \langle \xi' \rangle, |\xi_n - \mu_{j,\pm}| > \eta \langle \xi' \rangle, j = 1, 2\}$$

for arbitrary fixed  $\eta > 0$ . Here and below we use the notation  $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$  and we will denote by  $C$  some positive constants which may change from line to line.

For  $|\xi'| \leq C_0$  we have

$$|\operatorname{Im} \lambda_{j,\pm}(x, \xi', z)| \geq h^\epsilon / C, \quad |\operatorname{Im} \mu_{j,\pm}(x, \xi', z)| \geq h^\epsilon / C, \quad j = 1, 2.$$

Consider the domain

$$\Omega(x, \xi', z) = (\Omega_1 \cap \{\xi_n \geq \eta \langle \xi' \rangle, |\xi'| \geq C_0\}) \cup \Omega_2 \cup \Omega_3,$$

where

$$\Omega_2(x, \xi', z) = \{|\xi'| \leq C_0, |\xi_n| \leq C, 0 \leq \operatorname{Im} \xi_n \leq \frac{h^\epsilon}{C}\},$$

$$\Omega_3(x, \xi', z) = \{|\xi'| \leq C_0, |\xi_n| \leq C, \operatorname{Im} \xi_n \geq \frac{h^\epsilon}{C}, |\xi_n - \lambda_{j,+}| \geq C^{-1}, |\xi_n - \mu_{j,+}| \geq C^{-1}, j = 1, 2\}.$$

We have a holomorphic extension of  $R$  with respect to  $\xi_n$  in the domain  $\Omega(x', \xi', z)$ . Let

$$\gamma = \gamma(x', \xi') \subset \Omega(x, \xi', z)$$

be a simple curve which encircles  $\lambda_{j,+}$  and  $\mu_{j,+}$ ,  $j = 1, 2$ , in a positive sense. Introduce the operators  $\Pi_j : C_0^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\overline{\mathbb{R}_+^n})$ ,  $j = 0, 1, 2, 3$ , by

$$\Pi_j u(x) = (2\pi h)^{1-n} \int_{\mathbb{R}^{n-1}} \int_\gamma e^{i\langle x, \xi \rangle / h} R_1(x, \xi; z, h) \xi_n^{2+j} \hat{u}(\xi') d\xi' d\xi_n / (2\pi i), \quad j = 0, 1, \quad (3.1)$$

$$\Pi_j v(x) = (2\pi h)^{1-n} \int_{\mathbb{R}^{n-1}} \int_\gamma e^{i\langle x, \xi \rangle / h} R_2(x, \xi; z, h) \xi_n^j \hat{v}(\xi') d\xi' d\xi_n / (2\pi i), \quad j = 2, 3, \quad (3.2)$$



where

$$\hat{u}(\xi') = \int e^{-i\langle x', \xi' \rangle / h} u(x') dx'$$

is the semi-classical Fourier transform of  $u(x')$ . The reader may consult [22] for the properties of these operators as well as for the composition of a pseudodifferential operator with  $\Pi_j$ ,  $j = 0, 1, 2, 3$ . Notice that we have

$$Q_1(x, hD_x)\Pi_j \equiv 0, \quad j = 0, 1, \quad Q_2(x, hD_x)\Pi_j \equiv 0, \quad j = 2, 3.$$

Here  $Q_k(x, hD_x)$ ,  $k = 1, 2$ , are the operators in the local coordinates  $(x', x_n)$  and we use the same notation  $q_k(x, \xi)$  for the corresponding principal symbols of  $Q_k$ . The equality  $M_1 \equiv M_2$  means that the kernel  $\mathcal{K}(x, y; z, h)$  of the operator  $M_1 - M_2$  satisfies

$$|\partial_x^\alpha \partial_y^\beta \mathcal{K}(x, y; z, h)| \leq C_{\alpha, \beta, N} h^N, \quad \forall \alpha, \forall \beta, \forall N \in \mathbb{N}.$$

Given  $F = (f_1, f_2) \in C_0^\infty(\overline{\mathbb{R}_+^n} : \mathbb{C}^2)$ , we define by  $\tilde{F} = (\tilde{f}_1, \tilde{f}_2)$  the zero extension of  $(f_1, f_2)$  to  $\mathbb{R}^n$  and set  $R(\tilde{F}) = (R_1(\tilde{f}_1), R_2(\tilde{f}_2))$ .

Introduce the boundary operators

$$B_0(u, v) := \gamma_0(u - v),$$

$$B_1(u, v) := \gamma_0 h D_{x_n}(u - v),$$

$$B_2(u, v) := \gamma_0 \left( -(1 + m)^2 h^2 D_{x_n}^2 u + Y u - h^2 D_{x_n}^2 v \right),$$

$$B_3(u, v) := \gamma_0 \left( -(1 + m)^2 h^3 D_{x_n}^3 u + Y h D_{x_n} u - h^3 D_{x_n}^3 v \right),$$

where the operator  $Y(x', hD_{x'})$  has symbol  $Y(x', \xi')$  defined in the proof of Proposition 3. Let

$$B_j(R(\tilde{F})) = W_j, \quad j = 0, 1, 2, 3$$

and set  $W = (W_0, W_1, W_2, W_3)$ . We will search a parametrix  $E(z) = E(\tilde{F}, W; z)$  in the form

$$\begin{aligned} E(z) &= \begin{pmatrix} R_1(\tilde{f}_1) - (\Pi_0 A_0 + \Pi_1 A_1)(W) & 0 \\ 0 & R_2(\tilde{f}_2) - (\Pi_2 A_2 + \Pi_3 A_3)(W) \end{pmatrix} \\ &= \begin{pmatrix} R_1(\tilde{f}_1) & 0 \\ 0 & R_2(\tilde{f}_2) \end{pmatrix} - \Pi(W) \end{aligned}$$

with suitable tangential  $h$ - pseudodifferential operators  $A_j$ ,  $j = 0, 1, 2, 3$ , so that

$$B_j(R(\tilde{F}) - \Pi(W)) \equiv 0, \quad j = 0, 1, 2, 3.$$

The last relation is equivalent to

$$\begin{cases} B_0(\Pi_0 A_0 + \Pi_1 A_1, \Pi_2 A_2 - \Pi_3 A_3) \equiv W_0, \\ B_1(\Pi_0 A_0 + \Pi_1 A_1, \Pi_2 A_2 + \Pi_3 A_3) \equiv W_1, \\ B_2(\Pi_0 A_0 + \Pi_1 A_1, \Pi_2 A_2 + \Pi_3 A_3) \equiv W_2, \\ B_3(\Pi_0 A_0 + \Pi_1 A_1, \Pi_2 A_2 + \Pi_3 A_3) \equiv W_3. \end{cases} \quad (3.3)$$

To construct the parametrix close to the boundary, it is sufficient to prove that the system (3.3) for  $A_j$ ,  $j = 0, 1, 2, 3$ , is an elliptic one and for this purpose we examine the principal symbol of (3.3).

First we treat the case  $|\xi'| \geq C_0$ , where  $C_0 > 0$  is the constant of Proposition 4. Since  $|\operatorname{Im} z| \geq h^\epsilon$ , the roots  $\lambda_{j,\pm}$ ,  $\mu_{j,\pm}$ ,  $j = 1, 2$  are simple and we introduce

$$\frac{1}{\partial_{\xi_n} q_1(x', 0, \xi', \lambda_{j,+})} = r_j, \quad j = 1, 2, \quad \frac{1}{\partial_{\xi_n} q_2(x', 0, \xi', \mu_{j,+})} = r_{j+2}, \quad j = 1, 2.$$

For simplicity we omit the sign  $+$  in the notations below. By applying the theorem of residues, it is easy to see that the principal symbol of (3.3) is the matrix

$$M(x', \xi', z) = \begin{pmatrix} r_1 & r_2 & -r_3 & -r_4 \\ \lambda_1 r_1 & \lambda_2 r_2 & -\mu_1 r_3 & -\mu_2 r_4 \\ -(1+m)^2 \lambda_1^2 + Y) r_1 & -(1+m)^2 \lambda_2^2 + Y) r_2 & -\mu_1^2 r_3 & -\mu_2^2 r_4 \\ -(1+m)^2 \lambda_1^3 + Y \lambda_1) r_1 & -(1+m)^2 \lambda_2^3 + Y \lambda_2) r_2 & -\mu_1^3 r_3 & -\mu_2^3 r_4 \end{pmatrix} \begin{pmatrix} \lambda_1^2 & \lambda_1^3 & 0 & 0 \\ \lambda_2^2 & \lambda_2^3 & 0 & 0 \\ 0 & 0 & \mu_1^2 & \mu_1^3 \\ 0 & 0 & \mu_2^2 & \mu_2^3 \end{pmatrix}.$$

A simple calculation yields

$$\det M = -(\lambda_1 - \lambda_2)^2 (\mu_1 - \mu_2)^2 \left( \lambda_1 \lambda_2 \mu_1 \mu_2 \right)^2 \det \left[ \Lambda_1(x', \xi', z) \operatorname{diag} \{r_1, r_2, r_3, r_4\}(x', \xi', z) \right],$$

where  $\Lambda_1(x', \xi', z)$  is the matrix introduced in Section 2. On the other hand,

$$\begin{aligned} r_1 r_2 &= -(\lambda_1 - \lambda_2)^{-2} \prod_{\nu=1,2} (\lambda_{\nu,+} - \lambda_{1,-})^{-1} (\lambda_{\nu,+} - \lambda_{2,-})^{-1} \\ &= -(\lambda_1 - \lambda_2)^{-2} \frac{1}{4\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)^2}, \\ r_3 r_4 &= -(\mu_1 - \mu_2)^{-2} \prod_{\nu=1,2} (\mu_{\nu,+} - \mu_{1,-})^{-1} (\mu_{\nu,+} - \mu_{2,-})^{-1} \\ &= -(\mu_1 - \mu_2)^{-2} \frac{1}{4\mu_1 \mu_2 (\mu_1 + \mu_2)^2}. \end{aligned}$$

Thus we obtain

$$\det M = -\frac{\det \Lambda_1(x', \xi', z) \lambda_1 \lambda_2 \mu_1 \mu_2}{16(\lambda_1 + \lambda_2)^2 (\mu_1 + \mu_2)^2}.$$

According to Proposition 4 for  $|\xi'| \geq C_0$ , we have  $|\det \Lambda_1| \geq C_2$  and since  $\lambda_j \sim i\sqrt{s}$ ,  $\mu_j \sim i\sqrt{s}$ , as  $|\xi'| \rightarrow \infty$ , one obtains

$$|\det M(x', \xi', z)| \geq C_4 > 0, \quad |\xi'| \geq C_0. \quad (3.4)$$

Next we determine the symbols  $A_j$ ,  $j = 0, 1, 2, 3$ , from the system (3.3) and this completes the analysis for  $|\xi'| \geq C_0$ .

For the construction in the region  $z \in Z_0$ ,  $|\operatorname{Im} z| \geq h^\epsilon$ ,  $|\xi'| \leq C_0$ , we need a control of the inverse matrix  $M^{-1}(x', \xi', z)$  since we cannot apply Proposition 4. To do this, we follow the argument in [22] with some modifications.

Recall that

$$\begin{aligned} q_1(x, \xi', \xi_n) &= (\xi_n^2 + s(x, \xi') - \bar{\omega}_0)(\xi_n^2 + s(x, \xi') - \omega_0), \\ q_2(x, \xi', \xi_n) &= \frac{1}{(1 + m(x))^2} (\xi_n^2 + s(x, \xi') - (1 + m(x))\bar{\omega}_0) \\ &\quad \times (\xi_n^2 + s(x, \xi') - (1 + m(x))\omega_0) \end{aligned}$$

and let  $(q_j - z)^{-1}$ ,  $j = 1, 2$  be the inverse of the temporal distributions on the whole  $x_n$  axis. Introduce the distributions

$$u_j|_{x_n > 0} = \frac{1}{2\pi} \int_{\gamma} e^{ix_n \xi_n} (q_1 - z)^{-1} \xi_n^{j+1} d\xi_n, \quad j = 0, 1,$$

$$v_j|_{x_n > 0} = \frac{1}{2\pi} \int_{\gamma} e^{ix_n \xi_n} (q_2 - z)^{-1} \xi_n^{j-1} d\xi_n, \quad j = 2, 3,$$

where  $\gamma = \gamma(x', \xi')$  is the contour used in the definition of  $\Pi_j$ . We denote these distributions by  $u_j, v_j$ . Set

$$\begin{aligned} m_{0,0} &= u_0(0), \quad m_{0,1} = u_1(0), \quad m_{0,2} = -v_2(0), \quad m_{0,3} = -v_3(0), \\ m_{1,0} &= u_1(0), \quad m_{1,1} = (hD_{x_n} u_1)(0), \quad m_{1,3} = -v_3(0), \quad m_{1,4} = -(hD_{x_n} v_3)(0), \\ m_{2,k} &= -(1+m)^2 h^2 (D_{x_n}^2 u_k)(0) + Y u_k(0), \quad k = 0, 1, \quad m_{2,j} = -h^2 (D_{x_n}^2 v_j)(0), \quad j = 2, 3, \\ m_{3,k} &= -(1+m)^2 h^3 (D_{x_n}^3 u_k)(0) + Y h (D_{x_n} u_k)(0), \quad k = 0, 1, \quad m_{3,j} = -h^3 (D_{x_n}^3 v_j)(0), \quad j = 2, 3. \end{aligned}$$

For fixed  $(x', \xi')$  we obtain the matrix  $M(x', \xi', z) = \{m_{j,k}(x', \xi', z)\}_{j,k=0,1,2,3}$  introduced above. Given  $a = (a_0, a_1, a_2, a_3) \in \mathbb{C}^4$ , put

$$u = a_0 u_0 + a_1 u_1, \quad v = a_2 v_2 + a_3 v_3.$$

Then one gets

$$(q_1(x', 0, \xi', hD_{x_n}) - z)u = 0, \quad (q_2(x', 0, \xi', hD_{x_n}) - z)v = 0$$

and

$$\begin{aligned} u(0) &= m_{0,0}a_0 + m_{0,1}a_1, \quad -v(0) = m_{0,2}a_2 + m_{0,3}a_3. \\ h(D_{x_n} u)(0) &= m_{1,0}a_0 + m_{1,1}a_1, \quad -h(D_{x_n} v)(0) = m_{1,2}a_2 + m_{1,3}a_3. \end{aligned}$$

Similar equalities hold for the third and fourth boundary conditions. Setting  $U = (u, v)$ , we see that  $(u, v)$  satisfies a non-homogeneous boundary problem (2.16) with boundary data  $Ma$  and from (2.17) we deduce the estimate

$$\|U\|_4 \leq \frac{C}{|\operatorname{Im} z|} \|Ma\|_{\mathbb{C}^4}.$$

On the other hand,

$$u_j = (q_1 - z)^{-1} ((D_{x_n})^{j+1} \delta), \quad j = 0, 1, \quad v_j = (q_2 - z)^{-1} ((D_{x_n})^{j-1} \delta), \quad j = 2, 3,$$

and this yields

$$\begin{aligned} u &= \tilde{u}|_{x_n > 0} \text{ with } \tilde{u} = (q_1 - z)^{-1} (a_0 D_{x_n} \delta + a_1 D_{x_n}^2 \delta), \\ v &= \tilde{v}|_{x_n > 0} \text{ with } \tilde{v} = (q_2 - z)^{-1} (a_2 D_{x_n} \delta + a_3 D_{x_n}^2 \delta). \end{aligned}$$

This implies

$$\begin{aligned} \|a_0 D_{x_n} \delta + a_1 D_{x_n}^2 \delta\|_{-4} &\leq C_1 \|\tilde{u}\| \leq C_2 \|u\|_4, \\ \|a_2 D_{x_n} \delta + a_3 D_{x_n}^2 \delta\|_{-4} &\leq C_1 \|\tilde{v}\| \leq C_2 \|v\|_4, \end{aligned}$$

and we deduce

$$\|U\|_4 \geq C_3 \left( \|a_0 D_{x_n} \delta + a_1 D_{x_n}^2 \delta\|_{-4} + \|a_2 D_{x_n} \delta + a_3 D_{x_n}^2 \delta\|_{-4} \right).$$

Since  $\|a_0 D_{x_n} \delta + a_1 D_{x_n}^2 \delta\|_{-4} + \|a_2 D_{x_n} \delta + a_3 D_{x_n}^2 \delta\|_{-4}$  is equivalent to  $C_4 \|a\|_{\mathbb{C}^4}$ , we conclude that with a constant  $C_5 > 0$  which is uniform for  $|x'| \leq \rho, |\xi'| \leq C_0$  we have

$$\|M^{-1}(x', \xi', z)\| \leq C_5 / |\operatorname{Im} z|. \quad (3.5)$$

It is clear also that we have

$$\|M(x', \xi', z)\| \leq C_5/|\operatorname{Im} z| \quad (3.6)$$

uniformly with respect to  $|x'| \leq \rho$ ,  $|\xi'| \leq C_0$ .

Next, we will obtain some properties of the symbols  $A_j$ . Recall that  $A_j$  are tangential  $h$ -pseudodifferential operators on the boundary obtained as a solution of the system (3.3). In particular, the symbol of  $A_j$  is independent on  $x_n$  and  $\xi_n$ . It is easy to see that the symbol of  $B_j \Pi_k$  is in the set  $S_\epsilon^{\epsilon, -3+j+k+1}$  for  $k = 0, 1$  and in  $S_\epsilon^{\epsilon, -3+j+k-1}$  for  $k = 2, 3$ . This follows from the properties of the composition of  $B_j$  and  $\Pi_k$  explained in [22].

The system (3.3) can be written as

$$\mathcal{B} \circ \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \equiv \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix},$$

with a matrix operator  $\mathcal{B}$  having symbol

$$\mathcal{B} = \left( (b_{j,k}(x', \xi'; z, h))_{0 \leq j, k \leq 3} \right),$$

Moreover, the matrix  $M(x', \xi', z)$  is the principal symbol of  $\mathcal{B}$ . The above properties of the symbols  $B_j \Pi_k$  imply that for every cut-off function  $\psi \in C_0^\infty(\mathbb{R}^{n-1})$  we have  $\psi(\xi') b_{j,k}(x', \xi'; z, h) \in S_\epsilon^\epsilon$ .

Applying (3.5), (3.6), we deduce

$$\mathcal{B}^{-1} = \left( (d_{j,k}(x', \xi'; z, h))_{0 \leq j, k \leq 3} \right), \quad (3.7)$$

with  $\psi(\xi') d_{j,k}(x', \xi'; z, h) \in S_\epsilon^\epsilon$  for all  $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ . Therefore we can determine the symbols  $A_j$ ,  $j = 0, 1, 2, 3$ , by

$$A_j = \sum_{k=0}^3 d_{j,k}(x', \xi'; z, h) \#_h W_k(x', \xi'; z, h) = \sum_{k=0}^3 d_{j,k}(x', \xi'; z, h) \#_h B_k(R_1, R_2)(x', \xi'; z, h).$$

Next the distribution kernel of  $W_k = B_k(R_1, R_2)$ ,  $k = 0, 1, 2, 3$ , has the form

$$w_k(x', y; z, h) = (2\pi h)^{-n} \int e^{i\langle x' - y', \xi' \rangle / h} b_k(x', y, \xi'; z) d\xi',$$

with

$$|\partial_{x'}^{\alpha'} \partial_y^\beta \partial_{\xi'}^{\gamma'} (h D_{y_n})^j b_k(x', y, \xi'; z)| \leq C_{\alpha', \beta, \gamma', j} \langle \xi' \rangle^{-3+j+k-|\gamma'|} h^{-\epsilon} e^{-y_n h^\epsilon / h}. \quad (3.8)$$

For the reader convenience we sketch a proof of estimates (3.8) in the Appendix. From the above expression for  $A_j$  we conclude that for  $\psi \in C_0^\infty(\mathbb{R}^{n-1})$  we have

$$\psi(\xi') A_j(x', \xi'; z) \in S_\epsilon^{2\epsilon}, \quad j = 0, 1, 2, 3. \quad (3.9)$$

The reader should consult [11] for the composition of symbols in the spaces  $S_\epsilon^{m,k}$ . Finally, we can solve the system (3.3), determine  $A_j$  and this completes the construction of the parametrix for  $x$  close to the boundary and for  $|\xi'| \leq C_0$ ,  $|\operatorname{Im} z| \geq h^\epsilon$ .

To construct a global parametrix, choose a finite number of points  $x_k \in \bar{\Omega}$ ,  $k = 1, \dots, L$ , so that  $x_k$  with  $k = 1, \dots, L'$ , lie in the interior of  $\Omega$ , while  $x_k$  with  $k = L' + 1, \dots, L$ , belong to  $\partial\Omega$ . Let  $U_k$  be open neighborhoods of  $x_k$ . Thus one has a covering  $\cup_{k=1}^L U_k$  of  $\bar{\Omega}$  and we assume that in  $U_k$ ,  $k = L' + 1, \dots, L$ , we can introduce normal coordinates. Next, let  $\Phi_k \in C_0^\infty(U_k)$ ,  $k = 1, \dots, L$ ,

form a partition of unity on  $\bar{\Omega}$ . Introduce  $\Psi_k \in C_0^\infty(U_k)$ ,  $k = 1, \dots, L'$  with  $\Psi_k = 1$  near  $\text{supp } \Phi_k$ . We may choose  $\Psi_k(x)$ ,  $k = 1, \dots, L'$ , so that  $\text{supp } \Psi_k \cap \partial\Omega = \emptyset$ . Next let  $\Psi \in C_0^\infty(\mathbb{R}^n)$  be a function which is equal to 1 on  $\cup_{j=1}^L \text{supp } \Phi_k$  and let  $W_k(\tilde{F}) = \{B_j(R(\Phi_k \tilde{F}))\}_{j=0,1,2,3}$ ,  $k = L' + 1, \dots, L$ .

We define a global parametrix by

$$\mathcal{E}(\tilde{F}) = \sum_{k=1}^{L'} \Psi_k R(\Phi_k \tilde{F}) + \sum_{k=L'+1}^L \Psi \left( R(\Phi_k \tilde{F}) - \Pi(W_k(\tilde{F})) \right).$$

We get

$$(Q - z)\mathcal{E}(z) \equiv I,$$

$$B_j(\mathcal{E}(z)) \equiv 0, \quad j = 0, 1, 2, 3.$$

By a modification of  $\mathcal{E}(z)$  by an operator  $\equiv 0$ , we can arrange the boundary conditions

$$B_j(\mathcal{E}(z)) = 0, \quad j = 0, 1, 2, 3,$$

preserving the first relation which means that  $(Q - z)\mathcal{E}(z) - I$  has a kernel  $K(x, y; z, h)$  such that

$$|\partial_x^\alpha \partial_y^\beta K(x, y; z, h)| \leq C_{\alpha, \beta, N} h^N, \quad (x, y) \in (\Omega \times \Omega), \quad \forall \alpha, \forall \beta, \forall N.$$

Now we pass to the construction of a microlocal parametrix in the region (A). Here we must work in a region including an interval lying on the real axis, so the estimates with  $\mathcal{O}(|\text{Im } z|^{-1})$  are not usefull. The advantage is that for  $|\xi'| \geq C_0$ , where  $C_0 > 0$  is the constant of Propositions 4 and 5, we have uniform lower bounds for  $|\det \Lambda_1|$ ,  $|\det \Lambda_2|$ . More precisely, for  $k = 1, \dots, L'$ , we will take  $|\xi| \geq C_1 > 0$ , while for  $k = L' + 1, \dots, L$  we choose  $|\xi'| \geq C_0$ . Moreover, we will show that this microlocal parametrix depends *holomorphically* of  $z \in Z_0$ . Assume that  $z \in Z_0$ ,  $\text{Re } z \leq (\text{Im } \omega_0 + \delta)^2$ , where  $\delta$  and  $\omega_0$  satisfy the conditions mentioned in the beginning of the section. Choosing a big constant  $C_1 > 0$  (depending on  $Z_0$ ), let  $\psi(\xi) \in C_0^\infty(\mathbb{R}^n)$  be such that  $\psi(\xi) = 1$  for  $|\xi| \leq C_1$ . The construction of  $R((1 - \psi(hD_x))\Phi_k(x)\tilde{F})$ ,  $k = 1, \dots, L'$ , so that

$$(Q - z)\Psi_k R((1 - \psi(hD_x))\Phi_k) \equiv (1 - \psi(hD_x))\Phi_k, \quad k = 1, \dots, L' \quad (3.10)$$

is standard and it is clear that  $R((1 - \psi(hD_x))\Phi_k(x))$  depends holomorphically of  $z \in Z_0$ . Next, for  $k = L' + 1, \dots, L$ , we choose a fixed function  $\chi(\xi') \in C_0^\infty(\mathbb{R}^{n-1})$  such that  $\chi(\xi') = 1$  for  $|\xi'| \leq C_0$  and we construct  $R$  so that

$$(Q - z) \left[ \Psi R((1 - \chi(hD_{x'}))\Phi_k) \right] \equiv (1 - \psi(hD_{x'}))\Phi_k, \quad k = L' + 1, \dots, L.$$

Obviously, the terms

$$B_j \left( R((1 - \chi(hD_{x'}))\Phi_k) \right), \quad j = 0, 1, 2, 3$$

are holomorphic with respect to  $z$ . To treat the boundary terms  $\sum_{j=0,1} \Pi_j A_j$  and  $\sum_{j=2,3} \Pi_j A_j$ , we must know that the inverse matrix  $M^{-1}(x', \xi', z)$  is holomorphic in  $z$  for  $|\xi'| \geq C_0$ . As in [22], for  $|\xi'| \geq C_0$  we choose  $\gamma(x', \xi') \subset \Omega_1(x', \xi', z) \cap \{|\xi'| \geq C_0, \text{Im } \xi \geq \eta \langle \xi' \rangle\}$  with  $\eta > 0$  small enough. Then for  $|\xi'| \geq C_0$  the components  $m_{j,k}(x', \xi', z)$  of the matrix  $M(x', \xi', z)$  are holomorphic with respect to  $z \in Z_0$  and so is  $\det M(x', \xi', z)$ . In the case  $z \neq (\text{Im } \omega_0)^2$ , applying Proposition 4 for  $\det \Lambda_1(x', \xi', z)$ , we obtain (3.4). Since  $\det M(x', \xi', z)$  is continuous with respect to  $z$ , the estimate (3.4) remains true for  $z = (\text{Im } \omega_0)^2$  and  $|\xi'| \geq C_0$ . Another proof of this statement can be obtained by applying Proposition 5 combined with an expression of  $M(x', \xi', z)$  involving the double roots  $\lambda_+$ ,  $\mu_+$ .

The matrix  $M(x', \xi', z)$  is invertible and holomorphic in  $z$  for  $z \in Z_0$  and  $|\xi'| \geq C_0$ , hence we obtain immediately that  $M^{-1}(x', \xi', z)$  is holomorphic with respect to  $z \in Z_0$  for  $|\xi'| \geq C_0$ . We get the same property for the symbols  $A_j$ ,  $j = 0, 1, 2, 3$ , as well as for  $\Pi_j A_j + \Pi_{j+1} A_{j+1}$ . Finally, we obtain a microlocal parametrix

$$(Q - z)E_k(z) \equiv (1 - \chi(hD_{x'}))\Phi_k(x), \quad B_j(E_k(z)) = 0, \quad k = L' + 1, \dots, L, \quad j = 0, 1, 2, 3 \quad (3.11)$$

depending holomorphically of  $z \in Z_0$ .

#### 4. THE TRACE OF $f(Q)$

In this section we use the notations of the previous one and we assume (2.19) fulfilled. Let  $f \in C_0^\infty([-\infty, (\text{Im } \omega_0 + \delta)^2])$  be a cut-off function with sufficiently small  $\delta > 0$ . Let  $z$  belong to a fixed compact set and let  $\text{Re } z \leq (\text{Im } \omega_0 + \delta)^2$ . We choose an almost analytic extension  $\tilde{f}(z) \in C_0^\infty(\{z \in Z_0 : \text{Re } z < (\text{Im } \omega_0 + \delta)^2\})$  of  $f$  so that  $\bar{\partial}\tilde{f}(z) = \mathcal{O}(|\text{Im } z|^N)$ ,  $\forall N \in \mathbb{N}$ . Next, as in Section 4 of [22], we use the formula

$$f(Q) = -\frac{1}{\pi} \int \bar{\partial}\tilde{f}(z)(z - Q)^{-1}L(dz), \quad (4.1)$$

where  $L(dz)$  is the Lebesgue measure in  $\mathbb{C}$ . Notice that  $(z - Q)^{-1}$  is a matrix-valued operator and so is the operator  $f(Q)$ .

Following our construction in Section 3, the global parametrix is a sum of  $L$  terms. For  $k = 1, \dots, L'$ , from (3.10) we get

$$-\Psi_k R((1 - \psi(hD_x))\Phi_k) = (z - Q)^{-1}(1 - \psi(hD_x))\Phi_k + K_k(z)$$

with an operator  $K_k(z)$  having trace class norm  $[K_k(z)] = \mathcal{O}(|\text{Im } z|^{-1}h^\infty)$ . The operator on the left hand side is holomorphic with respect to  $z$  so the contribution of  $(z - Q)^{-1}(1 - \psi(hD_x))\Phi_k$  in (4.1) is negligible. For  $k = L' + 1, \dots, L$ , we apply the same argument exploiting now (3.11) and the fact that  $E_k(z)$  is holomorphic with respect to  $z$ . Thus we reduce the analysis to the examination of the sum

$$\begin{aligned} f(Q) &= -\frac{1}{\pi} \sum_{k=1}^{L'} \int \bar{\partial}\tilde{f}(z)(z - Q)^{-1}\psi(hD_x)\Phi_k(x)L(dz) \\ &\quad - \frac{1}{\pi} \sum_{k=L'+1}^L \int \bar{\partial}\tilde{f}(z)(z - Q)^{-1}\chi(hD_{x'})\Phi_k(x)L(dz) + K_L \end{aligned}$$

with  $[K_L] = \mathcal{O}(h^\infty)$  and  $\psi(\xi)$ ,  $\chi(\xi')$  introduced in Section 3.

Before going to a second reduction, it is necessary to study the traces of some operators. We start with a description of the distribution kernel  $\Pi_j(x, y'; z, h)$  of the operator  $\Pi_j$  given by the following

**Lemma 4.** *Fix  $0 < \epsilon < 1/2$  small enough. Let  $|\text{Im } z| \geq h^\epsilon$  and let  $z$  belong to a compact set. Modulo a term having order  $\mathcal{O}(h^\infty)$  in the trace class norm, we have the following representation*

$$\Pi_j(x, y'; z, h) = (2\pi h)^{-(n-1)} \int e^{i(x' - y')\xi'/h} m_j(x, y', \xi'; z, h) d\xi'$$

with

$$\begin{aligned}
& |\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \partial_{\xi'}^{\gamma'} (hD_{x_n})^k m_j(x, y', \xi'; z, h)| \\
& \leq C_{\alpha', \beta', \gamma', k} \langle \xi' \rangle^{-2+j+k-|\gamma'|} \frac{1}{|\operatorname{Im} z|} \left( e^{-x_n \operatorname{Im} \lambda_1/h} + e^{-x_n \operatorname{Im} \lambda_2/h} \right) \\
& \leq C_{\alpha', \beta', \gamma', k} \langle \xi' \rangle^{-2+j+k-|\gamma'|} h^{-\epsilon} e^{-x_n h^\epsilon/h}, \quad j = 0, 1
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
& |\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \partial_{\xi'}^{\gamma'} (hD_{x_n})^k m_j(x, y', \xi'; z, h)| \\
& \leq C_{\alpha', \beta', \gamma', k} \langle \xi' \rangle^{-4+j+k-|\gamma'|} \frac{1}{|\operatorname{Im} z|} \left( e^{-x_n \operatorname{Im} \mu_1/h} + e^{-x_n \operatorname{Im} \mu_2/h} \right) \\
& \leq C_{\alpha', \beta', \gamma', k} \langle \xi' \rangle^{-4+j+k-|\gamma'|} h^{-\epsilon} e^{-x_n h^\epsilon/h}, \quad j = 2, 3.
\end{aligned} \tag{4.3}$$

**Proof.** According to (3.1) and (3.2), we have

$$\begin{aligned}
m_j(x, y', \xi', z, h) &= \int_{\gamma} e^{i\langle x_n, \xi_n \rangle/h} R_1(x, \xi, z; h) \xi_n^{1+j} d\xi_n / (2\pi i), \quad j = 0, 1, \\
m_j(x, y', \xi', z, h) &= \int_{\gamma} e^{i\langle x_n, \xi_n \rangle/h} R_2(x, \xi, z; h) \xi_n^{j-1} d\xi_n / (2\pi i), \quad j = 2, 3.
\end{aligned}$$

First, we will treat the principal symbol  $m_{p,j}$  of  $m_j, j = 0, 1$ . Recall that  $\lambda_{j,+}$  are the roots of the equation  $(q_1 - z) = 0$  with respect to  $\xi_n$  for  $x_n$  close to 0 in the domain bounded by  $\gamma$ . Taking the integral over  $\gamma$ , we get for the principal symbol  $m_{p,j}$  the representation

$$m_{p,j}(x, y', \xi'; z, h) = \sum_{\nu=1,2} \lambda_{\nu,+}^{1+j} \left( \partial_{\xi_n} q_1(x', x_n, \xi', \lambda_{\nu,+}) \right)^{-1} e^{ix_n \lambda_{\nu,+}/h}, \quad j = 0, 1. \tag{4.4}$$

From the expression of  $\lambda_{j,+}$ , given in Section 2, we deduce

$$\partial_{\xi_n} q_1(x', x_n, \xi', \lambda_{\nu,+}) = 4\lambda_{\nu,+}(\lambda_{\nu,+}^2 + (s - (\operatorname{Re} \omega_0))) = 4(-1)^{\nu+1} \lambda_{\nu,+} \sqrt{\operatorname{Re} z - (\operatorname{Im} \omega_0)^2 + i \operatorname{Im} z}$$

and the estimate (4.2) with  $k = \alpha' = \gamma' = \beta' = 0$  follows from (4.4). For  $(\alpha', \gamma', \gamma', k') \neq (0, 0, 0, 0)$ , we obtain (4.2) by taking the derivatives of the right hand side term.

In general, modulo a term having order  $\mathcal{O}(h^\infty)$  in the trace class norm,  $R_1(x, \xi; z, h)$  is a sum of terms of the form  $h^N b_N(x, \xi; z) (q_1 - z)^{-2N-1}$  with  $b_N(x, \xi; z)$  a polynomial in  $z$  with smooth coefficients (see Proposition 8.6 and Proposition 8.7 in [7]). Applying the above arguments to each term, and using the fact that  $h|\operatorname{Im} z|^{-2} = \mathcal{O}(1)$ , we obtain (4.2). The proof of (4.3) is similar.  $\square$

**Lemma 5.** Let  $\chi_1(x, \xi') \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ . Under the assumptions of Lemma 4, the operators

$$(\Pi_j A_j(W) + \Pi_{j+1} A_{j+1}(W)) \chi_1(x, hD_{x'}), \quad j = 0, 2,$$

are trace class and

$$\operatorname{tr} \left( (\Pi_j A_j(W) + \Pi_{j+1} A_{j+1}(W)) \chi_1(x, hD_{x'}) \right) = \mathcal{O}(h^{1-n-4\epsilon}), \quad j = 0, 2. \tag{4.5}$$

**Proof.** Let  $\ell_j(x, y; z, h)$  be the kernel of  $(\Pi_j A_j(W) + \Pi_{j+1} A_{j+1}(W)) \chi_1(x, hD_{x'}), j = 0, 2$ . Since all terms of the operator  $(\Pi_j A_j(W) + \Pi_{j+1} A_{j+1}(W)) \chi_1(x, hD_{x'})$  are  $h$ -pseudodifferential operators

in  $x'$ , we deduce from (3.9), (4.2) and (4.3) that modulo a term of order  $\mathcal{O}(h^\infty)$  in trace class norm we have

$$\ell_j(x, y; z, h) = (2\pi h)^{-n} \int e^{i\langle x' - y', \xi' \rangle / h} c_j(x, y, \xi'; z) \chi_1(y, \xi') d\xi'$$

with

$$|\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \partial_{\xi'}^{\gamma'} (hD_{x_n})^k (hD_{y_n})^m c_j(x, y, \xi'; z)| \leq C_{\alpha', \beta', \gamma', k, m} \langle \xi' \rangle^{-N_j} h^{-3\epsilon} \exp(-(x_n + y_n)h^\epsilon / Ch).$$

This implies

$$|\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \partial_{\xi'}^{\gamma'} (hD_{x_n})^k (hD_{y_n})^m \ell_j(x, y; z, h)| \leq C_{\alpha', \beta', \gamma', k, m} h^{-n-3\epsilon} \exp(-(x_n + y_n)h^\epsilon / Ch),$$

which yields (4.5).  $\square$

We can repeat the arguments of the proofs of Lemma 4 and Lemma 5 in the case  $z = z_0$  with  $\text{Im } z_0 > 0$ . Then for the symbols  $m_j$  we obtain the estimates

$$|\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \partial_{\xi'}^{\gamma'} (hD_{x_n})^k m_j(x, y', \xi'; z, h)| \leq C_{\alpha', \beta', \gamma', k} \langle \xi' \rangle^{-2+j+k-|\gamma'|} e^{-x_n/Ch}, \quad j = 0, 1,$$

and a similar one for  $m_j, j = 2, 3$ , while for the kernel  $\ell_j(x, y; z, h)$  we get

$$|\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \partial_{\xi'}^{\gamma'} (hD_{x_n})^k (hD_{y_n})^m \ell_j(x, y; z, h)| \leq C_{\alpha', \beta', \gamma', k, m} h^{-n} \exp(-(x_n + y_n)/Ch) + \mathcal{O}(h^\infty).$$

Consequently, for the trace of  $\Pi_j A_j(W) + \Pi_{j+1} A_{j+1}(W) \chi_1(x, hD_{x'})$ ,  $j = 0, 2$ , we have

$$\text{tr} \left( (\Pi_j A_j(W) + \Pi_{j+1} A_{j+1}(W)) \chi_1(x, hD_{x'}) \right) = \mathcal{O}(h^{1-n}), \quad j = 0, 2. \quad (4.6)$$

To do a second reduction, consider a function  $\chi_k(x, \xi')$  with support in  $\{(x, \xi') : x \in U_k, |\xi'| \leq C_0\}$ ,  $U_k$  being a small neighborhood of  $x_k \in \partial\Omega$ . We fix a  $z_0$  with  $\text{Im } z_0 > 0$ ,  $\text{Re } z_0 < (\text{Im } \omega_0)^2$  and we construct a parametrix  $E(z_0)$  for the operator  $(Q - z_0)$  as in Section 3. Thus we obtain

$$(z_0 - Q)^{-1} \chi_k(x, hD_{x'}) = -E(z_0) \chi_k(x, hD_{x'}) + K_k \quad (4.7)$$

with trace class norm  $[K_k] = \mathcal{O}(h^\infty)$ .

Next, we are going to repeat the arguments in [22] with some minor modifications. The analysis of  $R(x, hD_x; z, h) \chi_k(x, hD_{x'})$  is the same as in [22] and we obtain a trace class operator with trace class norm  $\mathcal{O}(h^{-n})$ . On the other hand, for fixed  $z_0$  with  $\text{Im } z_0 > 0$  we apply the estimate (4.6). Next for  $\text{Im } z \neq 0$  and  $z$  in a compact set of  $\mathbb{C}$  we have

$$(z - Q)^{-1} \chi_k(x, hD_{x'}) = (I + (z_0 - z)(z - Q)^{-1})(z_0 - Q)^{-1} \chi_k(x, hD_{x'})$$

and taking into account (4.6), we get

$$\|(Q - z)^{-1} \chi_k(x, hD_{x'})\|_{\text{tr}} \leq C |\text{Im } z|^{-1} h^{-n}. \quad (4.8)$$

On the other hand, for  $\tilde{f}(z)$  we have the estimate  $\bar{\partial} \tilde{f}(z) = \mathcal{O}(|\text{Im } z|^N)$ ,  $\forall N$ , and combining this with (4.8) leads to

$$\int \bar{\partial} \tilde{f}(z) (z - Q)^{-1} \chi_k(x, hD_{x'}) L(dz) = \int_{|\text{Im } z| \geq h^\epsilon} \bar{\partial} \tilde{f}(z) (z - Q)^{-1} \chi_k(x, hD_{x'}) L(dz) + K_\epsilon, \quad (4.9)$$

with  $[K_\epsilon] = \mathcal{O}(h^\infty)$ .

Now we may apply the partition of unity  $\{\Phi_k(x)\}_{k=1, \dots, L}$  introduced in Section 3. Set

$$\chi_k(x, \xi) = \psi(\xi) \Phi_k(x), \quad k = 1, \dots, L', \quad \chi_k(x, \xi') = \chi(\xi') \Phi_k(x), \quad k = L' + 1, \dots, L$$



with the functions  $\psi(\xi)$ ,  $\chi(\xi')$  introduced in Section 3. Let  $E_k(z)$  be the parametric of  $(Q - z)$  constructed in Section 3 for  $z$  in a compact set  $Z_0$  and  $|\operatorname{Im} z| \geq h^\epsilon$  with  $\epsilon > 0$  small enough for which we have

$$(z - Q)^{-1} \chi_k(x, hD_{x'}) = -E_k(z) \chi_k(x, hD_{x'}) + K_k(z), \quad k = L' + 1, \dots, L \quad (4.10)$$

with  $[K_k(z)] = \mathcal{O}(h^\infty)$ , uniformly on  $z$ . For  $k = 1, \dots, L'$ , we have a similar expression and the parametrix  $E_k(z)$  has no boundary terms.

According to (4.10), modulo terms with trace class norm  $\mathcal{O}(h^\infty)$ , we get

$$\begin{aligned} \operatorname{tr}(f(Q)) &= \operatorname{tr} \left[ -\frac{1}{\pi} \sum_{\nu=1}^2 \sum_{k=1}^{L'} \int_{|\operatorname{Im} z| \geq h^\epsilon} \bar{\partial} \tilde{f}(z) R_{\nu,k}(x, hD_x; z, h) \chi_k(x, hD_x) L(dz) \right] \\ &\quad + \operatorname{tr} \left[ -\frac{1}{\pi} \sum_{\nu=1}^2 \sum_{k=L'+1}^L \int_{|\operatorname{Im} z| \geq h^\epsilon} \bar{\partial} \tilde{f}(z) R_{\nu,k}(x, hD_x; z, h) \chi_k(x, hD_{x'}) L(dz) \right] \\ &+ \operatorname{tr} \left[ \frac{1}{\pi} \sum_{\nu=0}^2 \sum_{k=L'+1}^L \int_{|\operatorname{Im} z| \geq h^\epsilon} \bar{\partial} \tilde{f}(z) \left( \Pi_\nu A_{\nu,k}(W) + \Pi_{\nu+1} A_{\nu+1,k}(W_k) \right) \chi_k(x, hD_{x'}) L(dz) \right] := (I) + (II) + (III). \end{aligned}$$

Here  $R_{\nu,k} = \Psi_k(x) R_\nu(x, hD_x; z, h)$  (see the definition of the global parametrix in Section 3) so the integration in (I) and (II) is over compact sets respectively in  $(x, \xi)$  and  $(x, \xi')$ . On the other hand, an application of Lemma 5 yields

$$(III) = \mathcal{O}(h^{-n+1-4\epsilon}).$$

Notice that by using (4.7) and (4.10) and the action of  $(Q - z)^{-1}$ , we are going to calculate the trace of

$$-\frac{1}{\pi} \int_{|\operatorname{Im} z| \geq h^\epsilon} \mathbf{1}_\Omega(x) R(x, hD_x; z, h) \chi_k(x, \xi') L(dz).$$

The principal symbol of  $-\frac{1}{\pi} \int_{|\operatorname{Im} z| \geq h^\epsilon} R_\nu(x, hD_x; z, h) L(dz)$  is  $f(q_\nu(x, \xi))$ , so the principal symbol of

$$-\frac{1}{\pi} \int_{|\operatorname{Im} z| \geq h^\epsilon} \mathbf{1}_\Omega(x) R_\nu(x, hD_x; z, h) \chi_k(x, \xi') L(dz)$$

is  $f(q_\nu(x, \xi)) \chi_k(x, \xi')$ ,  $x \in \Omega$ . The kernel  $K_{\nu,k}(x, y)$  of the operator with symbol  $f(q_\nu(x, \xi)) \chi_k(x, \xi')$  is continuous in  $\Omega \times \Omega$  and its trace is equal to  $\int_\Omega K_{\nu,k}(x, x) dx$ . Thus we conclude that

$$(I) + (II) = (2\pi h)^{-n} \left( \int_\Omega \int f(q_1(x, \xi)) dx d\xi + \int_\Omega \int f(q_2(x, \xi)) dx d\xi \right) + \mathcal{O}(h^{-n+1}), \quad 0 < h \leq h_0(\delta).$$

Consequently,

$$\begin{aligned} \operatorname{tr}(f(Q)) &= (2\pi h)^{-n} \left( \int_\Omega \int f(q_1(x, \xi)) dx d\xi + \int_\Omega \int f(q_2(x, \xi)) dx d\xi \right) \\ &\quad + \mathcal{O}(h^{-n+1-4\epsilon}), \quad 0 < h \leq h_0(\delta). \end{aligned} \quad (4.11)$$

The above argument implies easily that the spectrum of  $Q$  in  $[0, (\operatorname{Im} \omega_0 + \delta)^2]$  is formed by isolated eigenvalues of finite multiplicities. To prove this it is sufficient to show that the resolvent  $(Q - z_0)^{-1}$  for fixed  $z_0$  with  $\operatorname{Im} z_0 > 0$  is compact. Every term  $(Q - z_0)^{-1} \chi_k(x, \xi)$  is a trace class operator. On the other hand, for  $C_0 > 0$  large enough the ellipticity of boundary problem and our construction in Section 3 of a (microlocal) parametrix show that  $(Q - z_0)^{-1} (1 - \chi(hD_{x'})) \Phi_k(x)$  is a compact operator.

## 5. PROOF OF THEOREM 1

In this section we prove the estimate (1.4). Let  $\operatorname{Re} \omega_0 = 1$ ,  $\operatorname{Im} \omega_0 > 0$ . We fix  $0 < \operatorname{Im} \omega_0 < 1$  so that (2.19) holds and choose  $0 < \delta < \operatorname{Im} \omega_0$  small enough. Thus the construction of the semi-classical parametrix in Section 3 works and we may exploit the results of Section 4. Set  $(6\delta \operatorname{Im} \omega_0 + 9\delta^2) = \delta_1^2$  and let  $f \in C_0^\infty([-\infty, (\operatorname{Im} \omega_0 + 3\delta)^2]; [0, 1])$  be a function which is equal to 1 on  $[0, (\operatorname{Im} \omega_0 + 2\delta)^2]$ . Consider  $f(q_1(x, \xi))$  and  $f(q_2(x, \xi))$ , where the symbols  $q_j(x, \xi)$ ,  $j = 1, 2$ , were introduced in Section 3. On the support of  $f(q_1(x, \xi))$  we have

$$||\xi|^2 - 1|^2 + (\operatorname{Im} \omega_0)^2 \leq (\operatorname{Im} \omega_0 + 3\delta)^2$$

which yields

$$||\xi|^2 - 1| \leq ((\operatorname{Im} \omega_0) + 3\delta)^2 - (\operatorname{Im} \omega_0)^2)^{1/2} = \delta_1.$$

Thus on the support of  $f(q_1(x, \xi))$  we have  $1 - \delta_1 \leq |\xi|^2 \leq 1 + \delta_1$ . Similarly on the support of  $f(q_2(x, \xi))$  we have

$$1 - \delta_1 \leq \frac{1}{1+m} |\xi|^2 \leq 1 + \delta_1.$$

In the following we suppose that  $4\epsilon < 1/2$  so  $1 - n - 4\epsilon > 1/2 - n$ . Taking the integral over  $\Omega \times \mathbb{R}^n$ , we deduce

$$\int_{\Omega} \int f(q_1(x, \xi)) dx d\xi \leq \operatorname{vol}(\Omega) \omega_n \left( (1 + \delta_1)^{n/2} - (1 - \delta_1)^{n/2} \right),$$

$\omega_n$  being the volume of unit ball in  $\mathbb{R}^n$  and

$$\int_{\Omega} \int f(q_2(x, \xi)) dx d\xi \leq \int_{\Omega} (1 + m(x))^{n/2} dx \omega_n \left( (1 + \delta_1)^{n/2} - (1 - \delta_1)^{n/2} \right).$$

Consequently,

$$\begin{aligned} \operatorname{tr} f(Q) &= (2\pi h)^{-n} \left[ \int_{\Omega} \int f(q_1(x, \xi)) dx d\xi + \int_{\Omega} \int f(q_2(x, \xi)) dx d\xi \right] + \mathcal{O}(h^{1-n-4\epsilon}) \\ &= \frac{(1 + \delta_1)^{n/2} - (1 - \delta_1)^{n/2}}{(2\pi)^n} \omega_n \int_{\Omega} \left( 1 + (1 + m(x))^{n/2} \right) dx h^{-n} + \mathcal{O}(h^{1-n-4\epsilon}) \\ &= \left( (1 + \delta_1)^{n/2} - (1 - \delta_1)^{n/2} \right) C_2(\Omega, n) h^{-n} + \mathcal{O}(h^{1-n-4\epsilon}) := \mathcal{N}(2\delta; h). \end{aligned}$$

with  $C_2(\Omega, n) = \frac{\omega_n}{(2\pi)^n} \int_{\Omega} \left( 1 + (1 + m(x))^{n/2} \right) dx$ . This implies that the number  $M(2\delta; h)$  of the eigenvalues of  $|Q|^{1/2}$  smaller than  $\operatorname{Im} \omega_0 + 2\delta$  satisfies

$$M(2\delta; h) \leq \mathcal{N}(2\delta; h). \quad (5.1)$$

Let  $0 \leq \mu_1 \leq \mu_2 \leq \dots$  be the eigenvalues of  $|Q|^{1/2}$  followed by an infinite repetition of  $\inf \sigma(|Q|^{1/2})$  in the case if there are only finitely many eigenvalues. Since  $Q - z$  is related to an elliptic boundary problem for  $z < (\operatorname{Im} \omega_0)^2$ , we have

$$\mu_1 \geq \operatorname{Im} \omega_0 - o(1), \quad h \rightarrow 0.$$

Now let  $N = N(\delta; h)$  be the number of the eigenvalues  $\nu_1, \nu_2, \dots, \nu_N$  of  $h^2 \mathcal{P} - \omega_0$  with  $0 \leq |\nu_1| \leq \dots \leq |\nu_N| \leq \operatorname{Im} \omega_0 + \delta$ .

Applying the Weyl inequality for the non-selfadjoint operator  $h^2 \mathcal{P} - \omega_0$  (see Appendix a. in [20]), we obtain

$$\mu_1 \dots \mu_N \leq |\nu_1| \dots |\nu_N|. \quad (5.2)$$

As in [22], this leads to the estimate

$$N(\delta; h) \leq c_0 M(2\delta; h) \leq c_0 \mathcal{N}(2\delta; h), \quad 0 < h \leq h_0(\delta) \quad (5.3)$$

with a constant  $c_0 = c_0(\omega_0, \delta) > 0$  depending only on  $\omega_0$  and  $\delta$ . For reader convenience, we present the argument. Assume that  $M(2\delta; h) \leq N(\delta; h)$ . Then

$$\mu_1^M (\operatorname{Im} \omega_0 + 2\delta)^{N-M} \leq (\operatorname{Im} \omega_0 + \delta)^N$$

and we get

$$\left[ \frac{\operatorname{Im} \omega_0 + 2\delta}{\operatorname{Im} \omega_0 + \delta} \right]^N \leq \left[ \frac{\operatorname{Im} \omega_0 + 2\delta}{\mu_1} \right]^M.$$

Therefore for  $h \leq h_0(\delta)$  we have

$$N \log \left( \frac{\operatorname{Im} \omega_0 + 2\delta}{\operatorname{Im} \omega_0 + \delta} \right) \leq M \log \left( \frac{\operatorname{Im} \omega_0 + 2\delta}{\operatorname{Im} \omega_0 - \delta} \right).$$

We may estimate

$$c_0 = \frac{\log \left( 1 + \frac{3\delta}{\operatorname{Im} \omega_0 - \delta} \right)}{\log \left( 1 + \frac{\delta}{\operatorname{Im} \omega_0 + \delta} \right)}$$

as  $\delta \rightarrow 0$  and conclude that with  $c_0 = 3(1 + \mathcal{O}_{\omega_0}(\delta))$  we have

$$N(\delta; h) \leq c_0 M(2\delta; h).$$

Consider the disk  $D(\omega_0; \operatorname{Im} \omega_0 + \delta)$  with center  $\omega_0 \in \mathbb{C}$  and radius  $\operatorname{Im} \omega_0 + \delta < 1$ . Let  $S_0$  be the rectangle

$$S_0 = \{(x, y) \in \mathbb{R}^2 : 1 - \delta_2 \leq x \leq 1 + \delta_2, -X \leq y \leq \operatorname{Im} \omega_0\} \subset D(\omega_0; \operatorname{Im} \omega_0 + \delta),$$

where

$$\delta_2 = \sqrt{1 - \delta^2} \left( 2 \operatorname{Im} \omega_0 \delta + \delta^2 \right)^{1/2} < \delta_1,$$

and  $X = \left( (\operatorname{Im} \omega_0)^2 + \delta^2 (2 \operatorname{Im} \omega_0 \delta + \delta^2) \right)^{1/2} - \operatorname{Im} \omega_0 > 0$ . For  $\delta$  small enough we have  $X < \operatorname{Im} \omega_0$ .

We choose  $0 < \theta < \pi/2$  so that

$$\tan \theta = \frac{X}{1 + \delta_2} < X.$$

It is clear that when  $\theta \searrow 0$  we have  $\delta \searrow 0$ . According to the above result, the number of the eigenvalues of the operator  $h^2 \mathcal{P}$  in  $S_0$  can be estimated by  $c_0 \mathcal{N}(2\delta; h)$ , so the number of the eigenvalues of  $\mathcal{P}$  in  $h^{-2} S_0$  is estimated by  $c_0 \mathcal{N}(2\delta; h)$ .

Given  $r \gg 1$ , consider the set

$$S_{0,K} = \bigcup_{k=0}^K \left( \frac{1 + \delta_2}{1 - \delta_2} \right)^{-k} r S_0,$$

where  $K = K(r)$  is chosen so that  $\left( \frac{1 + \delta_2}{1 - \delta_2} \right)^K r^{-1} \leq h_0^2(\delta)$ . Therefore for

$$h = \left( \frac{1 + \delta_2}{1 - \delta_2} \right)^{k/2} r^{-1/2} \leq h_0(\delta)$$

we can apply the above estimate and the number of the eigenvalues of  $\mathcal{P}$  in  $S_{0,K}$  is estimated by

$$\begin{aligned} c_0 \sum_{k=0}^K \mathcal{N}\left(2\delta; \left(\frac{1+\delta_2}{1-\delta_2}\right)^{k/2} r^{-1/2}\right) &\leq \left((1+\delta_1)^{n/2} - (1-\delta_1)^{n/2}\right) c_0 C_2(\Omega, n) \sum_{k=0}^K \left(\frac{1-\delta_2}{1+\delta_2}\right)^{nk/2} r^{n/2} \\ &\quad + \sum_{k=0}^K \left(\frac{1-\delta_2}{1+\delta_2}\right)^{(n-1+4\epsilon)k/2} \mathcal{O}(r^{(n-1+4\epsilon)/2}) \\ &\leq c_0 C_2(\Omega, n) (1+\delta_2)^{n/2} \frac{(1+\delta_1)^{n/2} - (1-\delta_1)^{n/2}}{(1+\delta_2)^{n/2} - (1-\delta_2)^{n/2}} r^{n/2} + \mathcal{O}_\delta(r^{(n-1+4\epsilon)/2}), \quad r \geq r_0(\delta), \end{aligned}$$

Clearly,

$$\left((1+\delta_1)^{n/2} - (1-\delta_1)^{n/2}\right) = n\delta_1(1 + \mathcal{O}_{\omega_0}(\delta)),$$

$$\left((1+\delta_2)^{n/2} - (1-\delta_2)^{n/2}\right) = n\delta_2(1 + \mathcal{O}_{\omega_0}(\delta))$$

and

$$\frac{\delta_1}{\delta_2} = \frac{1}{\sqrt{1-\delta^2}} \left( \frac{6 \operatorname{Im} \omega_0 + 9\delta}{2 \operatorname{Im} \omega_0 + \delta} \right)^{1/2} \xrightarrow{\delta \rightarrow 0} \sqrt{3}.$$

Thus the number of the eigenvalues of  $\mathcal{P}$  in  $S_{0,K}$  can be estimated by

$$3\sqrt{3}(1 + \mathcal{O}_{\omega_0}(\delta)) \frac{\omega_n}{(2\pi)^n} \int_{\Omega} \left(1 + (1 + m(x))^{n/2}\right) dx r^{n/2}, \quad r \geq r_0(\delta),$$

where we increase  $r_0(\delta)$ , if it is necessary. On the other hand, it is easy to see that for small  $\theta > 0$  we have

$$\begin{aligned} \Lambda_{\theta,r} &= \{z \in \mathbb{C} : |\arg z| \leq \theta, |z| \leq r\} \subset \bigcup_{k=0}^{\infty} \left(\frac{1+\delta_2}{1-\delta_2}\right)^{-k} r S_0 \\ &= S_{0,K} \cup \bigcup_{k=K+1}^{\infty} \left(\frac{1+\delta_2}{1-\delta_2}\right)^{-k} r S_0 = S_{0,K} \cup Q_K. \end{aligned}$$

Here  $Q_K$  is a compact set independent on  $r$ . Indeed, for  $j \geq K(r) + 1$  we have

$$\left(\frac{1+\delta_2}{1-\delta_2}\right)^{-j} r < \frac{1}{h_0^2(\delta)},$$

hence the rectangles  $\left(\frac{1+\delta_2}{1-\delta_2}\right)^{-j} r S_0$  for  $j \geq K(r) + 1$  are included in a fixed compact set.

Finally, the number  $N(\theta, r)$  of the eigenvalues of  $\mathcal{P}$  in  $\Lambda_{\theta,r}$  is estimated by

$$N(\theta, r) \leq 3\sqrt{3}(1 + \mathcal{O}_{\omega_0}(\delta)) \frac{\omega_n}{(2\pi)^n} \left( \int_{\Omega} \left(1 + (1 + m(x))^{n/2}\right) dx \right) r^{n/2} + C_0(\delta), \quad r \geq r_0(\delta). \quad (5.4)$$

As we mentioned above, as  $\theta \rightarrow 0$ , we have  $\delta \rightarrow 0$ . Hence for fixed  $\omega_0$  we can replace  $\mathcal{O}_{\omega_0}(\delta)$  by  $\epsilon(\theta) \rightarrow 0$  as  $\theta \rightarrow 0$  and  $r_0(\delta)$  by  $r(\theta)$ . This completes the proof of Theorem 1.

## 6. APPENDIX

In this Appendix we sketch a proof of the estimate (3.8). Recall that we treat the region (B), where  $|\operatorname{Im} z| \geq h^\epsilon$ ,  $0 < \epsilon < 1/2$ .

For the operator  $W_j$ , we must study the kernels of the operators

$$\left( (hD_{x_n})^j R_k(x, hD_x; z, h) \right) \Big|_{x_n=0}, \quad k = 1, 2, \quad j = 0, 1, 2, 3.$$

The kernels of these operators have the form

$$(2\pi h)^{-n} \int_{\mathbb{R}_{\xi'}^{n-1}} e^{i(x'-y')\xi'/h} \int_{\mathbb{R}} e^{-iy_n \xi_n/h} \xi_n^j R_k(x', 0, \xi', \xi_n; z, h) d\xi_n.$$

For simplicity we treat below the case  $k = 1$ . In Section 3 it was shown that

$$R_1(x', 0, \xi', \xi_n; z, h) = \sum_{\nu=0}^N h^\nu c_\nu(x', 0, \xi', \xi_n; z) + h^{N+1} c_{N+1}(x', 0, \xi', \xi_n; z),$$

with  $c_{N+1} \in S_e^{\epsilon(N+2), -4-N-1}$  and  $c_\nu$  are holomorphic in the domain  $\Omega(x, \xi', z)$  defined in Section 3. Clearly,

$$(2\pi h)^{-n} \int_{\mathbb{R}_{\xi'}^{n-1}} e^{i(x'-y')\xi'/h} \int_{\mathbb{R}} e^{-iy_n \xi_n/h} \xi_n^k h^{N+1} c_{N+1}(x', 0, \xi', \xi_n; z) d\xi_n = \mathcal{O}(h^{M(N)}),$$

uniformly on  $|\operatorname{Im} z| > h^\epsilon$  with  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$ .

Let  $\gamma = \Gamma_\rho(x', \xi') = [-\rho, \rho] \cup \gamma_\rho(x', \xi')$  be a closed, simple loop, which encircles  $\lambda_{j,-}, \mu_{j,-}$  in the positive sense. Here  $\gamma_\rho(x', \xi')$  is a curve included in  $\{\xi_n \in \mathbb{C}; \operatorname{Im} \xi_n \leq 0, |\xi_n - \lambda_{j,-}| \geq \eta\langle \xi' \rangle, |\xi_n - \mu_{j,-}| \geq \eta\langle \xi' \rangle, j = 1, 2\}$ , and  $\rho > R_0(x', \xi')$  is a large positive constant. Set

$$G_{j,\nu}(x', y_n, \xi'; z, h) = \int_{\gamma_\rho(x', \xi')} e^{-iy_n \xi_n/h} \xi_n^j c_\nu(x', 0, \xi', \xi_n; z) d\xi_n.$$

Clearly  $G_{j,\nu}(x', y_n, \xi'; z, h)$  is independent of  $\rho > R_0(x', \xi')$ . Repeating the proof of Lemma 2, we see easily that  $G_{j,\nu}$  satisfies the estimate (3.8). Recall that  $y_n > 0$  and  $\operatorname{Im} \lambda_{j,-}, \operatorname{Im} \mu_{j,-} < -Ch^\epsilon$ . Now, to complete the proof of (3.8), it suffices to use the fact that

$$\begin{aligned} G_{j,\nu}(x', y_n, \xi'; z, h) &= \lim_{\rho \rightarrow \infty} \int_{\gamma_\rho(x', \xi')} e^{-iy_n \xi_n/h} \xi_n^j c_\nu(x', 0, \xi', \xi_n; z) d\xi_n \\ &= \int_{\mathbb{R}} e^{-iy_n \xi_n/h} \xi_n^j c_\nu(x', 0, \xi', \xi_n; z) d\xi_n. \end{aligned}$$

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