SPECTRAL ESTIMATES FOR RUELLE TRANSFER OPERATORS WITH TWO PARAMETERS AND APPLICATIONS

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Abstract. For C^2 weak mixing Axiom A flow $\phi_t : M \longrightarrow M$ on a Riemannian manifold M and a basic set Λ for ϕ_t we consider the Ruelle transfer operator $L_{f-s\tau+zg}$, where f and g are real-valued Hölder functions on Λ, τ is the roof function and $s, z \in \mathbb{C}$ are complex parameters. Under some assumptions about ϕ_t we establish estimates for the iterations of this Ruelle operator in the spirit of the estimates for operators with one complex parameter (see [4], [20], [21]). Two cases are covered: (i) for arbitrary Hölder f, g when $|\operatorname{Im} z| \leq B|\operatorname{Im} s|^\mu$ for some constants $B > 0$, $0 < \mu < 1$ ($\mu = 1$ for Lipschitz f, g), (ii) for Lipschitz f, g when $|\operatorname{Im} s| \leq B_1|\operatorname{Im} z|$ for some constant $B > 0$. Applying these estimates, we obtain a non zero analytic extension of the zeta function $\zeta(s, z)$ for $P_f - \epsilon < \operatorname{Re}(s) < P_f$ and $|z|$ small enough with simple pole at $s = s(z)$. Two other applications are considered as well: the first concerns the Hannay-Ozorio de Almeida sum formula, while the second deals with the asymptotic of the counting function $\pi_F(T)$ for weighted primitive periods of the flow ϕ_t.

1. Introduction

Let M be a C^2 complete (not necessarily compact) Riemannian manifold, and let $\phi_t : M \longrightarrow M$, $t \in \mathbb{R}$, be a C^2 weak mixing Axiom A flow (see [2], [11]). Let Λ be a *basic set* for ϕ_t, i.e. Λ is a compact ϕ_t- invariant subset of M, ϕ_t is hyperbolic and transitive on Λ and Λ is locally maximal, i.e. there exists an open neighborhood V of Λ in M such that $\Lambda = \cap_{t\in\mathbb{R}}\phi_t(V)$. The restriction of the flow ϕ_t on Λ is a hyperbolic flow [11]. For any $x \in M$ let $W^s_\epsilon(x), W^u_\epsilon(x)$ be the local stable and unstable manifolds through x, respectively (see [2], [6], [11]).

When M is compact and M itself is a basic set, ϕ_t is called an *Anosov flow*. It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x, y] \in \Lambda$ (cf. [6]). This means that there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\phi_t([x, y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x, y) = t$, defines the so called *temporal distance function*.

In the paper we will use the set-up and some arguments from [20]. First, as in [20], we fix a *(pseudo-) Markov partition* $\mathcal{R} = \{R_i\}_{i=1}^k$ of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$. Set $R = \cup_{i=1}^k R_i$, $U = \cup_{i=1}^k U_i$. Consider the *Poincaré map* $\mathcal{P} : R \longrightarrow R$, defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in R$. The function τ is the so called *first return time* associated with $\mathcal{R}$. Let $\sigma : U \longrightarrow U$ be the *shift map* given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. Let $\widehat{U}$ be the set of those points $x \in U$ such that $\mathcal{P}^m(x)$ is not a boundary point of a rectangle for any integer m. In a similar way define $\widehat{R}$. Clearly in general τ is not continuous on U, however under the assumption that the

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holonomy maps are Lipschitz (see Sect. 3) τ is *essentially Lipschitz* on U in the sense that there exists a constant $L > 0$ such that if $x, y \in U_i \cap \sigma^{-1}(U_j)$ for some i, j, then $|\tau(x) - \tau(y)| \leq L\, d(x, y)$. The same applies to $\sigma : U \longrightarrow U$.

The hyperbolicity of the flow on Λ implies the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma_0 > 1$ such that

$$c_0 \gamma_0^m\, d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma_1^m}{c_0} d(u_1, u_2) \tag{1.1}$$

whenever $\sigma^j(u_1)$ and $\sigma^j(u_2)$ belong to the same U_{i_j} for all $j = 0, 1 \ldots, m$.

Define a $k \times k$ matrix $A = \{A(i, j)\}_{i,j=1}^k$ by

$$A(i,j) = \begin{cases} 1 \text{ if } \mathcal{P}(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

It is possible to construct a Markov partition $\mathcal{R}$ so that A is irreducible and aperiodic (see [2]). Introduce $R^\tau = \{(x, t) \in R \times \mathbb{R} :\ 0 \leq t \leq \tau(x)\}/\sim$, where by $\sim$ we identify the points $(x, \tau(x))$ and $(\sigma x, 0)$. One defines the suspended flow $\sigma_t^\tau(x, s) = (x, s + t)$ on R^τ taking into account the identification $\sim$. For a Hölder continuous function f on R, the pressure $\Pr(f)$ with respect to σ is defined as

$$\Pr(f) = \sup_{m \in \mathcal{M}_\sigma} \{h(\sigma, m) + \int f dm\},$$

where $\mathcal{M}_\sigma$ denotes the space of all σ-invariant Borel probability measures and $h(\sigma, m)$ is the entropy of σ with respect to m. We say that f and g are cohomologous and we denote this by $f \sim g$ if there exists a continuous function w such that $f = g + w \circ \sigma - w$. For a function v on R one defines

$$v^n(x) := v(x) + v(\sigma(x)) + ... + v(\sigma^{n-1}(x)).$$

Let γ denote a primitive periodic orbit of ϕ_t and let $\lambda(\gamma)$ denote its least period. Given a Hölder function $F : \Lambda \longrightarrow \mathbb{R}$, introduce the weighted period $\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))dt$, where $x_\gamma \in \gamma$. Consider the weighted version of the dynamical zeta function (see Section 9 in [11])

$$\zeta_\phi(s, F) := \prod_\gamma \Big(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\Big)^{-1}.$$

Denote by $\pi(x, t) : R^\tau \longrightarrow \Lambda$ the semi-conjugacy projection which is one-to-one on a residual set and $\pi(t, x) \circ \sigma_t^\tau = \phi_t \circ \pi(t, x)$ (see [2]). Then following the results in [2], [3], a closed σ-orbit $\{x, \sigma x, ..., \sigma^{n-1}x\}$ is projected to a closed orbit γ in Λ with a least period

$$\lambda(\gamma) = \tau^n(x) := \tau(x) + \tau(\sigma(x)) + ... + \tau(\sigma^{n-1}(x)).$$

Passing to the symbolic model R (see [2], [11]), the analysis of $\zeta_\varphi(s, F)$ is reduced to that of the Dirichlet series

$$\eta(s) = \sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x)}.$$

with a Hölder continuous function $f(x) = \int_0^{\tau(x)} F(\pi(x, t))dt : R \longrightarrow \mathbb{R}$. On the other hand, to deal with certain problems (see Chapter 9 in [11] and [16]) it is necessary to study a more general series

$$\eta_g(s) = \sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} g^n(x) e^{f^n(x) - s\tau^n(x)}$$

with a Hölder continuous function $G : \Lambda \longrightarrow \mathbb{R}$ and $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt : R \longrightarrow \mathbb{R}$. For this purpose it is convenient to examine the zeta function

$$\zeta(s,z) := \prod_\gamma \Big(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma) + z\lambda_G(\gamma)}\Big)^{-1} = \exp\Big(\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x) + zg^n(x)}\Big) \tag{1.2}$$

depending on two complex variables $s, z \in \mathbb{C}$. Formally, we get

$$\eta_g(s) = \frac{\partial \log \zeta(s,z)}{\partial z}\Big|_{z=0}.$$

Example 1. If $G = 0$ we obtain the classical Ruelle *dynamical zeta function*

$$\zeta_\phi(s) = \prod_\gamma \Big(1 - e^{-s\lambda(\gamma)}\Big)^{-1}.$$

Then $Pr(0) = h$, where $h > 0$ is the topological entropy of ϕ_t and $\zeta_\phi(s)$ is absolutely convergent for $\operatorname{Re} s > h$ (see Chapter 6 in [11]).

Example 2. Consider the expansion function $E : \Lambda \longrightarrow \mathbb{R}$ defined by

$$E(x) := \lim_{t \to 0} \frac{1}{t} \log |\text{Jac }(D\phi_t|_{E^u(x)})|,$$

where the tangent space $T_x(M)$ is decomposed as $T_x(M) = E^s(x) \oplus E^0(x) \oplus E^u(x)$ with $E^s(x), E^u(x)$ tangent to stable and instable manifolds through x, respectively. Introduce the function $\lambda^u(\gamma) = \lambda_E(\gamma)$ and define $f : R \longrightarrow \mathbb{R}$ by

$$f(x) = -\int_0^{\tau(x)} E(\pi(x,t))dt.$$

Then we have $-\lambda^u(\gamma) = f^n(x)$, f is Hölder continuous function and $Pr(f) = 0$ (see [3]). Consequently, the series

$$\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x)} \tag{1.3}$$

is absolutely convergent for $\operatorname{Re} s > 0$ and nowhere zero and analytic for $\operatorname{Re} s \geq 0$ except for a simple pole at $\operatorname{Re} s = 0$ (see Theorem 9.2 in [11]). The roof functions $\tau(x)$ is constant on stable leaves of rectangles R_i of the Markov family $\mathcal{R}$, so we can assume that $\tau(x)$ depends only on $x \in U$. By a standard argument (see [11]) we can replace f in (1.3) by a Hölder function $\hat{f}(x)$ which depends only on $x \in U$ so that $f \sim \hat{f}$. Thus the series (1.3) can be written by functions $\hat{f}, \tau$ depending only on $x \in U$. We keep the notation f below assuming that f depends only on $x \in U$. The analysis of the analytic continuation of (1.3) is based on spectral estimates for the iterations of the Ruelle operator

$$L_{f - s\tau} v(x) = \sum_{\sigma y = x} e^{f(y) - s\tau(y)} v(y),\ v \in C^\alpha(U),\ s \in \mathbb{C}.$$

(see for more details [4], [15], [20], [21], [23]).

Example 3. Let f, τ be real-valued Hölder functions and let $P_f > 0$ be the unique real number such that $Pr(f - P_f\tau) = 0$. Let $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$, where $G : \Lambda \longrightarrow \mathbb{R}$ is a Hölder function. Then if the suspended flow σ_t^τ is weak-mixing, the function (1.2) is nowhere zero analytic function for $\operatorname{Re} s > P_f$ and z in a neighborhood of 0 (depending on s) with nowhere zero analytic extension to $\operatorname{Re} s = P_f$ ($s \neq P_f$) for small $|z|$. This statement is just Theorem 6.4 in [11]. To examine the analytic continuation of $\zeta(s, z)$ for $P_f - \eta_0 \leq \operatorname{Re} s$, $\eta_0 > 0$ and small $|z|$, it is necessary to establish and to exploit some spectral estimates for the iterations of the Ruelle operator

$$L_{f-s\tau+zg}v(x) = \sum_{\sigma y = x} e^{f(y)-s\tau(y)+zg(y)}v(y),\ v \in C^\alpha(U),\ s \in \mathbb{C}, z \in \mathbb{C}. \tag{1.4}$$

The analytic continuation of $\zeta(s, z)$ for small $|z|$ and that of $\eta_g(s)$ play a crucial role in the argument in [16] concerning the Hannay-Ozorio de Almeida sum formula for the geodesic flow on compact negatively curved surfaces. We deal with the same question for Axiom A flows on basic sets in Sect. 7.

Example 4. In the paper [7] the authors examine for Anosov flows the spectral properties of the Ruelle operator (1.4) with $f = 0$ and $z = \mathbf{i}w$, $w \in \mathbb{R}$, as well as the analyticity of the corresponding L-function $L(s, z)$. The properties of the Ruelle operator

$$L^n_{f-(P_f+a+\mathbf{i}b)\tau+\mathbf{i}w},\ w \in \mathbb{R},\ n \in \mathbb{N},$$

are also rather important in the paper [22] dealing with the large deviations for Anosov flows. Here as above $P_f \in \mathbb{R}$ is such that $Pr(f - P_f\tau) = 0$. However, it is important to note that in [7] and [22] the analysis of the Ruelle operators covers mainly the domain $\operatorname{Re} s \geq P_f$ and there are no results treating the spectral properties for $P_f - \eta_0 \leq \operatorname{Re} s < P_f$ and $z = \mathbf{i}w$, $w \in \mathbb{R}$. To our best knowledge the analytic continuation of the function $\zeta(s, z)$ for these values of s and z has not been investigated in the literature so far which makes it quite difficult to obtain sharper results.

In this paper under some hypothesis on the flow ϕ_t (see Sect. 3 for our standing assumptions) we prove spectral estimates for the iterations of the Ruelle operator $L^n_{f-s\tau+zg}$ with **two complex parameters** $s, z \in \mathbb{C}$. These estimates are in the spirit of those obtained in [4], [19], [20], [21] for the Ruelle operators with **one complex parameter** $s \in \mathbb{C}$. On the other hand, in this analysis some new difficulties appear when $|\operatorname{Im} s| \to \infty$ and $|\operatorname{Im} z| \to \infty$. First we prove in Theorem 5 spectral estimates in the case of arbitrary Hölder continuous functions f, g, when there exist constants $B > 0$ and $0 < \mu < 1$ such that $|\operatorname{Im} z| \leq B|\operatorname{Im} s|^\mu$ and $|\operatorname{Im} s| \geq b_0 > 0$. When f, g are Lipschitz one can take $\mu = 1$. This covers completely the case when $|z|$ is bounded and the estimates have the same form as those for operators with one complex parameter. Moreover, these estimates are sufficient for the applications in [11] and [16] when $|z|$ runs in a small neighborhood of 0 (see Sect. 6 and 7). In Sect. 5 we deal with the case when f, g are Lipschitz and there exists a constant $B_1 > 0$ such that $|\operatorname{Im} s| \leq B_1|\operatorname{Im} z|$ (see Theorem 6).

To study the analytic continuation of $\zeta(s, z)$ for $P_f - \eta_0 < \operatorname{Re} s < P_f$, we need a generalization of the so called Ruelle's lemma which yields a link between the convergence by packets of a Dirichlet series like (1.3) and the estimates of the iterations of the corresponding Ruelle operator. The reader may consult [23] for the precise result in this direction and the previous works ([18], [15], [9]), treating this question. For our needs in this paper we prove in Sect. 2 an analogue of Ruelle's lemma for Dirichlet series with two complex parameters following the approach in [23]. Combining Theorem 4 with the estimates in Theorem 5 (b), we obtain the following

Theorem 1. *Assume the standing assumptions in Sect.* 3 *fulfilled for a basic set* Λ. *Then for any Hölder continuous functions* $F, G : \Lambda \longrightarrow \mathbb{R}$ *there exists* $\eta_0 > 0$ *such that the function* $\zeta(s, z)$ *admits* *a non zero analytic continuation for*

$$(s,z) \in \{(s,z) \in \mathbb{C}^2 : \ P_f - \eta_0 \leq \operatorname{Re} s, \ s \neq s(z), \ |z| \leq \eta_0\}$$

with a simple pole at $s(z)$. The pole $s(z)$ is determined as the root of the equation $Pr(f - s\tau + zg) = 0$ with respect to s for $|z| \leq \eta_0$.

Applying the results of Sects. 4, 5, we study also the analytic continuation of $\zeta(s, \mathbf{i}w)$ for $P_f - \eta_0 < \operatorname{Re} s$ and $w \in \mathbb{R}$, $|w| \geq \eta_0$, in the case when $F, G : \Lambda \longrightarrow \mathbb{R}$ are Lipschitz functions (see Theorem 7). This analytic continuation combined with the arguments in [22] opens some new perspectives for the investigation of sharp large deviations for Anosov flows with exponentially shrinking intervals in the spirit of [12].

Our first application concerns the so called Hannay-Ozorio de Almeida sum formula (see [5], [10], [17]). Let $\phi_t : M \longrightarrow M$ be the geodesic flow on the unit-tangent bundle over a compact negatively curved surface M. In [17] it was proved that there exists $\epsilon > 0$ such that if $\delta(T) = \mathcal{O}(e^{-\epsilon T})$, for every Hölder continuous function $G : M \longrightarrow \mathbb{R}$, we have

$$\lim_{T \to +\infty} \frac{1}{\delta(T)} \sum_{T - \frac{\delta(T)}{2} \leq \lambda(\gamma) \leq T + \frac{\delta(T)}{2}} \lambda_G(\gamma) e^{-\lambda^u(\gamma)} = \int G d\mu, \tag{1.5}$$

where the notations $\lambda(\gamma), \lambda_G(\gamma)$ and $\lambda^u(\gamma)$ for a primitive periodic orbit γ are introduced above, while μ is the unique ϕ_t-invariant probability measure which is absolutely continuous with respect to the volume measure on M. The measure μ is called SRB (Sinai-Ruelle-Bowen) measure (see [3]). Notice that in the above case the Anosov flow ϕ_t is weak mixing and M is an attractor. Applying Theorem 1 and the arguments in [17], we prove the following

Theorem 2. *Let Λ be an attractor, that is there exists an open neighborhood V of Λ such that $\Lambda = \cap_{t \geq 0} \phi_t(V)$. Assume the standing assumptions of Sect.* 3 *fulfilled for the basic set Λ. Then there exists $\epsilon > 0$ such that if $\delta(T) = \mathcal{O}(e^{-\epsilon T})$, then for every Hölder function $G : \Lambda \longrightarrow \mathbb{R}$ the formula* (1.5) *holds with the SRB measure μ for ϕ_t.*

Our second application concerns the counting function

$$\pi_F(T) = \sum_{\lambda(\gamma) \leq T} e^{\lambda_F(\gamma)},$$

where γ is a primitive period orbit for $\phi_t : \Lambda \longrightarrow \Lambda$, $\lambda(\gamma)$ is the least period and $\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma)) dt$, $x_\gamma \in \gamma$. For $F = 0$ we obtain the counting function $\pi_0(T) = \#\{\gamma : \ \lambda(\gamma) \leq T\}$. These counting functions have been studied in many works (see [15] for references concerning $\pi_0(T)$ and [11], [14] for the function $\pi_F(T)$). The study of $\pi_F(T)$ is based on the analytic continuation of the function

$$\zeta_F(s) = \prod_\gamma \Big(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\Big)^{-1}, \ s \in \mathbb{C}$$

which is just the function $\zeta(s, 0)$ defined above. We prove the following

Theorem 3. *Let Λ be a basic set and let $F : \Lambda \longrightarrow \mathbb{R}$ be a Hölder function. Assume the standing assumptions of Sect.* 3 *fulfilled for Λ. Then there exists $\epsilon > 0$ such that*

$$\pi_F(T) = li(e^{Pr(F)T})(1 + \mathcal{O}(e^{-\epsilon T})), \ T \to \infty,$$

where $li(x) := \int_2^x \frac{1}{\log y} dy \sim \frac{x}{\log x}$, $x \to +\infty$.

In the case when $\phi_t : T^1(M) \longrightarrow T^1(M)$ is the geodesic flow on the unit tangent bundle $T^1(M)$ of a compact C^2 manifold M with negative section curvatures which are $\frac{1}{4}$-pinching the above result has been established in [14]. Following [20], [21], one deduces that the special case of a geodesic flow in [14] is covered by Theorem 3.

2. Ruelle lemma with two complex parameters

Let $B(\widehat{U})$ be the *space of bounded functions* $q : \widehat{U} \longrightarrow \mathbb{C}$ with its standard norm $\|q\|_0 = \sup_{x\in\widehat{U}} |g(x)|$. Given a function $q \in B(\widehat{U})$, the *Ruelle transfer operator* $L_q : B(\widehat{U}) \longrightarrow B(\widehat{U})$ is defined by $(L_q h)(u) = \sum_{\sigma(v)=u} e^{q(v)} h(v)$. If $q \in B(\widehat{U})$ is Lipschitz on $\widehat{U}$ with respect to the Riemann metric, then L_q preserves the space $C^{\mathrm{Lip}}(\widehat{U})$ of *Lipschitz functions* $q : \widehat{U} \longrightarrow \mathbb{C}$. Similarly, if q is ν-Hölder for some $\nu > 0$, the operator L_q preserves the *space* $C^\nu(\widehat{U})$ *of* ν*-Hölder functions* on $\widehat{U}$. In this section we assume that g, τ and f are real-valued $\nu-$Hölder continuous functions on $\hat{U}$. Then we can extend these functions as Hölder continuous on U.

We define the Ruelle operator $L_{g-sr+zf} : C^\nu(\hat{U}) \longrightarrow C^\nu(\hat{U})$ by

$$L_{f-s\tau+zg} v(x) = \sum_{\sigma y = x} e^{f(y)-s\tau(y)+zg(y)} v(y),\ s, z \in \mathbb{C}.$$

Next, for $\nu > 0$ define the ν-norm on a set $B \subset U$ by

$$|w|_\nu = \sup\Big\{\frac{|w(x) - w(y)|}{d(x,y)^\nu} : x, y \in B \cap U_i,\ i = 1, ..., k,\ x \neq y\Big\}.$$

Let

$$\|w\|_\nu = \|w\|_\infty + |w|_\nu,$$

and denote by $\|.\|_\nu$ be the corresponding norm for operators. Let $\chi_i(x)$ be the characteristic function of U_i.

Introduce the sum

$$Z_n(f - sr + zg) := \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x) + zg^n(x)}.$$

Our purpose is to prove the following statement which can be considered as Ruelle's lemma with two complex parameters.

Theorem 4. *For every Markov leaf* U_i *fix an arbitrary point* $x_i \in U_i$*. Then for every* $\epsilon > 0$ *and sufficiently small* $a_0 > 0, c_0 > 0$ *there exists a constant* $C_\epsilon > 0$ *such that*

$$\Big|Z_n(f - s\tau + zg) - \sum_{i=1}^k L^n_{f-s\tau+zg}\chi_i(x_i)\Big|$$

$$\leq C_\epsilon(1 + |s|)(1 + |z|) \sum_{m=2}^n \|L^{n-m}_{f-s\tau+zg}\|_\nu \gamma_0^{-m\nu} e^{m(\epsilon + Pr(f - a\tau + cg))},\ \forall n \in \mathbb{N} \tag{2.1}$$

for $s = a + \mathbf{i}b,\ z = c + \mathbf{i}w,\ |a| \leq a_0, |c| \leq c_0$.

The proof of this theorem follows that of Theorem 3.1 in [23] with some modifications. We have to take into account the presence of a second complex parameter z. Given a string $\alpha = (\alpha_0, ..., \alpha_{n-1})$

of symbols α_j taking the values in $\{1, ..., k\}$, we say that α is an admissible word if $A(\alpha_j, \alpha_{j+1}) = 1$ for all $0 \le j \le n-1$. Set $|\alpha| = n$ and define the cylinder of length n in the leaf U_{α_0} by

$$U_\alpha = U_{\alpha_0} \cap \sigma^{-1} U_{\alpha_1} \cap ... \cap \sigma^{-(n-1)} U_{\alpha_{n-1}}.$$

Each U_i is a cylinder of length 1. Next we introduce some other words (see Section in [23]). Given a word $\alpha = (\alpha_0, ..., \alpha_{n-1})$ and $i = 1, ..., k$, if $A(\alpha_{n-1}, i) = 1$ and $A(i, \alpha_0) = 1$, we define

$$\alpha i = (\alpha_0, ..., \alpha_{n-1}, i),\ i\alpha = (i, \alpha_0, ..., \alpha_{n-1}),\ \bar{\alpha} = (\alpha_0, ..., \alpha_{n-2}).$$

We have the following

Lemma 1. *Let w be a ν-Hölder real-valued function on U. Let x and y be on the same cylinder U_α with $|\alpha| = m$. Then there exists a constant $B > 0$ depending only on w, ν and the constants c_0 and γ_0 in (1.1) such that*

$$|w^m(x) - w^m(y)| \le B(d(\sigma^{m-1}x, \sigma^{m-1}y))^\nu.$$

The proof is a repetition of that of Lemma 2.5 in [23] and we leave the details to the reader.

Proposition 1. *Let $m \ge 1$ and let w be a function which is ν-Hölder continuous on all cylinder of length $m+1$. Then for the transfer operator $L_{f-s\tau+zg}$ we have*

$$L_{f-s\tau+zg} := \oplus_{|\alpha|=m+1} C^\nu(U_\alpha) \ni w \longrightarrow L_{f-s\tau+zg} w \in \oplus_{|\alpha|=m} C^\nu(U_\alpha).$$

Proof. Let w be ν-Hölder on $U_{i\alpha}$ for all i such that $A(i, \alpha_0) = 1$. Let $x, y \in \operatorname{Int} U_\alpha$ and let $|U| = \max_{i=1,,,k} \operatorname{diam}(U_i)$. Then

$$\begin{aligned}
&|L_{f-s\tau+zg}w(x) - L_{f-s\tau+zg}w(y)| \\
&= \Big| \sum_{A(i,\alpha_0)=1} e^{f(ix)-s\tau(ix)+zg(ix)} w(ix) - \sum_{A(i,\alpha_0)=1} e^{f(iy)-s\tau(iy)+zg(iy)} w(iy) \Big| \\
&\le \sum_{A(i,\alpha_0)=1} |e^{-s\tau(iy)}| \Big(|e^{s\tau(iy)-s\tau(ix)} - 1| |e^{f(iy)+zg(iy)} w(ix)| + |e^{f(iy)+zg(iy)} w(iy) - e^{f(ix)+zg(ix)} w(ix)| \Big) \\
&\le e^{a_0|\tau|_\infty} \sum_{A(i,\alpha_0)=1} \Big(|s||\tau|_\beta e^{a_0|\tau|_\nu |U|^\nu} e^{|f|_\infty + c_0|g|_\infty} |w|_\infty + |e^{f(iy)+zg(iy)} w(iy) - e^{f(ix)+zg(ix)} w(ix)| \Big).
\end{aligned}$$

Repeating this argument, we get

$$\begin{aligned}
&\sum_{A(i,\alpha_0)=1} |e^{f(iy)+zg(iy)} w(iy) - e^{f(ix)+zg(ix)} w(ix)| \\
&\le e^{c_0|g|_\infty} \sum_{A(i,\alpha_0)=1} \Big(|z||g|_\nu e^{c_0|g|_\nu |U|^\nu} e^{|f|_\infty} |w|_\infty + |e^{f(iy)} w(iy) - e^{f(ix)} w(ix)| \Big)
\end{aligned}$$

and we conclude that

$$|L_{f-s\tau+zg}w(x) - L_{f-s\tau+zg}w(y)| \le C|w|_\nu d(x,y)^\nu.$$

□

Now, as in [23], we will choose in every cylinder U_α a point $x_\alpha \in U_\alpha$. For the reader's convenience we recall the choice of x_α.
(1) If U_α has an n-periodic point, then we take $x_\alpha \in U_\alpha$ so that $\sigma^n x_\alpha = x_\alpha$.
(2) If U_α has no n-periodic point and $n > 1$ we choose $x_\alpha \in U_\alpha$ arbitrary so that $x_\alpha \notin \sigma(U_{\alpha_{n-1}})$.
(3) if $|\alpha| = n = 1$, then we take $x_\alpha = x_i$, where $i = \alpha_0$ and $x_i \in U_i$ is one of the points fixed in

Theorem 4.

Let χ_α be the characteristic function of U_α. Then Lemma 3.4 and Lemma 3.5 in [23] are applied without any change and we get

$$Z_n(f - s\tau + zg) = \sum_{|\alpha|=n} (L^n_{f-s\tau+zg}\chi_\alpha)(x_\alpha).$$

Proposition 2. *We have*

$$Z_n(f - s\tau + zg) - \sum_{i=1}^{k} L^n_{f-s\tau+zg}\chi_i(x_i)$$
$$= \sum_{m=2}^{n}\Big(\sum_{|\alpha|=m} L^n_{f-s\tau+zg}\chi_\alpha(x_\alpha) - \sum_{|\beta|=m-1} L^n_{f-s\tau+zg}\chi_\beta(x_\beta)\Big). \tag{2.2}$$

The proof is elementary by using the fact that

$$\sum_{i=1}^{k}(L^n_{f-s\tau+zg}\chi_{U_i})(x_i) = \sum_{|\alpha|=1} (L^n_{f-s\tau+zg}\chi_\alpha)(x_\alpha).$$

Now we repeat the argument in [23] and conclude that

$$\sum_{|\beta|=m-1} L^n_{f-s\tau+zg}\chi_\beta(x_\beta) = \sum_{|\alpha|=m} L^n_{f-s\tau+zg}\chi_\alpha(x_{\bar\alpha}).$$

Thus the proof of (2.1) is reduced to an estimate of the difference

$$L^n_{f-s\tau+zg}\chi_\alpha(x_\alpha) - L^n_{f-s\tau+zg}\chi_\alpha(x_{\bar\alpha}).$$

Observe that x_α and $x_{\bar\alpha}$ are on the same cylinder $U_{\bar\alpha}$. According to Proposition 1, the function $L^n_{f-s\tau+zg}\chi_\alpha$ is ν-Hölder continuous on $U_{\bar\alpha}$. Consequently, for every $n \geq 2$ we obtain

$$|L^n_{f-s\tau+zg}\chi_\alpha(x_\alpha) - L^n_{f-s\tau+zg}\chi_\alpha(x_{\bar\alpha})| \leq \|L^n_{f-s\tau+zg}\chi_\alpha\|_\nu d(x_\alpha, x_{\bar\alpha})^\nu,$$

where $\|.\|_\nu$ denotes the operator norm derived from the ν-Hölder norm. Going back to (2.2), we deduce

$$\Big|Z_n(f - s\tau + zg) - \sum_{i=1}^{k} L^n_{f-s\tau+zg}\chi_i(x_i)\Big|$$
$$\leq \sum_{m=2}^{n}\sum_{|\alpha|=m} \|L^{n-m}_{f-s\tau+zg}\|_\nu \|L^m_{f-s\tau+zg}\chi_\alpha\|_\nu d(x_\alpha, x_{\bar\alpha}). \tag{2.3}$$

This it makes possible to apply (1.1) and to conclude that

$$d(x_\alpha, x_{\bar\alpha}) \leq C^\nu \gamma_0^{-\nu(m-2)} d(\sigma^{m-2}x_\alpha, \sigma^{m-2}x_{\bar\alpha})^\nu \leq C_2\gamma_0^{-m\nu}.$$

To finish the proof we have to estimate the term $\|L^m_{g-sr+zf}\chi_\beta\|_\nu$. Given a word α of length $n > 1$ and $x \in \sigma(U_{\alpha_{n-1}}) \cap \operatorname{Int} U_i$, for any i with $A(\alpha_{n-1}, i) = 1$, we define $\sigma_\alpha^{-1}(x)$ to be the unique point y such that $\sigma^n(y) = x$ and $y \in U_\alpha$. For a symbol i we define $ix = \sigma_i^{-1}(x)$.

First we have

Lemma 2.
$$(L^m_{f-s\tau+zg}\chi_\beta)(x) = \begin{cases} e^{(f-s\tau+zg)^m(\sigma_\beta^{-1}x)}, \text{ if } x \in \sigma(U_{\beta_{m-1}}), \\ 0, \text{ otherwise.} \end{cases}$$

The proof is a repetition of that of Lemma 3.7 in [23] and it is based on the definition of σ_α^{-1} above and the fact that
$$(L^m_{f-s\tau+zg}\chi_\beta)(x) = \sum_{\sigma^m y = x} e^{f^m - s\tau^m + zg^m}(y)\chi_\beta(y).$$

For every admissible word β with $|\beta| = m$, we fix a point $y_\beta \in \sigma(U_{\beta_{m-1}})$ which will be chosen as in [23]. Define $z_\beta = \sigma_\beta^{-1}(y_\beta)$.

Lemma 3. *There exist constants $B_0 > 0, B_1 > 0, B_2 > 0$ such that we have the estimate*
$$\|L^m_{f-s\tau+zg}(\chi_\beta)\|_\nu \leq B_0\Big(e^{a_0|U|^\nu B_1} + B_1|s|e^{a_0|U|^\nu(1+\gamma_0^{-\nu})B_1}\Big)$$
$$\times\Big(e^{c_0|U|^\nu B_2} + B_2|z|e^{c_0|U|^\nu(1+\gamma_0^{-\nu})B_2}\Big)e^{(f^m - a\tau^m + cg^m)(z_\beta)}.$$

Proof. We will follow the proof of Lemma 3.8 in [23]. Let x and y be in the same Markov leaf. If $y \notin \sigma(U_{\beta_{m-1}})$, then $|L^m_{f-s\tau+zg}(\chi_\beta)(x)| = |L^m_{f-s\tau+zg}(\chi_\beta)(x) - L^m_{f-s\tau+zg}(\chi_\beta)(y)| = 0$. In the case when $x \notin \sigma(U_{\beta_{m-1}})$, we repeat the same argument. So we will consider the case when both x and y are in $\sigma(U_{\beta_{m-1}})$.

We have
$$|L^m_{f-s\tau+zg}(\chi_\beta)(x)| = |e^{(f^m-(a+ib)\tau^m+(c+id)g^m)(\sigma_\beta^{-1}x)}|$$
$$\leq \exp\Big((f^m - a\tau^m + cg^m)(\sigma_\beta^{-1}x) - (f^m - a\tau^m + cg^m)(\sigma_\beta^{-1}y)\Big)e^{(f^m-a\tau^m+cg^m)(z_\beta)}.$$
On the other hand, applying Lemma 1 with $w = \tau$, we get
$$|\tau^m(\sigma_\beta^{-1}x) - \tau^m(\sigma_\beta^{-1}y)| \leq B_1(d(\sigma^{m-1}\sigma_\beta^{-1}x, \sigma^{m-1}\sigma_\beta^{-1}y))^\nu \leq B_1|U|^\nu.$$
The same argument works for the terms involving f^m and g^m, applying Lemma 1 with $w = f, g$, respectively. Thus we obtain
$$|L^m_{f-s\tau+zg}(\chi_\beta)(x)| \leq e^{(C_0+a_0B_1+c_0B_2)|U|^\nu}e^{(f^m-a\tau^m+cg^m)(z_\beta)}.$$
and this implies an estimate for $|L^m_{f-s\tau+zg}(\chi_\beta)|_\infty$. Next,
$$|L^m_{f-s\tau+zg}(\chi_\beta)(x) - L^m_{f-s\tau+zg}(\chi_\beta)(y)|$$
$$\leq |e^{f^m(\sigma_\beta^{-1}(x))-f^m(\sigma_\beta^{-1}(y))} - 1||e^{f^m(\sigma_\beta^{-1}(y))}||e^{-s\tau^m(\sigma_\beta^{-1}(x))+s\tau^m(\sigma_\beta^{-1}(y))} - 1||e^{-s\tau^m(\sigma_\beta^{-1}(y))}|$$
$$\times|e^{zg^m(\sigma_\beta^{-1}(x))-zg^m(\sigma_\beta^{-1}(y))} - 1||e^{zg^m(\sigma_\beta^{-1}(y))}|.$$
As in [23], we have
$$|e^{-sr^m(\sigma_\beta^{-1}(x))+sr^m(\sigma_\beta^{-1}(y))} - 1||e^{-sr^m(\sigma_\beta^{-1}(y))}| \leq B_1\gamma_0^\nu|s|e^{a_0B_1(1+\gamma_0^{-\nu})|U|^\nu}e^{-ar^m(z_\beta)}d(x,y)^\nu.$$
For the product involving zg^m we have the same estimate with $B_2, |z|, c_0$ and c in the place of $B_1, |s|, a_0$ and a. A similar estimate holds for the term containing f^m with a constant B_3 in the place of B_1. Taking the product of these estimates, we obtain a bound for $|L^m_{f-s\tau+zg}(\chi_\beta)(x) - L^m_{f-s\tau+zg}(\chi_\beta)(y)|$, this implies the desired estimate for the ν-Hölder norm of $L_{f-ms\tau+zg}(\chi_\beta)$. This completes the proof. ∎

Now the proof of (2.1) is reduced to the estimate of
$$\sum_{|\beta|=m} e^{(f^m - a\tau^m + cg^m)(z_\beta)}.$$
Introduce the real-valued function $h = f - a\tau + cg$. Then we must estimate
$$\sum_{|\beta|=m} e^{h^m(z_\beta)}.$$
For this purpose we repeat the argument on pages 232-234 in [23] and deduce with some constant $d_0 > 0$ depending only on the matrix A and every $\epsilon > 0$ the bound
$$\sum_{|\beta|=m} e^{h^m(z_\beta)} \leq e^{d_0|h|_\infty} B_\epsilon e^{(m+d_0)(\epsilon + \mathrm{Pr}(h))}.$$
Combing this with the previous estimates, we get (2.1) and the proof of Theorem 4 is complete. $\square$

3. Ruelle operators – definitions and assumptions

For a contact Anosov flows ϕ_t with Lipschitz local stable holonomy maps it is proved in Sect. 6 in [20] that the following *local non-integrability condition* holds:

(LNIC): *There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and $\theta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ and any tangent vector $\eta \in E^u(\hat{z})$ to Λ at $\hat{z}$ with $\|\eta\| = 1$ there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z})$ with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta = \delta(\tilde{z}, \tilde{y}_1, \tilde{y}_2) > 0$ and $\epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon]$ such that*
$$|\Delta(\exp^u_z(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp^u_z(v), \pi_{\tilde{y}_2}(z))| \geq \delta \, \|v\|$$
for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$ and $v \in E^u(z; \epsilon')$ with $\exp^u_z(v) \in \Lambda$ and $\langle \frac{v}{\|v\|}, \eta_z \rangle \geq \theta_0$, where η_z is the parallel translate of η along the geodesic in $W^u_{\epsilon_0}(z_0)$ from $\hat{z}$ to z.

For any $x \in \Lambda$, $T > 0$ and $\delta \in (0, \epsilon]$ set
$$B^u_T(x, \delta) = \{y \in W^u_\epsilon(x) : d(\phi_t(x), \phi_t(y)) \leq \delta \ , \ 0 \leq t \leq T\}.$$

We will say that ϕ_t has a *regular distortion along unstable manifolds* over the basic set Λ if there exists a constant $\epsilon_0 > 0$ with the following properties:

(a) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that
$$\mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq R \, \mathrm{diam}(\Lambda \cap B^u_T(z, \delta))$$
for any $z \in \Lambda$ and any $T > 0$.

(b) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0, 1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\mathrm{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho \, \mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

A large class of flows on basic sets having regular distortion along unstable manifolds is described in [21].

In this paper we work under the following **Standing Assumptions:**

(A) ϕ_t has Lipschitz local holonomy maps over Λ,

(B) the local non-integrability condition (LNIC) holds for ϕ_t on Λ,

(C) ϕ_t has a regular distortion along unstable manifolds over the basic set Λ.

A rather large class of examples satisfying the above conditions is provided by imposing the following *pinching condition*:

(P): *There exist constants* $C > 0$ *and* $\beta \geq \alpha > 0$ *such that for every* $x \in M$ *we have*

$$\frac{1}{C}\, e^{\alpha_x\, t}\, \|u\| \leq \|d\phi_t(x)\cdot u\| \leq C\, e^{\beta_x\, t}\, \|u\| \quad , \quad u \in E^u(x)\ \ , t > 0$$

for some constants $\alpha_x, \beta_x > 0$ *with* $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ *and* $2\alpha_x - \beta_x \geq \alpha$ *for all* $x \in M$.

We should note that (P) holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so called $\frac{1}{4}$-pinching condition. (P) always holds when $\dim(M) = 3$.

Simplifying Assumptions: ϕ_t is a C^2 contact Anosov flow satisfying the condition (P).

As shown in [21] the pinching condition (P) implies that ϕ_t has Lipschitz local holonomy maps and regular distortion along unstable manifolds. Combining this with Proposition 6.1 in [20], shows that the Simplifying Assumptions imply the Standing Assumptions.

As in Sect. 1 consider a **fixed Markov family** $\mathcal{R} = \{R_i\}_{i=1}^k$ for the flow ϕ_t on Λ consisting of rectangles $R_i = [U_i, S_i]$ and let $U = \cup_{i=1}^k U_i$. The Standing Assumptions imply the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma_0 > 1$ such that (1.1) hold.

In what follows we will assume that f **and** g **are fixed real-valued functions in** $C^\alpha(\widehat{U})$ **for some fixed** $\alpha > 0$. Let $P = P_f$ be the unique real number so that $\Pr(f - P\,\tau) = 0$, where $\Pr(h)$ is the *topological pressure* of h with respect to the shift map σ defined in Section 2. Given $t \in \mathbb{R}$ with $t \geq 1$, following [4], denote by f_t the *average of* f *over balls in* U *of radius* $1/t$. To be more precise, first one has to fix an arbitrary extension $f \in C^\alpha(V)$ (with the same Hölder constant), where V is an open neighborhood of U in M, and then take the averages in question. Then $f_t \in C^\infty(V)$, so its restriction to U is Lipschitz (with respect to the Riemann metric) and:

(a) $\|f - f_t\|_\infty \leq |f|_\alpha / t^\alpha$;

(b) $\mathrm{Lip}(f_t) \leq \mathrm{Const}\ \|f\|_\infty t$;

(c) For any $\beta \in (0, \alpha)$ we have $|f - f_t|_\beta \leq 2\,|f|_\alpha / t^{\alpha-\beta}$.

In the special case $f \in C^{\mathrm{Lip}}(U)$ we set $f_t = f$ for all $t \geq 1$. Similarly for g. Let $\lambda_0 > 0$ be the largest eigenvalue of $L_{f-P\tau}$, and let $\hat{\nu}_0$ be the (unique) probability measure on U with $L^*_{f-P\tau}\hat{\nu}_0 = \hat{\nu}_0$. Fix a corresponding (positive) eigenfunction $h_0 \in \hat{C}^\alpha(U)$ such that $\int_U h_0\, d\hat{\nu}_0 = 1$. Then $d\nu_0 = h_0\, d\hat{\nu}_0$ defines a σ*-invariant probability measure* ν_0 on U. Setting

$$f_0 = f - P\,\tau + \ln h_0(u) - \ln h_0(\sigma(u)),$$

we have $L^*_{f^{(0)}}\nu_0 = \nu_0$, i.e. $\displaystyle\int_U L_{f^{(0)}} H\, d\nu_0 = \int_U H\, d\nu_0$ for any $H \in C(U)$, and $L_{f_0} 1 = 1$.

Given real numbers a and t (with $|a| + \frac{1}{|t|}$ small), denote by λ_{at} the *largest eigenvalue* of $L_{f_t-(P+a)\tau}$ on $C^{\mathrm{Lip}}(U)$ and by h_{at} the corresponding (positive) eigenfunction such that $\int_U h_{at}\, d\nu_{at} = 1$, where ν_{at} is the unique probability measure on U with $L^*_{f_t-(P+a)\tau}\nu_{at} = \nu_{at}$.

As is well-known the shift map $\sigma : \widehat{U} \longrightarrow \widehat{U}$ is naturally isomorphic to an one-sided subshift of finite type. Given $\theta \in (0, 1)$, a natural metric associated by this isomorphism is defined (for $x \neq y$) by $d_\theta(x, y) = \theta^m$, where m is the largest integer such that x, y belong to the same cylinder of length m. There exist $\theta = \theta(\alpha) \in (0, 1)$ and $\beta \in (0, \alpha)$ such that $(d(x, y))^\alpha \leq \mathrm{Const}\ d_\theta(x, y)$ and $d_\theta(x, y) \leq \mathrm{Const}\ (d(x, y))^\beta$ for all $x, y \in \widehat{U}$. One can then apply the Ruelle-Perron-Frobenius theorem to the sub-shift of fine type and deduce that $h_{at} \in C^\beta(\widehat{U})$. However this is not enough for our purposes – in Lemma 4 below we get a bit more.

Consider an arbitrary $\beta \in (0, \alpha)$. It follows from properties (a) and (c) above that there exists a constant $C_0 > 0$, depending on f and α but independent of β, such that

$$\|[f_t - (P+a)\tau] - (f - P\tau)\|_\beta \leq C_0 \,[|a| + 1/t^{\alpha-\beta}] \tag{3.1}$$

for all $|a| \leq 1$ and $t \geq 1$. Since $\Pr(f - P\tau) = 0$, it follows from the analyticity of pressure and the eigenfunction projection corresponding to the *maximal eigenvalue* $\lambda_{at} = e^{\Pr(f_t-(P+a)\tau)}$ of the Ruelle operator $L_{f_t-(P+a)\tau}$ on $C^\beta(U)$ (cf. e.g. Ch. 3 in [11]) that there exists a constant $a_0 > 0$ such that, taking $C_0 > 0$ sufficiently large, we have

$$|\Pr(f_t - (P+a)\tau)| \leq C_0 \,\left(|a| + \frac{1}{t^{\alpha-\beta}}\right) \quad , \quad \|h_{at} - h_0\|_\beta \leq C_0 \,\left(|a| + \frac{1}{t^{\alpha-\beta}}\right) \tag{3.2}$$

for $|a| \leq a_0$ and $1/t \leq a_0$. We may assume $C_0 > 0$ and $a_0 > 0$ are taken so that $1/C_0 \leq \lambda_{at} \leq C_0$, $\|f_t\|_\infty \leq C_0$ and $1/C_0 \leq h_{at}(u) \leq C_0$ for all $u \in U$ and all $|a|, 1/t \leq a_0$.

Given real numbers a and t with $|a|, 1/t \leq a_0$ consider the functions

$$f_{at} = f_t - (P+a)\tau + \ln h_{at} - \ln(h_{at} \circ \sigma) - \ln \lambda_{at}$$

and the operators

$$\mathcal{L}_{abt} = L_{f_{at} - \mathbf{i}\, b\, \tau} : C(U) \longrightarrow C(U) \;\; , \;\; \mathcal{M}_{at} = L_{f_{at}} : C(U) \longrightarrow C(U).$$

One checks that $\mathcal{M}_{at}\, 1 = 1$.

Taking the constant $C_0 > 0$ sufficiently large, we may assume that

$$\|f_{at} - f_0\|_\beta \leq C_0 \,\left[|a| + \frac{1}{t^{\alpha-\beta}}\right] \quad , \quad |a|, 1/t \leq a_0. \tag{3.3}$$

We will now prove a simple uniform estimate for $\mathrm{Lip}(h_{at})$. With respect to the usual metrics on symbol spaces this a consequence of general facts (see e.g. Sect. 1.7 in [1] or Ch. 3 in [11]), however here we need it with respect to the Riemann metric.

The proof of the following lemma is given in the Appendix.

Lemma 4. *Taking the constant $a_0 > 0$ sufficiently small, there exists a constant $T' > 0$ such that for all $a, t \in \mathbb{R}$ with $|a| \leq a_0$ and $t \geq 1/a_0$ we have $h_{at} \in C^{\mathrm{Lip}}(\widehat{U})$ and $\mathrm{Lip}(h_{at}) \leq T' t$.*

It follows from the above that, assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $T > 0$ (depending on $|f|_\alpha$ and a_0) such that

$$\|f_{at}\|_\infty \leq T \quad , \quad \|g_t\|_\infty \leq T \quad , \quad \mathrm{Lip}(h_{at}) \leq T\, t \quad , \quad \mathrm{Lip}(f_{at}) \leq T\, t \tag{3.4}$$

for $|a|, 1/t \leq a_0$. We will also assume that $T \geq \max\{\, \|\tau\|_0 \,, \mathrm{Lip}(\tau_{|\widehat{U}})\,\}$. From now on **we will assume that a_0, C_0, T, $1 < \gamma_0 < \gamma_1$ are fixed constants** with (1.1) and (3.1) – (3.4).

4. Ruelle operators depending on two parameters – the case when b is the leading parameter

Throughout this section we work under the Standing Assumptions made in Sect. 3 and with fixed real-valued functions $f, g \in C^\alpha(\widehat{U})$ as in Sect. 3. Throughout $0 < \beta < \alpha$ are fixed numbers.

We will study Ruelle operators of the form $L_{f-(P_f+a+\mathbf{i}b)\tau+zg}$, where $z = c + \mathbf{i}w$, $a, b, c, w \in \mathbb{R}$, and $|a|, |c| \leq a_0$ for some constant $a_0 > 0$. Such operators will be approximates by operators of the form

$$\mathcal{L}_{abtz} = L_{f_{at}-\mathbf{i}\,b\tau+zg_t} : C^\alpha(\widehat{U}) \longrightarrow C^\alpha(\widehat{U}).$$

In fact, since $f_{at} - \mathbf{i}b\tau + zg_t$ is Lipschitz, the operators $\mathcal{L}_{abtz}$ preserves each of the spaces $C^{\alpha'}(\widehat{U})$ for $0 < \alpha' \leq 1$ including the space $C^{\mathrm{Lip}}(\widehat{U})$ of Lipschitz functions $h : \widehat{U} \longrightarrow \mathbb{C}$. For such h we will denote by $\mathrm{Lip}(h)$ the Lipschitz constant of h. Let $\|h\|_0$ denote the *standard* sup *norm* of h on $\widehat{U}$. For $|b| \geq 1$, as in [4], consider the norm $\|.\|_{\mathrm{Lip},b}$ on $C^{\mathrm{Lip}}(\widehat{U})$ defined by $\|h\|_{\mathrm{Lip},b} = \|h\|_0 + \frac{\mathrm{Lip}(h)}{|b|}$. and also the norm $\|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_\beta}{|b|}$ on $C^\beta(U)$.

Our aim in this section is to prove the following

Theorem 5. *Let $\phi_t : M \longrightarrow M$ satisfy the Standing Assumptions over the basic set Λ, and let $0 < \beta < \alpha$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t over Λ as in Sect. 1. Then for any real-valued functions $f, g \in C^\alpha(\widehat{U})$ we have:*

(a) *For any constants $\epsilon > 0$, $B > 0$ and $\nu \in (0,1)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$, $A_0 > 0$ and $C = C(B,\epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f_{at}-\mathbf{i}b\tau+(c+\mathbf{i}w)g_t} h\|_{\mathrm{Lip},b} \leq C\,\rho^m\,|b|^\epsilon\,\|h\|_{\mathrm{Lip},b}$$

for all $h \in C^{\mathrm{Lip}}(\widehat{U})$, all integers $m \geq 1$ and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$, $1 \leq t \leq \frac{1}{A_0}\log|b|^\nu$ and $|w| \leq B\,|b|^\nu$.

(b) *For any constants $\epsilon > 0$, $B > 0$, $\nu \in (0,1)$ and $\beta \in (0,\alpha)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B,\epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f-(P_f+a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\beta,b} \leq C\,\rho^m\,|b|^\epsilon\,\|h\|_{\beta,b}$$

for all $h \in C^\beta(\widehat{U})$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B\,|b|^\nu$.

(c) *If $f, g \in C^{\mathrm{Lip}}(\widehat{U})$, then for any constants $\epsilon > 0$, $B > 0$ and $\beta \in (0,\alpha)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B,\epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f-(P_f+a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\mathrm{Lip},b} \leq C\,\rho^m\,|b|^\epsilon\,\|h\|_{\mathrm{Lip},b}$$

for all $h \in C^\beta(\widehat{U})$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B\,|b|$.

We will first prove part (a) of the above theorem and then derive part (b) by a simple approximation procedure. To prove part (a) we will use the main steps in Section 5 in [20] with necessary modifications. The proof of part (c) is just a much simpler version of the proof of (b).

Define a *new metric D* on $\widehat{U}$ by

$$D(x,y) = \min\{\mathrm{diam}(\mathcal{C}) : x, y \in \mathcal{C}\,,\ \mathcal{C} \text{ a cylinder contained in } U_i\}$$

if $x, y \in U_i$ for some $i = 1, \ldots, k$, and $D(x,y) = 1$ otherwise. Rescaling the metric on M if necessary, we will assume that $\mathrm{diam}(U_i) < 1$ for all i. As shown in [19], D is a metric on $\widehat{U}$ with $d(x,y) \leq D(x,y)$ for $x, y \in \widehat{U}_i$ for some i, and for any cylinder $\mathcal{C}$ in U the characteristic function $\chi_{\widehat{\mathcal{C}}}$ of $\widehat{\mathcal{C}}$ on $\widehat{U}$ is Lipschitz with respect to D and $\mathrm{Lip}_D(\chi_{\widehat{\mathcal{C}}}) \leq 1/\mathrm{diam}(\mathcal{C})$.

We will denote by $C^{\mathrm{Lip}}_D(\widehat{U})$ the *space of all Lipschitz functions $h : \widehat{U} \longrightarrow \mathbb{C}$ with respect to the metric D* on $\widehat{U}$ and by $\mathrm{Lip}_D(h)$ the *Lipschitz constant* of h with respect to D.

Given $A > 0$, denote by $K_A(\widehat{U})$ *the set of all functions* $h \in C_D^{\rm Lip}(\widehat{U})$ such that $h > 0$ and $\frac{|h(u)-h(u')|}{h(u')} \leq A\, D(u,u')$ for all $u, u' \in \widehat{U}$ that belong to the same $\widehat{U}_i$ for some $i = 1, \ldots, k$. Notice that $h \in K_A(\widehat{U})$ implies $|\ln h(u) - \ln h(v)| \leq A\ D(u,v)$ and therefore $e^{-A\ D(u,v)} \leq \frac{h(u)}{h(v)} \leq e^{A\ D(u,v)}$ for all $u, v \in \widehat{U}_i$, $i = 1, \ldots, k$.

We begin with a lemma of Lasota-Yorke type, which necessarily has a more complicated form due to the more complex situation considered. It involves the operators $\mathcal{L}_{abtz}$, and also operators of the form

$$\mathcal{M}_{atc} = L_{f_{at}+cg_t} : C^\alpha(\widehat{U}) \longrightarrow C^\alpha(\widehat{U}).$$

Fix arbitrary constants $\nu \in (0,1)$ **and** $\hat{\gamma}$ **with** $1 < \hat{\gamma} < \gamma_0$.

Lemma 5. *Assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $A_0 > 0$ such that for all $a, c, t \in \mathbb{R}$ with $|a|, |c| \leq a_0$ and $t \geq 1$ the following hold:*

(a) If $H \in K_E(\widehat{U})$ for some $E > 0$, then

$$\frac{|(\mathcal{M}_{atc}^m H)(u) - (\mathcal{M}_{atc}^m H)(u')|}{(\mathcal{M}_{atc}^m H)(u')} \leq A_0 \left[\frac{E}{\hat{\gamma}^m} + e^{A_0 t}\, t\right] D(u,u')$$

for all $m \geq 1$ and all $u, u' \in U_i$, $i = 1, \ldots, k$.

(b) If the functions h and H on $\widehat{U}$ and $E > 0$ are such that $H > 0$ on $\widehat{U}$ and $|h(v) - h(v')| \leq E\, H(v')\, D(v,v')$ for any $v, v' \in \widehat{U}_i$, $i = 1, \ldots, k$, then for any integer $m \geq 1$ and any $b, w, t \in \mathbb{R}$ with $|b|, t, |w| \geq 1$, for $z = c + \mathbf{i}w$ we have

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_0 \left(\frac{E}{\hat{\gamma}^m}(\mathcal{M}_{atc}^m H)(u') + (|b| + e^{A_0 t} t + t|w|)(\mathcal{M}_{atc}^m |h|)(u')\right) D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$. In particular, if

$$t \leq \frac{\log |b|^\nu}{A_0} \quad , \quad t \leq B|b|^{1-\nu} \quad , \qquad |w| \leq B|b|^\nu \tag{4.1}$$

for some constant $B > 0$, then

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_1 \left(\frac{E}{\hat{\gamma}^m}(\mathcal{M}_{atc}^m H)(u') + |b|(\mathcal{M}_{atc}^m |h|)(u')\right) D(u,u').$$

for some constant $A_1 > 0$.

A proof of this lemma is given in the Appendix.

From now on we will assume that a_0**,** η_0 **and** A_0 **are fixed with the properties in Lemma 5 above and** $a, b, c, w, t \in \mathbb{R}$ **are such that** $|a| \leq a_0$**,** $c \leq \eta_0$**,** $|b|, t, |w| \geq 1$ **and (4.1) hold. As before, set** $z = c + \mathbf{i}d$**.**

We will use the entire set-up and notation from Section 5 in [20]. In what follows we recall the main part of it.

Following Sect. 4 in [20], **fix an arbitrary point** $z_0 \in \Lambda$ **and constants** $\epsilon_0 > 0$ **and** $\theta_0 \in (0,1)$ **with the properties described in** (LNIC). Assume that $z_0 \in \mathrm{Int}_\Lambda(U_1)$, $U_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$ and $S_1 \subset \Lambda \cap W^s_{\epsilon_0}(z_0)$. Fix an arbitrary constant θ_1 such that

$$0 < \theta_0 < \theta_1 < 1 .$$

Next, fix an arbitrary orthonormal basis $e_1, \ldots, e_n$ in $E^u(z_0)$ and a C^1 parametrization $r(s) = \exp^u_{z_0}(s)$, $s \in V_0'$, of a small neighborhood W_0 of z_0 in $W^u_{\epsilon_0}(z_0)$ such that V_0' is a convex compact neighborhood of 0 in $\mathbb{R}^n \approx \mathrm{span}(e_1, \ldots, e_n) = E^u(z_0)$. Then $r(0) = z_0$ and $\frac{\partial}{\partial s_i} r(s)_{|s=0} = e_i$ for all $i = 1, \ldots, n$. Set $U_0' = W_0 \cap \Lambda$. Shrinking W_0 (and therefore V_0' as well) if necessary, we may assume that $\overline{U_0'} \subset \mathrm{Int}_\Lambda(U_1)$ and $\left| \left\langle \frac{\partial r}{\partial s_i}(s), \frac{\partial r}{\partial s_j}(s) \right\rangle - \delta_{ij} \right|$ is uniformly small for all $i, j = 1, \ldots, n$ and $s \in V_0'$, so that

$$\frac{1}{2}\langle \xi, \eta \rangle \leq \langle\, dr(s) \cdot \xi \ , \ dr(s) \cdot \eta \,\rangle \leq 2 \,\langle \xi, \eta \rangle \quad , \quad \xi, \eta \in E^u(z_0) \ , \ s \in V_0' \ ,$$

and $\frac{1}{2} \|s - s'\| \leq d(r(s), r(s')) \leq 2 \|s - s'\|$, $s, s' \in V_0'$.

Definitions ([20]): (a) For a cylinder $\mathcal{C} \subset U_0'$ and a unit vector $\xi \in E^u(z_0)$ we will say that a *separation by a ξ-plane occurs* in $\mathcal{C}$ if there exist $u, v \in \mathcal{C}$ with $d(u, v) \geq \frac{1}{2} \mathrm{diam}(\mathcal{C})$ such that $\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|} \ , \ \xi \right\rangle \geq \theta_1$.

Let $\mathcal{S}_\xi$ be the *family of all cylinders* $\mathcal{C}$ contained in U_0' such that a separation by an ξ-plane occurs in $\mathcal{C}$.

(b) Given an open subset V of U_0' which is a finite union of open cylinders and $\delta > 0$, let $\mathcal{C}_1, \ldots, \mathcal{C}_p$ ($p = p(\delta) \geq 1$) be the family of maximal closed cylinders in $\overline{V}$ with $\mathrm{diam}(\mathcal{C}_j) \leq \delta$. For any unit vector $\xi \in E^u(z_0)$ set $M_\xi^{(\delta)}(V) = \cup\{\mathcal{C}_j : \mathcal{C}_j \in \mathcal{S}_\xi \ , \ 1 \leq j \leq p\}$.

In what follows we will construct, amongst other things, a sequence of unit vectors $\xi_1, \xi_2, \ldots, \xi_{j_0} \in E^u(z_0)$. For each $\ell = 1, \ldots, j_0$ set $B_\ell = \{\eta \in \mathbf{S}^{n-1} : \langle \eta, \xi_\ell \rangle \geq \theta_0\}$. For $t \in \mathbb{R}$ and $s \in E^u(z_0)$ set $I_{\eta,t} g(s) = \frac{g(s + t\,\eta) - g(s)}{t}$, $t \neq 0$ (*increment* of g in the direction of η).

Lemma 6. ([20]) *There exist integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, a sequence of unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of U_0' which is a finite union of open cylinders of length n_1 such that setting $\mathcal{U} = \sigma^{n_1}(U_0)$ we have:*

(a) *For any integer $N \geq N_0$ there exist Lipschitz maps $v_1^{(\ell)}, v_2^{(\ell)} : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$) such that $\sigma^N(v_i^{(\ell)}(x)) = x$ for all $x \in \mathcal{U}$ and $v_i^{(\ell)}(\mathcal{U})$ is a finite union of open cylinders of length N ($i = 1, 2$; $\ell = 1, 2, \ldots, \ell_0$).*

(b) *There exists a constant $\hat{\delta} > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ with $s + h\,\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h} \left(\tau^N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \tau^N(v_1^{(\ell)}(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{\hat{\delta}}{2} \, .$$

(c) *We have $\overline{v_i^{(\ell)}(U)} \bigcap \overline{v_{i'}^{(\ell')}(U)} = \emptyset$ whenever $(i, \ell) \neq (i', \ell')$.*

(d) *For any open cylinder V in U_0 there exists a constant $\delta' = \delta'(V) > 0$ such that*

$$V \subset M_{\eta_1}^{(\delta)}(V) \cup M_{\eta_2}^{(\delta)}(V) \cup \ldots \cup M_{\eta_{\ell_0}}^{(\delta)}(V)$$

for all $\delta \in (0, \delta']$.

Fix U_0 and $\mathcal{U}$ with the properties described in Lemma 1; then $\overline{\mathcal{U}} = U$.

Set $\hat{\delta} = \min_{1 \leq \ell \leq \ell_0} \hat{\delta}_j$, $n_0 = \max_{1 \leq \ell \leq \ell_0} m_\ell$, and fix an arbitrary point $\hat{z}_0 \in U_0^{(\ell_0)} \cap \widehat{U}$.

Fix integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of W_0 with the properties described in Lemma 6. By the choice of U_0, $\sigma^{n_1} : U_0 \longrightarrow \mathcal{U}$ is one-to-one and has an inverse map $\psi : \mathcal{U} \longrightarrow U_0$, which is Lipschitz.

Set $E = \max\left\{4A_0 \, , \, \frac{2A_0 T}{\gamma - 1}\right\}$, where $A_0 \geq 1$ is the constant from Lemma 5.4, and **fix an integer** $N \geq N_0$ such that

$$\gamma^N \geq \max\left\{ 6A_0 \, , \, \frac{200\,\gamma_1^{n_1}\,A_0}{c_0^2} \, , \, \frac{512\,\gamma^{n_1}\,E}{c_0\,\hat{\delta}\,\rho} \right\} .$$

Then fix maps $v_i^{(\ell)} : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$, $i = 1, 2$) with the properties (a), (b), (c) and (d) in Lemma 6. In particular, (c) gives

$$\overline{v_i^{(\ell)}(U)} \cap \overline{v_{i'}^{(\ell')}(U)} = \emptyset \quad , \quad (i, \ell) \neq (i', \ell').$$

Since U_0 is a finite union of open cylinders, it follows from Lemma 6(d) that there exist a constant $\delta' = \delta'(U_0) > 0$ such that

$$M_{\eta_1}^{(\delta)}(U_0) \cup \ldots \cup M_{\eta_{\ell_0}}^{(\delta)}(U_0) \supset U_0 \quad , \quad \delta \in (0, \delta'].$$

Fix δ' with this property. Set

$$\epsilon_1 = \min\left\{ \frac{1}{32C_0} \, , \, c_1 \, , \, \frac{1}{4E} \, , \, \frac{1}{\hat{\delta}\,\rho^{p_0+2}} \, , \, \frac{c_0 r_0}{\gamma_1^{n_1}} \, , \, \frac{c_0^2(\gamma - 1)}{16T\gamma_1^{n_1}} \right\},$$

and let $b \in \mathbb{R}$ be such that $|b| \geq 1$ and

$$\frac{\epsilon_1}{|b|} \leq \delta'.$$

Let $\mathcal{C}_m$ ($1 \leq m \leq p$) be the family of *maximal closed cylinders* contained in $\overline{U_0}$ with $\mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|b|}$ such that $U_0 \subset \cup_{j=m}^{p} \mathcal{C}_m$ and $\overline{U_0} = \cup_{m=1}^{p} \mathcal{C}_m$. As in [20],

$$\rho\,\frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq m \leq p \, . \tag{4.2}$$

Fix an integer $q_0 \geq 1$ such that

$$\theta_0 < \theta_1 - 32\,\rho^{q_0 - 1}.$$

Next, let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all closed cylinders contained in $\overline{U_0}$ that are *subcylinders of co-length* $p_0\,q_0$ of some $\mathcal{C}_m$ ($1 \leq m \leq p$). Then $\overline{U_0} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q$. Moreover,

$$\rho^{p_0\,q_0+1} \cdot \frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{D}_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq j \leq q.$$

Given $j = 1, \ldots, q$, $\ell = 1, \ldots, \ell_0$ and $i = 1, 2$, set $\widehat{\mathcal{D}}_j = \mathcal{D}_j \cap \widehat{U}$, $Z_j = \overline{\sigma^{n_1}(\widehat{\mathcal{D}}_j)}$, $\widehat{Z}_j = Z_j \cap \widehat{U}$, $X_{i,j}^{(\ell)} = \overline{v_i^{(\ell)}(\widehat{Z}_j)}$, and $\widehat{X}_{i,j}^{(\ell)} = X_{i,j}^{(\ell)} \cap \widehat{U}$. It then follows that $\mathcal{D}_j = \psi(Z_j)$, and $U = \cup_{j=1}^{q} Z_j$. Moreover, $\sigma^{N-n_1}(v_i^{(\ell)}(x)) = \psi(x)$ for all $x \in \mathcal{U}$, and all $X_{i,j}^{(\ell)}$ are cylinders such that $X_{i,j}^{(\ell)} \cap X_{i',j'}^{(\ell')} = \emptyset$ whenever $(i, j, \ell) \neq (i', j', \ell')$, and

$$\mathrm{diam}(X_{i,j}^{(\ell)}) \geq \frac{c_0\,\rho^{p_0\,q_0+1}}{\gamma_1^N} \cdot \frac{\epsilon_1}{|b|}$$

for all $i = 1, 2$, $j = 1, \ldots, q$ and $\ell = 1, \ldots, \ell_0$. The *characteristic function* $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \longrightarrow [0, 1]$ of $\widehat{X}_{i,j}^{(\ell)}$ belongs to $C_D^{\mathrm{Lip}}(\widehat{U})$ and $\mathrm{Lip}_D(X_{i,j}^{(\ell)}) \leq 1/\mathrm{diam}(X_{i,j}^{(\ell)})$.

Let J be a *subset of the set* $\Xi = \{ \, (i, j, \ell) \; : \; 1 \leq i \leq 2 \, , \, 1 \leq j \leq q \, , \, 1 \leq \ell \leq \ell_0 \, \}$. Set

$$\mu_0 = \mu_0(N) = \min\left\{ \frac{1}{4} \, , \, \frac{c_0\,\rho^{p_0 q_0+2}\,\epsilon_1}{4\,\gamma_1^N} \, , \, \frac{1}{4\,e^{2TN}}\,\sin^2\left(\frac{\hat{\delta}\,\rho\,\epsilon_1}{256}\right) \right\},$$

and define the function $\omega = \omega_J : \widehat{U} \longrightarrow [0,1]$ by $\omega = 1 - \mu_0 \sum_{(i,j,\ell)\in J} \omega_{i,j}^{(\ell)}$. Clearly $\omega \in C_D^{\mathrm{Lip}}(\widehat{U})$ and $1-\mu \leq \omega(u) \leq 1$ for any $u \in \widehat{U}$. Moreover,

$$\mathrm{Lip}_D(\omega) \leq \Gamma = \frac{2\mu\,\gamma_1^N}{c_0\,\rho^{p_0 q_0+2}} \cdot \frac{|b|}{\epsilon_1}.$$

Next, define *the contraction operator* $\mathcal{N} = \mathcal{N}_J(a,b,t,c) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$ by

$$(\mathcal{N}h) = \mathcal{M}_{atc}^N(\omega_J \cdot h).$$

Using Lemma 5 above, the proof of the following lemma is the same as that of Lemma 5.6 in [20].

Lemma 7. *Under the above conditions for N and μ the following hold :*

(a) $\mathcal{N}h \in K_{E|b|}(\widehat{U})$ for any $h \in K_{E|b|}(\widehat{U})$;

(b) If $h \in C_D^{\mathrm{Lip}}(\widehat{U})$ and $H \in K_{E|b|}(\widehat{U})$ are such that $|h| \leq H$ in $\widehat{U}$ and $|h(v)-h(v')| \leq E|b|H(v')\,D(v,v')$ for any $v,v' \in U_j$, $j=1,\ldots,k$, then for any $i=1,\ldots,k$ and any $u,u' \in \widehat{U}_i$ we have

$$|(\mathcal{L}_{abtz}^N h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E|b|(\mathcal{N}H)(u')\,D(u,u').$$

Definition. A subset J of Ξ will be called *dense* if for any $m = 1,\ldots,p$ there exists $(i,j,\ell) \in J$ such that $\mathcal{D}_j \subset \mathcal{C}_m$.

Denote by $\mathsf{J} = \mathsf{J}(a,b)$ the *set of all dense subsets* J of Ξ.

Although the operator $\mathcal{N}$ here is different, the proof of the following lemma is very similar to that of Lemma 5.8 in [20].

Lemma 8. *Given the number N, there exist $\rho_2 = \rho_2(N) \in (0,1)$ and $a_0 = a_0(N) > 0$ such that $\int_{\widehat{U}} (\mathcal{N}_J H)^2 d\nu \leq \rho_2 \int_{\widehat{U}} H^2 d\nu$ whenever $|a|,|c| \leq a_0$, $t \geq 1/a_0$, J is dense and $H \in K_{E|b|}(\widehat{U})$.*

In what follows we assume that $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ are such that

$$H \in K_{E|b|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \widehat{U}\ , \tag{4.3}$$

and

$$|h(u) - h(u')| \leq E|b|H(u')\,D(u,u') \quad \text{whenever } u,u' \in \widehat{U}_i\ ,\ i=1,\ldots,k\ . \tag{4.4}$$

Let again $z = c + \mathbf{i}w$. Define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbb{C}$ ($\ell = 1,\ldots,j_0$, $i=1,2$) by

$$\chi_\ell^{(1)}(u) = \frac{\left|e^{(f_{at}^N - \mathbf{i}b\tau^N + zg_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{at}^N - \mathbf{i}b\tau^N + zg_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u))\right|}{(1-\mu)e^{f_{at}^N(v_1^{(\ell)}(u)) + cg_t^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f_{at}^N(v_2^{(\ell)}(u)) + cg_t^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left|e^{(f_{at}^N - \mathbf{i}b\tau^N + zg_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{at}^N - \mathbf{i}b\tau^N + zg_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u))\right|}{e^{f_{at}^N(v_1^{(\ell)}(u)) + cg_t^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1-\mu)e^{f_{at}^N(v_2^{(\ell)}(u)) + cg_t^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

and set $\gamma_\ell(u) = b\,[\tau^N(v_2^{(\ell)}(u)) - \tau^N(v_1^{(\ell)}(u))]$, $u \in \widehat{U}$.

Definitions. We will say that the cylinders $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are *adjacent* if they are subcylinders of the same $\mathcal{C}_m$ for some m. If $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ for some m and for some $\ell = 1,\ldots,\ell_0$

there exist $u \in \mathcal{D}_j$ and $v \in \mathcal{D}_{j'}$ such that $d(u,v) \geq \frac{1}{2}\,\mathrm{diam}(\mathcal{C}_m)$ and $\left\langle \frac{r^{-1}(v)-r^{-1}(u)}{\|r^{-1}(v)-r^{-1}(u)\|}\ ,\ \eta_\ell \right\rangle \geq \theta_1$, we will say that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are *η_ℓ-separable in $\mathcal{C}_m$*.

As a consequence of Lemma 6(b) one gets the following.

Lemma 9. (Lemma 5.9 in [20]) *Let $j, j' \in \{1, 2, \ldots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$ for some $m = 1, \ldots, p$ and $\ell = 1, \ldots, \ell_0$. Then $|\gamma_\ell(u) - \gamma_\ell(u')| \geq c_2\epsilon_1$ for all $u \in \widehat{Z}_j$ and $u' \in \widehat{Z}_{j'}$, where $c_2 = \dfrac{\hat{\delta}\,\rho}{16}$.*

The following lemma is the analogue of Lemma 5.10 in [20] and represents the main step in proving Theorem 1.

Lemma 10. *Assume $|b| \geq b_0$ for some sufficiently large $b_0 > 0$, $|a|, |c| \leq a_0$, and let* (4.1) *hold. Then for any $j = 1, \ldots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \ldots, q\}$ and $\ell \in \{1, \ldots, \ell_0\}$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$.*

To prove this we need the following lemma which coincides with Lemma 14 in [4] and its proof is almost the same.

Lemma 11. *If h and H satisfy* (4.3)-(4.4)*, then for any $j = 1, \ldots, q$, $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have:*

(a) $\dfrac{1}{2} \leq \dfrac{H(v_i^{(\ell)}(u'))}{H(v_i^{(\ell)}(u''))} \leq 2$ *for all $u', u'' \in \widehat{Z}_j$;*

(b) Either for all $u \in \widehat{Z}_j$ we have $|h(v_i^{(\ell)}(u))| \leq \frac{3}{4} H(v_i^{(\ell)}(u))$, or $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$ for all $u \in \widehat{Z}_j$.

Sketch of proof of Lemma 10. We use a modification of the proof of Lemma 5.10 in [20].

Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $\mathcal{D}_j \subset \mathcal{C}_m$. As in [20] we find $j', j'' = 1, \ldots, q$ such that $\mathcal{D}_{j'}, \mathcal{D}_{j''} \subset \mathcal{C}_m$ and $\mathcal{D}_{j'}$ and $\mathcal{D}_{j''}$ are η_ℓ-separable in $\mathcal{C}_m$.

Fix ℓ, j' and j'' with the above properties, and set $\widehat{Z} = \widehat{Z}_j \cup \widehat{Z}_{j'} \cup \widehat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \widehat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, i.e. $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4}\, H(v_i^{(\ell)}(u))$, $u \in \widehat{Z}$.

Since $\psi(\widehat{Z}) = \widehat{\mathcal{D}}_j \cup \widehat{\mathcal{D}}_{j'} \cup \widehat{\mathcal{D}}_{j''} \subset \mathcal{C}_m$, given $u, u' \in \widehat{Z}$ we have $\sigma^{N-n_1}(v_i^{(\ell)}(u)), \sigma^{N-n_1}(v_i^{(\ell)}(u')) \in \mathcal{C}_m$. Moreover, $\mathcal{C}' = v_i^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m))$ is a cylinder with $\mathrm{diam}(\mathcal{C}') \leq \frac{\epsilon_1}{c_0\,\gamma^{N-n_1}\,|b|}$. Thus, the estimate (8.3) in the Appendix below implies

$$|g_t^N(v_i^{(\ell)}(u)) - g_t^N(v_i^{(\ell)}(u'))| \leq \frac{C_1 t \epsilon_1}{c_0\, \gamma^{N-n_1}\, |b|}.$$

Using the above assumption, (4.1), (4.2) and (3.5), and assuming e.g.

$$e^{c g_t^N(v_i^{(\ell)}(u))} |h(v_i^{(\ell)}(u))| \geq e^{c g_t^N(v_i^{(\ell)}(u))} |h(v_i^{(\ell)}(u'))|,$$

we get[1]

$$
\begin{aligned}
&\frac{|e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u)) - e^{zg_t^N(v_i^{(\ell)}(u'))}h(v_i^{(\ell)}(u'))|}{\min\{|e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u))|, |e^{zg_t^N(v_i^{(\ell)}(u'))}h(v_i^{(\ell)}(u'))|\}}\\
= &\frac{|e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u)) - e^{zg_t^N(v_i^{(\ell)}(u'))}h(v_i^{(\ell)}(u'))|}{e^{cg_t^N(v_i^{(\ell)}(u'))}|h(v_i^{(\ell)}(u'))|}\\
\leq &\frac{|e^{zg_t^N(v_i^{(\ell)}(u))} - e^{zg_t^N(v_i^{(\ell)}(u'))}|}{e^{cg_t^N(v_i^{(\ell)}(u'))}} + \frac{e^{cg_t^N(v_i^{(\ell)}(u))}|h(v_i^{(\ell)}(u)) - h(v_i^{(\ell)}(u'))|}{e^{cg_t^N(v_i^{(\ell)}(u'))}|h(v_i^{(\ell)}(u'))|}\\
\leq &\frac{|e^{zg_t^N(v_i^{(\ell)}(u))} - e^{zg_t^N(v_i^{(\ell)}(u'))}|}{e^{cg_t^N(v_i^{(\ell)}(u'))}} + \frac{e^{c(g_t^N(v_i^{(\ell)}(u'))-g_t^N(v_i^{(\ell)}(u')))}\,E|b|H(v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|}D(v_i^{(\ell)}(u), v_i^{(\ell)}(u'))\\
\leq &\frac{|e^{cg_t^N(v_i^{(\ell)}(u))} - e^{cg_t^N(v_i^{(\ell)}(u'))}|}{e^{cg_t^N(v_i^{(\ell)}(u'))}} + |e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u))} - e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u'))}| + 4E|b|e^{2a_0NT}\,\mathrm{diam}(\mathcal{C}')\\
\leq &(e^{C_1t}C_1t + |w|C_1t)\,D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + 4E|b|e^{2Na_0T}\,\frac{\gamma^{n_1}\epsilon_1}{c_0\gamma^N}\\
\leq &\frac{(B+A_0)\gamma^{n_1}\epsilon_1}{c_0\gamma^N} + \frac{4E\gamma^{n_1}\epsilon_1}{c_0(e^{-2a_0T}\gamma)^N} < \frac{\pi}{12}
\end{aligned}
$$

assuming $a_0 > 0$ is is chosen sufficiently small and N sufficiently large. So, the angle between the complex numbers

$$e^{zg_t^N(v_i^{(\ell)}(u)}h(v_i^{(\ell)}(u)) \quad \text{and } e^{zg_t^N(v_i^{(\ell)}(u')}h(v_i^{(\ell)}(u'))$$

(regarded as vectors in $\mathbb{R}^2$) is $< \pi/6$. In particular, for each $i = 1, 2$ we can choose a real continuous function $\theta_i(u)$, $u \in \widehat{Z}$, with values in $[0, \pi/6]$ and a constant λ_i such that

$$e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u)) = e^{\mathbf{i}(\lambda_i+\theta_i(u))}e^{cg_t^N(v_i^{(\ell)}(u))}|h(v_i^{(\ell)}(u))|$$

for all $u \in \widehat{Z}$. Fix an arbitrary $u_0 \in \widehat{Z}$ and set $\lambda = \gamma_\ell(u_0)$. Replacing e.g λ_2 by $\lambda_2 + 2m\pi$ for some integer m, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \leq \pi$. Using the above, $\theta \leq 2\sin\theta$ for $\theta \in [0, \pi/6]$, and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2\sin|\theta_i(u) - \theta_i(u')| < \frac{c_2\epsilon_1}{8}$.

The difference between the arguments of the complex numbers

$$e^{\mathbf{i}\,b\,\tau^N(v_1^{(\ell)}(u))}e^{zg_t^N(v_1^{(\ell)}(u)}h(v_1^{(\ell)}(u)) \quad \text{and} \quad e^{\mathbf{i}\,b\,\tau^N(v_2^{(\ell)}(u))}e^{zg_t^N(v_2^{(\ell)}(u)}h(v_2^{(\ell)}(u))$$

is given by the function

$$\Gamma^{(\ell)}(u) = [b\,\tau^N(v_2^{(\ell)}(u)) + \theta_2(u) + \lambda_2] - [b\,\tau^N(v_1^{(\ell)}(u)) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1) + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u))\ .$$

Given $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$, since $\widehat{\mathcal{D}}_{j'}$ and $\widehat{\mathcal{D}}_{j''}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$, it follows from Lemma 9 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{c_2\epsilon_1}{2}\ .$$

Thus, $|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq \frac{c_2}{2}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$. Hence either $|\Gamma^{(\ell)}(u')| \geq \frac{c_2}{4}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ or $|\Gamma^{(\ell)}(u'')| \geq \frac{c_2}{4}\epsilon_1$ for all $u'' \in \widehat{Z}_{j''}$.

---

[1] Using some estimates as in the proof of Lemma 5(b) in the Appendix below and $\|cg_t^N\|_0 \leq a_0NT$ by (3.5).

Assume for example that $|\Gamma^{(\ell)}(u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in \widehat{Z}_{j'}$. Since $\widehat{Z} \subset \sigma^{n_1}(\mathcal{C}_m)$, as in [20] we have for any $u \in \widehat{Z}$ we get $|\Gamma_\ell(u)| < \frac{3\pi}{2}$. Thus, $\frac{c_2}{4}\epsilon_1 \leq |\Gamma^{(\ell)}(u)| < \frac{3\pi}{2}$ for all $u \in \widehat{Z}_{j'}$. Now as in [4] (see also [20]) one shows that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$. $\square$

Parts (a) and (b) of the following lemma can be proved in the same way as the corresponding parts of of Lemma 5.3 in [20], while part (c) follows from Lemma 5(b).

Lemma 12. *There exist a positive integer N and constants $\hat{\rho} = \hat{\rho}(N) \in (0,1)$, $a_0 = a_0(N) > 0$, $b_0 = b_0(N) > 0$ and $E \geq 1$ such that for every $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ such that (4.1) hold, there exists a finite family $\{\mathcal{N}_J\}_{J \in \mathsf{J}}$ of operators*

$$\mathcal{N}_J = \mathcal{N}_J(a,b,t,c) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U}),$$

where $\mathsf{J} = \mathsf{J}(a,b,t,c)$, with the following properties:

(a) The operators $\mathcal{N}_J$ preserve the cone $K_{E|b|}(\widehat{U})$;

(b) For all $H \in K_{E|b|}(\widehat{U})$ and $J \in \mathsf{J}$ we have $\int_{\widehat{U}} (\mathcal{N}_J H)^2 \, d\nu_0 \leq \hat{\rho} \int_{\widehat{U}} H^2 \, d\nu_0$.

(c) If $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ are such that $H \in K_{E|b|}(\widehat{U})$, $|h(u)| \leq H(u)$ for all $u \in \widehat{U}$ and $|h(u) - h(u')| \leq E|b|H(u')\, D(u,u')$ whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$, then there exists $J \in \mathsf{J}$ such that $|\mathcal{L}_{abw}^N h(u)| \leq (\mathcal{N}_J H)(u)$ for all $u \in \widehat{U}$ and for $z = c + \mathbf{i}w$ we have

$$|(\mathcal{L}_{abtz}^N h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E|b|(\mathcal{N}_J H)(u')\, D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$.

Proof of Theorem 5(a). Using an argument from [4] one derives from Lemma 12 that there exist a positive integer N and constants $\hat{\rho} \in (0,1)$ and $a_0 > 0$, $b_0 \geq 1$, $A_0 > 0$ such that for any $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ for which (4.1) hold, and for any $h \in C^{\mathrm{Lip}}(\widehat{U})$ with $\|h\|_{\mathrm{Lip},b} \leq 1$ we have

$$\int_U |\mathcal{L}_{abtz}^{Nm} h|^2 \, d\nu_0 \leq \hat{\rho}^m \quad , \quad m \geq 0. \tag{4.5}$$

Then the estimate claimed in Theorem 5(a) follows as in [4] (see also the proof of Corollary 3.3(a) in [19]). $\square$

The proof of Theorem 5(b) can be derived using an approximation procedure as in [4] – see the Appendix below for some details.

5. Spectral estimates when w is the leading parameter

Here we try to repeat the arguments from the previous section however changing the roles of the parameters b and w. We continue to use the assumptions made at the beginning of Sect. 4, however now we suppose that $f \in C^{\mathrm{Lip}}(\widehat{U})$. We will consider the case

$$|b| \leq B\,|w| \tag{5.1}$$

for an arbitrarily large (but fixed) constant $B > 0$.

Assume that $G : \Lambda \longrightarrow \mathbb{R}$ is a Lipschitz functions which is constant on stable leaves of $B_i = \{\phi_t(x) : x \in R_i, 0 \leq t \leq \tau(x)\}$ for each rectangle R_i of the Markov family and $A = \min_{x \in \Lambda} G(x) > 0$. Set

$$L = \mathrm{Lip}(G) \quad , \quad D = \mathrm{diam}(\Lambda)\,,$$

where without loss of generality we may assume that $D \geq 1$. We will also assume that

$$L \leq \hat{\mu}\, A \quad \text{, where} \quad \hat{\mu} = \frac{c_0\, \hat{\delta}}{128\, C_0\, C_1\, D}. \tag{5.2}$$

The function

$$g(x) = \int_0^{\tau(x)} G(\phi_t(x))\, dt \quad , \quad x \in R,$$

is constant on stable leaves of R, so it can be regarded as a function on U. Clearly $g \in C^{\mathrm{Lip}}(\widehat{U})$.

Remark. Notice that if we replace G by $G + d$ for some constant $d > 0$, then

$$g'(x) = \int_0^{\tau(x)} (G(\phi_t(x)) + d)\, dt = g(x) + d\, \tau(x),$$

so

$$\mathcal{L}_{f_a - \mathbf{i}\, b\tau + \mathbf{i}\, wg} = \mathcal{L}_{f_a - \mathbf{i}\, b\tau + \mathbf{i}\, w(g' - d\tau)} = \mathcal{L}_{f_a - \mathbf{i}\, (b+dw)\tau - \mathbf{i}\, wg'}.$$

Choose and fix $d > 0$ so that $\frac{\mathrm{Lip}(G)}{G_0 + d} \leq \hat{\mu}$. Then for $G' = G + d$ and $g' = g + d\tau$ we have $\frac{\mathrm{Lip}(G')}{\min G'} \leq \hat{\mu}$, and the operator $\mathcal{L}_{f_a - \mathbf{i}\, b\tau + \mathbf{i}\, wg} = \mathcal{L}_{f_a - \mathbf{i}\, b'\tau + \mathbf{i}\, wg'}$, where $b' = b + dw$. Thus, without loss of generality we may assume that $\frac{\mathrm{Lip}(G)}{\min G} \leq \hat{\mu}$, which is equivalent to (5.2). As in [12], this will imply a non-integrability property for g (see Lemma 10 below). In other words, dealing with an initial function G one has to first change it to arrange (5.2), and then with the new parameters b and w that appear in front of $\mathbf{i}\tau$ and $\mathbf{i}g$ consider the cases $|w| \leq B|b|$ (as in Theorem 5(c)) and $|b| \leq B|w|$, which is considered in this section.

As in Sect. 4, we will use the set-up and some arguments from [20]. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t over Λ as in Sect. 1.

Here we prove the following analogue of Theorem 5(c).

Theorem 6. *Let $\phi_t : M \longrightarrow M$ be a C^2 flow satisfying the Standing Assumptions over the basic set Λ. Assume in addition that* (5.2) *holds. Then for any real-valued functions $f, g \in C^{\mathrm{Lip}}(\widehat{U})$, any constants $\epsilon > 0$ and $B > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $w_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\| L^m_{f - (P_f + a + \mathbf{i}b)\tau + (c + \mathbf{i}w)g} h \|_{\mathrm{Lip},b} \leq C\, \rho^m\, |b|^{\epsilon}\, \|h\|_{\mathrm{Lip},b} \tag{5.3}$$

for all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|w| \geq w_0$ and $|b| \leq B\, |w|$.

Recall the definitions of $\lambda_0 > 0$, $\hat{\nu}_0$, h_0, f_0 from Sect. 3; now we have $h_0, f_0 \in C^{\mathrm{Lip}}(\widehat{U})$. Fix a small $a_0 > 0$. Given a real number a with $|a| \leq a_0$, denote by λ_a the *largest eigenvalue* of $L_{f-(P+a)\tau}$ on $C^{\mathrm{Lip}}(U)$ and by h_a the corresponding (positive) eigenfunction such that $\int_U h_a\, d\nu_a = 1$, where ν_a is the unique probability measure on U with $L^*_{f-(P+a)\tau} \nu_a = \nu_a$. Given real numbers a, b, c, w with $|a|, |c| \leq a_0$ consider the function

$$\tilde{f}_a = f - (P + a)\tau + \ln h_a - \ln(h_a \circ \sigma) - \ln \lambda_a$$

and the operators

$$\mathcal{L}_{abz} = L_{\tilde{f}_a - \mathbf{i}\, b\, \tau + zg} : C(U) \longrightarrow C(U) \quad , \quad \tilde{\mathcal{M}}_{ac} = L_{\tilde{f}_a + cg} : C(U) \longrightarrow C(U),$$

where $z = c + \mathbf{i}w$. Notice that $L_{\tilde{f}_a} 1 = 1$.

Taking the constant $C_0 > 0$ sufficiently large, we may assume that

$$\mathrm{Lip}(\tilde{f}_a - f_0) \leq C_0|a| \quad , \quad , \|\tilde{f}_a - f_0\|_0 \leq C_0\,|a| \quad , \quad |a| \leq a_0. \tag{5.4}$$

Thus, ssuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $T > 0$ (depending on f and a_0) such that

$$\|\tilde{f}_a\|_\infty \leq T \quad , \quad \mathrm{Lip}(h_a) \leq T \quad , \quad \mathrm{Lip}(\tilde{f}_a) \leq T \tag{5.5}$$

for $|a| \leq a_0$. As before, we will assume that $T \geq \max\{\,\|\tau\|_0\,,\,\mathrm{Lip}(\tau_{|\widehat{U}})\,\}$, and also that $\mathrm{Lip}(g) \leq T$ and $\|g\|_0 \leq T$.

Essentially in what follows we will repeat (a simplified version of) the proof of Theorem 5, so we will use the set-up in Sect. 4 – see the text after Lemma 6, up to and including the definition of ϵ_1.

Let $a, b, c, w \in \mathbb{R}$ be so that $|a|, |c| \leq a_0$, $|w| \geq w_0$, where w_0 is a sufficiently large constant defined as b_0 in Sect. 4, and $|b| \leq B|w|$. Set $z = c + \mathbf{i}w$.

Let $\mathcal{C}_m$ $(1 \leq m \leq p)$ be the family of *maximal closed cylinders* contained in $\overline{U_0}$ with $\mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|w|}$ such that $U_0 \subset \cup_{j=m}^p \mathcal{C}_m$ and $\overline{U_0} = \cup_{m=1}^p \mathcal{C}_m$. As before we have

$$\rho\,\frac{\epsilon_1}{|w|} \leq \mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|w|} \quad , \quad 1 \leq m \leq p.$$

Fix an integer $q_0 \geq 1$ as in Sect. 4, and let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all closed cylinders contained in $\overline{U_0}$ that are *subcylinders of co-length* $p_0\,q_0$ of some $\mathcal{C}_m$ $(1 \leq m \leq p)$. Then $\overline{U_0} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q$ and

$$\rho^{p_0\,q_0+1} \cdot \frac{\epsilon_1}{|w|} \leq \mathrm{diam}(\mathcal{D}_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|w|} \quad , \quad 1 \leq j \leq q.$$

Next, define the cylinders $Z_j = \overline{\sigma^{n_1}(\widehat{\mathcal{D}}_j)}$ and $X_{i,j}^{(\ell)} = \overline{v_i^{(\ell)}(\widehat{Z}_j)}$ as in Sect. 4, and consider the characteristic functions $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \longrightarrow [0,1]$. Let J be a *subset of the set* $\Xi = \Xi(a, w) = \{\ (i, j, \ell)\ :\ 1 \leq i \leq 2\ ,\ 1 \leq j \leq q\ ,\ 1 \leq \ell \leq \ell_0\ \}$. Define $\mu_0 > 0$ as in Sect. 4 and $\omega = \omega_J : \widehat{U} \longrightarrow [0,1]$ by $\omega = 1 - \mu_0 \displaystyle\sum_{(i,j,\ell)\in J} \omega_{i,j}^{(\ell)}$. Finally define $\mathcal{N} = \mathcal{N}_J(a, b, c) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$ by $(\mathcal{N}h) = \tilde{\mathcal{M}}_{ac}^N(\omega_J \cdot h)$.

Then we have the following analogue of Lemma 5.

Lemma 13. *Assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $A_0 > 0$ such that for all $a, c \in \mathbb{R}$ with $|a|, |c| \leq a_0$ the following hold:*

(a) If $H \in K_E(\widehat{U})$ for some $E > 0$, then

$$\frac{|(\tilde{\mathcal{M}}_{ac}^m H)(u) - (\tilde{\mathcal{M}}_{ac}^m H)(u')|}{(\tilde{\mathcal{M}}_{ac}^m H)(u')} \leq A_0\,\left[\frac{E}{\gamma_0^m} + 1\right]\,D(u, u')$$

for all $m \geq 1$ and all $u, u' \in U_i$, $i = 1, \ldots, k$.

(b) If the functions h and H on $\widehat{U}$ and $E > 0$ are such that $H > 0$ on $\widehat{U}$ and $|h(v) - h(v')| \leq E\,H(v')\,D(v, v')$ for any $v, v' \in \widehat{U}_i$, $i = 1, \ldots, k$, then for any integer $m \geq 1$ and any $b, w \in \mathbb{R}$ with $|b|, |w| \geq 1$, for $z = c + \mathbf{i}w$ we have

$$|(\mathcal{L}_{abw}^N h)(u) - (\mathcal{L}_{abw}^N h)(u')| \leq E|w|(\mathcal{N}H)(u')\,D(u, u').$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$.

The proof is a simplified version of that of Lemma 5 and we omit it.

Next, changing appropriately the definition of a dense subset $J$of Ξ, Lemma 8 holds again replacing $K_{E|b|}(\widehat{U})$ by $K_{E|w|}(\widehat{U})$.

Assume that $h, H \in C_D^{\rm Lip}(\widehat{U})$ are such that

$$H \in K_{E|w|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \widehat{U}, \tag{5.6}$$

and

$$|h(u) - h(u')| \leq E|w| H(u')\, D(u, u') \quad \text{whenever } u, u' \in \widehat{U}_i\ ,\ i = 1, \ldots, k. \tag{5.7}$$

Define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbb{C}$ by

$$\chi_\ell^{(1)}(u) = \frac{\left| e^{(\tilde{f}_a^N - {\bf i} b\tau^N + zg^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(\tilde{f}_a^N - {\bf i} b\tau^N + zg^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{(1-\mu) e^{\tilde{f}_a^N(v_1^{(\ell)}(u)) + cg^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{\tilde{f}_a^N(v_2^{(\ell)}(u)) + cg^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left| e^{(\tilde{f}_a^N - {\bf i} b\tau^N + zg^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(\tilde{f}_a^N - {\bf i} b\tau^N + zg^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{e^{\tilde{f}_a^N(v_1^{(\ell)}(u)) + cg^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1-\mu) e^{\tilde{f}_a^N(v_2^{(\ell)}(u)) + cg^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

and set $\gamma_\ell(u) = w\,[\tau_N(v_2^{(\ell)}(u)) - \tau_N(v_1^{(\ell)}(u))]$, $u \in \widehat{U}$. The crucial step in this section is to prove the following analogue of Lemma 9:

Lemma 14. *Let $j, j' \in \{1, 2, \ldots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$ for some $m = 1, \ldots, p$ and $\ell = 1, \ldots, \ell_0$. Then $|\gamma_\ell(u) - \gamma_\ell(u')| \geq c_3 \epsilon_1$ for all $u \in \widehat{Z}_j$ and $u' \in \widehat{Z}_{j'}$, where $c_3 = \dfrac{A\hat{\delta}\,\rho}{32}$.*

To prove the above we need the following.

Lemma 15. (Lemma 6 in [12]) *Assume that* (5.2) *holds. Under the assumptions and notation in Lemma 1, for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ so that $s + h\,\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h} \left(g^N(v_2^{(\ell)}(\tilde{r}(\cdot))) - g^N(v_1^{(\ell)}(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{A\hat{\delta}}{4}\,.$$

Proof of Lemma 14. This just a repetition of the proof of Lemma 5.9 in [20], where instead of using Lemma 6(b) we use the above Lemma 14. We omit the details. $\square$

Next, we need to prove the analogue of Lemma 10.

Lemma 16. *Assume $|w| \geq w_0$ for some sufficiently large $w_0 > 0$ and let $|b| \leq B|w|$. Then for any $j = 1, \ldots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \ldots, q\}$ and $\ell \in \{1, \ldots, \ell_0\}$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$.*

Sketch of proof of Lemma 16. We will use Lemma 11 which holds again with (4.3)-(4.4) replaced by (5.6)-(5.7).

Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $\mathcal{D}_j \subset \mathcal{C}_m$. As in [20] we find $j', j'' = 1, \ldots, q$ such that $\mathcal{D}_{j'}, \mathcal{D}_{j''} \subset \mathcal{C}_m$ and $\mathcal{D}_{j'}$ and $\mathcal{D}_{j''}$ are η_ℓ-separable in $\mathcal{C}_m$.

Fix ℓ, j' and j'' with the above properties, and set $\widehat{Z} = \widehat{Z}_j \cup \widehat{Z}_{j'} \cup \widehat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \widehat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, i.e. $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$, $u \in \widehat{Z}$.

Again we have $\psi(\widehat{Z}) = \widehat{\mathcal{D}}_j \cup \widehat{\mathcal{D}}_{j'} \cup \widehat{\mathcal{D}}_{j''} \subset \mathcal{C}_m$, and $\mathcal{C}' = v_i^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m))$ is a cylinder with $\mathrm{diam}(\mathcal{C}') \leq \frac{\epsilon_1}{c_0\, \gamma^{N-n_1}\, |w|}$. Thus, assuming e.g. $|h(v_i^{(\ell)}(u))| \geq |h(v_i^{(\ell)}(u'))|$, we get

$$\begin{aligned}
& \frac{|e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u)} h(v_i^{(\ell)}(u)) - e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u')} h(v_i^{(\ell)}(u'))|}{\min\{|h(v_i^{(\ell)}(u))|, |h(v_i^{(\ell)}(u'))|\}} \\
\leq \quad & |e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u)} - e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u')}| + \frac{E|w|\, H(v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|} D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) \\
\leq \quad & |b| C_1\, D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + 4E|w|\, D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) \\
\leq \quad & (B|w|\, C_1 + 4E|w|)\, \mathrm{diam}(\mathcal{C}') \leq \frac{(BC_1 + 4E)\epsilon_1}{\gamma_1^{N-n_1}} < \frac{\pi}{12}
\end{aligned}$$

assuming N is chosen sufficiently large. So, the angle between the complex numbers

$$e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u)} h(v_i^{(\ell)}(u)) \quad \text{and } e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u')} h(v_i^{(\ell)}(u'))$$

(regarded as vectors in $\mathbb{R}^2$) is $< \pi/6$. In particular, for each $i = 1, 2$ we can choose a real continuous function $\theta_i(u)$, $u \in \widehat{Z}$, with values in $[0, \pi/6]$ and a constant λ_i such that $h(v_i^{(\ell)}(u)) = e^{\mathbf{i}(\lambda_i + \theta_i(u))} |h(v_i^{(\ell)}(u))|$ for all $u \in \widehat{Z}$. Fix an arbitrary $u_0 \in \widehat{Z}$ and set $\lambda = \gamma_\ell(u_0)$. Replacing e.g λ_2 by $\lambda_2 + 2m\pi$ for some integer m, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \leq \pi$. Using the above, $\theta \leq 2 \sin \theta$ for $\theta \in [0, \pi/6]$, and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2 \sin |\theta_i(u) - \theta_i(u')| < \frac{c_2 \epsilon_1}{8}$.

The difference between the arguments of the complex numbers

$$e^{\mathbf{i}\, b\, \tau_N(v_1^{(\ell)}(u))} e^{\mathbf{i} w g_N(v_1^{(\ell)}(u)} h(v_1^{(\ell)}(u)) \quad \text{and} \quad e^{\mathbf{i}\, b\, \tau_N(v_2^{(\ell)}(u))} e^{\mathbf{i} w g_N(v_2^{(\ell)}(u)} h(v_2^{(\ell)}(u))$$

is given by the function

$$\Gamma^{(\ell)}(u) = [w\, g_N(v_2^{(\ell)}(u)) + \theta_2(u) + \lambda_2] - [w\, g_N(v_1^{(\ell)}(u)) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1) + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u)) \, .$$

Given $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$, since $\widehat{\mathcal{D}}_{j'}$ and $\widehat{\mathcal{D}}_{j''}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$, it follows from Lemma 9 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{c_3 \epsilon_1}{2} \, .$$

Thus, $|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq \frac{c_3}{2} \epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$. Hence either $|\Gamma^{(\ell)}(u')| \geq \frac{c_3}{4} \epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ or $|\Gamma^{(\ell)}(u'')| \geq \frac{c_3}{4} \epsilon_1$ for all $u'' \in \widehat{Z}_{j''}$.

Assume for example that $|\Gamma^{(\ell)}(u)| \geq \frac{c_2}{4} \epsilon_1$ for all $u \in \widehat{Z}_{j'}$. Since $\widehat{Z} \subset \sigma^{n_1}(\mathcal{C}_m)$, as in [20] we have for any $u \in \widehat{Z}$ we get $|\Gamma_\ell(u)| < \frac{3\pi}{2}$. Thus, $\frac{c_2}{4} \epsilon_1 \leq |\Gamma^{(\ell)}(u)| < \frac{3\pi}{2}$ for all $u \in \widehat{Z}_{j'}$. Now as in [4] (see also [20]) one shows that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$. □

Proof of Theorem 6. This is now the same as the proof of Theorem 5(a). □

6. Analytic continuation of the function $\zeta(s, z)$

Consider the function $\zeta(s, z)$ introduced in Section 1. Recall that $s = a + \mathbf{i}b, z = c + \mathbf{i}w$ with real $a, b, c, w \in \mathbb{R}$. First, we assume that f and g are functions in $C^\alpha(\Lambda)$ with some $0 < \alpha < 1$. Passing to the symbolic model defined by the Markov family $\mathcal{R}$ we obtain functions[2] in $C^\alpha(R)$ which we denote again by f and g. We assume that $Pr(f - P_f\tau) = 0$ and we set $s = P_f + a + \mathbf{i}b$. The functions f, g depend on $x \in R$. A second reduction is to replace f and g by functions $\hat{f}$, $\hat{g} \in C^{\alpha/2}(U)$ depending only on $x \in U$ so that $f = \hat{f} + h_1 - h_1 \circ \sigma$, $g = \hat{g} + h_2 - h_2 \circ \sigma$ (see Proposition 1.2 in [11]). Since for periodic points with $\sigma^n x = x$ we have $f^n(x) = \hat{f}^n(x)$, $g^n(x) = \hat{g}^n(x)$, we obtain the representation

$$\zeta(s, z) = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{\hat{f}^n(x) - (P_f + a + \mathbf{i}b)\tau^n(x) + (c + \mathbf{i}w)\hat{g}^n(x)}\Big).$$

In this section we will prove under the standing assumptions that there exists $\epsilon > 0$ and $\epsilon_0 > 0$ such that the function $\zeta(s, z)$ has a non non zero analytic continuation for $-\epsilon \leq a \leq 0$ and $|z| \leq \epsilon_0$ with a simple pole at $s = s(z)$, $s(0) = P_f$. Here $s(z)$ is determined from the equation $Pr(f - s\tau + zg) = 0$. For simplicity of the notation we denote below $\hat{f}$ and $\hat{g}$ again by f, g.

First consider the case $0 < \delta \leq |b| \leq b_0$. Since our standing assumptions imply that the flow ϕ_t is weak mixing, Theorem 6.4 in [11] says that for every fixed b lying in the compact interval $[\delta, b_0]$ there exists $\epsilon(b) > 0$ so that the function $\zeta(s, z)$ is analytic for $|s - P_f + \mathbf{i}b| \leq \epsilon(b)$, $|z| \leq \epsilon(b)$. This implies that there exists $\eta_0 = \eta_0(\delta, b_0) > 0$ such that $\zeta(s, z)$ is analytic for $P_f - \eta_0 \leq \operatorname{Re} s \leq P_f + \eta_0, \delta \leq |\operatorname{Im} s| \leq b_0$, $|z| \leq \eta_0$. Decreasing $\delta > 0$ and η_0, if it is necessary, we apply once more Theorem 6.4 in [11], to conclude that $\zeta(s, z)(1 - e^{Pr(f - s\tau + zg)})$ is analytic for

$$s \in \{s \in \mathbb{C} : |\operatorname{Re} s - P_f| \leq \eta, |\operatorname{Im} s| \leq \delta\}$$

and $|z| \leq \eta_0$. Consequently, the singularities of $\zeta(s, z)$ are given by (s, z) for which we have $Pr(f - s\tau + zg) = 0$ and, solving this equation, we get $s = s(z)$ with $s(0) = P_f$. It is clear that we have a simple pole at $s(z)$ since $\frac{d}{ds} Pr(f - s\tau + zg) \neq 0$ for $|z|$ small enough.

Now we pass to the case when $|\operatorname{Im} s| = |b| \geq b_0 > 0$, $|z| \leq \eta_0$. Then we fix a $\beta \in (0, \alpha/2)$ and we get with $0 < \mu < 1$ the inequality $|\operatorname{Im} b| \geq B_0 |z|^\mu$ with $B_0 = \frac{b_0}{\eta_0^\mu}$. Thus we are in position to apply the estimates of Theorem 5(b) saying that for every $\epsilon > 0$ there exist $0 < \rho < 1$ and $C_\epsilon > 0$ so that

$$\|L^m_{f - (P_f + a + \mathbf{i}b)\tau + zg}\|_{\beta, b} \leq C_\epsilon \rho^m |b|^\epsilon, \ \forall m \in \mathbb{N} \tag{6.1}$$

for $|a| \leq a_0, |b| \geq b_0$, $|z| \leq \eta_0$. Next we apply Theorem 4 with functions $f, g \in C^\beta(U)$. For $|\operatorname{Re} s - P_f| \leq \eta_0$, $|\operatorname{Im} s| \geq b_0$ and $|z| \leq \eta_0$ we deduce

$$|Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)| \leq \sum_{i=1}^{k} |L^n_{f - (P_f + \mathbf{i}a + b)\tau + zg}(\chi_i)(x_i)|$$

$$+ C(1 + |b|) \sum_{m=2}^{n} \|L^m_{f - (P_f + a + \mathbf{i}b)\tau + zg}\|_\beta \gamma_0^{-m\beta} e^{mPr(f - (P_f + a)\tau + (\operatorname{Re} z)g)}$$

[2] In fact, one has to define first f and g as functions in $C^\alpha(\hat{R})$ and then extend them as α-Hölder functions on R. In the same way one should proceed with Hölder functions on U.

$$\leq k\|L^n_{f-(P_f+a+\mathbf{i}b)\tau+zg}\|_\beta + C_\epsilon(1+|b|)|b|^\epsilon \sum_{m=2}^n \rho^{n-m}\gamma_0^{-m\beta} e^{m(\epsilon+Pr(f-(P_f+a)\tau+cg))}.$$

Taking η_0 and ϵ small, we arrange

$$\gamma_0^{-\beta} e^{\epsilon+Pr(f-(P_f+a)\tau+cg)} \leq \gamma_2 < 1$$

for $|a| \leq \eta_0$, $|c| \leq \eta_0$, since $Pr(f - P_f\tau) = 0$ and $\gamma_0^{-\nu} < 1$. Next increasing $0 < \rho < 1$, if it is necessary, we get $\frac{\gamma_2}{\rho} < 1$. Thus the sum above will be bounded by

$$C_\epsilon(1+|b|)|b|^\epsilon \rho^n \sum_{m=2}^\infty \Big(\frac{\gamma_2}{\rho}\Big)^m \leq C'_\epsilon |b|^{1+\epsilon}\rho^n$$

for $|a| \leq \eta_0$, $|z| \leq \eta_0$. The analysis of the term $\|L^n_{f-(P_f+a+\mathbf{i}b)+zg}\|_\beta$ follows the same argument and it is simpler. Finally, we get

$$|Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)| \leq B_\epsilon |b|^{1+\epsilon}\rho^n,\ \forall n \in \mathbb{N}$$

and the series

$$\sum_{n=1}^\infty \frac{1}{n} Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)$$

is absolutely convergent for $|a| \leq \eta_0, |b| \geq b_0, |z| \leq \eta_0$. This implies the analytic continuation of $\zeta(s,z)$ for $|\operatorname{Re} s - P_f| \leq \eta_0, |\operatorname{Im} s| \geq b_0$, $|z| \leq \eta_0$, thus completing the proof of Theorem 1.

To obtain a representation of the function $\eta_g(s) = \frac{\partial \log\zeta(s,z)}{\partial z}\big|_{z=0}$ for s sufficiently close to P_f, notice that for such values of s we have

$$\eta_g(s) = -\frac{\partial \log(1 - e^{Pr(f-s\tau+zg)})}{\partial z}\Big|_{z=0} + A_0(s)$$

$$= \frac{1}{s-P_f}\frac{\int g dm}{\int \tau dm} + A_1(s) = \frac{\int G d\mu_F}{s - P_f} + A_1(s),$$

where m is the equilibrium state of $f - P_f\tau$, μ_F is the equilibrium state of F and $A_0(s)$ and $A_1(s)$ are analytic in a neighborhood of P_f (see Chapter 6 in [11]). More precisely, μ_F is a σ_t^τ invariant probability measure on R^τ such that

$$Pr(F) = h(\sigma_1^\tau, \mu_F) + \int F(\pi(x,t))d\mu_F,$$

where $h(\sigma_1^\tau, \mu_F)$ is the metric entropy of σ_1^τ with respect to μ_F (see Chapter 6 in [11]).

Taking η_0 small enough, for $|z| \leq \eta_0$, $|\operatorname{Re} s - P_f| \leq \eta_0$ and $|\operatorname{Im} s| \geq \eta_0$ from the estimates for $Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)$ above, we deduce

$$|\log\zeta(s,z)| \leq C_\epsilon \max\Big(1, |\operatorname{Im} s|^{1+\epsilon}\Big).$$

To estimate $\eta_g(s)$, as in [16], we apply the Cauchy theorem for the derivative

$$\frac{\partial}{\partial z}\log\zeta(s,z)\big|_{z=0} = \frac{1}{2\pi\mathbf{i}\delta}\int_{|\xi|=\delta}\frac{\log\zeta(s,\xi)}{\xi^2}d\xi = \mathcal{O}(|\operatorname{Im} s|^{1+\epsilon}),\ |\operatorname{Im} s| \geq 1.$$

with $\delta > 0$ sufficiently small. Thus we obtain a $\mathcal{O}\Big(\max\Big(1, |\operatorname{Im} s|^{1+\epsilon}\Big)\Big)$ bound for the function

$$A(s) = \eta_g(s) - \frac{1}{s - P_f} \int G d\mu_F$$

which is analytic for $|\operatorname{Re} s - P_f| \leq \eta_0$. Decreasing η_0 and applying Phragmén-Lindelöf theorem, by a standard argument we obtain a bound $\mathcal{O}\Big(\max\Big(1, |\operatorname{Im} s|^{\alpha}\Big)\Big)$ with $0 < \alpha < 1$. Consequently, we have the following

Proposition 3. *Under the assumptions of Theorem* 1 *there exist* $\eta_0 > 0$ *and* $0 < \alpha < 1$ *such that for* $\operatorname{Re} s > P_f - \eta_0$ *we have*

$$\eta_g(s) = \frac{1}{s - P_f} \int G d\mu_F + A(s) \tag{6.2}$$

with an analytic function $A(s)$ *satisfying the estimate*

$$|A(s)| \leq C \max\Big(1, |\operatorname{Im} s|^{\alpha}\Big). \tag{6.3}$$

Next define $\mathcal{F}^\tau(\mathbb{C}) := \{F : R^\tau \longrightarrow \mathbb{C}\}$ and $\mathcal{F}^\tau(\mathbb{R}) := \{F : R^\tau \longrightarrow \mathbb{R}\}$ the spaces of complex-valued (real-valued) functions which are continuous. If $G \in \mathcal{F}^\tau(\mathbb{C})$ is Lipschitz continuous and if the standing assumptions for Λ are satisfied, the function

$$g(x) = \int_0^{\tau(x)} G(\pi(x, t)) dt$$

is Lipschitz continuous on R. Moreover, if the representative of G in the suspension space R^τ is constant on stable leaves, the function $g(x)$ depends only on $x \in U$. Now we introduce two definitions of independence.

Definition 1. *Two functions* $f_1, f_2 : U \to \mathbb{R}$ *are called* $\sigma-$ *independent if whenever there are constants* $t_1, t_2 \in R$ *such that* $t_1 f_1 + t_2 f_2$ *is co homologous to a function in* $C(U : 2\pi\mathbb{Z})$*, we have* $t_1 = t_2 = 0$.

For a function $G \in \mathcal{F}^\tau(\mathbb{R})$ consider the skew product flow S_t^G on $\mathbb{S}^1 \times R^\tau$ by

$$S_t^G(e^{2\pi \mathbf{i}\alpha}, y) = \Big(e^{2\pi \mathbf{i}(\alpha + G^t(y))}, \sigma_t^\tau(y)\Big).$$

Definition 2 ([8])**.** *Let* $G \in \mathcal{F}^\tau(\mathbb{R})$*. Then* G *and* σ_t^τ *are flow independent if the following condition is satisfied. If* $t_0, t_1 \in \mathbb{R}$ *are constants such that the skew product flow* S_t^H *with* $H = t_0 + t_1 G$ *is not topologically mixing, then* $t_0 = t_1 = 0$.

Notice that if G and σ_t^τ are flow independent, then the flow σ_t^τ is topologically weak mixing and the function G is not co homologous to a constant function. On the other hand, if G and σ_t^τ are flow independent, then $g(x) = \int_0^{\tau(x)} G(\pi(x, t)) dt$ and τ are $\sigma-$ independent.
Below we assume that g and τ are $\sigma-$ independent and we suppose that F, G is a Lipschitz functions Λ having representative in R^τ which are constant on stable leaves. Thus we obtain functions f, g which are in $C^{\mathrm{Lip}}(\widehat{U})$. We will now obtain an analytic continuation of $\zeta(s, z)$ for $P_f - \eta_0 < \operatorname{Re} s < P_f$ and $z = \mathbf{i}w$. Set $r(s, w) = f - (P_f + a + \mathbf{i}b)\tau + \mathbf{i}wg$. We choose $M > 0$ large enough so that we can apply Theorem 6 for $|w| \geq M$. We consider two cases.

Case 1. $\eta_0 \leq |w| \leq M$. We consider two sub cases: 1a) $|\operatorname{Im} s| \leq M_1$, 1$b$) $|\operatorname{Im} s| \geq M_1$. Here $M_1 > 0$ is chosen large enough so that Theorem 5 (b) holds with $|\operatorname{Im} s| \geq M_1$.

Let $|\operatorname{Im} s| \leq M_1$. Assume first that $\operatorname{Im} r(s_0, w_0)$ is cohomologous to $c + 2\pi Q$ with an integer-valued function $Q \in C(U; \mathbb{Z})$ and a constant $c \in [0, 2\pi)$. If $c = 0$, since g and τ are $\sigma-$ independent, from the fact that $b\tau + wg$ is co homologous to a function in $C(U; 2\pi\mathbb{Z})$, we deduce $b = w = 0$ which is impossible because $b = \operatorname{Im} s \neq 0$. Thus we have $c \neq 0$. Consequently, the operator $L_{f - s_0\tau + \mathbf{i}wg}$ has an eigenvalue $e^{\mathbf{i}c}$. Then there exists a neighborhood $U_1 \subset \mathbb{C} \times \mathbb{R}$ of (s_0, w_0) such that for $(s, w) \in U_1$ we have $Pr(r(s, w)) \neq 0$ and for $(s, w) \in U_2$ we have an analytic extension of $\log \zeta(s, w)$ given by

$$\log \zeta(s, w) = \frac{K_1(s, w)}{1 - e^{Pr(r(s,w))}} + J_1(s, w)$$

with functions $K_1(s, w), J_1(s, w)$ analytic with respect to s for $(s, w) \in U_1$. Second, let $\operatorname{Im} r(s_0, w_0)$ be not cohomologous to $c + 2\pi Q$. Then the spectral radius of $L_{f - s_0\tau + \mathbf{i}wg}$ is strictly less than 1 and this will be the case for (s, w) is a small neighborhood $U_2 \subset \mathbb{C} \times \mathbb{R}$ of (s_0, w_0). Applying Theorem 4, this implies easily that $\log \zeta(s, \mathbf{i}w)$ has an analytic continuation with respect to s.

Passing to the case 1b), we observe that $|\operatorname{Im} s| \geq \frac{M_1}{\eta_0}|w|$. Then, we apply Theorem 5, (c) combined with Theorem 4 to obtain an analytic continuation of $\log \zeta(s, \mathbf{i}w)$. Moreover, our argument works for $z = c + \mathbf{i}w$ with $|c| \leq \eta_0$ and $\eta_0 \leq |w| \leq M$ and we obtain an analytic continuation of $\log \zeta(s, z)$ for $P_f - \eta_0 \leq \operatorname{Re} s < P_f$, $|c| \leq \eta_0$, $\eta_0 \leq |w| \leq M$.

Case 2. $|w| \geq M$. We consider two sub cases: 2a) $|\operatorname{Im} s| \geq B|w|$, 2b) $|\operatorname{Im} s| \leq B|w|$, $B = \frac{M_1}{M}$. If we have 2a), we apply Theorem 5 (c). In the case 2b) we use the argument of Section 5 replacing $g(x)$ by $g'(x) = g(x) + d\tau(x)$, where the constant $d > 0$ is chosen so that for the function $G' = G + d$ we have

$$\frac{\operatorname{Lip} G'}{\min G'} \leq \hat{\mu},$$

where $\hat{\mu} > 0$ is the constant introduced in Section 5. Next we write

$$L_{f - (P_f + a + \mathbf{i}b)\tau + \mathbf{i}wg} = L_{f - (P_f + a + \mathbf{i}(b + dw)\tau + \mathbf{i}wg'}.$$

For the Ruelle operator involving g' we can apply Theorem 6 since $|b + dw| \leq (B + d)|w|$, $|w| \geq M$ and g is a Lipschitz function. An application of Theorem 4 implies the analytic continuation of $\log \zeta(s, \mathbf{i}w)$ for $P_f - \eta_0 \leq \operatorname{Re} s \leq P_f$ and $|w| \geq M$. From the above analysis we deduce the following

Theorem 7. *Assume the standing assumptions fulfilled for the basic set Λ. Let $F, G : \Lambda \longrightarrow \mathbb{R}$ be Lipschitz functions having representatives in R^τ which are constant on stable leaves. Assume that g and τ are σ-independent. Then there exists $\eta_0 > 0$ such that $\zeta(s, \mathbf{i}w)$ admits a non zero analytic continuation with respect to s for $P_f - \eta_0 \leq \operatorname{Re} s$, $w \in \mathbb{R}$ and $|w| \geq \eta_0$.*

7. Applications

7.1. **Hannay-Ozorio de Almeida sum formula.** The proof of (1.5) in [17] is based on the analytic continuation of the Dirichlet series

$$\eta(s) = \sum_{\gamma} \sum_{m=1}^{\infty} \lambda_G(\gamma) e^{m(-\lambda^u(\gamma) - (s-1)\lambda(\gamma))}, \; s \in \mathbb{C}$$

for $1-\eta_0 \leq \operatorname{Re} s < 1$. For this purpose the authors examine the analytic continuation of the symbolic function $\eta_g(s)$ with $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$ defined in Section 1 and they use the fact that the difference $\eta(s) - \eta_g(s)$ is analytic in a region $\operatorname{Re} s > 1-\epsilon'$, $\epsilon' > 0$. Next for the geodesic flow on surfaces with negative curvature they establish Proposition 3 with $P_f = 1$. Since M is an attractor, the equilibrium state of the function $-E(x)$ is just the SRB measure μ of ϕ_t (see [3]) and the residuum in (6.2) becomes $\int G d\mu$.

For the proof of Proposition 3 in [17] the authors exploit the link between the analytic continuation of $\zeta(s,z)$ and the spectral estimates of the Ruelle operator obtained by Dolgopyat [4]. However, in [17] Ruelle's lemma in [15] was used whose proof is rather sketchy and contains some steps which are not done in detail (see [23] for more information and comments concerning these steps and the gaps in their proofs). On the other hand, the estimates of Dolgopyat [4] are established only for Ruelle operators with one complex parameter, and to take into account the second parameter z some complementary analysis is necessary.

We would like to mention that [23] contains a correct and complete proof of Ruelle's lemma in the case of one complex parameter and Hölder function $\tau(x)$. A version of this lemma with two complex parameters is given in Section 2 above. Next, in Theorem 5 the spectral estimates for the Ruelle operator with two complex parameters are established for Axiom A flows on a basic set Λ of arbitrary dimension under the standing assumptions. If Λ is an attractor, according to [3], the equilibrium state of $-E(x)$ coincides with the SRB measure μ of ϕ_t. Thus we can apply Proposition 3 to obtain a representation of $\eta_g(s)$ with residue $\int G d\mu$. Using (6.2) and repeating the argument of Section 4 in [17], we obtain Theorem 2.

7.2. **Asymptotic of the counting function for period orbits.** As we mentioned in Sect. 1, the analysis of $\pi_F(T)$ is based on the analytic continuation of the function $\zeta(s,0)$ defined in Section 1. From the arguments in Section 6 with $z=0$ and the proof of Proposition 3 we get the following

Proposition 4. *Under the standing assumptions in Sect. 3 there exists $\eta_0 > 0$ such that $\frac{\zeta_F'(s)}{\zeta_F(s)}$ admits an analytic continuation for $Pr(F) - \eta_0 \leq \operatorname{Re} s$ with a simple pole at $s = Pr(F)$ with residue 1. Moreover, there exists $0 < \alpha < 1$ such that for $|\operatorname{Im} s| \geq 1$ we have*

$$\Big|\frac{\zeta_F'(s)}{\zeta_F(s)}\Big| \leq C|\operatorname{Im} s|^{\alpha}. \tag{7.1}$$

To obtain an asymptotic of $\pi_F(T)$, we examine the functions

$$\Psi(T) = \sum_{e^{nPr(F)\lambda(\gamma)} \leq T} \lambda(\gamma) e^{Pr(F)\lambda(\gamma)},\ \Psi_1(T) = \int_0^T \Psi(y)dy.$$

By a standard argument (see [15] and [14]) we obtain the representation

$$\psi_1(T) = \frac{T^2}{2} + \int_{\operatorname{Re} s = (1-\eta_0)Pr(F)} \Big(-\frac{\zeta_F'(s)}{\zeta_F(s)}\Big)\frac{T^s}{s(s+1)} ds = \frac{T^2}{2} + \mathcal{O}(T^{1+\alpha}),$$

where in the second equality the estimate (7.1) is used. This implies an asymptotic for $\Psi(T)$ and repeating the argument in [15], [14], one obtains Theorem 3.

8. Appendix: Proofs of some lemmas

Proof of Lemma 4. Denote by $\mathcal{F}_\theta(\widehat{U})$ the space of all functions $h : \widehat{U} \longrightarrow \mathbb{R}$ that are Lipschitz with respect to d_θ. Let $g \in C^{\mathrm{Lip}}(\widehat{U})$, and let $\theta = \theta_\alpha \in (0,1)$ be as in Sect. 3. Then $g \in \mathcal{F}_\theta(\widehat{U})$. Let $\lambda > 0$ be the maximal positive eigenvalue of L_g on $\mathcal{F}_\theta(\widehat{U})$ and let $h > 0$ be a corresponding normalized eigenfunction. By the Ruelle-Perron-Frobenius theorem, we have that $\frac{1}{\lambda^m} L_g^m 1$ converges uniformly to h. We will show that there exists a constant $C > 0$ such that $\frac{1}{\lambda^m}\mathrm{Lip}(L_g^m 1) \leq C$ for all m; this would then imply immediately that $h \in C^{\mathrm{Lip}}(\widehat{U})$ and $\mathrm{Lip}(h) \leq C$.

Take an arbitrary constant $K > 0$ such that $1/K \leq h(x) \leq K$ for all $x \in \widehat{U}$. Given $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$ and an integer $m \geq 1$ for any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. By (1.1) we have

$$|g_m(v) - g_m(v')| \leq \sum_{j=0}^{m-1} |g(\sigma^j(v)) - g(\sigma^j(v'))| \leq \mathrm{Lip}(g) \sum_{j=0}^{m-1} \frac{d(u,u')}{c_0 \gamma^m} \leq C' \,\mathrm{Lip}(g)\, d(u,u')$$

for some constant $C' > 0$. Thus,

$$\begin{aligned}
|(L_g^m 1)(u) - (L_g^m 1)(u')| &\leq \sum_{\sigma^m(v)=u} \left| e^{g_m(v)} - e^{g_m(v')} \right| = \sum_{\sigma^m(v)=u} e^{g_m(v)} \left| e^{g_m(v) - g_m(v')} - 1 \right| \\
&\leq e^{C' \,\mathrm{Lip}(g)} \sum_{\sigma^m(v)=u} e^{g_m(v)} \left| g_m(v) - g_m(v') \right| \\
&\leq e^{C' \,\mathrm{Lip}(g)}\, C' \mathrm{Lip}(g)\, d(u,u') \sum_{\sigma^m(v)=u} e^{g_m(v)} \\
&\leq e^{C' \,\mathrm{Lip}(g)}\, C' \mathrm{Lip}(g)\, d(u,u') \sum_{\sigma^m(v)=u} e^{g_m(v)}\, K h(v) \\
&= e^{C' \,\mathrm{Lip}(g)}\, C' K \mathrm{Lip}(g)\, d(u,u')\, (L_g^m h)(u) \\
&= e^{C' \,\mathrm{Lip}(g)}\, C' K \mathrm{Lip}(g)\, d(u,u')\, \lambda^m h(u) \\
&\leq \lambda^m\, C' K^2 e^{C' \,\mathrm{Lip}(g)}\, \mathrm{Lip}(g)\, d(u,u').
\end{aligned}$$

Thus, for every integer m the function $\frac{1}{\lambda^m} L_g^m 1 \in C^{\mathrm{Lip}}(\widehat{U})$ and $\frac{1}{\lambda^m}\mathrm{Lip}(L_g^m 1) \leq C' K^2 e^{C' \,\mathrm{Lip}(g)}\, \mathrm{Lip}(g)$. As mentioned above this proves that the eigenfunction $h \in C^{\mathrm{Lip}}(\widehat{U})$.

Using this with $g = f_t$ proves that $h_{at} \in C^{\mathrm{Lip}}(\widehat{U})$ for all $|a| \leq a_0$ and $t \geq 1/a_0$. However the above estimate for $\mathrm{Lip}(h_{at})$ would be of the form $\leq C\, e^{C\,t}\, t$ for some constant $C > 0$, which is not good enough.

We will now show that, taking $a_0 > 0$ sufficiently small, we have $\mathrm{Lip}(h_{at}) \leq Ct$ for some constant $C > 0$ independent of a and t.

Using (3.2) and choosing $a_0 > 0$ sufficiently small, we have $\lambda_{at}\gamma > \hat{\gamma}$ for all $|a| \leq a_0$ and $t > 1/a_0$. Fix an integer $m_0 \geq 1$ so large that $\frac{C_0^2}{c_0 \hat{\gamma}^m} < \frac{1}{2}$ for $m \geq m_0$. There exists a constant $d_0 > 0$ depending on m_0 such that for any u, u' belonging to the same U_i but not to the same cylinder of length m_0 we have $d(u,u') \geq d_0$. For such u, u' we have

$$\frac{|h_{at}(u) - h_{at}(u')|}{d(u,u')} \leq \frac{2\|h_{at}\|_0}{d_0} \leq \frac{2C_0}{d_0}.$$

So, to estimate $\mathrm{Lip}(h_{at})$ it is enough to consider pairs u, u' that belong to the same cylinder of length m_0.

Fix for a moment a, t with $|a| \leq a_0$ and $t \geq 1/a_0$. Set

$$L = \sup_{u \neq u'} \frac{|h_{at}(u) - h_{at}(u')|}{d(u,u')},$$

where the supremum is taken over all pairs $u \neq u'$ that belong to the same cylinder of length m_0. If $L < \mathrm{Lip}(h_{at})$, then the above implies

$$\mathrm{Lip}(h_{at}) \leq \frac{2C_0}{d_0} \leq \frac{2C_0}{d_0}\, t.$$

Assume that $L = \mathrm{Lip}(h_{at})$. Then there exist u, u' belonging to the same cylinder of length m_0 such that

$$\frac{3L}{4} < \frac{|h_{at}(u) - h_{at}(u')|}{d(u,u')}. \tag{8.1}$$

Fix such a pair u, u'. Let $m \geq m_0$ be an integer. For any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. By (1.1),

$$d(\sigma^j(v), \sigma^j(v')) \leq \frac{1}{c_0\, \gamma^{m-j}}\, d(u,u') \quad , \quad j = 0, 1, \ldots, m-1$$

so

$$|f_t^m(v) - f_t^m(v')| \leq \sum_{j=0}^{m-1} |f_t(\sigma^j(v)) - f_t(\sigma^j(v'))| \leq \mathrm{Const}\mathrm{Lip}(f_t)\, d(u,u') \leq \mathrm{Const}\, t\, d(u,u').$$

At the same time, by property (i), $\|f_t\|_0 \leq T''$ for some constant $T'' > 0$, so

$$|f_t^m(v) - f_t^m(v'(v))| \leq 2m\|f_t\|_0 \leq 2mT''.$$

Similarly,

$$|(P+a)\tau^m(v) - (P+a)\tau^m(v')| \leq \mathrm{Const}\, d(u,u') \leq T'',$$

assuming $T'' > 0$ is chosen sufficiently large. Thus,

$$\begin{aligned} &\left| e^{(f_t - (P+a)\tau)^m(v') - (f_t - (P+a)\tau)^m(v)} - 1 \right| \\ \leq \quad & e^{3mT''} \left| (f_t - (P+a)\tau)^m(v) - (f_t - (P+a)\tau)^m(v') \right| \leq e^{3mT''}\, \mathrm{Const}\, t\, d(u,u'). \end{aligned}$$

Using $L^m_{f_t-(P+a)\tau} h_{at} = \lambda^m_{at} h_{at}$, we obtain

$$\begin{aligned}
&\lambda^m_{at}\,|h_{at}(u) - h_{at}(u')| = \left| \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)}\, h_{at}(v) - \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v'(v))}\, h_{at}(v') \right| \\
\leq\ & \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)}\,|h_{at}(v) - h_{at}(v')| + \|h_{at}\|_0 \sum_{\sigma^m v = u} \left| e^{(f_t-(P+a)\tau)^m(v)} - e^{(f_t-(P+a)\tau)^m(v')} \right| \\
\leq\ & \frac{\mathrm{Lip}(h_{at})\, d(u,u')}{c_0\gamma^m} \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)} \\
& + C_0 \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)} \left| 1 - e^{(f_t-(P+a)\tau)^m(v') - (f_t-(P+a)\tau)^m(v)} \right| \\
\leq\ & \frac{L\, d(u,u')}{c_0\gamma^m} \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)} + C_0 e^{3mT''}\,\mathrm{Const}\, t\, d(u,u') \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)} \\
\leq\ & \left(\frac{L}{c_0\gamma^m} + C_0 e^{3mT''}\,\mathrm{Const}\, t \right) d(u,u') \sum_{\sigma^m v = u} e^{(f_t-(P+a)\tau)^m(v)}\, C_0 h_{at}(v) \\
=\ & \left(\frac{L}{c_0\gamma^m} + C_0 e^{3mT''}\,\mathrm{Const}\, t \right) d(u,u')\, C_0 \lambda^m_{at} h_{at}(u) \leq \left(\frac{L}{c_0\gamma^m} + C_0 e^{3mT''}\,\mathrm{Const}\, t \right) d(u,u')\, C_0^2 \lambda^m_{at}.
\end{aligned}$$

This, (8.1) and the choice of m_0 imply

$$\frac{3L}{4} < \frac{LC_0^2}{c_0\gamma^m} + C_0^3 e^{3mT''}\,\mathrm{Const}\, t \leq \frac{L}{2} + C_0^3 e^{3mT''}\,\mathrm{Const}\, t.$$

This is true for all $m \geq m_0$. In particular for $m = m_0$ we get

$$\frac{L}{4} < C_0^3 e^{3m_0T''}\,\mathrm{Const}\, t,$$

and so $\mathrm{Lip}(h_{at}) = L \leq \mathrm{Const}\ t$. ■

Proof of Lemma 5. (a) Let $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$ and let $m \geq 1$ be an integer. For any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. Then

$$|f^m_{at}(v) - f^m_{at}(v')| \leq \sum_{j=0}^{m-1} |f_{at}(\sigma^j(v)) - f_{at}(\sigma^j(v'))| \leq \frac{Tt}{c_0\,(\gamma - 1)}\, d(u,u') \leq C_1\, t\, D(u,u') \qquad (8.2)$$

for some constant $C_1 > 0$. Similarly,

$$|g^m_t(v) - g^m_t(v')| \leq C_1\, t\, D(u,u'). \qquad (8.3)$$

Also notice that if $D(u,u') = \mathrm{diam}(\mathcal{C}')$ for some cylinder $\mathcal{C}' = C[i_{m+1}, \ldots, i_p]$,then $v, v'(v) \in \mathcal{C}'' = C[i_0, i_1, \ldots, i_p]$ for some cylinder $\mathcal{C}''$ with $\sigma^m(\mathcal{C}'') = \mathcal{C}'$, so

$$D(v,v') \leq \mathrm{diam}(\mathcal{C}'') \leq \frac{1}{c_0\,\gamma^m}\,\mathrm{diam}(\mathcal{C}') = \frac{D(u,u')}{c_0\,\gamma^m}.$$

Using the above, $\mathrm{diam}(U_i) \le 1$, the definition of $\mathcal{M}_{atc}$, we get

$$\frac{|(\mathcal{M}^m_{atc}H)(u) - (\mathcal{M}^m_{atc}H)(u')|}{\mathcal{M}^m_{atc}H(u')} = \frac{\left|\sum_{\sigma^m v=u} e^{f^m_{at}(v)+cg^m_t(v)}\, H(v) - \sum_{\sigma^m v=u} e^{f^m_{at}(v')+cg^m_t(v')}\, H(v')\right|}{\mathcal{M}^m_{atc}H(u')}$$

$$\begin{aligned}
&\le \frac{\left|\sum_{\sigma^m v=u} e^{f^m_{at}(v)+cg^m_t(v)}\,(H(v)-H(v'))\right|}{\mathcal{M}^m_{atc}H(u')} + \frac{\sum_{\sigma^m v=u}\left|e^{f^m_{at}(v)+cg^m_t(v)} - e^{f^m_{at}(v')+cg^m_t(v')}\right|\, H(v')}{\mathcal{M}^m_{atc}H(u')}\\
&\le \frac{\sum_{\sigma^m v=u} e^{f^m_{at}(v)+cg^m_t(v)}\, E\, H(v')\, D(v,v')}{\mathcal{M}^m_{atc}H(u')}\\
&\quad + \frac{\sum_{\sigma^m v=u}\left|e^{[f^m_{at}(v)+cg^m_t(v)]-[f^m_{at}(v')+cg^m_t(v')]}-1\right|\, e^{f^m_{at}(v')+cg^m_t(v')}\, H(v')}{\mathcal{M}^m_{atc}H(u')}.
\end{aligned}$$

Using (8.2) and (8.3) and assuming $\eta_0 \le 1$, one obtains

$$|f^m_{at}(v) + cg^m_t(v)] - [f^m_{at}(v') + cg^m_t(v')| \le 2C_1 t\, D(u,u') \le 2C_1 t, \tag{8.4}$$

and therefore

$$\left|e^{[f^m_{at}(v)+cg^m_t(v)]-[f^m_{at}(v')+cg^m_t(v')]}-1\right| \le e^{2C_1 t} 2C_1 t\, D(u,u').$$

However (8.4) is not good enough to estimate the first term in the right-hand-side above. Instead we use (3.3) and (3.4) to get

$$\begin{aligned}
&|f^m_{at}(v) + cg^m_t(v)] - [f^m_{at}(v') + cg^m_t(v')|\\
\le\ & |f^m_t(v) - f^m_t(v)| + |P-a|\,|\tau^m(v) - \tau^m(v')| + |(h_{at}(v) - h_{at}(u)) - (h_{at}(v') - h_{at}(u')|\\
& + a_0|g^m_t(v) - g^m_t(v')|\\
\le\ & 2m\|f_t - f_0\|_0 + |f^m_0(v) - f^m_0(v')| + \mathrm{Const}\, D(u,u') + 4C_0 + 2ma_0\|g_t - g\|_0\\
\le\ & \mathrm{Const}\, D(u,u') + C_2 m a_0 \le C_2 + C_2 m\, a_0 \qquad (8.5)
\end{aligned}$$

for some constant $C_2 > 0$. We will now assume that $a_0 > 0$ is chosen so small that

$$e^{C_2 a_0} < \gamma/\hat{\gamma}. \tag{8.6}$$

Hence

$$\begin{aligned}
&\frac{|(\mathcal{M}^m_{atc}H)(u) - (\mathcal{M}^m_{atc}H)(u')|}{\mathcal{M}^m_{atc}H(u')}\\
\le\ & \frac{E\, D(u,u')}{c_0\gamma^m}\, \frac{\sum_{\sigma^m v=u} e^{[f^m_{at}(v)+cg^m_t(v)]-[f^m_{at}(v')+cg^m_t(v')]} e^{f^m_{at}(v')+cg^m_t(v')}\, H(v')}{\mathcal{M}^m_{atc}H(u')}\\
& + e^{2C_1 t}\, \frac{\sum_{\sigma^m v=u} 2C_1 t\, e^{f^m_{at}(v'(v))}\, H(v'(v))}{\mathcal{M}^m_{atc}H(u')}\\
\le\ & e^{C_2}\, e^{C_2 m a_0} \frac{E\, D(u,u')}{c_0\gamma^m} + 2C_1 t e^{2C_1 t}\, D(u,u') \le A_0\left[\frac{E}{\hat{\gamma}^m} + e^{A_0 t}\, t\right] D(u,u'),
\end{aligned}$$

for some constant $A_0 > 0$ independent of a, c, t, m and E.

(b) Let $m \geq 1$ be an integer and $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$. Using the notation $v' = v'(v)$ and the constant $C_2 > 0$ from part (a) above, where $\sigma^m v = u$ and $\sigma^m v' = u'$, and some of the estimates from the proof of part (a), we get

$$\begin{aligned}
& |\mathcal{L}^m_{abtz} h(u) - \mathcal{L}^m_{abtz} h(u')| \\
= & \left| \sum_{\sigma^m v = u} \left(e^{f^m_{at}(v) + c g^m_t(v) - \mathbf{i} b \tau^m(v) + \mathbf{i} w g^m_t(v)} \, h(v) - e^{f^m_{at}(v') + c g^m_t(v') - \mathbf{i} b \tau^m(v') + \mathbf{i} w g^m_t(v')} \, h(v') \right) \right| \\
\leq & \left| \sum_{\sigma^m v = u} e^{f^m_{at}(v) + c g^m_t(v) - \mathbf{i} b \tau^m(v) + \mathbf{i} w g^m_t(v)} \, [h(v) - h(v')] \right| \\
& + \sum_{\sigma^m v = u} \left| e^{f^m_{at}(v) + c g^m_t(v)} - e^{f^m_{at}(v') + c g^m_t(v')} \right| \, |h(v')| \\
& + \sum_{\sigma^m v = u} \left| e^{-\mathbf{i} b \tau^m(v) + \mathbf{i} w g^m_t(v)} - e^{-\mathbf{i} b \tau^m(v') - \mathbf{i} w g^m_t(v')} \right| \, e^{f^m_{at}(v') + c g^m_t(v')} |h(v')| \\
\leq & \sum_{\sigma^m v = u} e^{f^m_{at}(v) + c g^m_t(v)} \, |h(v) - h(v')| \\
+ & \sum_{\sigma^m v = u} \left| e^{[f^m_{at}(v) + c g^m_t(v)] - [f^m_{at}(v') + c g^m_t(v')]} - 1 \right| \, e^{f^m_{at}(v') + c g^m_t(v')} \, |h(v')| \\
& \sum_{\sigma^m v = u} \big(|b| \, |\tau^m(v) - \tau^m(v')| + |w| \, |g^m_t(v) - g^m_t(v')| \big) \, e^{f^m_{at}(v') + c g^m_t(v')} |h(v')|
\end{aligned}$$

Using the constants $C_1, C_2 > 0$ from the proof of part (a), (8.5) and (8.6) we get

$$\begin{aligned}
\sum_{\sigma^m v = u} e^{f^m_{at}(v) + c g^m_t(v)} \, |h(v) - h(v')| & \leq e^{C_2} \, e^{C_2 m a_0} \frac{E \, D(u, u')}{c_0 \gamma^m} \sum_{\sigma^m v = u} e^{f^m_{at}(v') + c g^m_t(v')} \, H(v') \\
& \leq \frac{e^{C_2} E}{c_0 \hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u') \, D(u, u').
\end{aligned}$$

This, (8.3) and (8.5) imply

$$\begin{aligned}
& |\mathcal{L}^m_{abtz} h(u) - \mathcal{L}^m_{abtz} h(u')| \\
\leq & \frac{e^{C_2} E}{c_0 \hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u') \, D(u, u') + e^{2C_1 t} 2 C_1 t \, D(u, u') \, (\mathcal{M}^m_{atc} |h|)(u') + (\mathrm{Const}\, |b| + |w| C_1 \, t) \, D(u, u')
\end{aligned}$$

Thus, taking the constant $A_0 > 0$ sufficiently large we get

$$|(\mathcal{L}^N_{abtz} h)(u) - (\mathcal{L}^N_{abtz} h)(u')| \leq A_0 \left(\frac{E}{\hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u') + (|b| + e^{A_0 t} t + t|w|)(\mathcal{M}^m_{atc} |h|)(u') \right) D(u, u'),$$

which proves the assertion. ■

As in [4] and [20] we need the following lemma whose proof is omitted here, since it is very similar to the proof of Lemma 5 given above.

Lemma 17. *Let $\beta \in (0,\alpha)$. There exists a constants $A_0' > 0$ such that for all $a,b,c,t,w \in \mathbb{R}$ with $|a|,|c|,1/|b|,1/t \leq a_0$ such that (4.1) hold, and all positive integers m and all $h \in C^\beta(U)$ we have*

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_0' \left[\frac{|h|_\beta}{\hat{\gamma}^{m\beta}} + |b|\,(\mathcal{M}_{atc}^m |h|)(u')\right] (d(u,u'))^\beta$$

for all $u,u' \in U_i$.

We will derive Theorem 5(b) from Theorem 5(a), proved in Sect. 4, and Lemma 17 above.

Proof of Theorem 5(b). We essentially repeat the proofs of Corollaries 2 and 3 in [4] (cf. also Sect. 3 in [19]).

Let $\epsilon > 0$, $B > 0$ and $\beta \in (0,\alpha)$. Take $\hat{\rho} \in (0,1)$, $a_0 > 0$, $b_0 > 0$, $A_0 > 0$ and N as in Theorem 2(a). We will assume that $\hat{\rho} \geq \frac{1}{\gamma_0}$. Let $a,b,c,w \in \mathbb{R}$ be such that $|a|,|c| \leq a_0$ and $|b| \geq b_0$. Let $t > 0$ be such that $1/t^{\alpha-\beta} \leq a_0$. Assume that (4.1) hold and set $z = c + \mathbf{i}w$.

First, as in [4] (see also Sect. 3 in [19]) one derives from Theorem 5(a) and Lemma 17 (approximating functions $h \in C^\beta(\widehat{U})$ by Lipschitz functions as in Sect. 3) that there exist constants $C_3 > 0$ and $\rho_1 \in (0,1)$ such that

$$\|\mathcal{L}_{abtz}^n h\|_{\beta,b} \leq C_3 |b|^\epsilon \rho_1^n \quad , \quad n \geq 0, \tag{8.7}$$

for all $h \in C^\beta(\widehat{U})$.

Next, given $h \in C^\beta(\widehat{U})$, we have

$$\mathcal{L}_{abtz}^n (h/h_{at}) = \frac{1}{\lambda_{at}^n\, h_{at}}\, L_{f_t-(P+a+\mathbf{i}b)\tau+zg_t} h,$$

so by (8.7) we get

$$\begin{aligned}
\|L_{f_t-(P+a+\mathbf{i}b)\tau+zg_t}^n h\|_{\beta,b} &\leq \lambda_{at}^n \|h_{at}\, \mathcal{L}_{abtz}^n (h/h_{at})\|_{\beta,b} \\
&\leq \mathrm{Const}(\lambda_{at}\rho_1)^n |b|^\epsilon\, \|h/h_{at}\|_{\beta,b} \leq \mathrm{Const}\,\rho_2^n\, |b|^\epsilon\, \|h\|_{\beta,b}\ ,
\end{aligned}$$

where $\lambda_{at}\rho_1 \leq e^{2C_0 a_0}\rho_2 = \rho_2 < 1$, provided $a_0 > 0$ is small enough.

We will now approximate $L_{f-(P+a+\mathbf{i}b)\tau+zg}$ by $L_{f_t-(P+a+\mathbf{i}b)\tau+g_t}$ in two steps. First, using the above it follows that

$$\begin{aligned}
\|L_{f-(P+a+\mathbf{i}b)\tau+cg+\mathbf{i}wg_t}^n h\|_{\beta,b} &= \left\| L_{f_t-(P+a+\mathbf{i}b)\tau+zg_t}^n \left(e^{(f^n-f_t^n)+c(g^n-g_t^n)} h \right) \right\|_{\beta,b} \\
&\leq \mathrm{Const}\,\rho_2^n\, |b|^\epsilon \left\| e^{(f^n-f_t^n)+c(g^n-g_t^n)} h \right\|_{\beta,b}.
\end{aligned}$$

Choosing the constant $C_4 > 0$ appropriately, $\|f - f_t\|_0 \leq C_4\, a_0$ and $|f - f_t|_\beta \leq C_4/t^{\alpha-\beta} \leq C_4 a_0$, so $\|f^n - f_t^n\|_0 \leq n\, \|f - f_t|_0 \leq C_4 n a_0$, and similarly $|f^n - f_t^n|_\beta \leq C_4 n a_0$. Similar estimates hold for $g^n - g_t^n$. Thus,

$$\|e^{(f^n-f_t^n)+c(g^n-g_t^n)} h\|_0 \leq e^{C_4 n a_0} \|h\|_0$$

and

$$\begin{aligned}
|e^{(f^n-f_t^n)+c(g^n-g_t^n)} h|_\beta &\leq \|e^{(f^n-f_t^n)+c(g^n-g_t^n)}\|_0\, |h|_\beta + |e^{(f^n-f_t^n)+c(g^n-g_t^n)}|_\beta\, \|h\|_\infty \\
&\leq e^{C_4 n a_0} |h|_\beta + e^{C_4 n a_0}\, |(f^n - f_t^n) + c(g^n - g_t^n)|_\beta\, \|h\|_\infty \\
&\leq C_5'\, n\, e^{C_4 n a_0}\, \|h\|_\beta.
\end{aligned}$$

Combining this with the previous estimate gives

$$\|e^{(f^n - f_t^n) + c(g^n - g_t^n)} h\|_{\beta,b} \leq C_5'' \, n \, e^{C_4 n a_0} \, \|h\|_\beta,$$

so

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau + cg + \mathbf{i}wg_t} h\|_{\beta,b} \leq C_5 \, \rho_2^n \, |b|^\epsilon \, n \, e^{C_4 n a_0} \, \|h\|_{\beta,b}.$$

Taking $a_0 > 0$ sufficiently small, we may assume that $\rho_2 \, e^{C_4 a_0} < 1$. Now take an arbitrary ρ_3 with $\rho_2 \, e^{C_4 a_0} < \rho_3 < 1$. Then we can take the constant $C_6 > 0$ so large that $n \, \rho_2^n \, e^{C_4 n a_0} \leq C_6 \rho_3^n$ for all integers $n \geq 1$. This gives

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau + cg + \mathbf{i}wg_t} h\|_{\beta,b} \leq C_6 \, \rho_3^n \, |b|^\epsilon \, \|h\|_{\beta,b} \quad , \quad n \geq 0. \tag{8.8}$$

Using the latter we can write

$$\begin{aligned} \|L^n_{f-(P+a+\mathbf{i}b)\tau + zg} h\|_{\beta,b} &= \left\| L^n_{f-(P+a+\mathbf{i}b)\tau + cg + \mathbf{i}wg_t} \left(e^{\mathbf{i}w(g^n - g_t^n)} h \right) \right\|_{\beta,b} \\ &\leq C_6 \, \rho_3^n \, |b|^\epsilon \, \left\| e^{\mathbf{i}w(g^n - g_t^n)} h \right\|_{\beta,b}. \end{aligned}$$

However, $\|e^{\mathbf{i}w(g^n - g_t^n)} h\|_0 = \|h\|_0$, $|g - g_t|_\beta \leq C_4 / t^{\alpha - \beta} \leq C_4 a_0 \leq 1$ (assuming $a_0 > 0$ is sufficiently small), and by (4.1), $|w| \leq B|b|^\mu \leq B|b|$, so

$$\begin{aligned} |e^{\mathbf{i}w(g^n - g_t^n)} h|_\beta &\leq \|e^{\mathbf{i}w(g^n - g_t^n)}\|_0 \, |h|_\beta + |e^{\mathbf{i}w(g^n - g_t^n)}|_\beta \, \|h\|_\infty \\ &\leq |h|_\beta + |w| \, |g^n - g_t^n|_\beta \, \|h\|_\infty \\ &\leq \|h\|_\beta + Bn|b| \|h\|_\infty. \end{aligned}$$

Thus,

$$\|e^{\mathbf{i}w(g^n - g_t^n)} h\|_{\beta,b} = \|e^{\mathbf{i}w(g^n - g_t^n)} h\|_0 + \frac{1}{|b|} |e^{\mathbf{i}w(g^n - g_t^n)} h|_\beta \leq 2Bn \|h\|_{\beta,b},$$

and therefore

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau + zg} h\|_{\beta,b} \leq C_7 \, \rho_3^n \, |b|^\epsilon \, n \, \|h\|_{\beta,b}.$$

Now taking an arbitrary ρ with $\rho_3 < \rho < 1$ and taking the constant $C_8 > C_7$ sufficiently large, we get

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau + zg} h\|_{\beta,b} \leq C_8 \, \rho^n \, |b|^\epsilon \, \|h\|_{\beta,b}$$

for all integers $n \geq 0$. ■

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