Residue Calculus and Effective Nullstellensatz

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1. Introduction.

Let $p_1, \ldots, p_M$ be polynomials in n variables with coefficients in an integral domain $\mathbf{A}$, and respective degrees $D_1 \geq D_2 \geq \ldots \geq D_M$, with no common zeros in an integral closure of the quotient field $\mathbf{K}$ of $\mathbf{A}$. It follows from effective versions of the Hilbert Nullstellensatz ([Br],[CGH],[Ko]) that one can find an element $r_0 \in \mathbf{A} \setminus \{0\}$ and polynomials $q_j \in \mathbf{A}[x]$ such that

$$r_0 = \sum_{j=1}^{M} q_j p_j \tag{1.1}$$

with a priori estimates on the degrees

$$\max_j \deg(q_j) \leq (3/2)^{\iota} D_1 \cdots D_\mu \,, \tag{1.2}$$

where $\mu = \min(n, M)$ and $\iota = \#\{j :\ 1 \leq j < \mu - 1,\, D_j = 2\}$.

When $\mathbf{A} = \mathbf{Z}$, the Arithmetic Bézout Theorem ([Ph2], [BGS, Theorem 5.4.4]) shows that the Faltings height H of the intersection of the arithmetic cycles X_j in $\mathbf{P}^n(\mathbf{Z})$ corresponding to the polynomials ${}^h p_j$ (homogeneous versions of the original polynomials) has the bound

$$H \leq c_n \mathcal{H} \left(\prod_{j=1}^{\nu} D_j \right) \left(\frac{1}{\mathcal{H}} + \sum_{j=1}^{\nu} \frac{1}{D_j} \right) ,$$

for some constant c_n, where $\mathcal{H} := \max_j(H(X_j))$, $\nu := \min(n+1, M)$. This implies that one can solve (1.1) with an r_0 such that

$$\log |r_0| \leq \tilde{c}_n h \left(\prod_{j=1}^{\nu} D_j \right) \left(\frac{1}{h} + \sum_{j=1}^{\nu} \frac{1}{D_j} \right) , \tag{1.3}$$

where h is the maximal size in the sense of Mahler of the p_j. There does not seem to exist so far an Arithmetic Division Theory that could provide good estimates for the Faltings heights of the cycles corresponding to ${}^h q_j$ or for the maximal Mahler size of the q_j. Nevertheless, using analytic methods based on the existence of integral representation formulas in Complex Analysis and multidimensional residues in $\mathbf{C}^n$, one can show ([BY1], [BY2], [El], [BGVY, Section 5]) that the system (1.1) can be solved with the estimates

$$\begin{cases} \max_j \deg(q_j) \leq n(2n+1)(3/2)^{\iota} \left(\prod_{j=1}^{\mu} D_j \right) \\ \max_j h(q_j) \leq \kappa(n) D_1^4 \left(\prod_{j=1}^{\mu} D_j \right)^8 (h + \log M + D_1 \log D_1) \end{cases} \tag{1.4}$$

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The problem that remained was to obtain size estimates similar to (1.3)-(1.4), in the case where $\mathbf{A}$ was an integral domain equipped with a size and whose quotient field is of positive characteristic. A typical example would be $\mathbf{A} = \mathbf{F}_p[\tau_1, \ldots, \tau_q]$, with size $\deg_\tau$. In order to solve this problem, as well as improve the exponents in (1.4), which we do in this paper (see Theorem 6.1 and Corollaries 6.1, 6.2, below), we had to get rid of all complex analytic tools involved in [BY1]. The way we proceed is to keep the structure of our original work, while eliminating all the *analytic artifacts.*

The main thing we do in Sections 2 and 3, which are independent of the Nullstellensatz, is to develop the algebraic theory of residues (as described in [L]) into a computational tool (see also [An], [AL], [H].) In fact, one can certainly extract from our work an algorithm to compute total sums of residues with respect to a dominant polynomial map, avoiding the search for Gröbner bases. It will become evident here that the key tool (from the computational point of view) is the Transformation Law and its variants, Propositions 2.2 and 2.3. For example, the algebraic substitute for Cauchy's formula, that is the Kronecker interpolation formula, is an immediate consequence of these properties of residues. In fact, already in the analytic context of [BY2], the key point of the proof was the use of the Cauchy-Weil representation formula (see also [BoH2]). Another consequence of the Transformation Law is that the analytic and algebraic definitions of residues coincide when $\mathbf{A}$ is the local ring of holomorphic functions $\mathcal{O}_n$ or any polynomial ring $\mathbf{F}[x_1, \ldots, x_n]$, for any subfield $\mathbf{F}$ of $\mathbf{C}$ [Bo]. It is interesting to point out that even for $\mathbf{A} = \mathbf{Z}$ we had already been compelled to develop the classical theory of residues in really novel ways in our work, see [BGVY] and references therein.

Analytic techniques have frequently inspired some results which are algebraic in nature. Such is the case for the Lipman-Teissier theorem ([LT], [LS], [HH]) about integral closures of ideals in regular local rings, which was originally proved in an analytic context by Briançon and Skoda in [BS] using Hörmander's estimates for the solution of the $\overline{\partial}$ equation. Then, it is not really a surprise, that our substitute for the use of integral representation formulas happens to be precisely Lipman-Teissier's result (as we will see in Section 3.) In fact, such a result seems to be closely connected to the vanishing theorems we prove in Section 3 for total sums of residues with respect to a proper polynomial map $P = (P_1, \ldots, P_n)$ from $\mathbf{K}^n$ to $\mathbf{K}^n$, with Lojasiewicz exponent δ, provided the quotient $\max(\deg(P_j))/\delta$ is close to 1. It is quite probable that such vanishing theorems will have interesting geometric consequences, as it is the case with the classical Jacobi vanishing theorem ([J], [G], [Ku2].) Note that also in [BY1], [BY2], this kind of vanishing theorem was crucial.

If Analysis remains present in this paper, it is in the use of the Lojasiewicz type inequality of [JKS] and its relation to properness in Section 4. For convenience of the reader, we have separated all the very technical estimates of sizes necessary to complete the proof of the effective Nullstellensatz into Section 5, which may be safely skipped on a first reading. In view of the length of this manuscript, we suggest the reader starts by glancing through Section 6 to get a global view of the of the proof of the main result and then appreciate the need for the different technical components. In fact, the proof can be summarized as follows. First we construct a convenient family of polynomials $P_1, \ldots, P_n$ belonging to the ideal generated by the original polynomials p_j and such that the collection of common zeros of the P_k is a finite set and we use them to write $1 = \sum p_j q_j$ using a Lagrange-type interpolation formula. It is at this point that we appeal to the residue theory developped in Sections 2 and 3. The Generalized Transformation Law for residues (proved in Section 2) and the general form of the Jacobi Vanishing Theorem of Sections 3 and 4 are then used to obtain good estimates of the degrees of the polynomials q_j. As we said, the estimates of the size of the coefficients appear in Section 5. As a short version of this paper has appeared recently in [BY3], it may also provide the reader with an alternate introduction to the subject and proofs of the present manuscript.

We hope that the tools we introduce here may help to solve some of the other open problems in this field, for example [Am], and the fact that there is a true residue calculus in Algebra, which may even extend to non-commutative bi-algebras [L, Section 1], suggests that effectivity results of this type could possibly be applied to more complicated algebraic situations, like the Weyl algebra (see [Gr].)

Bounds for the Nullstellensatz are related to problems of complexity, we refer to [SS] and references therein. A novel approach both to complexity and solvability of the Bézout equation is the use of the concept of straight-line programming, which was introduced in this context by Giusti, Heintz, and their collaborators, see for instance, [FGS], [GHMMP], or [GHHMMP]. Using these ideas, Krick and Pardo ([KP1], [KP2]) solved (1.1) when $\mathbf{A} = \mathbf{Z}$, finding polynomials q_j of degree at most $\kappa_n D_1^n$ and logarithmic size at most $\tilde{\kappa}_n D_1^{O(n)}(h + \log M)$. In [GHMMP, Theorem 5], an extension of this result to the case where the ring is substituted by a perfect field $\mathbf{K}$ is also stated.

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2. Residue symbols and transformation laws.

Let $\mathbf{R}$ be a commutative Noetherian ring. We recall from ([L, p. 44]) that a sequence $P = (P_1, \ldots, P_n)$ in $\mathbf{R}$ is *quasiregular* if and only if the Koszul complex over $\mathbf{R}$ determined by P is exact except possibly in degree 0. This is equivalent to say the following: let I denote the ideal generated by the P_j in $\mathbf{R}$, then whenever there is a relation of the form

$$\sum_{\substack{k \in \mathbf{N}^n \\ |k| = p}} a_k P^k \in I^{p+1},\ a_k \in \mathbf{R},\ p \in \mathbf{N},$$

then all a_k are in I (here $|k| := k_1 + \cdots + k_n$, $P^k = P_1^{k_1} \cdots P_n^{k_n}$). If the sequence P is *regular*, the Koszul complex is exact at all degrees. Note that the notion of regularity depends on the ordering of the sequence, while quasiregularity does not.

The following remark will be useful for us later.

Remark 2.1. Let $(h_1, \ldots, h_n)$ be a quasiregular sequence in a commutative Noetherian ring $\mathbf{R}$ and M an $n \times n$ matrix with coefficients in $\mathbf{R}$, then the sequence $(u, h - Mu) := (u_1, \ldots, u_n, h_1(x) - (Mu)_1, \ldots, h_n(x) - (Mu)_n)$ is a quasiregular sequence in $\mathbf{R}[u_1, \ldots, u_n]$. In order to see that, let us denote by I the ideal generated by the h_j and J is the ideal generated by the u_j and the $h_j-(Mu)_j$ in $\mathbf{R}[u]$. Let $p\in\mathbf{N}$ and a_{k_1,k_2} in $\mathbf{R}[u]$, $k_1,\ k_2\in\mathbf{N}^n$, $|k_1|+|k_2|=p$ such that

$$\sum_{|k_1|+|k_2|=p} a_{k_1,k_2}(u)u^{k_1}(h-Mu)^{k_2}\in J^{p+1} \tag{2.1}$$

Setting $u=0$ in (2.1) and using the fact that h is a quasiregular sequence, one gets that all $a_{0,k_2}(0)$ lie in ideal generated by $h_1,\dots,h_n$, which implies that $a_{0,k_2}(0)\in I$, and thus $a_{0,k_2}(u)\in J$, so that

$$\sum_{\substack{|k_1|+|k_2|=p\\ k_1\neq 0}} a_{k_1,k_2}(u)u^{k_1}(h-Mu)^{k_2}\in J^{p+1}. \tag{2.2}$$

Set $u_2=\cdots=u_n=0$, and denote $k_{1j}=(j,0,\dots,0)$ then (2.2) implies that

$$\sum_{j=1}^{p}\sum_{|k_2|=p-j} a_{k_{1j},k_2}(u_1,0)u_1^j(h-M(u_1,0))^{k_2}\in(I,u_1)^{p+1}\,.$$

Decomposing the two sides as polynomials in u_1 we see that

$$\sum_{|k_2|=p-1} a_{k_{1,1},k_2}(0)h^{k_2}\in I^p$$

so that $a_{k_{1,1},k_2}(0)\in I$ and $a_{k_{1,1},k_2}\in J$. We can repeat this reasoning to see that all a_{k_1,k_2} with $|k_1|=1$ belong to J. This procedure can be continued in an obvious way and the assertion in Remark 2.1 follows.

Remark 2.2. Note that when $\mathbf{R}=\mathbf{K}[x_1,\dots,x_n]$, $\mathbf{K}$ a field of arbitrary characteristic, then, if $P_1,\dots,P_n$ is a quasiregular sequence in $\mathbf{R}$ such that $(P_1,\dots,P_n)$ is a proper ideal I, it follows that the P_j are algebraically independent over $\mathbf{K}$. In fact, assume one has a non trivial relation

$$\sum_{|k|\le M} a_kP_1^{k_1}\dots P_n^{k_n}\equiv 0,\ a_k\in\mathbf{K}\,;$$

rewrite it as

$$\sum_{|k|=q_0} a_kP^k\equiv-\sum_{|k|>q_0} a_kP^k\in I^{q_0+1}\,.$$

From the definition of quasiregularity, all the a_k, $|k|=q_0$ are in the ideal, since this ideal is proper, they must be zero.

Suppose now that $\mathbf{R}$ is a Noetherian $\mathbf{K}$-algebra, where $\mathbf{K}$ is a commutative field. Given arbitrary $x_1,\dots,x_n$ in $\mathbf{R}$ and $h_1,\dots,h_n$, also in $\mathbf{R}$, such that $(h_1,\dots,h_n)$ is a quasiregular sequence (generating an ideal $(h)=I$ in $\mathbf{R}$) such that $\mathbf{P}:=\mathbf{R}/(h)\mathbf{R}$ is a finite dimension $\mathbf{K}$-vector space, we follow Lipman [L, Chapter 3] (see also [Hu]) to define the residue symbols

$$\operatorname{Res}\begin{bmatrix}Qdx_1\wedge\cdots\wedge dx_n\\ h_1^{k_1+1},\dots,h_n^{k_n+1}\end{bmatrix}=\operatorname{Res}\begin{bmatrix}Qdx\\ h^{k+\underline{1}}\end{bmatrix},\quad Q\in\mathbf{R},k\in\mathbf{N}^n.$$

Let $\mathbf{E} = Hom_{\mathbf{K}}(\mathbf{P}, \mathbf{P})$. Let σ be any $\mathbf{K}$-linear map from $\mathbf{P}$ to $\mathbf{R}$ such that $\pi \circ \sigma = Id_{\mathbf{P}}$, where π is the quotient map from $\mathbf{R}$ to $\mathbf{P}$. For instance, if $\mathbf{R} = \mathbf{K}[x_1, \ldots, x_n]$, one can choose $\sigma(r)$ to be the remainder in the division algorithm with respect to a Gröbner basis of any representative of r modulo I. From the quasiregularity, it follows that any element Q in $\mathbf{R}$ has a formal expansion

$$Q = \sum_{k \in \mathbf{N}^n} \sigma(q_k) h^k, \tag{2.3}$$

where the $q_k \in \mathbf{P}$ are uniquely determined (depending on the choice of σ) and the series in (2.3) converges in the I-adic completion $\widehat{\mathbf{R}}$ of $\mathbf{R}$, with the topology associated to the pseudodistance

$$d(Q_1, Q_2) = exp(-v_I(Q_1 - Q_2)),$$

where

$$v_I(Q) := sup(\{p \in \mathbf{N},\, Q \in I^p\}).$$

One can define linear operator $Q \mapsto Q^\sharp$ from $\mathbf{R}$ into $\mathbf{E}[[h]]$ as follows: given any r in $\mathbf{P}$, one can write in $\widehat{\mathbf{R}}$

$$Q \cdot \sigma(r) = \sum_{k \in \mathbf{N}^n} \sigma(r_k(Q, r)) h^k, \tag{2.4}$$

where the $r_k(Q, r) \in \mathbf{P}$ are uniquely determined. Since σ is $\mathbf{K}$-linear, each map

$$q_k^\sharp : r \mapsto r_k(Q, r)$$

defines an element in $\mathbf{E}$. We now define $Q^\sharp$ in $\mathbf{E}[[h]]$ as the formal series of operators

$$Q^\sharp := \sum_{k \in \mathbf{N}^n} q_k^\sharp h^k.$$

One can expand, as a product of formal series of operators. the product

$$Q^\sharp \det \left[\frac{\partial x_i^\sharp}{\partial h_j} \right] = \sum_{k \in \mathbf{N}^n} \delta_k h^k, \tag{2.5}$$

where the determinant in (2.5) is computed using the standard product rule, keeping track of the noncommutativity of the multiplication in $\mathbf{E}$ (see [L, 1.10.3, p.21]). It is clear that the previous constructions depend on the choice of the section σ. Nevertheless, it is important to remark that the traces (in fact, the characteristic polynomials) of the operators δ_k do not depend on the choice of the section σ. As done by Lipman, we define the residual symbols by

$$\operatorname{Res} \left[\begin{matrix} Q dx \\ h^{k+\underline{1}} \end{matrix} \right] := \operatorname{Tr}(\delta_k) \in \mathbf{K}. \tag{2.6}$$

Note that if $Q \in I$ then the expansion (2.5) of $Q^\sharp$ does not contain a term with index 0, and so the residue symbol (2.6) is zero for $k = 0$. Another important and immediate consequence of the definition of the residual symbol is the following lemma, similar to the Fubini theorem for integrals (see [Ho].)

Lemma 2.1. *Let $\mathbf{R} := \mathbf{K}[x_1, \dots, x_L, y_1, \dots, y_K]$, where $L, K \in \mathbf{N}$ and $\mathbf{K}$ is a commutative field. Let $P_1(x), \dots, P_L(x)$ be L polynomials defining a quasiregular sequence in $\mathbf{K}[x]$ and $Q_1[y], \dots, Q_K[y]$, K polynomials defining a quasiregular sequence in $\mathbf{K}[y]$. Then, for any multiindices $l \in \mathbf{N}^L$, $k \in \mathbf{N}^K$, one has*

$$\mathrm{Res}\begin{bmatrix} x^l y^k dx \wedge dy \\ P_1(x), \dots, P_L(x), Q_1(y), \dots, Q_K(y) \end{bmatrix} = \mathrm{Res}\begin{bmatrix} x^l dx \\ P_1, \dots, P_L \end{bmatrix} \mathrm{Res}\begin{bmatrix} y^k dy \\ Q_1, \dots, Q_K \end{bmatrix},$$

with the standard notations $x^l := x_1^{l_1} \cdots x_L^{l_L}$, $y^K =: y_1^{k_1} \cdots y_K^{k_K}$, $dx := dx_1 \wedge \cdots \wedge dx_L$, $dy := dy_1 \wedge \cdots \wedge dy_K$.

As a simple example, let us consider in the algebra $\mathbf{K}(y_1, \dots, y_n)[x_1, \dots, x_n]$ the quasiregular sequence $(x_1 - y_1, \dots, x_n - y_n)$. Then, one can easily verify from the definitions and the elementary properties mentioned above that for $Q \in \mathbf{K}[x_1, \dots, x_n]$ one has the identity

$$Q(y) = \mathrm{Res}\begin{bmatrix} Q dx_1 \wedge \cdots \wedge dx_n \\ x_1 - y_1, \dots, x_n - y_n \end{bmatrix} \tag{2.7}$$

which is the algebraic version of Cauchy's formula. In fact, one just uses that $Q(x) \equiv Q(y) \bmod (x_1 - y_1 \dots, x_n - y_n)$ and the invariance of the residue under translation in the variables x.

Another important formula, when $\mathbf{R} = \mathbf{K}[x_1, \dots, x_n]$ and $\mathbf{K}$ is infinite, is the Jacobi vanishing theorem [KK, Theorem 4.8], that is, if $P_1, \dots, P_n \in \mathbf{R}$ have no common zeros at infinity in the projective space $\mathbf{P}^n_{\overline{\mathbf{K}}}$, where $\overline{\mathbf{K}}$ is an integral closure of $\mathbf{K}$, then

$$\mathrm{Res}\begin{bmatrix} Q dx_1 \wedge \cdots \wedge dx_n \\ P_1, \dots, P_n \end{bmatrix} = 0 \tag{2.8}$$

for any $Q \in \mathbf{R}$ such that

$$\deg(Q) \leq \sum_{j=1}^{n} \deg P_j - n - 1 \tag{2.9}$$

In the case of a single variable, for polynomial of degree D $P(x) = a_0 x^D + \cdots + a_D$, we have that

$$\{\overline{1}, \overline{x}, \dots, \overline{x}^{D-1}\}$$

is a basis for the quotient space $\mathbf{P}$. If $Q \in \mathbf{R} = \mathbf{K}[x]$, let

$$\overline{Q} = \sum_{k=0}^{D-1} \alpha_k \overline{x}^k.$$

Then, as shown in [Ho, Example 2, p.519], one has

$$\mathrm{Res}\begin{bmatrix} Q dx \\ P \end{bmatrix} = \frac{\alpha_{D-1}}{a_0}. \tag{2.10}$$

One of the main properties of the residue symbols is the Transformation Law [L, Corollary 2.8, p.40], namely,

Proposition 2.1. *Let $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$ be two quasiregular sequences in $\mathbf{R}$, such that $g = Af$, where A is a $n \times n$ matrix with coefficients in $\mathbf{R}$, and such that the quotients $\mathbf{R}/(f)$ and $\mathbf{R}/(g)$, are finite dimensional $\mathbf{K}$-vector spaces. Then for any $x_1, \dots, x_n, Q \in \mathbf{R}$,*

$$\operatorname{Res}\begin{bmatrix} Q dx_1 \wedge \cdots \wedge dx_n \\ f_1, \dots, f_n \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q\Delta dx_1 \wedge \cdots \wedge dx_n \\ g_1, \dots, g_n \end{bmatrix}, \tag{2.11}$$

where Δ is the determinant of the matrix A.

Later on, we will use the following variant of this proposition.

Proposition 2.2. *Let $f = (f_0, f_1, \dots, f_n)$ be a regular sequence in some order in $\mathbf{R}$ and let $g = (g_1, \dots, g_n)$ be such that the sequence $f_0, g_1, \dots, g_n$ is quasiregular and the quotients $\mathbf{R}/(f)$ and $\mathbf{R}/(f_0, g)$ are finite dimensional. Assume that there are nonnegative integers $s_1, \dots, s_n$ and an $n \times n$ matrix $A = [a_{jl}]$ of elements in $\mathbf{R}$ such that*

$$f_0^{s_j} g_j = \sum_{l=1}^{n} a_{jl} f_l \quad j = 1, \dots, n. \tag{2.12}$$

Then, for any $k_0 \in \mathbf{N}$ and any $x_0, \dots, x_n, Q \in \mathbf{R}$, one has

$$\operatorname{Res}\begin{bmatrix} Q dx_0 \wedge \cdots \wedge dx_n \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q\Delta dx_0 \wedge \cdots \wedge dx_n \\ f_0^{k_0+1+|s|}, g_1, \dots, g_n \end{bmatrix} \tag{2.13}$$

where $|s| = s_1 + \cdots + s_n$ and Δ is the determinant of the matrix A.

Proof. Let N be an integer strictly larger than $|s| + k_0$. It follows that the sequence $f_0^N, f_1, \dots, f_n$ is also quasiregular. Hence, from the relations (2.12) we conclude that there is an $n \times (n+1)$ matrix $\widetilde{A}$ with entries in $\mathbf{R}$ such that

$$g_j = \tilde{a}_{j0} f_0^N + \sum_{l=1}^{n} \tilde{a}_{jl} f_l = \tilde{a}_{j0} f_0^{N-k_0-1} f_0^{k_0+1} + \sum_{l=1}^{n} \tilde{a}_{jl} f_l. \tag{2.14}$$

Let A' be the $n \times n$ matrix obtained from $\widetilde{A}$ by deleting the first column. Using the Transformation Law (2.11) for the sequences $f_0^{k_0+1}, f_1, \dots, f_n$ and $f_0^{k_0+1}, g_1, \dots, g_n$ we obtain for any Q, x

$$\operatorname{Res}\begin{bmatrix} Q dx \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q \det(A') dx \\ f_0^{k_0+1}, g_1, \dots, g_n \end{bmatrix}.$$

A second application of the Transformation Law yields

$$\operatorname{Res}\begin{bmatrix} Q dx \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q \det(A'') dx \\ f_0^{k_0+1+|s|}, g_1, \dots, g_n \end{bmatrix}, \tag{2.15}$$

where A'' is obtained from A' by multiplying the jth-line by $f_0^{s_j}$.

In order to finish the proof we need to show that the difference $\det(A'') - \Delta$ is in the ideal I' generated by $f_0^{k_0+1+|s|}, g_1, \ldots, g_n$. If that were the case, then, as pointed out following (2.6), the corresponding residue symbol would be zero. This fact will clearly imply the identity (2.13).

Note that the sequence $f_0^N, f_1, \ldots, f_n$ is regular for some convenient order, as follows from the original hypotheses on $f_0, \ldots, f_n$. Moreover, from (2.12) and (2.14) we obtain the relations

$$\Big(\sum_{l=1}^{n}(a_{jl} - f_0^{s_j}\tilde{a}_{jl})f_l\Big) - \tilde{a}_{j0}f_0^{N+s_j} = 0 \quad j = 1, \ldots, n.$$

Since the sequence $f_0^N, f_1, \ldots, f_n$ is regular, the module of relations in $\mathbf{R}^{n+1}$ is generated by the elements of the form

$$(0, \;\cdots\;, \underset{i}{-f_j}, \; 0, \;\cdots\;, \underset{j}{f_i}, \; 0, \;\cdots\;, 0) \qquad 1 \leq i < j \leq n,$$

$$(f_i, \; 0, \;\cdots\;, \underset{i}{-f_0^N}, \; 0, \;\cdots\;, 0) \qquad 1 \leq i \leq n,$$

Observe that the difference between the j lines of the matrices A and A'' is in the projection of this module of relations onto the last n coordinates. Thus, the difference between $\det(A)$ and $\det(A'')$ is a sum of determinants of the following form: the l first lines are either $(0, \ldots, -bf_j, 0, \ldots, bf_i, 0, \ldots)$, or $(0, \ldots, bf_0^N, 0, \ldots)$, for some $b \in \mathbf{R}$ which may change from line to line. The remaining $n - l$ last lines are of the form: $(f_0^{s_j}\tilde{a}_{j1}, \ldots, f_0^{s_j}\tilde{a}_{jn})$. Any determinant that contains a line $(0, \ldots, bf_0^N, 0, \ldots)$ can be ignored since it gives an element in I' as soon as N is sufficiently big. Consider then a determinant among those remaining, for example

$$\begin{vmatrix} -bf_2 & bf_1 & 0 & \ldots & 0 \\ \ldots & -b'f_j & \ldots & b'f_i & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_0^{s_j}\tilde{a}_{j1} & \ldots & \ldots & \ldots & f_0^{s_j}\tilde{a}_{jn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_0^{s_{j'}}\tilde{a}_{j'1} & \ldots & \ldots & \ldots & f_0^{s_{j'}}\tilde{a}_{j'n} \end{vmatrix} \tag{2.16}$$

A simple algebraic manipulation (formally just replace the first column by the linear combination of columns $C_1 + \frac{f_2}{f_1}C_2 + \cdots + \frac{f_n}{f_1}C_n$, where in fact the division by f_1 is just an artificial trick to justify the transformation, since everything is multiplied again by f_1 later) shows that the determinant (2.16) also equals

$$\begin{vmatrix} 0 & b & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & b'f_i & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_0^{s_j}(g_j - \tilde{a}_{j0}f_0^N) & \tilde{a}_{j2}f_0^{s_j} & \ldots & \ldots & f_0^{s_j}\tilde{a}_{jn} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_0^{s_{j'}}(g_{j'} - \tilde{a}_{j'0}f_0^N) & \tilde{a}_{j'2}f_0^{s_{j'}} & \ldots & \ldots & f_0^{s_{j'}}\tilde{a}_{j'n} \end{vmatrix}$$

which is in the ideal I', since the first column contains only elements of I'. In fact, these are just the standard computations for the Koszul complex. Note that it is here where the exactness of this complex played a fundamental role. This completes the proof of the proposition. ∎

Let us now explain an idea which we will use extensively later: the introduction of additional parameters in order to compute residue symbols. As we have seen in Remark 2.1, if h is a quasiregular sequence in $\mathbf{R}$, then $(u, h-u)$ is a quasiregular sequence in $\mathbf{R}[u]$. So, as shown in [L, (3.2,c)], for any $\mu, \nu \in (\mathbf{N}^*)^n$, the sequence $(u_1^{\mu_1}, \dots, u_n^{\mu_n}, (h_1 - u_1)^{\nu_1}, \dots, (h_n - u_n)^{\nu_n})$ is also quasiregular. Let us show that we have the following property.

Lemma 2.2. *Let $h := (h_1, \dots, h_n)$ be a quasiregular sequence in $\mathbf{R}$ such that the quotient $\mathbf{R}/(h)$ is a finite dimensional $\mathbf{K}$-vector space. For any $x_1, \dots, x_n, Q$ in $\mathbf{R}$ and any $k \in \mathbf{N}^n$,*

$$\begin{aligned} &\operatorname{Res}\begin{bmatrix} Q(x)du \wedge dx \\ u_1^{k_1+1}, \dots, u_n^{k_n+1}, h_1 - u_1, \dots, h_n - u_n \end{bmatrix} = \\ &= \operatorname{Res}\begin{bmatrix} Q(x)du \wedge dx \\ u_1, \dots, u_n, (h_1 - u_1)^{k_1+1}, \dots, (h_n - u_n)^{k_n+1} \end{bmatrix} \\ &= \operatorname{Res}\begin{bmatrix} Q(x)dx \\ h^{k+\underline{1}} \end{bmatrix}. \end{aligned} \tag{2.17}$$

Proof. We write

$$h_j^{k_j+1} = u_j^{k_j+1} + (h_j - u_j)(\sum_{l=0}^{k_j} u_j^l h_j^{k_j-l}), \ j = 1, \dots, n.$$

From the transformation law in $\mathbf{K}[u, x]$ applied to the pairs $(u^{k+\underline{1}}, h-u)$ and $(u^{k+\underline{1}}, h^{k+\underline{1}})$, one gets

$$\operatorname{Res}\begin{bmatrix} Q(x)du \wedge dx \\ u^{k+\underline{1}}, h-u \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q(x)\prod_{j=1}^n \left(\sum_{l=0}^{k_j} u_j^l h_j^{k_j-l}\right) du \wedge dx \\ u^{k+\underline{1}}, h^{k+\underline{1}} \end{bmatrix}.$$

From Lemma 2.1, one has then

$$\begin{aligned} \operatorname{Res}\begin{bmatrix} Q(x)du \wedge dx \\ u_1^{k_1+1}, \dots, u_n^{k_n+1}, h_1 - u_1, \dots, h_n - u_n \end{bmatrix} &= \operatorname{Res}\begin{bmatrix} u^k du \\ u^{k+\underline{1}} \end{bmatrix} \operatorname{Res}\begin{bmatrix} Q(x)dx \\ h^{k+\underline{1}} \end{bmatrix} \\ &= \operatorname{Res}\begin{bmatrix} Q(x)dx \\ h^{k+\underline{1}} \end{bmatrix}. \end{aligned}$$

Let us also write

$$h_j^{k_j+1} - (h_j - u_j)^{k_j+1} = u_j \left(\sum_{l=0}^{k_j} h_j^l (h_j - u_j)^{k_j-l}\right), \ j = 1, \dots, n,$$

and

$$u_j^{k_j+1} = u_j^{k_j} u_j, \ j = 1, \ldots, n.$$

From the transformation law in $\mathbf{K}[u,x]$ with the pairs $(u, (h-u)^{k+\underline{1}})$ and $(u^{k+\underline{1}}, h^{k+\underline{1}})$, we also get

$$\begin{aligned}\operatorname{Res}\begin{bmatrix} Q(x)du \wedge dx \\ u_1, \ldots, u_n, (h_1-u_1)^{k_1+1}, \ldots, (h_n-u_n)^{k_n+1}\end{bmatrix} &= \operatorname{Res}\begin{bmatrix} Q(x)u^k du \wedge dx \\ u^{k+\underline{1}}, h^{k+\underline{1}}\end{bmatrix} \\ &= \operatorname{Res}\begin{bmatrix} Q(x)dx \\ h^{k+\underline{1}}\end{bmatrix}\end{aligned},$$

which concludes the proof of the lemma. ∎

The Transformation Law has the following extension [Ky]. The proof in [Ky], based on the same ideas than the proof of the Transformation Law given in [GH], is not complete. In the analytic case, an immediate and complete proof of this generalized transformation law (with the formulation we propose here), was given by [BoH1]; their proof is based on the representation of residues by Bochner-Martinelli formulas [BGVY]. We need here to give a completely algebraic proof, which is in fact valid under the general hypotheses in [L, Chapter 3].

Proposition 2.3. *Let $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ be two quasiregular sequences in $\mathbf{R}$, such that the quotients $\mathbf{R}/(f)$ and $\mathbf{R}/(g)$ are finite dimensional $\mathbf{K}$-vector spaces and*

$$g_j = \sum_{l=1}^{n} a_{jl} f_l, \ j = 1, \ldots, n,$$

where the coefficients a_{jl} are in $\mathbf{R}$ and we let Δ be the determinant of the matrix $A = [a_{jl}]$. Then, for any $x_1, \ldots, x_n, Q \in \mathbf{R}$, any $k \in \mathbf{N}^n$, and any $n \times n$ matrix $[a_{jl}]$ with coefficients in $\mathbf{A}$, we have

$$\operatorname{Res}\begin{bmatrix} Qdx \\ f^{k+\underline{1}}\end{bmatrix} = \sum_{\substack{|q_{;j}|=k_j \\ 1\le j\le n}} \prod_{i=1}^{n} \binom{\mu_i}{q_{i;}} \operatorname{Res}\begin{bmatrix} Q\Delta \prod_{1\le i,j\le n}(a_{ij})^{q_{i,j}} dx \\ g_1^{\mu_1+1}, \ldots, g_n^{\mu_n+1}\end{bmatrix}, \tag{2.18}$$

where we have introduced the following notations for the matrix of indices $q_{i,j} \in \mathbf{N}$

$$q_{;j} = (q_{1,j}, \ldots, q_{n,j}), \quad q_{i;} = (q_{i,1}, \ldots, q_{i,n}), \quad \mu_i = |q_{i;}|$$

and

$$\binom{\mu_i}{q_{i;}} = \frac{\mu_i!}{q_{i,1}! \cdots q_{i,n}!}.$$

Proof. As a consequence of Remark 2.1, we know that the sequence $(u, f-u) := (u_1, \ldots, u_n, f_1-u_1, \ldots, f_n-u_n)$ is quasiregular in $\mathbf{R}[u]$. From Lemma 2.2 one obtains

$$\operatorname{Res}\begin{bmatrix} Qdx \\ f^{k+\underline{1}}\end{bmatrix} = \operatorname{Res}\begin{bmatrix} Qdu \wedge dx \\ u, (f-u)^{k+\underline{1}}\end{bmatrix} = \operatorname{Res}\begin{bmatrix} Qdu \wedge dx \\ u^{k+\underline{1}}, f-u\end{bmatrix}.$$

We know from Remark 2.1 that the sequence $(u, g-Au)$ is also quasiregular in $\mathbf{R}[u]$. Using the Transformation Law one has

$$\operatorname{Res}\begin{bmatrix} Qdx \\ f^{k+\underline{1}} \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q\Delta du\wedge dx \\ u^{k+1}, g-Au \end{bmatrix},$$

where Δ is the determinant of the matrix A. For any $j\in\{1,\dots,n\}$, one has

$$g_j^{|k|+1} - ((Au)_j)^{|k|+1} = (g_j-(Au)_j)\left[\sum_{l=0}^{|k|} g_j^{|k|-l}((Au)_j)^l\right]. \tag{2.19}$$

Since the polynomials $((Au)_j)^{|k|+1}$ are in the ideal generated by the $u_i^{k_i+1}$, $i=1,\dots,n$, one can apply the Transformation Law (with the systems $(u_1^{k_1+1},\dots,u_n^{k_n+1}, g-Au)$ and $(u_1^{k_1+1},\dots,u_n^{k_n+1}, g_1^{|k|+1},\dots,g_n^{|k|+1})$. Thus (2.19) implies that

$$\operatorname{Res}\begin{bmatrix} Qdx \\ f^{k+\underline{1}} \end{bmatrix} = \operatorname{Res}\begin{bmatrix} \Delta Q\prod_{j=1}^n\left(\sum_{l=0}^{|k|} g_j^{|k|-l}((Au)_j)^l\right)du\wedge dx \\ u_1^{k_1+1},\dots,u_n^{k_n+1}, g_1^{|k|+1},\dots,g_n^{|k|+1} \end{bmatrix}. \tag{2.20}$$

Let $r\in\mathbf{R}$ denote the coefficient of u^k in the development of

$$\Delta Q\prod_{j=1}^n\left(\sum_{l=0}^{|k|} g_j^{|k|-l}((Au)_j)^l\right),$$

so that

$$r = \sum_{0\le l_i\le |k|} Q_l g_1^{|k|-l_1}\cdots g_n^{|k|-l_n}, \tag{2.21}$$

for some convenient $Q_l\in\mathbf{R}$. We now appeal to Lemma 2.1 to rewrite (2.20) as

$$\operatorname{Res}\begin{bmatrix} Qdx \\ f^{k+\underline{1}} \end{bmatrix} = \operatorname{Res}\begin{bmatrix} rdx \\ g_1^{|k|+1},\dots,g_n^{|k|+1} \end{bmatrix}.$$

Using the previous representation (2.21) of r, the linearity of the residual symbol,

$$H\mapsto \operatorname{Res}\begin{bmatrix} Hdx \\ g_1^{|k|+1},\dots,g_n^{|k|+1} \end{bmatrix},$$

and the Transformation Law, one obtains

$$\begin{aligned}\operatorname{Res}\begin{bmatrix} Qdx \\ f^{k+\underline{1}} \end{bmatrix} &= \sum_{0\le l_i\le |k|}\operatorname{Res}\begin{bmatrix} Q_l g_1^{|k|-l_1}\cdots g_n^{|k|-l_n}dx \\ g_1^{|k|+1},\dots,g_n^{|k|+1} \end{bmatrix} \\ &= \sum_{0\le l_i\le |k|}\operatorname{Res}\begin{bmatrix} Q_l dx \\ g^{l+\underline{1}} \end{bmatrix}.\end{aligned}$$

Taking into account the precise value of Q_l, we get (2.18). This completes the proof of Proposition 2.3. ■

Remark 2.3. When $\mathbf{K}$ is a field of characteristic zero, the generalized transformation law of the last proposition can be understood as follows. For f and g related as in Proposition 2.3, consider the left $\mathbf{R}$-module $\mathcal{K}$ of $\mathbf{K}$-linear operators of $\mathbf{R}$ into $\mathbf{K}$ of the form

$$Q \longmapsto \sum_{\substack{k\in\mathbf{N}^n \\ |k|\leq q}} \left(\operatorname{Res}\begin{bmatrix} H_k Q dx \\ f^{k+\underline{1}} \end{bmatrix} + \operatorname{Res}\begin{bmatrix} J_k Q dx \\ g^{k+\underline{1}} \end{bmatrix}\right)$$

where the coefficients $H_k, J_k \in \mathbf{R}$ and the length q are arbitrary. One can consider the two homomorphisms of $\mathbf{R}$-modules between $\mathbf{R}[x_1,\dots,x_n]$ and $\mathcal{K}$ defined by

$$\sigma_f : \sum_k H_k x^k \mapsto \sum_k k! \operatorname{Res}\begin{bmatrix} H_k \bullet dx \\ f^{k+\underline{1}} \end{bmatrix}$$

$$\sigma_g : \sum_k H_k x^k \mapsto \sum_k k! \operatorname{Res}\begin{bmatrix} H_k \bullet dx \\ g^{k+\underline{1}} \end{bmatrix}$$

where we we have used $\bullet$ to represent the operators

$$\operatorname{Res}\begin{bmatrix} H \bullet dx \\ h \end{bmatrix}(r) = \operatorname{Res}\begin{bmatrix} Hr\, dx \\ h \end{bmatrix}.$$

Then, one has for any $P \in \mathbf{R}[x]$,

$$\sigma_f(P) = \Delta\, \sigma_g(x \mapsto P({}^tA \cdot x)),$$

where ${}^tA$ is the transposed of A. When $P \in \mathbf{R}$, this is the Transformation Law (2.11).

Let now $\mathbf{R} = \mathbf{K}[x_1,\dots,x_n]$. Given a quasiregular sequence $P_1,\dots,P_n$ in $\mathbf{R}$, one can extend the action of the corresponding residue symbol to rational functions Q_1/Q_2, whenever Q_1, Q_2 are two elements in $\mathbf{R}$ such that the ideal $(P_1,\dots,P_n,Q_2)$ is the whole ring $\mathbf{R}$. Namely, we define

$$\operatorname{Res}\begin{bmatrix} \frac{Q_1}{Q_2} dx_1\wedge\cdots\wedge dx_n \\ P_1,\dots,P_n \end{bmatrix} := \operatorname{Res}\begin{bmatrix} Q_1 V dx_1\wedge\cdots\wedge dx_n \\ P_1,\dots,P_n \end{bmatrix}, \tag{2.22}$$

where V is any polynomial such that for some $U_1,\dots,U_n$ in $\mathbf{R}$ one has $1 = U_1P_1 + \cdots + U_nP_n + VQ_2$. This definition does not depend on the choice of V, since if

$$1 = \sum_{j=1}^n U_jP_j + VQ_2 = \sum_{j=1}^n U_j'P_j + V'Q_2$$

then $V - V'$ belongs to the ideal $(P_1,\dots,P_n)$. (In fact, $V - V'$ belongs to the localization of this ideal at any maximal ideal in $\mathbf{R}$.)

In this context, the following lemma will be useful later.

Lemma 2.3. *Let $P_{ij}\,, 1\le i\le n, 1\le j\le m$ be a collection of polynomials in $\mathbf{R}$ such that the polynomials $\Theta_i := \prod_{j=1}^m P_{ij}$ define a quasiregular sequence. Assume additionally that the ideal generated by all the possible products*

$$\Xi_{j_1,\ldots,j_n} := \prod_{\substack{1\le i\le n\\ l_1\neq j_1,\cdots,l_n\neq j_n}} P_{i,l_i}\,,\ 1\le j_1,\ldots,j_n\le m,$$

is the whole ring $\mathbf{R}$. Then for any rational function $Q\in\mathbf{K}(x)$ with no poles on the set of common zeros of the Θ_i in $\overline{\mathbf{K}}^n$, we have

$$\operatorname{Res}\begin{bmatrix} Qdx \\ \Theta_1,\ldots,\Theta_n\end{bmatrix} = \sum_{1\le j_1,\ldots,j_n\le m} \operatorname{Res}\begin{bmatrix} \Big(Q\Big/\prod\limits_{\substack{1\le i\le n\\ j\neq j_i}} P_{ij}\Big)dx \\ P_{1,j_1},\ldots,P_{n,j_n}\end{bmatrix}. \tag{2.23}$$

Proof. Let $Q=Q_1/Q_2$ be an irreducible representation. Since Q_2 and the Θ_i have no common zeros over $\overline{\mathbf{K}}$, there exist polynomials $V_0,V_1,\ldots,V_n$ in $\mathbf{R}$ such that $1=V_0Q_2+\sum V_j\Theta_j$. The second hypothesis implies there are polynomials $W_{j_1,\ldots,j_n}$ such that

$$1=\sum_{1\le j_1,\cdots,j_n\le m} \Xi_{j_1,\ldots,j_n}W_{j_1,\ldots,j_n}\,.$$

Using the definition (2.22) of the residue symbol of a rational function and the Transformation Law we have

$$\begin{aligned}
\operatorname{Res}\begin{bmatrix} Qdx \\ \Theta_1,\ldots,\Theta_n\end{bmatrix} &= \operatorname{Res}\begin{bmatrix} Q_1V_0dx \\ \Theta_1,\ldots,\Theta_n\end{bmatrix}\\
&= \operatorname{Res}\begin{bmatrix} Q_1V_0\Big(\sum\limits_{1\le j_1,\ldots,j_n\le m}\Xi_{j_1,\ldots,j_n}W_{j_1,\ldots,j_n}\Big)dx \\ \Theta_1,\ldots,\Theta_n\end{bmatrix}\\
&= \sum_{1\le j_1,\ldots,j_n\le m}\operatorname{Res}\begin{bmatrix} Q_1V_0\Xi_{j_1,\ldots,j_n}W_{j_1,\ldots,j_n}dx \\ \Theta_1,\ldots,\Theta_n\end{bmatrix}\\
&= \sum_{1\le j_1,\ldots,j_n\le m}\operatorname{Res}\begin{bmatrix} Q_1V_0W_{j_1,\ldots,j_n}dx \\ P_{1,j_1},\ldots,P_{n,j_n}\end{bmatrix}\\
&= \sum_{1\le i_1,\ldots,i_n\le m}\operatorname{Res}\begin{bmatrix} \Big(Q\Big/\prod\limits_{\substack{1\le l\le n\\ j\neq i_l}} P_{lj}\Big)dx \\ P_{1,i_1},\ldots,P_{n,i_n}\end{bmatrix}
\end{aligned}$$

To obtain the last line, for every multiindex i we have used the Bézout identity

$$\begin{aligned}1 &= V_0\Big(\sum_{1\le j_1,\cdots,j_n\le m} \Xi_{j_1,\dots,j_n} W_{j_1,\dots,j_n}\Big)Q_2 + \sum_j V_j\Theta_j \\ &= V_0 W_{i_1,\dots,i_n}(\Xi_{i_1,\dots,i_n}Q_2) + \sum_{k=1}^n U_{ik}P_{k,i_k}\end{aligned}$$

and the definition (2.22) in order to transform each term in the previous sum. ■

The Transformation Laws remain valid for the residue symbols of rational functions, as shown in the following proposition.

Proposition 2.4. *Let $(f_1,\dots,f_n)$ and $(g_1,\dots,g_n)$ be two quasiregular sequences in the polynomial ring $\mathbf{K}[x_1,\dots,x_n]$ such that*

$$g_j = \sum_{k=1}^n a_{jl} f_l\,,\ j=1,\dots,n\,. \tag{2.24}$$

Then, for any rational function Q_1/Q_2 such that $(f_1,\dots,f_n,Q_2) = (g_1,\dots,g_n,Q_2) = \mathbf{K}[x_1,\dots,x_n]$, and for any multiindex $k\in\mathbf{N}^n$, one has

$$\mathrm{Res}\begin{bmatrix}(Q_1/Q_2)dx\\ f^{k+\underline{1}}\end{bmatrix} = \sum_{\substack{|q_{;j}|=k_j\\ 1\le j\le n}} \prod_{i=1}^n \binom{\mu_i}{q_{i;}} \mathrm{Res}\begin{bmatrix}(Q_1/Q_2)\Delta\prod_{1\le i,j\le n}(a_{ij})^{q_{i,j}}dx\\ g_1^{\mu_1+1},\dots,g_n^{\mu_n+1}\end{bmatrix},$$

with the same notations as in Proposition 2.3.

Proof. One has just to notice that if one takes $q=|k|+1$, then we have a Bézout identity

$$1 = \sum_{j=1}^n u_{q,j} g_j^q + V_M Q_2$$

which can also be written (thanks to the relations (2.24)) as

$$1 = \sum_{j=1}^n \tilde{u}_{q,j} f_j^{k_j+1} + V_M Q_2\,.$$

We then have, by definition of the extended residue symbol, that for any $\mu\in\mathbf{N}$ such that $|\mu|=q$,

$$\mathrm{Res}\begin{bmatrix}(Q_1/Q_2)\Delta\prod_{1\le i,j\le n}(a_{ij})^{q_{i,j}}dx\\ g_1^{\mu_1+1},\dots,g_n^{\mu_n+1}\end{bmatrix} = \mathrm{Res}\begin{bmatrix}Q_1V_M\Delta\prod_{1\le i,j\le n}(a_{ij})^{q_{i,j}}dx\\ g_1^{\mu_1+1},\dots,g_n^{\mu_n+1}\end{bmatrix}$$

and also

$$\mathrm{Res}\begin{bmatrix}(Q_1/Q_2)dx\\ f^{k+\underline{1}}\end{bmatrix} = \mathrm{Res}\begin{bmatrix}Q_1V_Mdx\\ f^{k+\underline{1}}\end{bmatrix}.$$

Then the conclusion of the proposition follows from formula (2.18). ■

In the same vein, we have also the following proposition.

Proposition 2.5. *Let $f_0, f_1, \dots, f_n$ be a regular sequence in $\mathbf{K}[x_0, \dots, x_n]$. Let $g_1, \dots, g_n$ in $\mathbf{K}[x_0, \dots, x_n]$ such that the sequence $(f_0, g_1, \dots, g_n)$ is quasiregular. Assume that there are nonnegative integers $s_1, \dots, s_n$ and an $n \times n$ matrix A of elements in $\mathbf{K}[x_0, \dots, x_n]$ such that*

$$f_0^{s_j} g_j = \sum_{l=1}^{n} a_{jl} f_l \,, \quad j = 1, \dots, n.$$

Let Q_1/Q_2 be a rational function such that $(f_0, f_1, \dots, f_n, Q_2) = (f_0, g_1, \dots, g_n, Q_2) = \mathbf{K}[x_0, x_1, \dots, x_n]$. Then, for any $k_0 \in \mathbf{N}$, one has

$$\mathrm{Res} \begin{bmatrix} (Q_1/Q_2) dx_0 \wedge \cdots \wedge dx_n \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} = \mathrm{Res} \begin{bmatrix} (Q_1/Q_2) \Delta dx_0 \wedge \cdots \wedge dx_n \\ f_0^{k_0+1+|s|}, g_1, \dots, g_n \end{bmatrix}$$

where $|s| = s_1 + \cdots + s_n$ and Δ is the determinant of the matrix A.

Proof. It is similar to the last proof. Let us consider $u_0, \dots, u_n, V$ such that

$$1 = u_0 f_0^{k_0+1+|s|} + \sum_{j=1}^{n} u_j g_j + V Q_2 \,. \tag{2.25}$$

and $v_0, \dots, v_n, W$ such that

$$1 = v_0 f_0^{k_0+1} + \sum_{j=1}^{n} v_j f_j + W Q_2 \,. \tag{2.26}$$

Multiplying (2.25) and (2.26) by $f_0^{|s|}$ and comparing the identities, we conclude that $f_0^{|s|}(W - V)$ is in the ideal generated by $f_0^{k_0+1+|s|}, f_1, \dots, f_n$. Therefore,

$$\begin{aligned} \mathrm{Res} \begin{bmatrix} (Q_1/Q_2) dx \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} &= \mathrm{Res} \begin{bmatrix} Q_1 W dx \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} = \mathrm{Res} \begin{bmatrix} Q_1 f_0^{|s|} W dx \\ f_0^{k_0+1+|s|}, f_1, \dots, f_n \end{bmatrix} \\ &= \mathrm{Res} \begin{bmatrix} Q_1 f_0^{|s|} V dx \\ f_0^{k_0+1+|s|}, f_1, \dots, f_n \end{bmatrix} = \mathrm{Res} \begin{bmatrix} Q_1 V dx \\ f_0^{k_0+1}, f_1, \dots, f_n \end{bmatrix} \\ &= \mathrm{Res} \begin{bmatrix} Q_1 V \Delta dx \\ f_0^{k_0+1+|s|}, g_1, \dots, g_n \end{bmatrix} = \mathrm{Res} \begin{bmatrix} (Q_1/Q_2) \Delta dx \\ f_0^{k_0+1+|s|}, g_1, \dots, g_n \end{bmatrix} , \end{aligned}$$

if one uses formula (2.13). ■

3. Residue symbols and properness.

In this section, we consider an infinite algebraically closed field $\mathbf{K}$ (any characteristic), equipped with a non trivial absolute value $|\ |$. We will consider the norms, defined respectively on $\mathbf{K}^n$ and $\mathbf{K}^{n+1}$ by,

$$\begin{aligned} |x| &= \max_{1 \le i \le n} |x_i| \,, \ x = (x_1, \dots, x_n) \in \mathbf{K}^n \\ |X| &= \max_{0 \le i \le n} |X_i| \,, \ X = (X_0, \dots, X_n) \in \mathbf{K}^{n+1} \,. \end{aligned}$$

Definition 3.1. *Let $P_j \in \mathbf{K}[x_1, \ldots, x_n]$, $1 \le j \le n$, the polynomial map $P = (P_1, \ldots, P_n)$ from $\mathbf{K}^n$ to $\mathbf{K}^n$ is proper if and only if $\mathbf{K}[x_1, \ldots, x_n]$ is a finitely generated $\mathbf{K}[P_1, \ldots, P_n]$-module.*

Due to the following proposition, one can check properness by means of inequalities.

Proposition 3.1. *Let $P = (P_1, \ldots, P_n)$ be a polynomial map from $\mathbf{K}^n$ to $\mathbf{K}^n$. The morphism P is proper if and only if there exist three constants K, γ, $\delta > 0$ such that*

$$|x| \ge K \Longrightarrow |P(x)| \ge \gamma |x|^{\delta} \,. \tag{3.1}$$

Any exposant $\delta > 0$ such that (3.1) holds for convenient constants K, γ is called a Lojasiewicz exponent for the map P.

Proof. We are greatly indebted to Q. Liu for the proof of this statement in the case of positive characteristic. The most interesting part of the proof is the fact that condition (3.1) implies properness. This can be shown as follows. One can assume that $\mathbf{K}$ is complete (otherwise, take a completion of $\mathbf{K}$.) It is clear from (3.1) that for any point $z \in \mathbf{K}^n$, the set $P^{-1}(z)$ is an algebraic set which is closed and bounded, thus finite; this means that P is a quasi-finite morphism. It follows from Zariski's Main Theorem [Mu] that one can factorize P as $P = g \circ f$, where $f : \mathbf{K}^n \mapsto \mathcal{X}$ is an open immersion from $\mathbf{K}^n$ into some affine variety $\mathcal{X}$, and $g : \mathcal{X} \mapsto \mathbf{K}^n$ is a finite morphism (therefore proper.) When $\mathbf{K}$ has characteristic is 0 and can be topologically embedded in $\mathbf{C}$ (when the absolute value is not ultrametric), f (as P) is proper in the topological sense, so that $f(\mathbf{K}^n)$ is a closed subset (in the topological sense) in $\mathcal{X}$, that is $f(\mathbf{K}^n) = \mathcal{X}$ and we are done. When the characteristic is positive or when the absolute value is ultrametric, one can show that, under the hypothesis (3.1), P is proper in the rigid sense (see [Ki]), which implies that in the decomposition $P = g \circ f$, f is also proper in the rigid sense. Therefore $f(\mathbf{K}^n)$ is closed (in the rigid analytic sense) and equals $\mathcal{X}$, so we are done in this case.

Let now suppose that P is a proper morphism from $\mathbf{K}^n$ to $\mathbf{K}^n$. We can write down the integral dependency relations satisfied by the x_j, $j = 1, \ldots, n$ over $\mathbf{K}[P_1, \ldots, P_n]$, that is

$$x_j^{N_j} \equiv \sum_{k=1}^{N_j} A_{j,k}(P_1, \ldots, P_n) x_j^{N_j - k} \,, \; j = 1, \ldots, n, \tag{3.2}$$

where $A_{j,k} \in \mathbf{K}[x_1, \ldots, x_n]$. One gets from (3.2) inequalities of the form

$$|x_j|^{N_j} \le C_j (1 + |x_j|)^{N_j - 1} (1 + |P(x)|)^{q_j} \,, \; j = 1, \ldots n, \tag{3.3}$$

where $C_j > 0$, $q_j \in \mathbf{N}$. From these inequalities, it is immediate to deduce that (3.1) holds for some convenient choice of K, γ, δ (depending on the C_j, q_j, N_j, $1 \le j \le n$.) ■

Since for any $x \in \mathbf{K}^n$, one has $|P(x)| \le C(1 + |x|)^D$, where $D := \max_{1 \le j \le n} \deg P_j$ and C is a positive constant depending on the coefficients of the P_j. It follows that, if δ satisfies (3.1) (with corresponding constants K, γ), then $\delta \le D$ (just take x such that $|x|$ is arbitrarily large, which is possible since $|\ |$ is not the trivial absolute value on $\mathbf{K}$.) Moreover, let ${}^hP_1, \ldots, {}^hP_n$ be the homogeneous polynomials in $n+1$ variables corresponding to $P_1, \ldots, P_n$, namely,

$$ {}^hP_j(X_0, \ldots, X_n) := X_0^{\deg P_j} P_j(\frac{X_1}{X_0}, \ldots \frac{X_n}{X_0}),\ j = 1, \ldots, n\,. $$

Then, one has the following proposition

Proposition 3.2. *Let $P = (P_1, \ldots, P_n)$ be a proper polynomial map from $\mathbf{K}^n$ to $\mathbf{K}^n$ such that (3.1) is fullfilled with constants K, γ, δ. Let $D := \max_{1\le j\le n} \deg P_j$. Then, for any j, $1 \le j \le n$, one can find a positive constant Γ_j and some homogeneous polynomial $\mathcal{R}_j$ in two variables, with coefficients in $\mathbf{K}$ and total degree $r_j \ge \delta$ such that, for any $X = (X_0, \ldots, X_n) \in \mathbf{K}^{n+1}$,*

$$ |\mathcal{R}_j(X_0, X_j)|\,|X_0|^{D-\delta} \le \Gamma_j |X|^{r_j-\delta} \left(\sum_{k=1}^n |X_0|^{D-\deg P_k}\, |{}^hP_k(X)| \right). \tag{3.4} $$

Proof. Let us write (3.1) for $x = (X_1/X_0, \ldots, X_n/X_0)$, where $X = (X_0, \ldots, X_n) \in \mathbf{K}^{n+1}$, $X_0 \ne 0$. Then, whenever $\max\limits_{1\le j\le n} |X_j| \ge K|X_0|$, we get

$$ (\max_{1\le j\le n} |X_j|)^{\delta} |X_0|^{D-\delta} \le \frac{1}{\gamma} \left(\sum_{k=1}^n |X_0|^{D-\deg P_k}\, |{}^hP_k(X)| \right). \tag{3.5} $$

As we have already seen in the proof of Proposition 3.1, the algebraic set $P^{-1}(0)$ is finite since P is a proper map. From the Hilbert Nullstellensatz it follows that one can find polynomials $R_1(x_1), \ldots, R_n(x_n)$ in one variable, such that $R_j(x_j)$ lies in the ideal generated by $P_1, \ldots, P_n$ in $\mathbf{K}[x_1, \ldots, x_n]$. One can assume that $r_j := \deg R_j \ge \delta$. Let us define

$$ \mathcal{R}_j(X_0, X_j) := X_0^{r_j} R_j(X_j/X_0),\ X \in \mathbf{K}^{n+1}\,. $$

For any $x = (x_1, \ldots, x_n)$ such that $|x| \le 2K$, one has $|R_j(x_j)| \le \kappa_j |P(x)|$ for some $\kappa_j = \kappa_j(K) > 0$. One has also, for some $\tilde{\kappa_j} > 0$, $|\mathcal{R}_j(X)| \le \tilde{\kappa_j} |X|^{r_j}$. Therefore, for any $X = (X_0, \ldots, X_n) \in \mathbf{K}^{n+1}$, $X_0 \ne 0$, such that $\max\limits_{1\le j\le n} |X_j| \le 2K|X_0|$, one gets

$$ |\mathcal{R}_j(X_0, X_j)| \le \kappa_j |X_0|^{r_j-D} \left(\sum_{k=1}^n |X_0|^{D-\deg P_k}\, |{}^hP_k(X)| \right). \tag{3.6} $$

So, if $X \in \mathbf{K}^{n+1}$ and $X_0 \ne 0$, we have, either

$$ \begin{aligned} |\mathcal{R}_j(X_0, X_j)||X_0|^{D-\delta} &\le \kappa_j |X_0|^{r_j-\delta} \left(\sum_{k=1}^n |X_0|^{D-\deg P_k}\, |{}^hP_k(X)| \right) \\ &\le \kappa_j |X|^{r_j-\delta} \left(\sum_{k=1}^n |X_0|^{D-\deg P_k}\, |{}^hP_k(X)| \right), \end{aligned} \tag{3.7} $$

when $\max\limits_{1\leq j\leq n} |X_j| \leq 2K|X_0|$, or

$$\begin{aligned}
(\max_{1\leq j\leq n} |X_j|)^{\delta}|X_0|^{D-\delta}|\mathcal{R}_j(X_0,X_j)| &\leq \frac{1}{\gamma}|\mathcal{R}_j(X_0,X_j)|\left(\sum_{k=1}^{n}|X_0|^{D-\deg P_k}\,|^hP_k(X)|\right)\\
&\leq \frac{\tilde{\kappa}_j}{\gamma}|X|^{r_j}\left(\sum_{k=1}^{n}|X_0|^{D-\deg P_k}\,|^hP_k(X)|\right)\\
&\leq \frac{\tilde{\kappa}_j}{\gamma\min(1,K)}(\max_{1\leq j\leq n}|X_j|)^{r_j}\left(\sum_{k=1}^{n}|X_0|^{D-\deg P_k}\,|^hP_k(X)|\right)
\end{aligned}$$

which, together with (3.5), implies

$$|X_0|^{D-\delta}|\mathcal{R}_j(X_0,X_j)| \leq \frac{\tilde{\kappa}_j}{\gamma\min(1,K)}|X|^{r_j-\delta}\left(\sum_{k=1}^{n}|X_0|^{D-\deg P_k}\,|^hP_k(X)|\right), \tag{3.8}$$

when $\max\limits_{1\leq j\leq n} |X_j| \geq K|X_0|$. Note that (3.8) is similar to (3.7). In fact we just proved that for any $X\in\mathbf{K}^{n+1}$, $X_0\neq 0$, then

$$|\mathcal{R}_j(X_0,X_j)||X_0|^{D-\delta} \leq \Gamma_j|X|^{r_j-\delta}\left(\sum_{k=1}^{n}|X_0|^{D-\deg P_k}\,|^hP_k(X)|\right),$$

where $\Gamma_j = \max(\kappa_j, \tilde{\kappa_j}/\gamma\min(1,K))$. The inequality remains valid when $X_0 = 0$, so Proposition 3.2 is completely proved. ■

The following proposition is a corollary of the Lipman- Teissier theorem ([LT], [LS]) about integral closure of ideals.

Proposition 3.3. *Let $\mathcal{P}_1,\ldots,\mathcal{P}_m$ be homogeneous polynomials of degree D in the $n+1$ variables $X_0,\ldots,X_n$, with coefficients in the field $\mathbf{K}$. Let $\mathcal{Q}$ be another homogeneous polynomial in $\mathbf{K}[X_0,\ldots,X_n]$, of $\deg\mathcal{Q}\geq D$, such that, for some positive constant Γ,*

$$|\mathcal{Q}(X)| \leq \Gamma|X|^{\deg\mathcal{Q}-D}\max_{1\leq j\leq m}|\mathcal{P}_j(X)|\,,\ X\in\mathbf{K}^{n+1}\,. \tag{3.9}$$

Then $\mathcal{Q}^{n+1}$ lies in the ideal generated by $\mathcal{P}_1,\ldots,\mathcal{P}_m$ in $\mathbf{K}[X_0,\ldots,X_n]$.

Proof. Let us consider the regular local ring $\mathbf{K}[X_0,\ldots,X_n]_{\mathcal{M}}$ (of dimension $n+1$), where $\mathcal{M}$ denotes the maximal ideal $(X_0,\ldots,X_n)$. Let $\mathcal{I}$ be the ideal generated by $\mathcal{P}_1,\ldots,\mathcal{P}_m$ in this local ring.

Fix $s>D$, such that s is coprime with the characteristic of $\mathbf{K}$, and consider the ideal in $\mathbf{K}[X_0,\ldots,X_n]_{\mathcal{M}}$, $\mathcal{I}_s := \mathcal{I}+\mathcal{M}^s$. We want to show that $\mathcal{Q}$ is in the integral closure of $\mathcal{I}_s$ in $\mathbf{K}[X_0,\ldots,X_n]_{\mathcal{M}}$. This can be done following the ideas in [LT].

First, since $\sqrt{\mathcal{I}_s}=\mathcal{M}$ (see [NR]), one can find a regular sequence $(p_1,\ldots,p_{n+1})$ such that the ideal $\mathcal{J}_s := (p_1,\ldots,p_{n+1})$ is a reduction of $\mathcal{I}+\mathcal{M}^s$ in $\mathbf{K}[X_0,\ldots,X_n]_{\mathcal{M}}$. The

p_j, $1 \leq j \leq n+1$, are linear combinations of the $\mathcal{P}_j$ and of all the monomials generating $\mathcal{M}^s$. Since $s > D$ and s is coprime with the characteristic of the field, one can assume that the homogeneous parts of higher degree (in fact s) of the p_j have the origin as only common zero in $\mathbf{K}^{n+1}$ and that the Jacobian of $(p_1, \ldots, p_{n+1})$ is not identically zero. The p_j, $j = 1, \ldots, n+1$, define a zero dimensional algebraic variety $V_s = V$ in $\mathbf{K}^{n+1}$, containing the origin. Since $\mathcal{J}_s$ is a reduction of $\mathcal{I}_s$, $\mathcal{P}_1, \ldots \mathcal{P}_m$, which are in $\mathcal{I}_s$, are also in the integral closure of $\mathcal{J}_s$ in the local ring $\mathbf{K}[X_0, \ldots, X_n]_{\mathcal{M}}$. This implies, by means of integral dependency relations, that for any $X \in \mathbf{K}^{n+1}$ such that $|X| \leq \epsilon$ (for a convenient choice of $\epsilon > 0$), one has

$$\max_{1 \leq j \leq m} |\mathcal{P}_j(X)| \leq C \max_{1 \leq j \leq n+1} |p_j(X)| \tag{3.10}$$

for some positive constant C, so that, for $|X| \leq \epsilon$,

$$|\mathcal{Q}(X)| \leq \Gamma_\epsilon \max_{1 \leq j \leq n+1} |p_j(X)|\,. \tag{3.11}$$

Let $q_s \in \mathbf{K}[X_0, \ldots, X_n]$ be a polynomial in $\mathbf{K}[X_0, \ldots, X_n]$ such that $q_s(0) \neq 0$ and q_s is in the ideal generated by $p_1, \ldots, p_{n+1}$ in all the localizations $\mathbf{K}[X_0, \ldots, X_n]_{\mathcal{M}_\alpha}$, where α is any point in $V \setminus \{0\}$. Therefore, for each $R > 0$ one can find a positive constant $\widetilde{\Gamma}(R)$ such that, for any $X \in \mathbf{K}^{n+1}$, with $|X| \leq R$, one has

$$|\mathcal{Q}(X) q_s(X)| \leq \widetilde{\Gamma}(R) \max_{1 \leq j \leq n+1} |p_j(X)|\,. \tag{3.12}$$

Since the homogeneous parts of higher degree of $p_1, \ldots, p_{n+1}$ have the origin as only common zero in $\mathbf{K}^{n+1}$, it follows from Proposition 3.1 that the polynomial map $(p_1, \ldots, p_{n+1})$ is proper (in the algebraic sense), with $[\mathbf{K}[X_0, \ldots, X_n] : \mathbf{K}[p_1, \ldots, p_{n+1}]] = s^{n+1}$ (by Bézout's theorem.) This implies that one can find a relation of integral dependency

$$(\mathcal{Q} q_s)^{s^{n+1}} \equiv \sum_{l=1}^{s^{n+1}} A_l(p_1, \ldots, p_{n+1}) (\mathcal{Q} q_s)^{s^{n+1} - l}\,, \tag{3.13}$$

which can be obtained just writing that the multiplication operator corresponding to $\mathcal{Q} q_s$, acting on the finite dimensional $\mathbf{K}(u_1, \ldots, u_{n+1})$-vector space

$$\frac{\mathbf{K}(u_1, \ldots, u_{n+1})[X_0, \ldots, X_n]}{(p_1(X) - u_1, \ldots, p_{n+1}(X) - u_{n+1})}\,,$$

annihilates its characteristic polynomial. From (3.12), we deduce that for u in $\mathbf{K}^{n+1}$ such that $|u| \leq 1$ and any $1 \leq l \leq s^{n+1}$, there is a constant $C_l > 0$ such that

$$|A_l(u)| \leq C_l |u|^l\,.$$

(Since $A_l(u)$ corresponds to the l-elementary symmetric polynomial in the $[\mathcal{Q} q_s](\alpha_j(u))$, where $\alpha_1(u), \ldots, \alpha_{s^{n+1}}(u)$ are the zeroes of $(p_1 - u_1, \ldots, p_{n+1} - u_{n+1})$.) Therefore, the polynomial $A_l(p_1,\dots,p_{n+1})$ is in $\mathcal{J}_s^l$ and (3.13) provides a relation of integral dependency for $\mathcal{Q}$ over the ideal $\mathcal{J}_s$ in the local ring $\mathbf{K}[X_0,\dots,X_n]_{\mathcal{M}}$. Since $\mathcal{J}_s$ is a reduction of $\mathcal{I}_s$, the polynomial $\mathcal{Q}$ is in the integral closure of $\mathcal{I}_s$ in the local ring $\mathbf{K}[X_0,\dots,X_n]_{\mathcal{M}}$.

Applying the Lipman-Teissier theorem in the regular local ring $\mathbf{K}[X_0,\dots,X_n]_{\mathcal{M}}$, we conclude that $\mathcal{Q}^{n+1}$ is in $\mathcal{I}_s$. Since this is true for any $s>D$, from

$$\mathcal{I} = \bigcap_{s>D} \mathcal{I}_s \,,$$

we conclude that $\mathcal{Q}^{n+1}\in\mathcal{I}$. Because $\mathcal{Q}$ is homogeneous and the $\mathcal{P}_j$, $1\le j\le m$, are homogeneous with the same degree, $\mathcal{Q}^{n+1}$ is in the ideal generated by $\mathcal{P}_1,\dots,\mathcal{P}_m$ in $\mathbf{K}[X_0,\dots,X_n]$. This concludes the proof of our assertion. ■

The Jacobi vanishing theorem (2.8) was extended, using analytic methods, to proper polynomial maps in $\mathbf{C}^n$ in [BY1] and [BY2] as follows. Let us assume the degrees D_j of the polynomials P_j are in decreasing order and that δ is an exponent such that (3.1) holds. Then, for any polynomial $Q\in\mathbf{C}[x]$ and multiindex k, one has

$$(|k|+2n-1)\delta > \deg Q + D_1+\cdots+D_{n-1}+n \Longrightarrow \operatorname{Res}\begin{bmatrix} Q dx \\ P^{k+\underline{1}}\end{bmatrix} = 0\,. \tag{3.14}$$

This statement was crucial in the proof of the effective Nullstellensatz over $\mathbf{C}$ given in [BY1]. On the other hand, one can see that this vanishing theorem is not the best one could expect. For example, (3.14) implies that

$$\deg Q < (2n-1)\delta - (D_1+\cdots+D_{n-1}) - n \Longrightarrow \operatorname{Res}\begin{bmatrix} Q dx \\ P_1,\dots,P_n\end{bmatrix} = 0\,. \tag{3.15}$$

A more careful analysis of the Bochner-Martinelli representation of the residue current (see [Y1], [Y2]) yields the statement

$$\deg Q \le n\delta - n - 1 \Longrightarrow \operatorname{Res}\begin{bmatrix} Q dx \\ P_1,\dots,P_n\end{bmatrix} = 0\,. \tag{3.16}$$

The point here is that this result depends on the Lojasiewicz exponent δ, related to the properness condition, but not on the degrees of the P_j. We do not know how to prove such result when $\mathbf{K}$ has positive characteristic, though it is possibly true. Nevertheless, we have the following result that will be enough to prove the effective Nullstellensatz theorem below.

Proposition 3.4. *Let $P_1,\dots,P_n$ be polynomials in $\mathbf{K}[x_1,\dots,x_n]$, such that $\deg P_j = D$, $1\le j\le n$, and such that there exist strictly positive constants K, γ, and a strictly positive integer δ such that*

$$|x|\ge K \Longrightarrow \max_{1\le j\le n} |P_j(x)| \ge \gamma |x|^{\delta}\,. \tag{3.17}$$

Assume that

$$(1-\epsilon_n)D<\delta\,,\ \text{ for } \epsilon_n:=\frac{1}{n(n+1)}\,. \tag{3.18}$$

Then, for any $(k_1,\dots,k_n)\in\mathbf{N}^n$*, one has*

$$\deg Q\le n(n+1)(|k|+n)(\delta-(1-\epsilon_n)D)-n-1\Longrightarrow \operatorname{Res}\begin{bmatrix} Q dx\\ P^{k+\underline{1}}\end{bmatrix}=0\,. \tag{3.19}$$

Moreover, under the stronger hypothesis that

$$(1-\frac{\epsilon_n}{n+1})D<\delta\,, \tag{3.20}$$

one has

$$\begin{cases}\deg Q\le n(D-1)\\ k\neq 0\end{cases}\Longrightarrow \operatorname{Res}\begin{bmatrix} Q dx\\ P^{k+\underline{1}}\end{bmatrix}=0\,. \tag{3.21}$$

Proof. It is clear that global residual symbols in $\mathbf{R}=\mathbf{K}[x_1,\dots,x_n]$ are well defined since the sequence $(P_1,\dots,P_n)$ is quasi-regular in $\mathbf{K}[x_1,\dots,x_n]$ (because of the properness of the map $(P_1,\dots,P_n)$, see Proposition 3.1.), and therefore the same is true for all the sequences $(P_1^{k_1+1},\dots,P_n^{k_n+1})$. Let $\mathcal{R}_1,\dots,\mathcal{R}_n$, be the polynomials associated to $(P_1,\dots,P_n)$ by Proposition 3.2. From Proposition 3.3 it follows that, for any $1\le j\le n$,

$$[X_0^{D-\delta}\mathcal{R}_j(X_0,X_j)]^{n+1}\in(\mathcal{P}_1,\dots,\mathcal{P}_n)\,,$$

where $\mathcal{P}_j(X)={}^hP_j(X)$, $1\le j\le n$. Note that we can choose $\mathcal{R}_j$ to be distinguished in X_j. For any multi-index k, we have then

$$[X_0^{D-\delta}\mathcal{R}_j(X_0,X_j)]^{(n+1)(|k|+n)}\in(\mathcal{P}_1^{k_1+1},\dots,\mathcal{P}_n^{k_n+1})\,.$$

Let $\mathcal{R}_j^k(X_0,X_j):=[\mathcal{R}_j(X_0,X_j)]^{(n+1)(|k|+n)}$, $1\le j\le n$. One can write

$$\mathcal{R}_j^k(X_0,X_j)X_0^{(n+1)(D-\delta)(|k|+n)}=\sum_{l=1}^n\mathcal{R}_{j,l}^k(X)\mathcal{P}_l^{k_l+1}(X)\,,$$

where $\mathcal{R}_{j,l}^k$ is homogeneous, with degree

$$\deg\mathcal{R}_{j,l}^k=\deg\mathcal{R}_j^k+(n+1)(|k|+n)(D-\delta)-(k_l+1)D\,,\ 1\le j,l\le n\,.$$

Let $R_j^k(x_j):=\mathcal{R}_j^k(1,x_j)$. Then, one has the polynomial identities

$$R_j^k(x_j)=\sum_{l=1}^n R_{j,l}^k(x)P_l^{k_l+1}(x)\,, \tag{3.22}$$

where $R^k_{j,l}(x) = \mathcal{R}^k_{j,l}(1,x)$. Let Δ_k be the determinant of the matrix $[R^k_{j,l}]_{\substack{1\le j\le n\\ 1\le l\le n}}$. The degree of Δ_k is at most

$$\deg \Delta_k \le \sum_{j=1}^{n} \deg R^k_j + (|k|+n)[n(n+1)(D-\delta)-D]\,.$$

From the Transformation Law in $\mathbf{R} = \mathbf{K}[x_1,\dots,x_n]$ (Proposition 2.1), applied to the two quasi-regular sequences $(P_1,\dots,P_n)$ and $(R^k_1,\dots,R^k_n)$, one has, for any $Q \in \mathbf{K}[x_1,\dots,x_n]$,

$$\operatorname{Res}\begin{bmatrix} Q dx_1 \wedge \cdots \wedge dx_n \\ P_1^{k_1+1},\dots,P_n^{k_n+1}\end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q\Delta_k dx_1 \wedge \cdots \wedge dx_n \\ R^k_1,\dots,R^k_n\end{bmatrix}.$$

Since the homogeneous parts of highest degree in $R^k_1,\dots,R^k_n$ have no common zeroes except at the origin, one can apply the Jacobi vanishing theorem (2.8) and get, if we define $\rho_k := \sum_{j=1}^n \deg R^k_j$,

$$\rho_k + \deg Q - n(n+1)(|k|+n)[\delta-(1-\epsilon_n)D] \le \rho_k - n - 1 \Longrightarrow \operatorname{Res}\begin{bmatrix} Qdx \\ P^{k+\underline{1}}\end{bmatrix} = 0\,,$$

which gives the conclusion (3.19). In order to check (3.21), we have just to check that condition (3.20) implies that

$$n(D-1) < n(n+1)^2[\delta-(1-\epsilon_n)D] - n\,,$$

that is

$$(n(n+1)^2-1)D < n(n+1)^2\delta\,,$$

which is exactly the condition (3.20). ∎

As a corollary of this result, let us state the following proposition (that will be crucial for our purposes later on.)

Proposition 3.5. *Let $P_1,\dots,P_n$ be n polynomials in $\mathbf{K}[x_1,\dots,x_n]$, of degree D with the property that there exist strictly positive constants K, γ, such that (3.17) holds for some integer $\delta > 0$ satisfying $1-\frac{1}{n(n+1)^2} < \frac{\delta}{D} \le 1$. Suppose that the g_{jl}, $1 \le j,l \le n$ are elements in $\mathbf{K}[x_1,\dots,x_n,y_1,\dots,y_n]$, with degree less or equal than $D-1$, such that*

$$P_j(x) - P_j(y) = \sum_{l=1}^{n} (x_l - y_l) g_{jl}(x,y)\,,\ 1 \le j \le n\,,\ x,\, y \in \mathbf{K}^n\,; \tag{3.23}$$

Then, if $\Delta(x,y) := \det[g_{jl}(x,y)]_{\substack{1\le j\le n\\ 1\le l\le n}}$ (such a Δ is called a Bézoutian for the map P), the following polynomial identity holds

$$1 = \operatorname{Res}\begin{bmatrix} \Delta(x,y)dx \\ P_1(x),\dots,P_n(x)\end{bmatrix}\,,\ y \in \mathbf{K}^n\,. \tag{3.24}$$

Proof. The first remark one can make here is that, since $(P_1, \dots, P_n)$ is proper (from Proposition 3.1), it defines a quasiregular sequence in $\mathbf{K}[x_1, \dots, x_n]$. On the other hand, there are integral dependence relations of the form

$$x_j^{N_j} = \sum_{l=1}^{N_j} A_{jl}(P_1, \dots, P_n) x_j^{N_j - l}\,,\ j = 1, \dots, n\,,$$

which can be rewritten in the form

$$x_j^{N_j} - \sum_{l=1}^{N_j} A_{jl}(u_1, \dots, u_n) x_j^{N_j - l} = \sum_{l=1}^{n} A_j^l(x_j, P, u)(P_l - u_l)\,,\ j = 1, \dots, n\,, \tag{3.25}$$

where the A_j^l are polynomials in $2n+1$ variables. Such relations show that, for any $u = (u_1, \dots, u_n) \in \mathbf{K}^n$, the sequence $(P_1 - u_1, \dots, P_n - u_n)$ remains quasiregular. Remark 2.1 shows that one can also consider such a sequence as a quasiregular sequence in $\mathbf{K}(u)[x_1, \dots, x_n]$ and compute for any polynomial $Q \in \mathbf{K}[x_1, \dots, x_n]$, the residue symbol with values in $\mathbf{K}(u)$

$$\Phi(u) := \mathrm{Res}\begin{bmatrix} Q dx \\ P_1 - u_1, \dots, P_n - u_n \end{bmatrix}.$$

Applying Proposition 2.1, together with the identities (3.25), one gets

$$\Phi(u) = \mathrm{Res}\begin{bmatrix} Q(x) \det[A_j^l(x_j, P, u)]_{\substack{1\le j\le n\\ 1\le l\le n}}\, dx \\ x_1^{N_1} - \sum_{l=1}^{N_1} A_{1l}(u) x_1^{N_1 - l}, \dots, x_n^{N_n} - \sum_{l=1}^{N_n} A_{nl}(u) x_n^{N_n - l} \end{bmatrix}.$$

While Φ is a priori a rational function, it follows from Lemma 2.1 and (2.10) that $\Phi \in \mathbf{K}[u_1, \dots, u_n]$. We want to show now that Φ is in fact a constant that belongs to $\mathbf{K}$, when $\deg Q \le n(D-1)$, provided the hypothesis on the ratio δ/D is satisfied. This will be done in two steps: first, we will show that for any polynomial Q, one has

$$\mathrm{Res}\begin{bmatrix} Q dx \\ P_1 - u_1, \dots, P_n - u_n \end{bmatrix} = \sum_{0 \le k_1, \dots, k_n \le \kappa(Q)} \mathrm{Res}\begin{bmatrix} Q dx \\ P_1^{k_1+1}, \dots, P_n^{k_n+1} \end{bmatrix} u_1^{k_1} \cdots u_n^{k_n} \tag{3.26}$$

for $\kappa(Q)$ large enough and independent of u. (This will hold in fact under the weaker hypothesis $1 - (\epsilon_n/2) < \delta/D \le 1$, where, as before, $\epsilon_n := 1/n(n+1)$.) Then, in the second step, we will use statement (3.21) from Proposition 3.4 to conclude that Φ is an element in $\mathbf{K}$ provided $\deg Q \le n(D-1)$ and $1 - (\epsilon_n/(n+1)) < \delta/D \le 1$.

• Let us prove the first step. Let $u \in \mathbf{K}^n$. Then the morphism $(P_1(P_1 - u_1), \dots, P_n(P_n - u_n))$ is also proper (see Proposition 3.1) and such that, for $|x| \ge K(u)$,

$$\max_{1\le j\le n} |P_j(x)(P_j(x) - u_j)| \ge \frac{\gamma^2}{2} |x|^{2\delta}\,.$$

Then, the statement (3.19) in Proposition 3.4 implies that for any $\widetilde{Q} \in \mathbf{K}[x_1, \ldots, x_n]$ one has

$$\begin{aligned} &\deg \widetilde{Q} \leq 2n(n+1)(|k|+n)(\delta - (1-\epsilon_n)D) - n - 1 \Longrightarrow \\ &\Longrightarrow \operatorname{Res}\left[\begin{matrix} \widetilde{Q}dx \\ (P_1(P_1-u_1))^{k_1+1}, \ldots, (P_n(P_n-u_n))^{k_n+1} \end{matrix}\right] = 0 \end{aligned} \tag{3.27}$$

This implies that for any polynomial Q such that

$$\deg Q \leq 2n(n+1)(|k|+n)(\delta - (1-\frac{\epsilon_n}{2})D) - n - 1\,, \tag{3.28}$$

and any choice of $\eta_1, \ldots \eta_n$ in $\{0,1\}$, one has

$$\begin{aligned} &\operatorname{Res}\left[\begin{matrix} Qdx \\ P_1^{k_1+1}(P_1-u_1)^{\eta_1}, \ldots, P_n^{k_n+1}(P_n-u_n)^{\eta_n} \end{matrix}\right] = \\ &= \operatorname{Res}\left[\begin{matrix} Q\prod_{j=1}^{n}(P_j-u_j)^{k_j+1-\eta_j}\,dx \\ (P_1(P_1-u_1))^{k_1+1}, \ldots, (P_n(P_n-u_n))^{k_n+1} \end{matrix}\right] = 0\,. \end{aligned} \tag{3.29}$$

The first equality is just a consequence of Proposition 2.1, while the second follows from the fact that condition (3.28) is equivalent to

$$\deg Q + (|k|+n)D \leq 2n(n+1)(|k|+n)(\delta - (1-\epsilon_n)D) - n - 1\,,$$

which implies that if

$$\widetilde{Q} = \widetilde{Q}_k := Q\prod_{j=1}^{n}(P_j-u_j)^{k_j+1-\eta_j}\,,$$

then

$$\deg \widetilde{Q} \leq 2n(n+1)(|k|+n)(\delta - (1-\epsilon_n)D) - n - 1$$

and we can therefore apply (3.27). Clearly, since $1-(\epsilon_n/2) < \delta/D \leq 1$, for k such that condition (3.28) holds, (for instance, if $|k| \geq \kappa = \kappa(Q)$, depending on the degree of Q, but not on the choice of u,) one has

$$\operatorname{Res}\left[\begin{matrix} Qdx \\ P_1^{k_1+1}(P_1-u_1)^{\eta_1}, \ldots, P_n^{k_n+1}(P_n-u_n)^{\eta_n} \end{matrix}\right] = 0 \tag{3.30}$$

for any choice of $\eta_1, \ldots, \eta_n \in \{0,1\}$, and any $u \in \mathbf{K}^n$. Now, we note that applying the Transformation Law again, for any $u \in \mathbf{K}^n$ one has

$$\begin{aligned} &\sum_{0 \leq k_1, \ldots, k_n \leq \kappa} \operatorname{Res}\left[\begin{matrix} Qdx \\ P_1^{k_1+1}, \ldots, P_n^{k_n+1} \end{matrix}\right] u_1^{k_1}\cdots u_n^{k_n} = \operatorname{Res}\left[\begin{matrix} Q\prod_{j=1}^{n}\dfrac{P_j^{\kappa+1}-u_j^{\kappa+1}}{P_j-u_j}dx \\ P_1^{\kappa+1}, \ldots, P_n^{\kappa+1} \end{matrix}\right] \\ &= \operatorname{Res}\left[\begin{matrix} Q\prod_{j=1}^{n}(P_j^{\kappa+1}-u_j^{\kappa+1})dx \\ P_1^{\kappa+1}(P_1-u_1), \ldots, P_n^{\kappa+1}(P_n-u_n) \end{matrix}\right] \end{aligned}$$

Now, one can rewrite

$$\operatorname{Res}\left[\begin{array}{c} Q\prod_{j=1}^{n}(P_j^{\kappa+1}-u_j^{\kappa+1})dx \\ P_1^{\kappa+1}(P_1-u_1),\ldots,P_n^{\kappa+1}(P_n-u_n)\end{array}\right]=$$

$$=\operatorname{Res}\left[\begin{array}{c} Q\prod_{j=2}^{n}(P_j^{\kappa+1}-u_j^{\kappa+1})dx \\ P_1-u_1,P_2^{\kappa+1}(P_2-u_2),\ldots,P_n^{\kappa+1}(P_n-u_n)\end{array}\right]-$$

$$-u_1^{\kappa+1}\operatorname{Res}\left[\begin{array}{c} Q\prod_{j=2}^{n}(P_j^{\kappa+1}-u_j^{\kappa+1})dx \\ P_1^{\kappa+1}(P_1-u_1),\ldots,P_n^{\kappa+1}(P_n-u_n)\end{array}\right]=$$

$$=\operatorname{Res}\left[\begin{array}{c} Q\prod_{j=2}^{n}(P_j^{\kappa+1}-u_j^{\kappa+1})dx \\ P_1-u_1,P_2^{\kappa+1}(P_2-u_2),\ldots,P_n^{\kappa+1}(P_n-u_n)\end{array}\right]$$

since

$$\operatorname{Res}\left[\begin{array}{c} Q\prod_{j=2}^{n}(P_j^{\kappa+1}-u_j^{\kappa+1})dx \\ P_1^{\kappa+1}(P_1-u_1),\ldots,P_n^{\kappa+1}(P_n-u_n)\end{array}\right]=0$$

because it can be written as a sum of expressions of the form

$$\operatorname{Res}\left[\begin{array}{c} Qdx \\ P_1^{k_1+1}(P_1-u_1)^{\eta_1},\ldots,P_n^{k_n+1}(P_n-u_n)^{\eta_n}\end{array}\right]=0$$

where $|k|+n\geq k_1+1=\kappa+1$. Iterating this procedure, we get the required polynomial identity (3.26). We would like to point out that these computations are the algebraic counterpart of the manipulations of the kernel of the Cauchy-Weil formula that appear in [BT, Section 1].

• We now apply (3.21) in Proposition 3.4 to get, as announced, that Φ is a constant in $\mathbf{K}$, provided that $\deg Q\leq n(D-1)$ (we are asumming in this case that the stronger hypothesis $1-(\epsilon_n/n+1)<\delta/D\leq 1$ holds.)

We are now ready to conclude the proof of our proposition. Recall from Cauchy's formula (2.7) that one has for any $Q\in\mathbf{K}[x]$,the polynomial identity in $\mathbf{K}[y_1,\ldots,y_n]$,

$$1=\operatorname{Res}\left[\begin{array}{c} Qdx \\ x_1-y_1,\ldots,x_n-y_n\end{array}\right].$$

We can now apply to this formula the Transformation Law to the two regular sequences (in $\mathbf{K}(y)[x_1,\ldots,x_n]$), $(x_1-y_1,\ldots,x_n-y_n)$ and $(P_1-P_1(y),\ldots,P_n-P_n(y))$. The identities (3.23) imply the following identity in $\mathbf{K}(y_1,\ldots,y_n)$:

$$1=\operatorname{Res}\left[\begin{array}{c} \Delta(x,y)dx \\ P_1-P_1(y),\ldots,P_n-P_n(y)\end{array}\right].$$

From what we have just proved, as $\deg_x \Delta(x,y) \leq n(D-1)$, we conclude that

$$\mathrm{Res}\begin{bmatrix} \Delta(x,y)dx \\ P_1 - P_1(y), \ldots, P_n - P_n(y) \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \Delta(x,y)dx \\ P_1, \ldots, P_n \end{bmatrix},$$

which completes the proof of (3.24). ∎

In order to complete this section, we need a few complements about computations of residue symbols for polynomial maps of the form $(P_1, \ldots, P_n)$, whenever $(P_1, \ldots, P_n)$ is a quasiregular sequence in $\mathbf{K}[x_1, \ldots, x_n]$ generating a proper ideal. Applying Remark 2.2 and [Pe, Satz 56] we conclude that the corresponding polynomial map P is dominant, that is, one can find n relations of the form

$$A_{j0}(P)x_j^{N_j} = \sum_{l=1}^{N_j} A_{jl}(P)x_j^{N_j-l}, \; j = 1, \ldots, n,$$

with coefficients that are polynomials in n variables and $A_{j0} \not\equiv 0$. This equality can be rewritten in the form

$$A_{j0}(u)x_j^{N_j} - \sum_{l=1}^{N_j} A_{jl}(u)x_j^{N_j-l} = \sum_{l=1}^{n} A_j^l(x_j, P, u)(P_l - u_l), \; j = 1, \ldots, n, \tag{3.31}$$

where the A_j^l are polynomials in $2n+1$ variables.

Relations of the form (3.31) show that, for any $u = (u_1, \ldots, u_n) \in \mathbf{K}^n$ outside the hypersurface $\prod_j A_{j0}(u) = 0$, thus for u generic, the sequence $(P_1 - u_1, \ldots, P_n - u_n)$ remains quasiregular. In particular, the set of common zeros of this sequence of polynomials is a zero dimensional variety or it could be empty. For such u, using the Transformation Law (2.11), the relations (3.31), Lemma 2.1, and the explicit computation (2.10) in the one variable case, one can show that for any $Q \in \mathbf{K}[x]$

$$\mathrm{Res}\begin{bmatrix} Q(x)dx \\ P(x) - u \end{bmatrix} = \Psi(u) \in \mathbf{K}(u)$$

(see also [Bi] for an analytic proof of this result.) The main difficulty one has when P is not a proper map over the origin, (that is, when $\prod_j A_{j0}(0) = 0$ [Je],) is that it is in general impossible to compute the different residue symbols

$$\mathrm{Res}\begin{bmatrix} Qdx \\ P^{k+\underline{1}} \end{bmatrix}, \; k \in \mathbf{N}^n,$$

from the rational function Ψ. For example, if $P_1 = x(1+x^2yz), P_2 = y(1+x^2yz), P_3 = z$ and $Q = 1$, then one can see that $\Psi \equiv 0$ while

$$\mathrm{Res}\begin{bmatrix} 1\,dx \\ P \end{bmatrix} = 1.$$

We overcome this difficulty by means of the following interesting lemma.

Lemma 3.1. *Let $P_1,\ldots,P_n$ be a quasiregular sequence in $\mathbf{K}[x]$, then for any $q\in\mathbf{N}$ and for any $\alpha\in\mathbf{K}^n$, the sequence $(t^{q+1},P_1(x)-\alpha_1 t,\ldots,P_n(x)-\alpha_n t)$ is a quasiregular sequence in $\mathbf{K}[x,t]$. Moreover, we have the formula*

$$\mathrm{Res}\left[\begin{matrix} Q(x)dt\wedge dx \\ t^{q+1},P_1(x)-\alpha_1 t,\ldots,P_n(x)-\alpha_n t\end{matrix}\right]=\sum_{|k|=q}\mathrm{Res}\left[\begin{matrix} Qdx \\ P^{k+\underline{1}}\end{matrix}\right]\alpha^k\,. \tag{3.32}$$

Proof. From Remark 2.1 we obtain the quasiregularity in $\mathbf{K}[x,t]$ of the longer sequence $(t^{q+1},P_1-\alpha_1 t,\ldots,P_n-\alpha_n t)$. To compute the residue symbol in the left hand side of (3.32), let us consider the identities

$$P_j^{q+1}=(\alpha_j t)^{q+1}+(P_j-\alpha_j t)\sum_{k_j=0}^{q}(\alpha_j t)^{k_j}P_j^{q-k_j}\,.$$

We apply the Transformation Law replacing $P_j-\alpha_j t$ by P_j^{q+1} in the left hand side of (3.32) to obtain

$$\mathrm{Res}\left[\begin{matrix} Qdt\wedge dx \\ t^{q+1},P-\alpha t\end{matrix}\right]=\mathrm{Res}\left[\begin{matrix} Q\prod_{j=1}^{n}\big(\sum_{k_j=0}^{q}(\alpha_j t)^{k_j}P_j^{q-k_j}\big)dt\wedge dx \\ t^{q+1},P_1^{q+1},\ldots,P_n^{q+1}\end{matrix}\right].$$

We use the linearity of the residue symbol, the Transformation Law in order to simplify common factors in both lines of the symbol, and, finally, the Fubini property (Lemma 2.1), to obtain the desired formula (3.32). ■

This lemma is usually applied in the following form.

Proposition 3.6. *Let $P_1,\ldots,P_n$ be a quasiregular sequence defining a proper ideal in $\mathbf{K}[x_1,\ldots,x_n]$. Consider a system of integral dependency relations for the coordinates, of the form*

$$B_j(x_j,u):=A_{j0}(u)x_j^{N_j}-\sum_{l=1}^{N_j}A_{jl}(u)x_j^{N_j-l}=\sum_{l=1}^{n}A_j^l(x_j,P,u)(P_l-u_l)\,,\ j=1,\ldots,n$$

and let s_j be the valuation (in u) of the polynomial B_j, thus

$$B_j(x_j,\alpha_1 t,\ldots,\alpha_n t)=t^{s_j}\big(R_j(x_j,\alpha)-tS_j(x_j,\alpha,t)\big)\,,\quad R_j\not\equiv 0\,. \tag{3.33}$$

Let $\alpha\in\mathbf{K}^n$ be such that, for any $j\in\{1,\ldots,n\}$, one has $R_j(.,\alpha)\not\equiv 0$. Then, for any $q\in\mathbf{N}$, one has

(3.34)

$$\mathrm{Res}\left[\begin{matrix} Qdt\wedge dx \\ t^{q+1},P-\alpha t\end{matrix}\right]=\sum_{0\le|k|\le q+|s|}\mathrm{Res}\left[\begin{matrix} Q\Delta(x,P,\alpha t)\prod_{j=1}^{n}S_j^{k_j}(x_j,\alpha,t)dt\wedge dx \\ t^{q+1+|s|-|k|},R_1^{k_1+1}(x_1,\alpha),\ldots,R_n^{k_n+1}(x_n,\alpha)\end{matrix}\right]$$

where $\Delta(x,P,u)$ is the determinant of the matrix $[A_j^l(x,P,u)]_{\substack{1\le j\le n\\ 1\le l\le n}}$.

Proof. Since the sequence $P_1,\dots,P_n$ is quasiregular, so is the sequence $t, P_1-\alpha_1 t,\dots,P_n-\alpha_n t$ in $\mathbf{K}[t,x]$. Moreover, since the base field $\mathbf{K}$ is infinite, there is an $n\times n$ invertible matrix $\mathcal{A}=[a_{jl}]_{\substack{1\le j\le n\\ 1\le l\le n}}$ with coefficients in $\mathbf{K}$ such that the sequence $\mathcal{A}P$ is regular for the increasing order. Then the sequence

$$(t,\sum_{l=1}^n a_{1l}P_l - t\sum_{l=1}^n a_{1l}\alpha_l,\dots,\sum_{l=1}^n a_{nl}P_l - t\sum_{l=1}^n a_{nl}\alpha_l)$$

is also regular. Hence, we can use Proposition 2.2 with $\mathbf{R}=\mathbf{K}[t,x]$, and $f_0=t$, $f_j(t,x)=\sum_{l=1}^n a_{jl}P_j - t\sum_{l=1}^n a_{jl}\alpha_l$, $g_j(x,t)=t^{-s_j}B_j(x_j,\alpha t)$, $1\le j\le n$. Then, we have, from formula (2.13) and an additional application of the Transformation Law

$$\mathrm{Res}\begin{bmatrix} Qdt\wedge dx \\ t^{q+1}, P-\alpha t\end{bmatrix} = \mathrm{Res}\begin{bmatrix} Q\Delta(x,P,\alpha t)dt\wedge dx \\ t^{q+1+|s|}, R(x,\alpha)-tS(x,\alpha,t)\end{bmatrix}$$

where we denote by $R(x,\alpha)-tS(x,\alpha,t)$ the sequence

$$R_1(x_1,\alpha)-tS_1(x_1,\alpha,t),\dots,R_n(x_n,\alpha)-tS_n(x_n,\alpha,t)\,.$$

We now use the identities

$$R_j^{q+1+|s|} = t^{q+1+|s|}S_j^{q+1+|s|} + (R_j - tS_j)\big(\sum_{k_j=0}^{q+|s|}(tS_j)^{k_j}R_j^{q+|s|-k_j}\big)\,,$$

(where the variables have been left implicit), together with the Transformation Law (2.11) and the linearity of the residue symbols, in order to obtain formula (3.34). ∎

Remark 3.1. Note that if one lets $q=0$ and chooses a convenient α, the last proposition yields a formula to compute the residue symbol

$$\mathrm{Res}\begin{bmatrix} Qdx \\ P_1,\dots,P_n\end{bmatrix}$$

from the knowledge of relations of dependency for the coordinates x_j over $\mathbf{K}(P_1,\dots,P_n)$. In fact, as it follows from Lemma 3.1, the right-hand side of (3.34) is a polynomial in α, though it would seem to be a rational function if one just looks at its expression.

Remark 3.2. Lemma 3.1 and Proposition 3.6 remain valid if one replaces Q by Q_1/Q_2, where $(P_1,\dots,P_n,Q_2)=\mathbf{K}[x_1,\dots,x_n]$, with the residue symbols understood in the generalized sense of (2.22). This follows from Propositions 2.4 and 2.5.

4. Lojasiewicz inequalities.

In this section, the ring $\mathbf{R}$ will be $\mathbf{K}[x_1,\dots,x_n]$, where $\mathbf{K}$ is an algebraically closed field of arbitrary characteristic, equipped with a non trivial absolute value $|\ |$. The corresponding norm in $\mathbf{K}^n$ was defined at the beginning of Section 3. Given n integers $D_1 \geq D_2 \ldots \geq D_n \geq 1$ we define, as in [JKS],

$$B := B(D_1, \ldots, D_n) = (\frac{3}{2})^\iota D_1 \cdots D_n \tag{4.1}$$

where $\iota = \#\{j < n-1 | D_j = 2\}$. The main result of this section is based on the arguments in [BY1] and [BGVY, Propositions 5.7 and 5.8]. Small modifications are required by the fact that we are now working with fields of arbitrary characteristic. We recall that a sequence of polynomials $P_1, \ldots, P_n$ is said to be normal if it is a regular sequence for any ordering.

Proposition 4.1. *Let $P_1, \ldots, P_n$ be a quasiregular sequence in $\mathbf{K}[x_1, \ldots, x_n]$, then one can find n linear combinations (with coefficients in $\mathbf{K}$) of the P_i, namely $\widetilde{P}_1, \ldots, \widetilde{P}_n$, n linearly independent $\mathbf{K}$-linear forms $L_1, \ldots, L_n$, and a positive constant K such that for any $N \in \mathbf{N}^*$ and any $x \in \mathbf{K}^n$ with $|x| > K$ one has*

$$\max_{1 \leq i \leq n} |L_i(x)|^{NB} |\widetilde{P}_i(x)| \geq \gamma_N |x|^{(N-1)B} \tag{4.2}$$

for some constant $\gamma_N > 0$.

Proof. Since the base field is infinite and the sequence of P_j is quasiregular, using the pigeonhole principle as in [MW] we can find a triangular, invertible matrix M_0, with coefficients in $\mathbf{K}$, such that the sequence of polynomials P'_j, $j = 1, \ldots, n$ defined by the system of linear equations $P' = M_0 P$ is a regular sequence. Note that $\deg P'_j = D_j$. Using the same principle, one can find an invertible matrix M_1 with coefficients in $\mathbf{K}$ so that the new system $P'' = M_1 P'$ is normal and every minor of M_1 is non zero [BY1, Lemma 5.2]. Let J be any subset of $\{1, \ldots, n\}$ of cardinality k, $1 \leq k \leq n-1$. As in the proof of [El],[BGVY, Proposition 5.8, p.125], one can find a collection of polynomials $\widetilde{P}_{J,j}$, $j \in J$, $\deg \widetilde{P}_{J,j} \leq D_j$, which are linear combinations with coefficients in $\mathbf{K}$ of the P''_j, given by an invertible matrix, so that one has

$$\kappa_J \max_{j \in J} |\widetilde{P}_{J,j}| \leq \max_{j \in J} |P''_j| \leq \kappa'_J \max_{j \in J} |\widetilde{P}_{J,j}|$$

for some strictly positive constants κ_J, κ'_J. Clearly, the polynomials $\widetilde{P}_{J,j}$ define the same algebraic variety as the P''_j, $j \in J$. That is, a variety of codimension at least $n-k$. From the Noether Normalization Theorem [ZS, vol 1, Chapter 5, p. 266] applied to all possible systems with different J, we can show there is an invertible matrix M_2 and two positive constants C_0, K_0 such that for any $x \in \mathbf{K}^n$ with $|x| \geq K_0$, any $k \in \{1, \ldots, n-1\}$, and any subset in $\{1, \ldots, n\}$ with $\#J = k$,

$$\begin{aligned} \widetilde{P}_{J,j}(M_2 x) = 0, \forall j \in J &\Longleftrightarrow P''_j(M_2 x) = 0, \forall j \in J \\ &\Longrightarrow \sum_{l=1}^{k} |x_l| \leq C_0 \sum_{l=k+1}^{n} |x_l| \,. \end{aligned} \tag{4.3}$$

Using the global Lojasiewicz inequality proved in [JKS, Corollary 6], we get that, for any $k \in \{1, \ldots, n-1\}$, there exists $\epsilon_k > 0$ such that, for any subset J, $\#J = k$, the set

$$\mathcal{X}_J^{(\epsilon_k)} := \{x \in \mathbf{K}^n,\, |x| \geq K_0 + 1,\, \max_{j \in J} |\widetilde{P}_{J,j}(M_2 x)| \leq \frac{\epsilon_k}{(1+|x|)^B}\}$$

is included in the cone

$$\mathcal{Y}_k := \{x \in \mathbf{K}^n,\, \sum_{l=1}^{k} |x_l| \leq (C_0 + 1) \sum_{l=k+1}^{n} |x_l|\}\,.$$

We associate to this fan of cones $\mathcal{Y}_k$, $1 \leq k \leq n-1$, a collection of linear forms $\Lambda_1, \ldots, \Lambda_n$, as follows. Let $M = [m_{jl}]$ be some element in $\mathcal{M}_n(\mathbf{K})$ (the space of $n \times n$ matrices with entries in $\mathbf{K}$) such that all the minors of M have a norm bigger than 1. We let ρ be the maximum of the norm of all these minors. Since the absolute value is non trivial, one can always find some element $\alpha \in \mathbf{K}$ such that $|\alpha| > (C_0 + 1)n\rho$. This guarantees that the linear forms Λ_j defined by

$$\Lambda_j(x) = \sum_{l=1}^{n} m_{jl} \alpha^{l-1} x_l \quad j = 1, \ldots, n,$$

are linearly independent. It follows from Cramer's rule that there is a constant $\epsilon_0 > 0$ such that for any $k < n$, the inequality

$$|x_1| + \ldots + |x_k| \leq (C_0 + 1)(|x_{k+1}| + \ldots + |x_n|) \tag{4.4}$$

implies that

$$\min_{\{J' : \#(J') = n-k\}} \Big(\sum_{j \in J'} |\Lambda_j(x)| \Big) \geq \epsilon_0 |x|. \tag{4.5}$$

Note that because the Λ_j were chosen to be independent, the inequality (4.5) is valid even if $J = \emptyset$. Since the P_j'' are linear combinations of the P_j, and conversely, it follows from [JKS, Corollary 6], applied to the sequence $P_1, \ldots, P_n$, that for convenient constants C_1, and ϵ_n, one has for $|x| > C_1$,

$$\max_{1 \leq j \leq n} |P_j''(M_2 x)| \geq \frac{\epsilon_n}{(1+|x|)^B}\,.$$

Hence, for $\eta \ll \epsilon_n$, the set $\{|x| > C_1\}$ can be written as the disjoint union of the sets

$$\mathcal{Z}_J := \Big\{ |x| > C_1 : |P_j''(M_2 x)| < \frac{\eta}{(1+|x|)^B} \quad \text{if} \quad j \in J$$
$$\text{and} \quad |P_j''(M_2 x)| \geq \frac{\eta}{(1+|x|)^B} \quad \text{if} \quad j \notin J \Big\}.$$

Fix J of cardinal $\#(J) = k$. Then, if $x \in \mathcal{Z}_J$, one has

$$\max_{j \in J} |\widetilde{P}_{J,j}(M_2 x)| \leq \frac{\kappa'_J \eta}{(1+|x|)^B} \leq \frac{\epsilon_k}{(1+|x|)^B}.$$

Hence, $x \in \mathcal{X}_J^{(\epsilon_k)}$, and so $x \in \mathcal{Y}_k$, thus it satisfies (4.4), hence also (4.5) and

$$\sum_{j \notin J} |\Lambda_j(x)| \geq \epsilon_0 |x| \,.$$

Therefore, we have for $x \in \mathcal{Z}_J$

$$\max_{j \notin J} |\Lambda_j(x)| \geq \frac{\epsilon_0}{n-k} |x| \,.$$

Hence, for any $N \in \mathbf{N}^*$,

$$\begin{aligned}
\sum_{j=1}^{n} |\Lambda_j(x)|^{NB} |P''_j(M_2 x)| &\geq \sum_{j \notin J} |\Lambda_j(x)|^{NB} |P''_j(M_2 x)| \\
&\geq (\max_{j \notin J} |\Lambda_j(x)|^{NB})(\min_{j \notin J} |P''_j(M_2 x)|) \\
&\geq \eta (\frac{\epsilon_0}{n})^{NB} \frac{|x|^{NB}}{(1+|x|)^B}
\end{aligned}$$

The fact that the sets $\mathcal{Z}_J$ form a partition of the set $\{|x| \geq C_1\}$ implies that, for a convenient choice of $\gamma_N > 0$, for any x with $|x| \geq K$ one has the inequality

$$\max_{1 \geq j \geq n} |\Lambda_j(x)|^{NB} |P''_j(M_2 x)| \geq \gamma_N |x|^{(N-1)B} \,.$$

This concludes the proof of Proposition 4.1, as we can choose for the matrix of linear forms $L = \Lambda M_2^{-1}$, where Λ is the matrix of the linear forms Λ_j. ■

Remark 4.1. It is clear in the previous that the only restriction on the choice of the matrices M_0, M_1, M_2 is that they lie outside some algebraic variety in $\mathbf{K}^{3n^2}$. Moreover, any choice for the coefficients of the linear forms L_j can be slightly perturbed, in fact, we can also consider affine perturbations of the L_j, that is perturbations of the form $x \mapsto u_{j0} + L_j(x)$. One can also keep, for N fixed, the same constant γ_N for all small perturbations.

One can combine this result with Proposition 3.5 and get the following technical but important result.

Proposition 4.2. *Let $P_1, \ldots, P_n$ be a quasiregular sequence in $\mathbf{K}[x_1, \ldots, x_n]$, with $D_1 \geq D_2 \geq \cdots \geq D_n$, $D_j := \deg P_j$. Then one can find a polynomial Φ in $n(n+1) + n^2$ variables*

u_{jl}, v_{jk}, $1 \le j,k \le n, 0 \le l \le n$, with coefficients in $\mathbf{K}$, and $\deg \Phi \le 2^{n+1}(n+1)^4 D_1^n$, such that, for any $(U,V) \in \mathbf{K}^{n\times(n+1)} \oplus \mathcal{M}_n(\mathbf{K})$, with $\Phi(U,V) \neq 0$, the polynomials

$$\Pi^j_{U,V}(x) := U^j(x) < V^j, P(x) >:= \left(u_{j,0} + \sum_{l=1}^n u_{jl}x_l\right)\left(\sum_{l=1}^n v_{jl}P_l\right) \tag{4.6}$$

have degree exactly D_1+1, define a quasiregular sequence in $\mathbf{K}[x_1,\dots,x_n]$, and moreover, if $N \in \mathbf{N}^*$ is such that

$$\frac{B+D_1}{NB+D_1} < \frac{1}{n(n+1)^2}\,, \tag{4.7}$$

then the following polynomial identity holds in $\mathbf{K}[y_1,\dots,y_n]$

$$1 = \mathrm{Res}\left[\begin{matrix}\Delta_{N,U,V}(x,y)dx_1\wedge\cdots\wedge dx_n\\ U^1(x)^{NB} < V^1,P>,\dots,U^n(x)^{NB} < V^n,P>\end{matrix}\right]. \tag{4.8}$$

This formula (which is a polynomial identity in y), holds whenever $\Delta_{N,U,V}(x,y)$ is the determinant of an arbitrary $n\times n$ matrix whose coefficients δ_{jl} in $\mathbf{K}[U,V,x,y]$ have degree in x,y at most $NB+D_1-1$ and satisfy the relations
(4.9)

$$U^j(x)^{NB} < V^j, P(x) > - U^j(y)^{NB} < V^j,P(y) >= \sum_{l=1}^n (x_l-y_l)\delta_{jl}(x,y)\,,\ j=1,\dots,n.$$

Remark 4.2. For example, one can construct δ_{jl} as

$$\delta_{jl} = U^j(x)^{NB} < V^j, g^l(x,y) > + < V^j, P(y) > \varphi^{(U,N)}_{jl}(x,y)\,,$$

where the g^l is a vector of components g^l_j, which are polynomials in the $2n$ variables (x,y) of degree D_1-1 such that

$$P_j(x) - P_j(y) = \sum_{l=1}^n (x_l-y_l)g^l_j(x,y)\,,\ j=1,\dots,n.$$

and the $\varphi^{(U,N)}_{jl}$ are polynomials in the variables (U,x,y) such that

$$\deg_{(x,y)} \varphi^{(N)}_{jl} = NB-1$$

and

$$U^j(x)^{NB} - U^j(y)^{NB} = \sum_{l=1}^n (x_l-y_l)\varphi^{(U,N)}_{jl}(x,y)\,,\ j=1,\dots,n.$$

Proof. For a generic choice of $V^1,\dots,V^n$ in $\mathbf{K}^n$, the sequence $< V^j, P >$, $j=1,\dots,n$, is a normal sequence. (This follows from the pigeonhole principle, since the field is supposed to be infinite.) Choose V so that it is the case. For any subset $J \subset \{1,\ldots,n\}$, $\#(J)=k$, $0\leq k\leq n$, the polynomials $< V^j, P >$, $j\in J$, define an algebraic set in $\mathbf{K}^n$ with dimension $n-k$. Therefore, for any generic choice of the U^j, $j\notin J$, the polynomials $< V^j, P >$, $j\in J$, together with the affine functions $U^j(x)$, $j\notin J$, define a zero dimensional set in $\mathbf{K}^n$, that is, they define a quasiregular sequence. We conclude from these remarks that the polynomials $U^j(x) < V^j, P(x) >$, $j=1,\ldots,n$, define a quasiregular sequence in $\mathbf{K}(U,V)[x_1,\ldots,x_n]$, so that for any $j\in\{1,\ldots,n\}$, there exist a polynomial $\mathcal{Q}_j(U,V,x_j)\in\mathbf{K}[U,V,x_j]$, such that $\mathcal{Q}_j$ lies in the ideal generated by the $U^j(x) < V^j, P(x) >$ in $\mathbf{K}[U,V,x_1,\ldots,x_n]$. Of course, one could find such polynomials just by using elimination theory, as done in [vdW], later on we shall do it in a more constructive way, in order to obtain sharper estimates for the degree in U,V. Let us write for the moment

$$\mathcal{Q}_j(U,V,x_j) = Q_{j0}(U,V)x_j^{\nu_j} + \sum_{l=1}^{\nu_j} Q_{jl}(U,V)x_j^{\nu_j - l}\,,\ j=1,\ldots,n\,. \tag{4.10}$$

We then define

$$\Psi(U,V) := \prod_{j=1}^{n} Q_{j0}(U,V). \tag{4.11}$$

and

$$\Phi(U,V) := \Psi(U,V)\prod_{j=1}^{n} v_{j1}. \tag{4.12}$$

For any choice U,V such that $\Phi(U,V)\neq 0$, all the polynomials (in x) $\Pi^j_{U,V}$ have degree D_1+1 and define a quasiregular sequence in $\mathbf{K}[x]$. To construct $\mathcal{Q}_j$ with good degree estimates, let us proceed as follows. Choose a subset $J\subseteq\{1,\ldots,n\}$ and consider the n polynomials in $\mathbf{K}[U,V,x_1][x_2,\ldots,x_n]$, $U^j(x)$, $j\in J$, $< V^j, P >$, $j\notin J$. The ring $\mathbf{A} := \mathbf{K}[U,V,x_1]$ is a factorial regular ring, with Krull dimension $2n^2+n+1$, which can be equipped with a size, in the sense of [Ph1, Section 1, p. 3-4]. Namely, we can take for the definition of the size on $\mathbf{A}$ the map $\mathbf{t}$ defined by

$$\mathbf{t}(\Theta) = \deg_{U,V}\Theta,\ \ \Theta\in \mathrm{Pol}(\mathbf{A})\,, \Theta\neq 0\,,\ \mathbf{t}(0) = -\infty$$

where $\mathrm{Pol}(\mathbf{A})$ is the $\mathbf{A}$ module $\mathbf{A}[(Y_i)_{i\in\mathbf{N}}]$ of polynomials in some arbitrary number of variables with coefficients in $\mathbf{A}$. From [Ph1, Theorem 4], one can find a polynomial $\mathcal{Q}_{J,1}(U,V,x_1)\in\mathbf{A}$ which belongs to the ideal generated in $\mathbf{A}[x_2,\ldots,x_n]$ by the polynomials $U^j(x)$, $j\in J$, and $< V^j, P >$, $j\notin J$ and such that

$$\deg_{U,V}\ \mathcal{Q}_{J,1} \leq (2(n^2+1)+2n)D_1^n(n+1) \leq 2(n+1)^3 D_1^n\,.$$

One can do this construction for any subset J of $\{1,\ldots,n\}$ and get a family of relations

$$\mathcal{Q}_{J,1}(U,V,x_1) = \sum_{k\in J} a_{1,J,k}(U,V,x)U^j(x) + \sum_{k\notin J} a_{1,J,k}(U,V,x) < V^j(x), P(x) >\,,$$

where the $a_{J,k}$ are in $\mathbf{A}[U,V,x_1] = \mathbf{K}[U,V,x]$. Let us consider the product

$$\begin{aligned} \mathcal{Q}_1(U,V,x_1) &= \prod_{J\subseteq\{1,\dots,n\}} \mathcal{Q}_{J,1}(U,V,x_1) \\ &= \prod_{J\subseteq\{1,\dots,n\}} \Big(\sum_{k\in J} a_{1,J,k}(U,V,x) U^j(x) + \sum_{k\notin J} a_{1,J,k}(U,V,x) < V^j(x), P(x) > \Big) . \end{aligned} \tag{4.13}$$

We claim that the development of this product leads to

$$\mathcal{Q}_1(U,V,x_1) = \sum_{j=1}^{n} b_{1,j}(U,V,x) U^j(x) < V^j, P(x) > \tag{4.14}$$

for some $b_{1,j}$ in $\mathbf{K}[U,V,x]$. This follows from a simple combinatorial argument: consider all sequences with length n formed with 0 or 1. There is a correspondence between factors in the right-hand side of (4.13) and such sequences: namely, the factor

$$\sum_{k\in J} a_{1,J,k}(U,V,x) U^j(x) + \sum_{k\notin J} a_{1,J,k}(U,V,x) < V^j(x), P(x) >$$

corresponds to the sequence $(\epsilon_1,\dots,\epsilon_n)$ where $\epsilon_k = 0$ if and only if $k \in J$. Consider these sequences as the successive rows of a $2^n \times n$ matrix, the first row being $(0,\dots,0)$, the last one being $(1,\dots,1)$. For example, for $n = 3$, we get the matrix

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

One can see that it is impossible to select inductively (starting with the first row) a coefficient *in each row*, according to the following rule: any time one selects a 0 in the position (k,l) (k for row, l for column), it is forbidden to choose a 1 which is in the position (k',l) with $k' > k$. For example, for $n = 3$, if we choose the 0 which is in the position $(1,1)$, we have to take a 0 again in the second row (for example in the second column), which prevents us from taking a 1 in the two first columns any longer. We have therefore to take the 0 in the position $(5,3)$ and the 1 in the position $(8,3)$, which does not fit with our rule. The impossibility to find such a path shows that there is always a pairing $(U^j, < V^j, P >)$ (for some $j \in \{1,\dots,n\}$) in each term in the expansion of the product in (4.13), and this observation proves (4.14). Thus, the polynomial $\mathcal{Q}_1$ is in the ideal generated by the $\Pi^j_{U,V}$, $1 \le j \le n$, in $\mathbf{K}[U,V,x]$. The degree in U,V of $\mathcal{Q}_1$ is at most $2^{n+1}(n+1)^3 D_1^n$. One can repeat this construction for the other indices $\neq 1$, in this case, the polynomial Φ associated to this sequence $\mathcal{Q}_j$ via (4.13) and (4.14) has degree at most $2^{n+1}(n+1)^4 D_1^n$.

Let us fix now an integer N such that the condition (4.7) is satisfied. For any U, V for which $\Phi(U, V) \neq 0$, we can rewrite the residue symbol in (4.8) as

$$\operatorname{Res}\left[\begin{array}{c}\Delta_{N,U,V}(x,y)\left(\displaystyle\prod_{j=1}^{n} < V^j, P(x) >^{NB-1}\right)dx \\ \Pi^1_{U,V}(x)^{NB}, \ldots, \Pi^n_{U,V}(x)^{NB}\end{array}\right].$$

Using the form (4.10) of the polynomials $\mathcal{Q}_j$, the generalized transformation law of Proposition 2.3, and formula (2.10) for the computation of residues in one variable, one can show that the residue symbol (4.8) is a rational function in U, V of the form $\mathcal{N}/\Psi^T$, where $\mathcal{N}$ is a polynomial in (U, V) and T is a nonnegative integer. Consider now the point $((0, U_0'), V_0)$, such that U_0' and V_0 correspond respectively to the coefficients of the linear forms $\widetilde{L}_j$ in the x_j, $j = 1, \ldots, n$, (resp. $\widetilde{P}_j$ in the P_j, $j = 1, \ldots, n$) found in Proposition 4.1. It follows from Remark 4.1 that one can choose $\Phi(U, V) \neq 0$, close to $((0, U_0'), V_0)$, and such that the polynomial map

$$x \mapsto (U^1(x)^{NB} < V^1, P >, \ldots, U^n(x)^{NB} < V^n, P >) \tag{4.15}$$

is a proper map with Lojasiewicz exponent at least $\delta = \delta_N = (N-1)B$. Moreover, every entry of the map (4.15) has the same degree, namely, $D = D_1 + NB$. If the condition (4.7) is satisfied, we can apply Proposition 3.5 so that, from (3.24), we conclude that

$$\mathcal{N}(U, V) = \Psi(U, V)^T. \tag{4.16}$$

This follows from the fact that the determinant $\Delta_{N,U,V}(x, y)$ is a Bézoutian for the polynomial map (4.15). Therefore, the identity (4.16), which originally holds only in a neighborhood of $((0, U_0'), V_0)$ and outside the hypersurface $\Phi(U, V) = 0$, is valid everywhere. So, for any (U, V) outside the locus $\Phi = 0$, one can rewrite (4.16) as

$$1 = \frac{\mathcal{N}(U, V)}{\Psi(U, V)^T},$$

which is the identity (4.8). Thus we have completed the proof of the proposition. ∎

Remark 4.3. Since for any generic choice of V, the sequence $< V, P_1 >, \ldots, < V, P_n >$ is a normal sequence, any subfamily of these polynomials with cardinal $1 \leq k \leq n$ defines, for V generic, an algebraic variety in $\mathbf{K}^n$ with codimension at least k. Therefore, for any pair of subsets $\mathcal{J}, \mathcal{J}' \subset \{1, \ldots, n\}$ such that $\#\mathcal{J} + \#\mathcal{J}' = n + 1$, the polynomials

$$< V^j, P(x) >,\ j \in \mathcal{J};\ U^{j'}(x),\ j' \in \mathcal{J}'$$

(considered in $\mathbf{K}(U, V)[x_1, \ldots, x_n]$) define a non proper ideal in this ring. From [Ph1, Theorem 4], one can find, for any such pair of subsets $\mathcal{J}$, $\mathcal{J}'$, a polynomial $\Phi_{\mathcal{J},\mathcal{J}'} \in$

$\mathbf{K}[U,V]\setminus\{0\}$, of degree at most $2(n+1)^3D_1^n$, such that, whenever $\Phi_{\mathcal{J},\mathcal{J}'}(U,V)\neq 0$, then the polynomials

$$< V^j, P >,\ j\in\mathcal{J};\ U^{j'}(x),\ j'\in\mathcal{J}',$$

have no common zeros in $\mathbf{K}^n$. We will define the polynomial $\widetilde{\Phi}$ as the product of all the polynomials $\Phi_{\mathcal{J},\mathcal{J}'}$ for all possible choices of $\mathcal{J}$, $\mathcal{J}'$ such that $\#\mathcal{J}+\#\mathcal{J}'=n+1$. The degree of the polynomial $\widetilde{\Phi}$ is at most $2^{2n+1}(n+1)^3D_1^n$. It will be important for us later on the fact that if we choose (U,V) such that

$$\Phi(U,V)\widetilde{\Phi}(U,V)\neq 0\,, \tag{4.17}$$

then, the polynomials $\Theta_j(x):=U^j(x)< V^j,P(x)>$ satisfy the hypothesis required in Lemma 2.3 and the identity (4.8) for a convenient choice of N.

5. Size estimates.

In this section, $\mathbf{A}$ will be a unitary factorial regular integral domain with a size, its quotient field will be denoted by $\mathbf{K}$ and assumed to be infinite. The basic examples in the characteristic zero case are $\mathbf{A}=\mathbf{Z}[\tau_1,\dots,\tau_q]\,,q\in\mathbf{N}$ and in the characteristic $p>0$ case, $\mathbf{A}=\mathbf{F}_p[\tau_1,\dots,\tau_q]$.

Let us recall from [Ph1] that a size in $\mathbf{A}$ is a map $\mathbf{t}$ from $\mathrm{Pol}(\mathbf{A})=\mathbf{A}[(Y_i)_{i\in\mathbf{N}}]$ into $\{-\infty\}\cup[0,\infty[$ such that:

(**T0**) For any bijection σ of $\mathbf{N}$ into itself and for any $f\in\mathrm{Pol}(\mathbf{A})$ one has $\mathbf{t}(\tilde\sigma(f))=\mathbf{t}(f)$, where $\tilde\sigma$ is the isomorphism of the $\mathbf{A}$-module $\mathrm{Pol}(\mathbf{A})$ such that $\tilde\sigma(Y_i)=Y_{\sigma(i)}$.

(**T1**) $\mathbf{t}(0)=-\infty$, $\mathbf{t}(v)=0$ for any $v\in\mathbf{A}^*$, and $\mathbf{t}(Y_i)=0$ for any indeterminate Y_i.

(**T2**) $\mathbf{t}(fg)=\mathbf{t}(f)+\mathbf{t}(g)$.

(**T3**) There are constants $c,c'>0$ such that if $F=f_1Y^{\alpha_1}+\cdots+f_kY^{\alpha_k}$, where the Y^{α_j} are different monomials which do not appear in any of the elements $f_l\in\mathrm{Pol}(\mathbf{A})$, then

$$\mathbf{t}(F)\leq c\max_{1\leq l\leq k}(\mathbf{t}(f_l)+c'\deg(f_l)\log(m(f_l)+1))+c'\log k\,, \tag{5.1}$$

where $m(f_l)$ denotes the number of indeterminates Y_i that actually appear in f_l.

(**T4**) There is an additional constant $c''>0$ such that if $F=v_1f_1Y^{\alpha_1}+\cdots+v_kf_kY^{\alpha_k}$, with $v_1,\dots,v_k\in\mathbf{A}^*$ and the Y^{α_j} are pairwise distinct monomials of degree at most d in m indeterminates which do not appear in any of the elements $f_l\in\mathrm{Pol}(\mathbf{A})$, then

$$\max_{1\leq l\leq k}(\mathbf{t}(f_l))\leq c''\mathbf{t}(F)+c'd\log(m+1)\,. \tag{5.2}$$

To simplify the estimations in this paper , we shall assume that $c\geq c''$, and $c\geq 1$.

We will also need the following lemma, which is a simple consequence of these properties.

Lemma 5.1. *Let* $\mathbf{A}$ *be a ring with a size and* f *be an element in* $\mathbf{A}[\xi_1, \dots, \xi_L, Y_1, \dots, Y_K]$, *with size* $\mathbf{t}$ *(when considered as an element of* $\operatorname{Pol}(\mathbf{A})$ *and total degree* d *in all the variables* (ξ, Y). *There exist elements* $f_1, \dots, f_K$ *in* $\mathbf{A}[\xi, Y_1, \dots, Y_K, Z_1, \dots, Z_K]$ *such that*

$$f(\xi, Y_1 + Z_1, \dots, Y_K + Z_K) - f(\xi, Y_1, \dots, Y_K) = \sum_{i=1}^{K} f_i(\xi, Y, Z) Z_i$$

and

$$\max_{1 \le i \le K} \mathbf{t}(f_i) \le c^4 \big(c\mathbf{t} + 7c' d \log(L + 2K + 1) \big) . \tag{5.3}$$

Proof. Let $\tilde{f}$ be the element of $\operatorname{Pol}(\mathbf{A})$ defined as the polynomial in the $L + 2K$ variables (ξ, Y, Z) by

$$\tilde{f}(\xi, Y, Z) := f(\xi, Y_1 + Z_1, \dots, Y_K + Z_K) - f(\xi, Y_1, \dots, Y_K) .$$

If $\breve{f}(\xi, Y, Z) := f(\xi, Y_1 + Z_1, \dots, Y_K + Z_K)$, then it follows from (5.1) that

$$\mathbf{t}(\tilde{f}) \le c\big(\max(\mathbf{t}(f), \mathbf{t}(\breve{f})) + c' d \log(L + 2K + 1) \big) + c' \log 2 .$$

On the other hand, it follows from (5.2) that, if we develop $\tilde{f}$ as a polynomial in Z,

$$\tilde{f}(\xi, Y, Z) = \sum_{\beta \in (\mathbf{N}^K)^*} \tilde{f}_\beta(\xi, Y) Z^\beta ,$$

and we have the size estimates

$$\max_\beta \mathbf{t}(\tilde{f}_\beta) \le c\mathbf{t}(\tilde{f}) + c' d \log(K + 1) .$$

In order to estimate the size of $\tilde{f}$, we need to estimate the size of $\breve{f}$. For that purpose we develop f as a polynomial in Y.

$$f(\xi, Y) = \sum_{J \in \mathbf{N}^K} f_J(\xi) Y^J .$$

One has, again from (5.2),

$$\max_J \mathbf{t}(f_J) \le c\mathbf{t}(f) + c' d \log(K + 1) .$$

We also have

$$\mathbf{t}\Big(\prod_{i=1}^{K} (Y_i + Z_i)^{J_i} \Big) = \sum_{i=1}^{K} J_i \mathbf{t}(Y_i + Z_i) \le c' |J| \log 2 \le c' d \log 2 .$$

Therefore, we have

$$\mathbf{t}\Big(f_J(\xi)\prod_{i=1}^{K}(Y_i+Z_i)^{J_i}\Big) \le c\mathbf{t}(f)+c'd\log(2K+2)\,,$$

and so,

$$\mathbf{t}(\breve{f}) \le c^2\mathbf{t}+2cc'd\log(L+2K+1)+c'\binom{K+d}{K} \le c^2\mathbf{t}+3cc'd\log(L+2K+1)\,.$$

We have then

$$\mathbf{t}(\tilde{f}) \le c^2(c\mathbf{t}+4c'd\log(L+2K+1))+c'\log 2\,.$$

We now construct the f_j as follows: first, let

$$f_1(Y,Z) := \frac{1}{Z_1}\sum_{\beta,\beta_1>0}\tilde{f}_\beta(\xi,Y)Z^\beta\,.$$

Then, for $2\le j\le n$, we define

$$f_j(Y,Z) := \frac{1}{Z_j}\sum_{\substack{\beta,\beta_1=\cdots=\beta_{j-1}=0\\ \beta_j>0}}\tilde{f}_\beta(\xi,Y)Z^\beta\,.$$

The size estimate for f_j is given by (5.1), namely

$$\begin{aligned}\mathbf{t}(f_j) &\le c\big(\max_\beta \mathbf{t}(\tilde{f}_\beta)+c'(d-1)\log(L+2K+1)\big)+c'\log\binom{K+d}{K}\\ &\le c\big(c\mathbf{t}(\tilde{f})+3c'd\log(L+2K+1)\big)\,.\end{aligned}$$

These inequalities combine to give the conclusion of the lemma. ■

Let us now introduce the function ϑ_0 from $[0,\infty[$ to $\mathbf{N}\cup\{+\infty\}$ defined by

$$\vartheta_0(\xi) := \#\{a\in\mathbf{A}\,:\,\mathbf{t}(a)\le\xi\}\,.$$

This function is increasing, so we can consider its one-side inverse ϑ defined for $k\in\mathbf{N}\cup\{+\infty\}$ as

$$\vartheta(k) := \inf\{\xi\in[0,\infty[:\,\vartheta_0(\xi)\ge k\}\,.$$

This function will play a role in the estimates of sizes in the following way. If $0\ne\Phi\in\mathbf{A}[y_1,\ldots,y_q]$ has total degree D in the y variables, one can find elements $a_1,\ldots,a_q\in\mathbf{A}$ such that $\mathbf{t}(a_j)\le\vartheta(D+1)$ and $\Phi(a_1,\ldots,a_q)\ne0$. This is immediate by induction on q.

Example 5.1. If $\mathbf{A}[\mathcal{U}]$ is a polynomial ring, $\mathcal{U}$ being a finite set of indeterminates, a size $\mathbf{t}$ on $\mathbf{A}$ induces in a natural way a size on the polynomial ring $\mathbf{A}[\mathcal{U}]$: let τ be any homomorphism of $\mathbf{A}$-algebras between $\mathbf{A}[\mathcal{U}]$ and $\mathrm{Pol}(\mathbf{A})$ which injects $\mathcal{U}$ into $\{Y_i, i\in\mathbf{N}\}$;

such an homomorphism τ can be extended as an homomorphism from $\mathrm{Pol}(\mathbf{A}[\mathcal{U}])$ into $\mathrm{Pol}(\mathbf{A})$. One defines a size on $\mathbf{A}[\mathcal{U}]$ as

$$\mathbf{t}(f) = \mathbf{t}(\tau(f)),\ f \in \mathrm{Pol}(\mathbf{A}[\mathcal{U}])\,.$$

Example 5.2. On $\mathbf{A}[\mathcal{U}]$, where $\mathcal{U} = \{u_1, \ldots, u_q\}$ is a finite set of indeterminates, there is another way to define a size, completely independent of the fact that $\mathbf{A}$ may be equipped with a size, namely

$$\breve{\mathbf{t}}(f) = \deg_{\mathcal{U}} f,\ f \in \mathrm{Pol}(\mathbf{A}[\mathcal{U}])\,.$$

When $\mathbf{A}$ is equipped with a size, one can combine on $\mathbf{A}[\mathcal{U}]$ the size $\mathbf{t}$ in Example 5.1 and the size $\breve{\mathbf{t}}$ of Example 5.2. For any positive constant C,

$$\mathbf{t}_C : f \in \mathrm{Pol}(\mathbf{A}[\mathcal{U}]) \mapsto C\breve{\mathbf{t}}(f) + \mathbf{t}(f) \tag{5.4}$$

is a size on $\mathbf{A}[\mathcal{U}]$. Moreover, one can see that conditions T3 and T4 for this size are satisfied with constants independent of C. For instance, to verify T3 we let

$$f = \sum_{l=1}^{k} f_l v_l Y^{\alpha_l} = \sum_{l=1}^{k} \sum_{\beta \in \mathbf{N}^q} f_{l\beta} v_l u^\beta Y^{\alpha_l}$$

where $v_l \in \mathbf{A}[\mathcal{U}]^* = \mathbf{A}^*$, $f_l = \sum_{\beta\in\mathbf{N}^q} f_{l\beta} u^\beta \in \mathbf{A}[\mathcal{U}]$ and the Y^{α_l} do not contain any coordinate involved in one of the f_l. It follows from (5.1) and (5.2) that, if c, c', c'' are the constants relative to the size $\mathbf{t}$,

$$\begin{aligned}\mathbf{t}_C(f) &\le c \max_{l,\beta} \big(\mathbf{t}(f_{l\beta}) + c' \deg f_{l\beta} \log(m(f_{l\beta}) + 1)\big) + c' \log\Big[k \binom{\deg_{\mathcal{U}} f + q}{q}\Big] + C \deg_{\mathcal{U}} f \\ &\le cc'' \max_l \Big(\mathbf{t}(f_l) + \frac{c'}{c''} \deg f_l \log(m(f_l) + 1)\Big) + (C + c'(c+1)\log(q+1)) \deg_{\mathcal{U}} f + c' \log k\end{aligned}$$

Therefore, provided that $C \ge c'(c+1)\log(q+1)$,

$$\mathbf{t}_C(f) \le (cc''+2) \max_l \left(\mathbf{t}_C(f_l) + \frac{cc'}{cc''+2} \deg f_l\ \log(m(f_l)+1)\right) + c' \log k\,. \tag{5.5}$$

On the other hand, since

$$\deg_{\mathcal{U}} \left(\sum_{l=1}^{k} v_l f_l Y^{\alpha_l}\right) = \max_l (\deg_{\mathcal{U}} f_l)\,,$$

when the monomials Y^{α_l} are distinct and do not involve any $\mathcal{U}$ variable, any size $\mathbf{t}_C$ on $\mathbf{A}[\mathcal{U}]$ satisfies condition T4, namely, if $F = \sum_{l=1}^{k} v_l f_l Y^{\alpha_l}$ with distinct monomials Y^{α_l} which do not involve coordinates appearing in one of the f_l,

$$\max_{1\le l\le k} (\mathbf{t}_C(f_l)) \le (c''+1)\mathbf{t}_C(F) + c' d \log(m+1)\,. \tag{5.6}$$

where m is the number of variables involved in the Y^{α_l}, d is the maximum of the degrees of these monomials, and the constants c', c'', are the constants related to the size $\mathbf{t}$ on $\mathbf{A}$ by (5.3).

Example 5.3. On $\mathbf{F}_p[\tau_1, \ldots, \tau_q]$ there are several natural choices of size, for instance, given a multiindex $m \in \mathbf{N}^q$ we define

$$\mathbf{t}_m(f) = \deg_\tau(f(\tau_1^{m_1}, \ldots, \tau_q^{m_q})) \tag{5.7}$$

whenever $f \in \mathrm{Pol}(\mathbf{F}_p[\tau])$. The constants for such a size are $c = c'' = 1, c' = 0$. In this example it is easy to compute the function ϑ_0 explicitly. For instance when $m = (1, \ldots, 1)$, if $[\xi]$ denotes the integral part of ξ, then

$$\vartheta_0(\xi) = p^{d(\xi)} = e^{d(\xi)\log p}, \quad \text{where } d(\xi) = \binom{q + [\xi]}{q}.$$

Thus, it is finite-valued and

$$\vartheta(k) \simeq (\log k / \log p)^{1/q}.$$

The computation for other values of m is similar.

Example 5.4. For $\mathbf{A}[\mathcal{U}] = \mathbf{Z}[u_1, \ldots, u_q]$ and a positive constant C, we have a size $\mathbf{t}_C$, associated to the Mahler measure $\mathbf{t}$ over $\mathbf{Z}$, as done in [BY1, (4.9)]. The Mahler measure is defined as follows, for $f \in \mathrm{Pol}(\mathbf{Z})$ depending on m variables, we integrate on the torus $\mathbf{T}^m$, with respect to the normalized measure $d\xi$, and let

$$\mathbf{t}(f) = \int_{\mathbf{T}^m} \log|f(\xi)| d\xi.$$

In this particular case, if $C \geq 2\log(q+1)$ then we can take $c = 3$, $c' = 1$, and $c'' = 2$ for the constants corresponding to the size $\mathbf{t}_C$. Similarly, the function ϑ_0 corresponding to the Mahler measure is approximately the exponential function, so $\vartheta(k) \simeq \log k$.

Notation. From now on (in this section and the following one), when we consider the constants c, c', c'' relative to a size $\mathbf{t}$, we will take them as the constants relative to the size $\mathbf{t}_C$ in example 5.3 (for C large enough.) Generally speaking, this means we replace the original values of c and c'' by $cc'' + 2$ and $c'' + 1$ (see Example 5.2.); the constant c' remains unchanged. Of course, in some particular situations, one can make better choices for c, c', c'' (see Remarks 5.2 and 5.3 below.)

As a consequence of Theorem 4 in [Ph1], we have the following result.

Lemma 5.2. *Let $\mathbf{A}$ be any factorial regular integral domain, with size $\mathbf{t}$, of Krull dimension κ and quotient field $\mathbf{K}$. Assume that $p_1, \ldots, p_n$ are elements in $\mathbf{A}[x_1, \ldots, x_n]$, which are algebraically independent over $\mathbf{K}$ and such that $x \mapsto (p_1(x), \ldots, p_n(x))$ is a dominant polynomial map from $\mathbf{K}^n$ to $\mathbf{K}^n$. Let also q be a given polynomial in $\mathbf{A}[x_1, \ldots, x_n]$. Let*

$$h := \max(\mathbf{t}(p_j),\ 1 \leq j \leq n, \mathbf{t}(q), c' \log(n+2))$$

and

$$\sigma := \sum_{l=1}^{n} \frac{1}{\deg p_l} + \frac{1}{\deg q}\,.$$

Then, there exists a polynomial $S_q \in \mathbf{A}[x_0, x_1, \dots, x_n]$ and a positive constant ϖ, which depends on n, κ, c, c', c'', such that

$$S_q(q, p_1, \dots, p_n) = 0\,, \tag{5.8}$$

$$\deg S_q \leq \varpi(2\sigma + 1) \deg q \prod_{l=1}^{n} \deg p_l \tag{5.9}$$

and

$$\mathbf{t}(S_q) \leq \varpi\big(1 + 2\sigma c(h + c' D \log(2n+2))\big) \deg q \prod_{l=1}^{n} \deg p_l \tag{5.10}$$

where $D := \max(\deg q,\, \deg p_j)$.

Remark 5.1. The fact that there exists a polynomial S_q satisfying (5.8) and (5.9) with $\varpi = 1, \sigma = 0$, is a consequence of the theorem of Perron [Pe, Satz 57]. Since we are interested in size estimates, this theorem is not sufficient for us.

Proof. We consider the polynomial ring $\mathbf{A}[u_0, \dots, u_n]$ equipped with the size $\mathbf{t}_C$ from Example 5.2, with C to be chosen later sufficiently large. Since the polynomials $p_1, \dots, p_n, q$ are algebraically dependent, the ideal generated by the polynomials $q - u_0, p_1 - u_1, \dots, p_n - u_n$ in $\mathbf{A}(\mathcal{U})[x]$ is a non proper ideal, and therefore, by [Phi, Theorem 4], one can find an element $S_q(u_0, u_1, \dots, u_n) \in \mathbf{A}[\mathcal{U}]$ which belongs to the ideal generated by $q - u_0, p_1 - u_1, \dots, p_n - u_n$ in $\mathbf{A}[\mathcal{U}, x]$ and such that

$$\mathbf{t}_C(S_q) \leq \varpi\big(1 + \sigma \max(\mathbf{t}_C(p_j - u_j)\,, 1 \leq j \leq n, \mathbf{t}_C(q - u_0))\big) \deg q \prod_{l=1}^{n} \deg p_l\,, \tag{5.11}$$

where ϖ depends only on c, c', c'', κ. From the inequality (5.1) we conclude that

$$\begin{aligned}\max(\mathbf{t}_C(p_j - u_j)\,, 1 \leq j \leq n, \mathbf{t}_C(q - u_0)) &\leq c(h + c' D \log(n+1)) + c' \log 2 + C \\ &\leq c(h + c' D \log(2n+2)) + C\,.\end{aligned}$$

From (5.11) we have

$$\mathbf{t}(S_q) + C \deg(S_q) \leq \varpi\big(1 + \sigma(c(h + c' D \log(2n+2)) + C)\big) \deg q \prod_{l=1}^{n} \deg p_l\,, \tag{5.12}$$

that is,

$$\deg(S_q) \le \varpi \left(\sigma + \frac{1 + \sigma c(h + c' D \log(2n+2))}{C}\right) \deg q \prod_{l=1}^{n} \deg p_l \,.$$

If we choose $C \simeq c(h + c' D \log(2n+2))$ then $C \ge c'(c+1)\log(n+2)$ provided $h \ge c' \log(n+2)$, and we have then

$$\deg(S_q) \le \varpi(2\sigma + 1) \deg q \deg p_1 \cdots \deg p_n \,.$$

With this choice of C we get also

$$\mathbf{t}(S_q) \le \varpi\big(1 + 2\sigma c(h + c' D \log(2n+2))\big) \deg q \prod_{l=1}^{n} \deg p_l \,.$$

This concludes the proof of the lemma. ■

Remark 5.2. When $\mathbf{A} = \mathbf{F}_p[\tau_1, \ldots, \tau_q]$ we can take as in Example 5.3 the size $\mathbf{t}_C$ in $\mathbf{A}[u_0, \ldots, u_n]$ defined by

$$\mathbf{t}_C(f) = \deg_{\tau,u} f(\tau_1, \ldots, \tau_q, u_0^C, \ldots, u_n^C) \text{ and } C \in \mathbf{N} \,.$$

As we have pointed out before, the constants relative to this size are independent of C and thus coincide with those for $C = 0$, that is, $c = c'' = 1$, $c' = 0$. We can take here $h = \max(\mathbf{t}(p_j),\, 1 \le j \le n, \mathbf{t}(q))$ and $\varpi = 2n + q + 1$ in Lemma 5.2.

Remark 5.3. When $\mathbf{A} = \mathbf{Z}$ and $\mathbf{t}$ is the size corresponding to the Mahler measure, we have $c = c' = c'' = 1$ and the constants corresponding to $\mathbf{t}_C$ are respectively, $c = 3$, $c' = 1$, and $c'' = 2$. In this case, $\varpi = 9(n+1)2^{n+2}\big(1 + 4\log(n+1)\big)^{n+2}$.

Remark 5.4. The last two remarks use the estimates in [Ph1, Theorem 4], but for $\mathbf{A} = \mathbf{Z}$ we could have also used the height estimates from the Arithmetic Bézout Theorem from [BGS, Section 5.4]. The estimates in the last paper are more natural but they are expressed in terms of Faltings heights instead of the Mahler measure. We refer to [Ph2] for a comparison of these two points of view.

Lemma 5.2 will be crucial for the estimates of multidimensional residues. Such estimates are given by the following result.

Lemma 5.3. *Let $m \ge n$ and $p_1, \ldots, p_m$ a family of polynomials in $\mathbf{A}[x]$, where $\mathbf{A}$ is a regular factorial domain (with infinite quotient field $\mathbf{K}$) equipped with a size $\mathbf{t}$. Let their ordering by degrees $\deg p_j = D_j$ satisfy $D_m \ge D_{m-1} \ge \ldots \ge D_{n+1}$ and $D_m \ge D_1 \ge D_2 \ge \cdots \ge D_n$. Assume also that $p_1, \ldots, p_n$ is a quasiregular sequence and that the product $\prod_{j=n+1}^{m} p_j$ does not vanish on the set of common zeros of $p_1, \ldots, p_n$. Let h and $\widetilde{D}$ be defined by*

$$h := \max(\mathbf{t}(p_j),\, 1 \le j \le m,\, c' \log(n+2))\,, \quad \widetilde{D} = D_1 \cdots D_n \,.$$

Then, for any multiindices $J \in \mathbf{N}^n$ and $k \in \mathbf{N}^m$ we have

$$\text{Res}\begin{bmatrix} x^J dx \Big/ \Big(\prod_{j=n+1}^{m} p_j^{k_j+1} \Big) \\ p_1^{k_1+1}, \ldots, p_n^{k_n+1} \end{bmatrix} = \frac{r_1}{r_2}, \tag{5.13}$$

where $r_1 = r_1(J,k)$, $r_2 = r_2(J,k) \in \mathbf{A}$ satisfy

(5.14′)

$$\begin{aligned} \mathbf{t}(r_1) \le\ & C_0 \varpi^4 n^7 c^{12} \widetilde{D}(|k| + \widetilde{D} + m)(|J| + D_m^2 \widetilde{D}(|k| + \widetilde{D} + m))(h + c' D_m \log(2n+2)) \\ & + c^2 c' C_0 \varpi^4 n^7 (|k| + m) D_m^2 \widetilde{D}^4 \log(n+1)\,. \end{aligned}$$

and

$$\mathbf{t}(r_2) \le C_0 \varpi^4 n^7 c^4 \widetilde{D}(|k|+\widetilde{D}+m)(|J|+D_m^2 \widetilde{D}(|k|+\widetilde{D}+m))(h+c' D_m \log(2n+2))\,, \tag{5.14″}$$

where C_0 is an absolute constant (independent of n and of the size.) Moreover, the same denominator $0 \ne r_2$ can be used if one replaces $k_1, \ldots, k_n$ by any $n-$uplet $(l_1, \ldots, l_n)$ of integers such that $|l| \le |k|$ and J by any multiindex J' such that $|J'| \le |J|$.

Proof. All along the proof of this lemma, C_0 will denote an absolute constant (independent of n and of the size.) Since the proof is rather technical, we will never make this constant explicit. Nevertheless, a careful look at the estimates shows that this constant remains below 10^3.

One can assume that the p_j, $j = 1, \ldots, n$ generate a proper ideal (otherwise all residue symbols (5.13) would be 0). In order to compute the residue symbols (5.13) we use Lemma 3.1 and Remark 3.2, which imply the following: if $\alpha \in \mathbf{A}^n$, and $k' = (k_1, \ldots, k_n)$, one has

$$\begin{aligned} & \text{Res}\begin{bmatrix} x^J dt \wedge dx \Big/ \Big(\prod_{j=n+1}^{m} p_j^{k_j+1} \Big) \\ t^{|k'|+1}, p_1(x) - \alpha_1 t, \ldots, p_n(x) - \alpha_n t \end{bmatrix} \\ & = \sum_{\substack{l \in \mathbf{N}^n \\ |l| = |k'|}} \text{Res}\begin{bmatrix} x^J dx \Big/ \Big(\prod_{j=n+1}^{m} p_j^{k_j+1} \Big) \\ p_1(x)^{l_1+1}, \ldots, p_n(x)^{l_n+1} \end{bmatrix} \alpha^l\,. \end{aligned} \tag{5.15}$$

We now rewrite the left hand side of (5.15) (for α fixed) using Lemma 5.2. We first apply this lemma to the polynomials $p_1, \ldots, p_n$ and x_1. (Later, x_1 will be replaced by the other coordinates x_j.) Since the ideal generated by the p_j is a proper ideal and the sequence is quasiregular, we know that these polynomials are algebraically independent. One can find a polynomial $\mathcal{Q}_1$ in $\mathbf{A}[u_0, u_1, \ldots, u_n]$ which contains at least two monomials with distinct powers of u_0 and is such that

$$\mathcal{Q}_1(x_1, p_1(x), \ldots, p_n(x)) \equiv 0\,. \tag{5.16}$$

The total degree of $\mathcal{Q}_1$ is at most $\varpi(2\sigma+1)\widetilde{D}$ and its size is at most $\varpi[1+2\sigma c(h+D_1\log(2n+2))]\widetilde{D}$, where $\sigma = 1+\sum_{j=1}^{n}\frac{1}{D_j}$ and ϖ is the constant associated to the size as in Lemma 5.2. Let s_1 be the valuation in $u_1,\dots,u_n$ of the polynomial $\mathcal{Q}_1$, and write

$$\mathcal{Q}_1(u) = \sum_{|l|=s_1} u_1^{l_1}\cdots u_n^{l_n} a_{1l}(u_0) + \widetilde{\mathcal{Q}_1}(u)\,,$$

where $\widetilde{\mathcal{Q}_1}$ contains all the monomial terms whose degrees in the last n variables exceed s_1. Clearly, we can do the same for the other variables $x_j\,, j>1$. The polynomial $\mathcal{Q}_j$ we construct will satisfy the following estimates (taking $\sigma \le n+1$)

$$\left\{\begin{array}{l}\deg(\mathcal{Q}_j) \le \varpi(2n+3)\widetilde{D}\\ \mathbf{t}(\mathcal{Q}_j) \le c\varpi(2n+3)\widetilde{D}(h+c'D_1\log(2n+2))\end{array}\right. \qquad 1\le j\le n\,. \tag{5.17}$$

In order to simplify the estimates, we replace below $2n+3$ by $3(n+1)$, and perform other similar simplifications, they do not affect the order of magnitude of the estimates except for a multiplicative constant.

Similarly, for any $1\le j\le n$, we can rewrite the polynomials $\mathcal{Q}_j$ as

$$\mathcal{Q}_j(u) = \sum_{|l|=s_j} u_1^{l_1}\cdots u_n^{l_n} a_{jl}(u_0) + \widetilde{\mathcal{Q}_j}(u)\,,\ j=1,\dots,n.$$

For any $\alpha\in\mathbf{A}^n$, one has

$$\mathcal{Q}_j(u_0,\alpha_1 t,\dots,\alpha_n t) = t^{s_j}\left(R_j(u_0,\alpha) - tS_j(u_0,\alpha,t)\right),\ j=1,\dots,n. \tag{5.18}$$

where

$$R_j(u_0,\alpha) = \sum_{|l|=s_j}\alpha^l a_{jl}(u_0)\,,\ j=1,\dots,n.$$

Moreover, as we have seen in Section 3, we can rewrite (5.16) as

$$\mathcal{Q}_j(x_j,u') = \mathcal{Q}_j(x_j,u'-p'+p') = \sum_{l=1}^{n}\mathcal{Q}_{jl}(x_j,p',u')(p_l-u_l)$$

where $p' := (p_1,\dots,p_n)$, $u' := (u_1,\dots,u_n)$. We will denote as $\Delta(x,p',u')$ the determinant of the matrix $[\mathcal{Q}_{jl}]_{\substack{1\le j\le n\\ 1\le l\le n}}$.

One can also apply Lemma 5.2 with $q=p_j$, $j=m+1,\dots,n$. For any such j, there exists a polynomial $\mathcal{Q}_j$ in $\mathbf{A}[u_0,u_1,\dots,u_n]$ which contains at least two monomials with distinct powers of u_0 and is such that

$$\mathcal{Q}_j(p_j,p_1,\dots,p_n)\equiv 0\,,\ j=n+1,\dots,m. \tag{5.19}$$

The total degree of $\mathcal{Q}_j$, $n+1 \le j \le m$, is at most $\varpi(2\sigma_j+1)\widetilde{D}D_j$ and its size is at most $\varpi\big(1+2\sigma_j c(h+c'\widetilde{D}_j\log(2n+2))\big)D_1\cdots D_n D_j$, where $\widetilde{D}_j := \max(D_j, D_1, \ldots, D_n)$, $\sigma_j := \dfrac{1}{D_j} + \displaystyle\sum_{l=1}^{n}\dfrac{1}{D_l}$. Moreover, we can assume (if not, divide (5.19) by a power of p_j), that $\mathcal{Q}_j(0, u_1, \ldots, u_n) \not\equiv 0$. The estimates we will use later for these polynomials are

$$\begin{cases} \deg(\mathcal{Q}_j) \le 3\varpi(n+1)D_m\widetilde{D} \\ \mathbf{t}(\mathcal{Q}_j) \le 3c\varpi(n+1)D_m\widetilde{D}(h+c'D_m\log(2n+2)) \end{cases} \qquad n+1 \le j \le m\,. \tag{5.20}$$

Let s_j, $j = n+1, \ldots, m$ be the valuation (in $u_1, \ldots, u_n$) of the polynomial $\mathcal{Q}_j(0, u_1, \ldots, u_n)$ and define the polynomials T_j of $n+1$ variables by

$$\mathcal{Q}_j(0, \alpha_1 t, \ldots, \alpha_n t) = t^{s_j} T_j(\alpha, t)\,. \tag{5.21}$$

Let $\alpha \in \mathbf{A}^n$ be such that the polynomial $T_j(\alpha, 0) \neq 0$. This condition is generic. For $n+1 \le j \le m$, let

$$\mathcal{Q}_j(u) = \sum_{l=0}^{d_j-1} q_{jl}(u_1, \ldots, u_n)u_0^{d_j-l} + \mathcal{Q}_j(0, u_1, \ldots, u_n)\,.$$

For such a generic α, the corresponding polynomials in the $n+1$ variables $t, x_1, \ldots, x_n$, defined by

$$t^{s_j}T_j(\alpha, t) - \sum_{l=0}^{d_j-1} q_{jl}(\alpha_1 t, \ldots, \alpha_n t)p_j(x)^{d_j-l}\,,$$

are in the ideal generated by $p_1(x) - \alpha_1 t, \ldots, p_n(x) - \alpha_n t$ in $\mathbf{A}[t, x]$.

For $\alpha \in \mathbf{A}^n$ generic, we can assume also that the polynomial in u_0

$$\prod_{j=1}^{n} R_j(u_0, \alpha)$$

is not identically zero. The left hand side of (5.15) can be rewritten, using Proposition 3.6 and Remark 3.2, as

(5.22)

$$\operatorname{Res}\begin{bmatrix} x^J dt\wedge dx \Big/ \Big(\displaystyle\prod_{j=n+1}^{m} p_j^{k_j+1}\Big) \\ t^{|k'|+1}, p_1(x)-\alpha_1 t, \ldots, p_n(x)-\alpha_n t \end{bmatrix} =$$

$$= \operatorname{Res}\begin{bmatrix} x^J \displaystyle\prod_{j=n+1}^{m}\Big(\frac{\sum_{i=0}^{d_j-1} q_{ji}(\alpha t)p_j^{d_j-i-1}}{T_j(\alpha, t)}\Big)^{k_j+1} dt\wedge dx \\ t^{\kappa+1}, p_1(x)-\alpha_1 t, \ldots, p_n(x)-\alpha_n t \end{bmatrix}$$

$$= \sum_{\substack{l\in\mathbf{N}^n \\ |l|\le\kappa'}} \operatorname{Res}\begin{bmatrix} x^J\Delta(x, p', \alpha t) \displaystyle\prod_{j=n+1}^{m}\Big(\frac{\sum_{i=0}^{d_j-1} q_{ji}(\alpha t)p_j^{d_j-i-1}}{T_j(\alpha, t)}\Big)^{k_j+1} \prod_{j=1}^{n} S_j^{l_j}(x_j, \alpha, t)dt\wedge dx \\ t^{\kappa+1+|s'|-|l|}, R_1(x_1, \alpha)^{l_1+1}, \ldots, R_n(x_n, \alpha)^{l_n+1} \end{bmatrix}$$

where $s' := (s_1, \ldots, s_n)$, $k' := (k_1, \ldots, k_n)$, $\kappa := |k'| + \sum_{j=n+1}^m s_j(k_j + 1)$, and $\kappa' := |k'| + |s'|$. Formula (5.22) leads to computations in one variable, which are easy to perform thanks to formula (2.10). These computations show that for α generic we can write

$$\mathrm{Res}\left[\begin{matrix} x^J dt \wedge dx \Big/ \Big(\prod_{j=n+1}^m p_j^{k_j+1} \Big) \\ t^{|k'|+1}, p_1(x) - \alpha_1 t, \ldots, p_n(x) - \alpha_n t \end{matrix}\right] =: \frac{R_{1,J,k}(\alpha)}{R_{2,J,k}(\alpha)} ,$$

where $R_{1,J,k}$ and $R_{2,J,k}$ are in $\mathbf{A}[\alpha_1, \ldots, \alpha_n]$ and they are totally explicit. Later on, we shall give estimates for their sizes. From formula (5.15), we have

$$\frac{R_{1,J,k}(\alpha)}{R_{2,J,k}(\alpha)} = \sum_{\substack{l \in \mathbf{N}^n \\ |l| = |k'|}} \mathrm{Res}\left[\begin{matrix} x^J dx \Big/ \Big(\prod_{j=n+1}^m p_j^{k_j+1} \Big) \\ p_1(x)^{l_1+1}, \ldots, p_n(x)^{l_n+1} \end{matrix}\right] \alpha^l .$$

Let r_2 be a common denominator for all the residue symbols

$$\mathrm{Res}\left[\begin{matrix} x^J dx \Big/ \Big(\prod_{j=n+1}^m p_j^{k_j+1} \Big) \\ p_1(x)^{l_1+1}, \ldots, p_n(x)^{l_n+1} \end{matrix}\right] , \ |l| = |k'|.$$

We have

$$\frac{R_{1,J,k}(\alpha)}{R_{2,J,k}(\alpha)} = \frac{R_{J,k}(\alpha)}{r_2} \tag{5.23}$$

where

$$R_{J,k}(\alpha) := \sum_{\substack{l \in \mathbf{N}^n \\ |l| = |k'|}} r_2 \mathrm{Res}\left[\begin{matrix} x^J dx \Big/ \Big(\prod_{j=n+1}^m p_j^{k_j+1} \Big) \\ p_1(x)^{l_1+1}, \ldots, p_n(x)^{l_n+1} \end{matrix}\right] \alpha^l \in \mathbf{A}[\alpha] .$$

We can rewrite (5.23) as the polynomial identity

$$r_2 R_{1,J,k}(\alpha) = R_{J,k}(\alpha) R_{2,J,k}(\alpha)$$

in the factorial ring $\mathbf{A}[\alpha]$. Since one can assume that $R_{J,k}/r_2$ is in reduced form, r_2 divides $R_{2,J,k}$ in $\mathbf{A}[\alpha]$. Therefore, one has

$$\mathbf{t}(r_2) \leq \mathbf{t}(R_{2,J,k}). \tag{5.24}$$

This implies that $R_{J,k}$ divides $R_{1,J,k}$, which gives $\mathbf{t}(R_{J,k}) \leq \mathbf{t}(R_{1,J,k})$. From the condition T4 (inequality (5.2)), one has

$$\mathbf{t}\left(r_2 \mathrm{Res}\left[\begin{matrix} x^J dx \Big/ \Big(\prod_{j=n+1}^m p_j^{k_j+1} \Big) \\ p_1(x)^{k_1+1}, \ldots, p_n(x)^{k_n+1} \end{matrix}\right] \right) \leq c'' \mathbf{t}(R_{1,J,k}) + c'|k'| \log(n+1) .$$

As a consequence of this inequality and (5.24), we get

$$\operatorname{Res}\left[\begin{array}{c} x^J dx\Big/\Big(\prod_{j=n+1}^{m} p_j^{k_j+1}\Big) \\ p_1(x)^{k_1+1},\ldots,p_n(x)^{k_n+1}\end{array}\right] = \frac{r_1}{r_2},$$

with

$$\max(\mathbf{t}(r_1),\mathbf{t}(r_2)) \le \max(\mathbf{t}(R_{2,J,k}), c''\mathbf{t}(R_{1,J,k})) + c'|k'|\log(n+1)\,. \tag{5.25}$$

It remains to give estimates for the sizes of the polynomials $R_{1,J,k}$ and $R_{2,J,k}$, which we will do now. Note that one can use the same denominator r_2 for all the residue symbols

$$\operatorname{Res}\left[\begin{array}{c} x^J dx\Big/\Big(\prod_{j=n+1}^{m} p_j^{k_j+1}\Big) \\ p_1(x)^{l_1+1},\ldots,p_n(x)^{l_n+1}\end{array}\right]$$

with $|l| = |k'|$.

The final estimates for $R_{1,J,k}$ and $R_{2,J,k}$ are done using formula (5.22). We need to compute explicitly a residue symbol of the form

$$\begin{aligned}&\operatorname{Res}[l] := \\ &= \operatorname{Res}\left[\begin{array}{c} x^J\Delta(x,p',\alpha t)\prod_{j=n+1}^{m}\Big(\frac{\sum_{i=0}^{d_j-1} q_{ji}(\alpha t)p_j^{d_j-i-1}}{T_j(\alpha,t)}\Big)^{k_j+1}\prod_{j=1}^{n} S_j^{l_j}(x_j,\alpha,t)dt\wedge dx \\ t^{\kappa+1+|s'|-|l|}, R_1^{l_1+1}(x_1,\alpha),\ldots,R_n^{l_n+1}(x_n,\alpha)\end{array}\right]\end{aligned} \tag{5.26}$$

where $l \in \mathbf{N}^n$ and $|l| = l_1+\cdots+l_n \le |k'|+|s'|$. We will keep l fixed for the moment. Let us estimate first the degrees in the variables x and t, of the polynomial

$$\Upsilon(\alpha,x,t) := x^J\Delta(x,p',\alpha t)\prod_{j=n+1}^{m}\Big(\sum_{i=0}^{d_j-1} q_{ji}(\alpha t)p_j^{d_j-i-1}\Big)^{k_j+1}\prod_{j=1}^{n} S_j^{l_j}(x_j,\alpha,t)\,.$$

In terms of the $\tilde{d}_j := \deg \mathcal{Q}_j$ and $d := \max(\tilde{d}_j)$, the degree of Υ in any x_i is at most

$$|J| + D_1\sum_{j=1}^{n}(\tilde{d}_j-1) + \sum_{j=n+1}^{m}\tilde{d}_jD_j(k_j+1) + \sum_{j=1}^{n} l_j\tilde{d}_j \le |J| + ndD_1 + dD_m(|k|+m)\,.$$

Since $d \le \varpi(2n+3)D_m\widetilde{D}$ by (5.20), we obtain

$$\deg_{x_i}\Upsilon \le |J| + C_0\varpi^2n^3D_m^2\widetilde{D}(|k|+\widetilde{D}+m)\,. \tag{5.27}$$

Similarly,

$$\deg_t \Upsilon \leq \sum_{j=1}^{n}(\tilde{d}_j - 1) + \sum_{j=n+1}^{m} \tilde{d}_j(k_j+1) + \sum_{j=1}^{n}(\tilde{d}_j - s_j - 1)l_j \leq d(|k|+m) + n^2 d^2 \,.$$

Therefore,

$$\deg_t \Upsilon \leq C_0 \varpi^2 n^4 D_m \widetilde{D}(D_m \widetilde{D} + |k| + m)\,. \tag{5.28}$$

We need now to estimate the size of the polynomial, in α, x_j, $R_j^{l_j+1}(x_j,\alpha)$. If the a_{jl} are the coefficients in the expansion of R_j as a polynomial in α, then, it follows from property (5.2) of the size that we have

$$\mathbf{t}(a_{ji}(x_j)) \leq c\mathbf{t}(\mathcal{Q}_j) + c' d_j \log(n+1)\,, \tag{5.29}$$

so that using property (5.1) we obtain

$$\begin{aligned} \mathbf{t}(R_j) &\leq c \max_i(\mathbf{t}(a_{ji}) + c' d_j \log 2) + c' \log \binom{s_j+n}{s_j} \\ &\leq c^2 \mathbf{t}(\mathcal{Q}_j) + cc' d_j \log(2n+2) + c' n \log(s_j+1)\,. \end{aligned}$$

Therefore, from the estimate (5.17) we conclude that

$$\begin{aligned} \mathbf{t}(R_j^{l_j+1}) &\leq (|k'| + n \max_{1\leq\iota\leq n}(d_\iota) + 1)(c^2\mathbf{t}(\mathcal{Q}_j) + cc'd_j\log(2n+2) + c'n\log(s_j+1)) \\ &\leq (|k| + nd + 1)\big(c\varpi(2n+3)\widetilde{D}(c^2 h + c^2 c' D_1 \log(2n+2)) + \\ &\quad + cc'\widetilde{D}\log(2n+2) + c'n\log[\varpi(2n+3)\widetilde{D}]\big) \\ &\leq C_0 \varpi^2 n^3 c^3 \widetilde{D}(|k| + \widetilde{D})\big(h + c' D_1 \log(2n+2)\big) \end{aligned} \tag{5.30}$$

Similar size estimates hold for the polynomials $S_j^{l_j}$ (as polynomials in x_j, α, and t, of total degree $2(d_j - s_j - 1)l_j$.) Namely,

$$\mathbf{t}(S_j^{l_j}) \leq C_0 \varpi^2 c^3 n^3 \widetilde{D}(|k| + \widetilde{D})\big[h + c' D_1 \log(2n+2)\big]\,. \tag{5.31}$$

We can do the same for the sizes of the polynomials T_j, $n+1 \leq j \leq m$. Suppose first that for $n+1 \leq j \leq m$,

$$\mathcal{Q}_j(0,u') = \sum_i b_{ji} u_1^{i_1} \cdots u_n^{i_n}\,.$$

From the definition (5.21) and property (5.2) of the size we have

$$\mathbf{t}(b_{ji}) \leq c\mathbf{t}(\mathcal{Q}_j) + c'\tilde{d}_j \log(n+2)\,,$$

so that using (5.2) we obtain

$$\begin{aligned}\mathbf{t}(T_j) &\le c\max_i(\mathbf{t}(b_{ji})+c'\log 2)+c'\log\binom{2\tilde{d}_j+n+1}{2\tilde{d}_j}\\&\le c^2\mathbf{t}(\mathcal{Q}_j)+cc'\tilde{d}_j\log(2n+2)+c'(n+1)\log(2\tilde{d}_j+1)\,.\end{aligned}$$

Therefore, recalling (5.20), we conclude that, for $n+1\le j\le m$,

$$\begin{aligned}\mathbf{t}(T_j^{k_j+1}) &\le (k_j+1)(c^2\mathbf{t}(\mathcal{Q}_j)+cc'\tilde{d}_j\log(2n+2)+c'(n+1)\log(2\tilde{d}_j+1))\\&\le C_0\varpi n(k_j+1)c^3D_m\widetilde{D}\big(h+c'D_m\log(2n+2)\big)\,.\end{aligned}\tag{5.32}$$

For $n+1\le j\le m$, let us write $T_j(\alpha,t)=v_{j0}(\alpha)+t\widetilde{T}_j(\alpha,t)$. Let us define the polynomial $\breve{T}_j$ by

$$(v_{j0}(\alpha)-T_j(\alpha,t))^{\kappa+|s'|}=v_{j0}^{\kappa+|s'|}(\alpha)-T_j(\alpha,t)\breve{T}_j(\alpha,t)\,.$$

Hence,

$$1=T_j(\alpha,t)\frac{\breve{T}_j(\alpha,t)}{v_{j0}^{\kappa+|s'|}(\alpha)}+t^{\kappa+|s'|}\frac{\breve{T}_j(\alpha,t)}{v_{j0}^{\kappa+|s'|}(\alpha)}\,,$$

where $\breve{T}_j(\alpha,t)=(-\widetilde{T}_j(\alpha,t))^{\kappa+|s'|}$. Using the definition (2.22) of the residue symbol of a rational function, one can replace in formula (5.26) the product $\prod_{j=1}^n T_j(\alpha,t)^{-k_j-1}$ by the product

$$\prod_{j=n+1}^m\left(\frac{\breve{T}_j(\alpha,t)}{v_{j0}^{\kappa+|s'|}(\alpha)}\right)^{k_j+1}=\left(\prod_{j=n+1}^m\frac{1}{v_{j0}^{k_j+1}(\alpha)}\right)^{\kappa+|s'|}\prod_{j=n+1}^m\breve{T}_j^{k_j+1}(\alpha,t)\,.$$

We will also need later an estimate for the size of $\breve{T}_j$. This estimate can be obtained easily using (5.1) once we note that

$$T_j(\alpha,t)\breve{T}_j(\alpha,t)=v_{j0}^{\kappa+|s'|}(\alpha)-(v_{j0}(\alpha)-T_j(\alpha,t))^{\kappa+|s'|}$$

and that T_j and $v_{j0}-T_j$ have similar size estimates. The size of v_{j0} is at most $c\mathbf{t}(T_j)+c'd_j\log 2$, so that using (5.20) we have, if $\kappa=|k'|+\sum_{j=n+1}^m s_j(k_j+1)$,

$$\begin{aligned}\mathbf{t}(\breve{T}_j) &\le c^2(\kappa+|s'|)(\mathbf{t}(T_j)+2c'd_j\log(2n+2))+c'\log 2\\&\le C_0\varpi^2c^5n^2(|k|+m)D_m^2\widetilde{D}^2(h+c'D_m\log(2n+2))\,.\end{aligned}\tag{5.33}$$

Now we are able to estimate the polynomial $\mathbf{R}_{2,J,k}$. Let

$$R_j(x_j,\alpha)=\sum_{i=0}^{\delta_j}\rho_{ji}(\alpha)x_j^{\delta_j-i}\,,\ j=1,\ldots,n.$$

Then

$$R_j^{l_j+1}(x_j,\alpha)=\rho_{j0}^{l_j+1}(\alpha)x_j^{\delta_j(l_j+1)}+\rho_{l,j,1}(\alpha)x_j^{\delta_j(l_j+1)-1}+\cdots.$$

For $1\le j\le n$, the $(l_j+1)\delta_j\times(l_j+1)\delta_j$ companion matrix of the multiplication operator by x_j in $\mathbf{A}(\alpha)[x_j]/R_j^{l_j+1}$ is

$$\Gamma_j:=\frac{1}{\rho_{j0}^{l_j+1}(\alpha)}\begin{pmatrix}0&0&\cdot&\cdot&\cdot&0&\cdot\\1&0&\cdot&\cdot&\cdot&0&\cdot\\0&1&\cdot&\cdot&\cdot&0&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&0&-\rho_{l,j,2}(\alpha)\\0&\cdot&\cdot&\cdot&0&1&-\rho_{l,j,1}(\alpha)\end{pmatrix}$$

By (2.10), the residue symbol

$$\operatorname{Res}\begin{bmatrix}x_j^{i_j}\,dx\\R_j^{l_j+1}(x_j,\alpha)\end{bmatrix}$$

is one of the coefficients of the matrix $\Gamma_j^{i_j}/\rho_{j0}^{l_j+1}$. (Namely, the last coefficient in the first column.) On the other hand, we have

$$\operatorname{Res}\begin{bmatrix}t^{i_0}dt\\t^{\kappa+1+|s'|-|l|}\end{bmatrix}=\begin{cases}1 \text{ if } i_0=\kappa+|s'|-|l|\\0 \text{ if not.}\end{cases}$$

If we use these auxiliary computations in formula (5.26), we see that the residue symbol is an element of $\mathbf{A}(\alpha)$ with denominator

$$\left(\prod_{j=1}^{n}\rho_{j0}^{l_j+1}(\alpha)\right)^{\deg_{x_j}\Upsilon+1}\left(\prod_{j=n+1}^{m}v_{j0}(\alpha)^{k_j+1}\right)^{\kappa+|s'|}.$$

A common denominator valid for all indices l such that $l_j\le\kappa'$, $1\le j\le n$, will therefore be

$$R_{2,J,k}(\alpha)=\left(\prod_{j=1}^{n}\rho_{j0}(\alpha)\right)^{(\kappa'+1)(\deg_{x_j}\Upsilon+1)}\left(\prod_{j=n+1}^{m}v_{j0}(\alpha)^{k_j+1}\right)^{\kappa+|s'|}.$$

Since $\kappa+|s'|\le d(|k|+m)\le\varpi 3(n+1)D_m\widetilde{D}(|k|+m)$ and also (see (5.2))

$$\begin{aligned}\mathbf{t}(\rho_{j0}^{l_j+1})&\le c\mathbf{t}(R_j^{l_j+1})+c'(l_j+1)d_j\log 2\,,\quad 1\le j\le n,\\\mathbf{t}(v_{j0}^{k_j+1})&\le c\mathbf{t}(T_j^{k_j+1})+c'(k_j+1)d_j\log 2\,,\quad n+1\le j\le m\,,\end{aligned}$$

we obtain the final estimate for the size of r_2, since using (5.30) and (5.32) one finds that

$$\mathbf{t}(R_{2,J,k})\le C_0\varpi^4n^7c^4\widetilde{D}(|k|+\widetilde{D}+m)(|J|+D_m^2\widetilde{D}(|k|+\widetilde{D}+m))(h+c'D_m\log(2n+2))\,. \tag{5.34}$$

By (5.24) we have that $\mathbf{t}(r_2) \le \mathbf{t}(R_{2,J,k})$, so that the estimate (5.34) is also valid for $\mathbf{t}(r_2)$, which gives the estimate (5.14'). Note here that the $R_{2,J,k}$ we found does not depend on l and is a common denominator for all residue symbols Res$[l]$. Moreover, we can use the same denominator when we replace J by any multiindex J' such that $|J'| \le |J|$.

In order to estimate the size of a numerator for Res$[l]$, we need first to estimate the coefficients involved in any of the matrices $\Gamma_j^{i_j}$, where $1 \le i_1, \dots, i_n \le \max_j \deg_{x_j} \Upsilon$. More precisely, let us write

$$\Gamma_j^{i_j} = \Big(\frac{1}{\rho_{j0}^{l_j+1}(\alpha)}\Big)^{i_j} \widetilde{\Gamma}_j^{i_j}\,.$$

If we define the size of a matrix as the maximum size of its coefficients, then, as $\rho_{j0}^{l_j+1}(\alpha)$ divides $R_{2,J,k}(\alpha)$, we obtain

$$\mathbf{t}\bigg(R_{2,J,k}\prod_{j=1}^{n}\Gamma_j^{i_j}\bigg) \le \mathbf{t}(R_{2,J,k}) + \mathbf{t}\bigg(\prod_{j=1}^{n}\widetilde{\Gamma}_j^{i_j}\bigg)\,, \tag{5.35}$$

so that our first objective will be to estimate the size of any element in a matrix of the form $\widetilde{\Gamma}_j^{i_j}$, $1 \le i_j \le \deg_{x_j}\Upsilon$. The contribution of the residue symbol corresponding to the multiindex $(i_1, \dots, i_n)$ in the development of $R_{1,J,k}$ is precisely a particular element of this matrix. The size of the matrix $\widetilde{\Gamma}_j$ is estimated by

$$\mathbf{t}(\widetilde{\Gamma}_j) \le c\mathbf{t}(R_j^{l_j+1}) + c'(l_j+1)d_j\log 2 \le c\mathbf{t}(R_j^{l_j+1}) + c'(\kappa + |s'| + 1)d_j \log 2\,.$$

Since by (5.1), $\mathbf{t}(\sum_{\iota=1}^{p} f_\iota) \le \max(\mathbf{t}(f_\iota) + \mathit{deg}(f_\iota)\log(m(f_\iota)+1)) + c'\log p$, we have, since the matrix $\widetilde{\Gamma}_j$ involves only $2(l_j+1)d_j - 1$ non zero coefficients and any coefficient of $\widetilde{\Gamma}_j^{i_j}$ is a sum of at most $(2d_j(l_j+1))^{i_j}$ products of coefficients of $\widetilde{\Gamma}_j$,

$$\begin{aligned}
\mathbf{t}(\widetilde{\Gamma}_j^{i_j}) &\le \big(c\mathbf{t}(R_j^{l_j+1}) + c'(l_j+1)d_j\log(2n+2)\big)\deg_{x_j}\Upsilon + c' i_j \log(2d_j(l_j+1)) \\
&\le \big(c\mathbf{t}(R_j^{l_j+1}) + c'(|k'|+|s'|+1)d_j\log(2n+2)\big)\deg_{x_j}\Upsilon + 3c' i_j d_j \log(l_j + d_j + 2) \\
&\le \big(c\mathbf{t}(R_j^{l_j+1}) + 3c'(|k'|+|s'|+1)d_j\log(2n+2)\big)\deg_{x_j}\Upsilon \\
&\le C_0 \varpi^4 n^6 c^4 \widetilde{D}(|k| + \widetilde{D})(|J| + D_m^2 \widetilde{D}(|k| + \widetilde{D} + m))(h + c' D_m \log(2n+2))\,.
\end{aligned}$$

Such an estimate, combined with (5.34) and (5.35), provides an estimate for the size of

$$R_{2,J,k}\prod_{j=1}^{n}\Gamma_j^{i_j}\,.$$

We need also to estimate the size of the polynomial Υ' in x, α, t defined as

$$\Upsilon'(x,\alpha,t) := \Upsilon(\alpha, x, t)\prod_{j=n+1}^{m}\check{T}_j^{k_j+1}(\alpha,t)\,.$$

In the factorization of this polynomial, the factors we still need to estimate are $\Delta(x, p', \alpha t)$ and all the terms of the form $\sum_{i=0}^{d_j-1} q_{ji}(\alpha t) p_j^{d_j-i-1}$, for $n+1 \le j \le m$. In both cases we need estimates for the maximum size of the coefficients of the polynomials $\mathcal{Q}_j, 1 \le j \le m$, and to simplify the notation, we will denote them by $\mathbf{t}_j$. Using (5.2) we have

$$\mathbf{t}_j \le c\mathbf{t}(\mathcal{Q}_j) + c'\tilde{d}_j \log(n+2)\,. \tag{5.36}$$

Then, for $n+1 \le j \le m, 0 \le i \le d_j$, we have, for the size of the polynomials $q_{ji}(\alpha t)$ considered as polynomials in the $n+1$ variables (α, t),

$$\mathbf{t}(q_{ji}) \le c\mathbf{t}_j + c' \log \binom{2d_j + n + 1}{2d_j} \le c\mathbf{t}_j + c'(n+1)\log(2d_j + 1)\,.$$

Using now (5.1) and (5.36), we obtain an estimate for the second type of terms:

$$\begin{aligned}
\mathbf{t}\Big(\sum_{i=0}^{d_j-1} q_{ji} p_j^{d_j-i-1}\Big) &\le c \max_i \big(\mathbf{t}(q_{ji}) + d_j h + c'(D_m d_j + 2d_j)\log(2n+2)\big) + c' \log d_j \\
&\le c^2 \mathbf{t}_j + c d_j (h + c'(D_m + 2)\log(2n+2)) + c'(c(n+1)+1)\log(2d_j+1) \\
&\le C_0 \varpi n c^3 D_m \widetilde{D}(h + c' D_m \log(2n+2))\,.
\end{aligned} \tag{5.37}$$

We turn now to the estimation of Δ. Recall that

$$\mathcal{Q}_j(x_j, \alpha t) = \sum_{i=1}^{n} (P_j - \alpha_j t)\mathcal{Q}_{ji}(x_j, p', \alpha t)\,, \quad 1 \le j \le n\,,$$

and $\Delta = \det[\mathcal{Q}_{ji}]$.

From Lemma 5.1 we know that we can chose the $\mathcal{Q}_{ji}$ so that

$$\max_i \mathbf{t}(\mathcal{Q}_{ji}) \le c^4\big(c\mathbf{t}(\mathcal{Q}_j) + 7c'\tilde{d}_j \log(2n+2)\big)\,. \tag{5.38}$$

Expanding each $\mathcal{Q}_{ji}$ as a polynomial in $u' = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$, we get, for $1 \le i, j \le n$,

$$\mathcal{Q}_{ji}(u, v) = \sum_{K_1, K_2} q_{ji}^{(K_1, K_2)}(u_0) u'^{K_1} v^{K_2}\,,$$

with the size estimates (from (5.2))

$$\mathbf{t}(q_{ji}^{(K_1,K_2)}) \le c\mathbf{t}(\mathcal{Q}_{ji}) + c'\tilde{d}_j \log(2n+1)\,.$$

Therefore, the size (as a polynomial in u_0, α, t) of each term $q_{ji}^{K_1,K_2}(u_0) t^{|K_2|} p'^{K_1} \alpha^{K_2}$ is at most $c\mathbf{t}(\mathcal{Q}_{ji}) + c'\tilde{d}_j \log(2n+1) + |K_1|h$, that is at most $c\mathbf{t}(\mathcal{Q}_{ji}) + \tilde{d}_j(h + c'\log(2n+1))$. We conclude (using (5.1) again) that

$$\mathbf{t}(\mathcal{Q}_{ji}(x_j, p', \alpha t)) \le c^2\mathbf{t}(\mathcal{Q}_{ji}) + c\tilde{d}_j(h + 2c'D_1 \log(2n+2) + c'\log(2n+1)) + 2c'\tilde{d}_j \log(2n+2)\,. \tag{5.39}$$

We then have, combining (5.38) and (5.39), that for any $1 \leq i, j \leq n$,

$$\mathbf{t}(\mathcal{Q}_{ji}(x_j, p', \alpha t)) \leq c^7 \mathbf{t}(\mathcal{Q}_j) + c d_j (h + 10 c' c^5 D_1 \log(2n+2)) + 2c' d_j \log(2n+2) \,. \tag{5.40}$$

We get immediately from that the final estimate for the size of $\Delta(x, p', \alpha t)$, that is, if $d' = \max_{1 \leq j \leq n} \tilde{d}_j$,

$$\begin{aligned}
\mathbf{t}(\Delta(x, p', \alpha t)) &\leq n c^8 \max_j \mathbf{t}(\mathcal{Q}_j) + n c^2 d' \big(h + 10 c' c^5 D_1 \log(2n+2)\big) + \\
&\quad + 2ncc' d' \log(2n+2) + 2nc' D_1 d' \log(2n+2) + c' \log n! \\
&\leq n c^8 \max_j \mathbf{t}(\mathcal{Q}_j) + n c^2 d' \big(h + 15 c' c^5 D_1 \log(2n+2)\big) \\
&\leq C_0 \varpi n^2 c^9 \widetilde{D} (h + c' D_1 \log(2n+2)) \,. \qquad (5.41)
\end{aligned}$$

In order to summarize the estimates for the size of the polynomial Υ' we need to put together the estimates (5.31), (5.33), (5.37), and (5.41). If we add the sizes of all factors, we get

$$\mathbf{t}(\Upsilon') \leq C_0 \varpi^2 n^3 c^9 D_m^2 \widetilde{D}^2 (|k| + m)^2 (h + c' D_m \log(2n+2)) \,. \tag{5.42}$$

The total degree of Υ' as a polynomial in α, t, x can be estimated from (5.27), (5.28), and the estimate of the degrees of the $\breve{T}_j$,

$$\deg_{\alpha,t}(\breve{T}_j) \leq 2 d_j (\kappa + |s'|) \leq C_0 n^2 \varpi^2 D_m^2 \widetilde{D}^2 (|k| + m) \,.$$

We have then

$$\deg(\Upsilon') \leq |J| + C_0 \varpi^2 n^4 D_m^2 \widetilde{D}^2 (|k| + m) \,. \tag{5.43}$$

Let

$$\Upsilon'(x, \alpha, t) := \sum_{i \in \mathbf{N}^{n+1}} \Upsilon'_i(\alpha) t^{i_0} x^{i_1} \cdots x^{i_n} \,.$$

We get, by (5.2),

$$\begin{aligned}
\max_i \mathbf{t}(\Upsilon'_i) &\leq c \mathbf{t}(\Upsilon') + c' \deg(\Upsilon') \log(n+2) \\
&\leq C_0 \varpi^2 n^3 c^{10} D_m^2 \widetilde{D}^2 (|k| + m)(|k| + \widetilde{D} + m)(h + c' D_m \log(2n+2)) \,.
\end{aligned}$$

As we have pointed it before (see formula (2.10)), one can write

$$R_{2,J,k}(\alpha) \mathrm{Res}[l](\alpha) = \sum_{i \in \mathbf{N}^{n+1}} \Upsilon'_i(\alpha) \mathrm{Res} \left[\frac{t^{i_0} dt}{t^{\kappa + 1 + |s'|}} \right] \xi_{l, i_1, \ldots, i_n}(\alpha) \,,$$

where $\xi_{l,i_1,\ldots,i_n}(\alpha)$ is one of the coefficients of the matrix $R_{2,J,k} \widetilde{\Gamma}_1^{i_1} \ldots \widetilde{\Gamma}_n^{i_n}$. One can see easily, as a consequence of (5.1), that the size estimate for each polynomial $\Upsilon'_i(\alpha) \xi_{l,i_1,\ldots,i_n}(\alpha)$ is at most

$$C_0 \varpi^4 n^7 c^{10} \widetilde{D} (|k| + \widetilde{D} + m)(|J| + D_m^2 \widetilde{D} (|k| + \widetilde{D} + m))(h + c' D_m \log(2n+2)) \,.$$

On the other hand, the degree in α of the polynomial $R_{2,J,k}\mathrm{Res}[l]$ is at most

$$\begin{aligned}&\max_{1\le j\le n} \deg(R_j^{|k'|+|s'|+1})\deg(\Upsilon')+\deg(R_{2,J,k})\\&\le C_0\varpi^2 n^3\widetilde{D}^2(|J|+\varpi^2n^4D_m^2\widetilde{D}^2(|k|+m))\,,\end{aligned}$$

Applying (5.1), it follows that the size of

$$\begin{aligned}R_{1,J,k}&=\sum_{0\le l_i\le|k'|+|s'|} R_{2,J,k}\mathrm{Res}[l]\\&=\sum_{0\le l_i\le|k'|+|s'|}\sum_{i\in\mathbf{N}^{n+1}}\mathrm{Res}\begin{bmatrix}t^{i_0}dt\\t^{\kappa+1+|s'|}\end{bmatrix}\Upsilon'_i\xi_{l,i_1,\dots,i_n}\end{aligned}$$

is at most

(5.44)
$$\begin{aligned}\mathbf{t}(R_{1,J,k})&\le C_0\varpi^4n^7c^{11}\widetilde{D}(|k|+\widetilde{D}+m)(|J|+D_m^2\widetilde{D}(|k|+\widetilde{D}+m))(h+c'D_m\log(2n+2))\\&\quad+cc'C_0\varpi^4n^7(|k|+m)D_m^2\widetilde{D}^4\log(n+1)\,.\end{aligned}$$

The conclusion of the lemma follows from (5.25), (5.34) and (5.44).

■

We will use in the next section a variant of this lemma.

Lemma 5.4. *Let $\tilde{p}_1,\dots,\tilde{p}_n,p_{n+1},\dots,p_m$ be m polynomials in $\mathbf{A}[x]$ with size at most h and degree at most D, such that $\tilde{p}_1,\dots,\tilde{p}_n$ is a quasi regular sequence and the ideal $\big(\tilde{p}_1,\dots,\tilde{p}_n,\prod_{j=n+1}^m p_j\big)$ is $\mathbf{A}[x]$. Assume that $h\ge c'\log(n+2)$ and there is an $n\times n$ matrix $A=[a_{jl}]_{\substack{1\le j\le n\\1\le l\le n}}$ with coefficients in $\mathbf{A}$, of size at most a, which is invertible in $\mathcal{M}_n(\mathbf{K})$, and such that the polynomials*

$$p_j:=\sum_{l=1}^n a_{jl}\tilde{p}_l\,,\ j=1,\dots,n\,,$$

have respective degrees $D_1,\dots,D_n$. Then, for any multiindices $J\in\mathbf{N}^n$ and $k\in\mathbf{N}^m$ we have

$$\mathrm{Res}\begin{bmatrix}x^Jdx\Big/\Big(\prod_{j=n+1}^m p_j^{k_j+1}\Big)\\ \tilde{p}_1^{k_1+1},\dots,\tilde{p}_n^{k_n+1}\end{bmatrix}=\frac{r_1}{r_2}\,,\quad r_1=r_1(J,k),\ r_2=r_2(J,k)\in\mathbf{A}\,. \tag{5.45}$$

Moreover,

$$\begin{aligned}\mathbf{t}(r_1)&\le C_0\varpi^4n^7c^{13}\widetilde{D}(|k|+\widetilde{D}+m)(|J|+D^2\widetilde{D}(|k|+\widetilde{D}+m))(h+a+c'D\log(2n+2))\\&\quad+c^4c'C_0\varpi^4n^7(|k|+m)D_m^2\widetilde{D}^4\end{aligned}$$

and

$$\mathbf{t}(r_2) \leq C_0 \varpi^4 n^7 c^4 \widetilde{D}(|k| + \widetilde{D} + m)(|J| + D^2 \widetilde{D}(|k| + \widetilde{D} + m))(h + a + c' D \log(2n+2))$$

where C_0 is an absolute constant (independent of n and of the size), $\widetilde{D} = D_1 \cdots D_n$. Furthermore, we can use the same denominator r_2 if one replaces $k_1, \ldots, k_n$ by any $n-$uplet $(l_1, \ldots, l_n)$ of integers such that $|l| \leq |k|$ and J by any multiindex J' such that $|J'| \leq |J|$.

Proof. Since the matrix A is invertible in $\mathcal{M}_n(\mathbf{K})$, the polynomials $p_1, \ldots, p_n$ satisfy the conditions of Lemma 5.3. From (5.1) one has

$$\mathbf{t}(p_j) \leq c(a + h + c' D \log(n+1)) + c' \log n\,. \tag{5.46}$$

We rewrite the residue symbol (5.45) using Proposition 2.4. Let δ be the determinant of the matrix A. Then
(5.47)

$$\operatorname{Res}\left[\begin{matrix} x^J dx \Big/ \prod\limits_{j=n+1}^{m} p_j \\ \tilde{p}^{k+\underline{1}} \end{matrix}\right] = \delta \sum_{\substack{|q_{;j}|=k_j \\ 1\leq j\leq n}} \prod_{i=1}^{n} \binom{\mu_i}{q_{i;}} \Big(\prod_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} a_{ij}^{q_{ij}}\Big) \operatorname{Res}\left[\begin{matrix} x^J dx \Big/ \prod\limits_{j=n+1}^{m} p_j \\ p_1^{\mu_1+1}, \cdots p_n^{\mu_n+1}, \end{matrix}\right].$$

where we recall from Proposition 2.3 the following notation for the matrix of indices $q_{i,j}$

$$q_{;j} = (q_{1,j}, \ldots, q_{n,j}), \quad q_{i;} = (q_{i,1}, \ldots, q_{i,n}) \quad \mu_i = |q_{i;}|$$

and

$$\binom{\mu_i}{q_{i;}} = \frac{\mu_i!}{q_{i,1}! \cdots q_{i,n}!}\,.$$

It follows from Lemma 5.3 that each residue symbol that appears in the right hand side of (5.47) is of the form r_{1q}/r_2, where $r_{1q}, r_2 \in \mathbf{A}$, $r_2 \neq 0$ and

$$\begin{aligned}\mathbf{t}(r_{1,q}) \leq\ & C_0 \varpi^4 n^7 c^{12} \widetilde{D}(|k| + \widetilde{D} + m)(|J| + D^2 \widetilde{D}(|k| + \widetilde{D} + m))(h + a + c' D \log(2n+2)) \\ & + c^2 c' C_0 \varpi^4 D_m^2 \widetilde{D}^4\end{aligned}$$

and

$$\mathbf{t}(r_2) \leq C_0 \varpi^4 n^7 c^4 \widetilde{D}(|k| + \widetilde{D} + m)(|J| + D^2 \widetilde{D}(|k| + \widetilde{D} + m))(h + a + c' D \log(2n+2))\,,$$

(We use here that r_2 can be chosen to be independent of q.) The number of terms (taking into account repetitions) involved in the sum in (5.47) is at most

$$\prod_{j=1}^{n} \binom{n + k_j - 1}{n-1} (1 + |k|)^{n(n-1)}\,.$$

Therefore, the size of the element

$$r_1 = \delta \sum_{\substack{|q_{;j}|=k_j \\ 1\le j\le n}} \prod_{i=1}^{n} \binom{\mu_i}{q_{i;}} \Big(\prod_{\substack{1\le i\le n \\ 1\le j\le n}} a_{ij}^{q_{ij}} \Big) r_{1q}$$

is at most

$$\begin{aligned} \mathbf{t}(r_1) &\le \mathbf{t}(\delta) + c(\max_q(\mathbf{t}(r_{1q})) + |k|a) + c'n(n-1)\log(1+|k|)) \\ &\le cna + c(\max_q(\mathbf{t}(r_{1q})) + |k|a) + c'n(n-1)\log(1+|k|)) + c'\log n!\,. \end{aligned}$$

The conclusion of the lemma immediately follows from the size estimates for the r_{1q}. ■

We will use extensively in the next section the following simple consequence of the last result.

Lemma 5.5. *Let $\tilde{p}_1, \dots, \tilde{p}_n, p_{n+1}, \dots, p_m$ be m polynomials in $\mathbf{A}[x]$ as in Lemma 5.4. Let $f \in \mathbf{A}[\xi_1, \dots, \xi_L, x_1, \dots, x_n]$ of degree $\mathbf{d}$ and size $\mathbf{t}$. Then, for any multiindex $k = (k_1, \dots, k_m)$ one has*

(5.48)

$$\operatorname{Res}\left[\begin{matrix} f(\xi,x)dx \Big/ \Big(\prod_{j=n+1}^{m} p_j^{k_j+1} \Big) \\ \tilde{p}_1^{k_1+1}, \dots, \tilde{p}_n^{k_n+1} \end{matrix}\right] = \frac{r_1(\xi)}{r_2}\,, \quad r_1 = r_1(f,k) \in \mathbf{A}[\xi],\ r_2 = r_2(k) \in \mathbf{A}$$

with

(5.49)

$$\begin{aligned} &\mathbf{t}(r_i) \le \\ &c^2\mathbf{t} + C_0\varpi^4 n^7 c^{14}\widetilde{D}(|k| + \widetilde{D} + m)(\mathbf{d} + D^2\widetilde{D}(|k| + \widetilde{D} + m))(h + a + c'D\log(2n+2)) \\ &+ c^4c'C_0\varpi^4 n^7(|k|+m)D_m^2\widetilde{D}^4\,,\ i = 1, 2. \end{aligned}$$

Proof. Let

$$f = \sum_\beta f_\beta(\xi)x^\beta\,.$$

Using (5.2) we have

$$\max_\beta \mathbf{t}(f_\beta) \le c\mathbf{t} + c'\mathbf{d}\log(n+1)\,.$$

Let $r_2 = r_2(k)$ be the common denominator for all residue symbols of the form (5.45) with $|J| \le \mathbf{d}$. We know from Lemma 5.4 that

(5.50) $$\mathbf{t}(r_2) \le C_0\varpi^4 n^7 c^{13}\widetilde{D}(|k|+\widetilde{D}+m)(\mathbf{d}+D^2\widetilde{D}(|k|+\widetilde{D}+m))(h+a+c'D\log(2n+2))\,.$$

Let now $r_{1,\beta}$ be the numerator of the residue symbol (5.48) when f is replaced by x^β and we same denominator $r_2 = r_2(k)$. Then, we can write

$$r_1 = \sum_\beta f_\beta(\xi) r_{1,\beta}\,,$$

where the estimates for the sizes of $r_{1,\beta}$ are also given by (5.50), with the extra term

$$c^3 c' C_0 \varpi^4 n^7 (|k| + m) D_m^2 \widetilde{D}^4 .$$

Therefore, to estimate $\mathbf{t}(r_1)$ we can use (5.1) and get exactly the statement of the lemma, modulo a change of the value of C_0. ∎

Remark 5.5. It is possible to separate the degrees $\mathbf{d}_x$ and $\mathbf{d}_\xi$ in the statement of the preceding lemma. In this case, the estimates for the size of the numerator and denominator of f are

$$\begin{aligned} &c^2\mathbf{t} + C_0\varpi^4 n^7 c^{14}\widetilde{D}(|k| + \widetilde{D} + m)(\mathbf{d}_x + D^2\widetilde{D}(|k| + \widetilde{D} + m))(h + a + c'D\log(2n+2)) \\ &\quad + c^4 c' C_0 \varpi^4 n^7 (|k| + m) D_m^2 \widetilde{D}^4 + cc'\mathbf{d}_\xi \log(L+1) + c' n \log \mathbf{d}_\xi . \end{aligned}$$

6. Effective Nullstellensatz.

We provide now the solution of the Bézout identity with good degree and size estimates.

Theorem 6.1. *Let $p_1, \dots, p_M \in \mathbf{A}[x_1, \dots, x_n]$ and $\mathbf{A}$ be an integral domain with infinite quotient field $\mathbf{K}$. The ring $\mathbf{A}$ is assumed to be a factorial regular ring with Krull dimension κ and equipped with a size $\mathbf{t}$ (with corresponding c, c', ϑ .) The degrees $D_j = \deg(p_j)$ are assumed to be in decreasing order and $h := \max(\mathbf{t}(p_j), c' \log(n+2))$. If $p_1, \dots, p_M$ have no common zero in some algebraic closure $\overline{\mathbf{K}}$ of $\mathbf{K}$, there exists $r_0 \in \mathbf{A}$, and $q_1, \dots, q_M \in \mathbf{A}[x]$, such that*

$$r_0 = \sum_{j=1}^{M} q_j p_j ,$$

with the estimates

$$\begin{cases} \deg(p_j q_j) \le n(n+1)^3 B(D_1, \dots, D_n) + n(D_1 - 1) \\ \mathbf{t}(p_j q_j) \le C_0 \varpi^4 2^n n^{17} c^{16} B^4 D_1^2 \big(h + \vartheta[(\gamma_0 D_1)^{2n}] + c' \log M + c' D_1 \log(2n+2)\big) \\ \mathbf{t}(r_0) \le C_0 \varpi^4 2^n n^{17} c^{16} B^4 D_1^2 \big(h + \vartheta[(\gamma_0 D_1)^{2n}] + c' \log M + c' D_1 \log(2n+2)\big) \end{cases}$$

where γ_0, C_0 are absolute integral constants, $B = B(D_1, \dots, D_n)$ is defined by (4.1), and ϖ is the constant depending on n, κ and on the size $\mathbf{t}$ that appears in Lemma 5.2.

Remark 6.1. The case $\mathbf{A} = \mathbf{Z}$ has been studied in [BY1] using analytic methods and we obtained there a similar result with slightly worse estimates (for the values of c, c', ϖ in this case, we refer to our previous Remark 5.3.)

Our main example will be $\mathbf{A} = \mathbf{F}_p[\tau_1, \dots, \tau_q]$. In this case, $c = 1$, $c' = 0$ and $\varpi = 2n + q + 1$, as seen in Remark 5.3. The function ϑ in this case (see Example 5.3) is $\vartheta(k) \simeq (\log k / \log p)^{1/q}$.

Proof. We may repeat the same polynomial several times in the sequence $p_1, \dots, p_M$, so that one can always assume that $M > n$. First, we use the pigeonhole principle to find an $n \times M$ triangular matrix (a_{ij}) with coefficients in $\mathbf{A}$, $a_{ii} = 1$ and $a_{ij} = 0$ for $i > j$, such that the polynomials P_i defined by $P_i = \sum a_{ij} p_j$ form a quasiregular sequence. We start with $P_1 = p_1$. Then, since P_1 has at most D_1 irreducible distinct factors in $\mathbf{K}[x]$ and the field $\mathbf{K}$ is infinite, there are elements $a_{2j} \in \mathbf{A}$, $j = 3, \dots, M$ such that $\mathbf{t}(a_{2j}) \leq \vartheta(D_1 + 1)$ and, if $P_2 := p_2 + \sum_{j=3}^{M} a_{2j} p_j$, then (P_1, P_2) defines a quasiregular sequence (even regular whenever the ideal (P_1, P_2) is proper.) In order to see that, let $\mathcal{P}_1$ be the set of distinct irreducible components of the variety $\{P_1 = 0\}$ in $\overline{\mathbf{K}}^n$; for each $\gamma \in \mathcal{P}_1$, we choose a point $\alpha_\gamma \in \gamma$ of and consider the non zero homogeneous polynomial in $N - 2$ variables $w_3, \dots, w_M$

$$T_1(w) := \prod_{\gamma \in \mathcal{P}_1} \left(p_2(\alpha_\gamma) + \sum_{j=3}^{M} w_j p_j(\alpha_\gamma) \right).$$

This polynomial has total degree at most D_1 and we can find, from the definition of ϑ, a point $w^0 \in \mathbf{A}^{M-2}$ such that $T_1(w^0) \neq 0$, with $\max(\mathbf{t}(w_j^0)) \leq \vartheta(D_1 + 1)$; we take $a_{2j} = w_j^0$, $j = 3, \dots, M$. Then, once P_2 is constructed, we go on and use the same idea to construct P_3, considering this time the set of irreducible components in $\{P_1 = P_2 = 0\}$ (whose cardinal is at most $D_1 D_2$ by Bézout theorem since the sequence (P_1, P_2) is regular whenever the ideal is proper.) The new coefficients a_{3j} have their sizes estimated by $\vartheta(D_1 D_2 + 1)$. Proceeding in the same way, we obtain a quasiregular sequence $P_1, \dots, P_n$. The maximal size of the P_i is at most

$$\mathbf{t}(P_i) \leq c(h + \vartheta(\widetilde{D}) + c' D_1 \log(n+1)) + c' \log M \,,\ i = 1, \dots, n \,. \tag{6.1}$$

In the second step, consider the polynomials Φ and $\widetilde{\Phi}$ of $2n^2 + n$ variables associated to $P_1, \dots, P_n$ by Proposition 4.2 and Remark 4.3. Choose $(U, V) \in \mathbf{K}^{n \times (n+1)} \oplus \mathcal{M}_n(\mathbf{R})$ such that $\Phi(U, V)\widetilde{\Phi}(U, V)\Sigma(V) \neq 0$, where $\Sigma(V)$ is the product of all minors of the matrix $[v_{jl}]_{\substack{1 \leq j \leq n \\ 1 \leq l \leq n}}$. The degree of Σ is at most $n 2^{2n}$, so that the total degree of the polynomial $\Phi\widetilde{\Phi}\Sigma$ is at most $2^{2(n+1)}(n+1)^4 D_1^{2n} \leq (\gamma_0 D_1)^{2n}$, where γ_0 is an absolute constant. Using the definition of the function ϑ and the comment following that definition, one can find (U, V) such that

$$\begin{cases} \Phi(U, V)\widetilde{\Phi}(U, V)\Sigma(U, V) \neq 0 \\ \\ \max(\mathbf{t}(u_{jl}), \mathbf{t}(v_{jl})) \leq \vartheta[(\gamma_0 D_1)^{2n}] \end{cases} \tag{6.2}$$

For $N = (n+1)^3$ the condition

$$\frac{B + D_1}{NB + D_1} < \frac{1}{n(n+1)^2}$$

holds, and so, by Proposition 4.2, the following polynomial identity holds in $\mathbf{K}[x_1, \dots, x_n]$

$$1 = \operatorname{Res} \begin{bmatrix} \Delta_{N,U,V}(x, y) dx_1 \wedge \cdots \wedge dx_n \\ \\ (U^1(x))^{NB} < V^1, P >, \dots, (U^n(x))^{NB} < V^n, P > \end{bmatrix}. \tag{6.3}$$

From now on, we assume that the variables U, V have been fixed and so we will drop them, as well as N, from the notation, when convenient. In particular, we will denote by $\Theta_j(x) = (U^j(x))^{NB} < V^j, P(x) >$, $j = 1, \ldots, n$. Correspondingly, we let $\Delta(x, y) = \Delta_{N,U,V}(x, y)$, and δ_{ij} the entries of the corresponding matrix. We can choose the entries of the matrix Δ in accordance to Lemma 5.1, so that they have good size estimates. We have

$$\begin{aligned}\max_j(\mathbf{t}(< V^j, P >) &\leq c\big(\max_i(\mathbf{t}(P_i)) + \vartheta[(\gamma_0 D_1)^{2n}] + c' D_1 \log(n+1)\big) + c' \log n \\ &\leq c^2 \left(h + 2c'\big(D_1 \log(n+1) + \vartheta[(\gamma_0 D_1)^{2n}] + \log M\big) \right).\end{aligned}$$

At this point, we have the following size estimates for the polynomials Θ_j, $1 \leq j \leq n$,

$$\begin{aligned}\mathbf{t}(\Theta_j) &\leq c^2 \left(h + 2c'\big(D_1 \log(n+1) + \vartheta[(\gamma_0 D_1)^{2n}] + \log M\big) \right) \\ &\quad + c(n+1)^3 B\big(\vartheta[(\gamma_0 D_1)^{2n}] + c' \log(n+1)\big) \\ &\leq c^2 \left(h + B(n+1)^3 \big(\vartheta[(\gamma_0 D_1)^{2n}] + \gamma_0 c' \log(n+1)\big) + 2c' \log M \right)\end{aligned} \tag{6.4}$$

so that the entries of Δ satisfy the estimates

$$\max_{i,j}(\mathbf{t}(\delta_{i,j})) \leq c^7 \left(h + (n+1)^3 B\big(\vartheta[(\gamma_0 D_1)^{2n}] + \gamma_0 c' \log(2n+1)\big) + 2c' \log M \right), \tag{6.5}$$

as shown in Lemma 5.1.

Using exactly the same argument we used at the beginning of the proof, we find coefficients $a_{n+1,1}, \ldots, a_{n+1,M}$ of size smaller than $\vartheta[(\gamma_0 D_1)^{2n}]$ such that

$$q = \sum_{j=1}^{M} a_{n+1,j} p_j$$

does not vanish on the algebraic variety defined by the polynomials Θ_j. We have the following estimates for $\mathbf{t}(q)$,

$$\mathbf{t}(q) \leq c\big(h + \vartheta[(\gamma_0 D_1)^{2n}] + c' D_1 \log(n+1)\big) + c' \log M\,. \tag{6.6}$$

One can now rewrite the identity (6.3) by decomposing it into a sum of residues of rational functions. This can be done here because the ideal generated by $\Theta_1, \ldots, \Theta_n, q$ is $\mathbf{K}[x]$. In order to do this we introduce the Hefer divisors $g_{n+1,j}$ for q, that is, the polynomials in $2n$ variables defined by the successive divided differences,

$$g_{n+1,j}(x, y) = \frac{q(x_1, \ldots, x_{j-1}, x_j, y_{j+1}, \ldots, y_n) - q(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n)}{x_j - y_j}, \quad j = 1, \ldots, n.$$

We rewrite the determinant $\Delta(x,y)$ as

$$(6.7)\qquad \frac{1}{q(x)}\begin{vmatrix} \delta_{11} & \cdot\ \cdot\ \cdot & \delta_{1n} & g_{n+1,1}(x,y) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \delta_{n1} & \cdot\ \cdot\ \cdot & \delta_{nn} & g_{n+1,n}(x,y) \\ \Theta_1(y)-\Theta_1(x) & \cdot\ \cdot\ \cdot & \Theta_n(y)-\Theta_n(x) & q(y) \end{vmatrix}$$

We can develop the $(n+1)\times(n+1)$ determinant in (6.7) along the last row and obtain

$$(6.8)\qquad \Delta(x,y)=\frac{1}{q(x)}\Bigg(\Big(\sum_{j=1}^{n}(\Theta_j(y)-\Theta_j(x))\Delta_j(x,y)\Big)+q(y)\Delta(x,y)\Bigg).$$

Since the residue symbol is annihilated by the ideal, we can rewrite (6.7) as a Bézout identity

$$(6.9)\qquad 1=\sum_{j=1}^{n}\operatorname{Res}\begin{bmatrix}\Delta_j(x,y)dx/q(x)\\ \\ \Theta_1(x),\dots,\Theta_n(x)\end{bmatrix}\Theta_j(y)+\operatorname{Res}\begin{bmatrix}\Delta(x,y)dx/q(x)\\ \\ \Theta_1(x),\dots,\Theta_n(x)\end{bmatrix}q(y)\,.$$

(in order to unify our notations, we used $\Delta_0(x,y):=\Delta(x,y)$.) It is clear that (6.9) is an identity of the form

$$1=\sum_{j=1}^{M}p_j(y)q_j(y)\,,$$

where the q_j are in $\mathbf{K}[x]$. This is going to be the formula that solves the effective Nullstellensatz with good estimates. Note that, up to this point, we already have the estimates for the degrees.

Let us now consider the problem of size estimates. The size of the polynomials Δ_j, $j=0,\dots,n$, can be obtained immediately from the estimates (6.5) and Lemma 5.1 applied to $g_{n+1,j}$, $j=1,\dots,n$. We have

(6.10)

$$\mathbf{t}(\Delta_j(x,y))\le nc^8\Big(h+B(n+1)^3\big(\vartheta[(\gamma_0D_1)^{2n}]+\gamma_0c'\log(2n+1)\big)+2c'\log M\Big)\,,\ j=0,\dots,n\,.$$

We now introduce

$$P_{j,1}(x):=<V^j,P(x)>,\ P_{j,2}(x):=(U^j(x))^{NB}\,,\ j=1,\dots,n\,,$$

and apply Lemma 2.3 in order to express differently the residue symbols

$$\operatorname{Res}\begin{bmatrix}\Delta_j(x,y)dx/q(x)\\ \\ \Theta_1(x),\dots,\Theta_n(x),\end{bmatrix}\,,\ j=0,\dots,n\,.$$

The hypotheses of the lemma are fullfilled since $\Phi(U,V)\widetilde{\Phi}(U,V) \neq 0$. Then, we have, for any $0 \leq s \leq n$,

$$
\text{(6.11)} \qquad \mathrm{Res}\begin{bmatrix} \Delta_s(x,y)dx/q(x) \\ \Theta_1,\ldots,\Theta_n \end{bmatrix} = \sum_{1\leq j_1,\ldots,j_n\leq 2} \mathrm{Res}\begin{bmatrix} \Big(\Delta_s(x,y)\Big/ q(x) \prod\limits_{\substack{1\leq i\leq n \\ j\neq j_i}} P_{ij}\Big)dx \\ \\ P_{1,j_1},\ldots,P_{n,j_n} \end{bmatrix}.
$$

Any of the residue symbols in the right hand side of (6.11) can be computed using Lemma 5.5. More precisely, for s fixed in $0,\ldots,n$, consider the residue symbol

$$
\mathrm{Res}\begin{bmatrix} \Big(\Delta_s(x,y)\Big/ q(x) \prod\limits_{\substack{1\leq i\leq n \\ j\neq j_i}} P_{ij}\Big)dx \\ \\ P_{1,j_1},\ldots,P_{n,j_n} \end{bmatrix}.
$$

Up to sign, one can rewrite it as

$$
\text{(6.12)} \qquad \mathrm{Res}\begin{bmatrix} \Delta_s(x,y)dx\Big/ q(x)\Big(\prod_{j\in\mathcal{J}} < V^j, P(x) > \Big)\Big(\prod_{i\in\mathcal{I}} U^i(x)^{NB}\Big) \\ < V^{i_1}, p(x) >, \ldots, < V^{i_\mu}, p(x) >, (U^{j_1}(x))^{NB}, \ldots, (U^{j_{n-\mu}}(x))^{NB} \end{bmatrix},
$$

where $\mathcal{I} = \{i_1,\ldots,i_\mu\}$ and $\mathcal{J} = \{j_1,\ldots,j_{n-\mu}\}$ define a partition of $\{1,\ldots,n\}$.

We use now an argument due to M. Elkadi (see [El], [BGVY, p. 125-126]). For any subset $\mathcal{I} \subset \{1,\ldots,n\}$ of cardinal μ, one can find, since $\Sigma(V) \neq 0$, μ linear combinations $q_{\mathcal{I},1},\ldots,q_{\mathcal{I},\mu}$ of $< V^{i_1}, P >, \ldots, < V^{i_\mu}, P >$, of the form

$$
q_{\mathcal{I},j} = \sum_{l=j}^{n} \rho_{\mathcal{I},j,l} P_l\,, \quad j = 1,\ldots,\mu,
$$

with the $\rho_{\mathcal{I},j,j}$ all different from zero. Moreover, the coefficients involved in such linear combinations $q_{\mathcal{I},j}$ are product of at most μ minors of the matrix $[v_{jl}]_{\substack{1\leq j\leq n \\ 1\leq l\leq n}}$. So the maximal size a of such coefficients is at most

$$
cn^2 \max(\mathtt{t}(v_{jl})) + c'n\log(n!) \leq cn^2(\vartheta[(\gamma_0 D_1)^n] + \log(n+1))\,.
$$

Therefore, if $\mathcal{I} = \{i_1,\ldots,i_\mu\}$, we can apply Lemma 5.5 with $m = 2n+1$, $\tilde{p}_{i_j}(x) = q_{\mathcal{I},j}$ for $j = 1,\ldots,\mu$, $\tilde{p}_i(x) = U^i(x)$ if $i \notin \mathcal{I}$, and $p_{n+1},\ldots,p_{2n+1}$ are the polynomials

$$
< V^i, P(x) >, i \notin \mathcal{I};\; U^i(x), i \in \mathcal{I};\; q\,.
$$

The matrix A in the statement of this lemma is the matrix which changes the system of n polynomials

$$
(< V^i, P >,\ i \in \mathcal{I};\ U^i(x),\ i \in \{1,\ldots,n\} \setminus \mathcal{I})
$$

into the system

$$(q_{\mathcal{I},j},\, j=1,\ldots,\mu;\, U^i(x),\, i\in\{1,\ldots,n\}\setminus\mathcal{I})\,.$$

We let k be the multiindex corresponding to the exponents of these polynomials as they appear in the residue symbol (6.12). We have $|k|\leq 2n(n+1)^3B$. As a consequence of Lemma 5.5, the residue symbol (6.12) can be written as $r_{\mathcal{I},1}/r_{\mathcal{I},2}$, where $r_{\mathcal{I},1}\in\mathbf{A}[y]$, $r_{\mathcal{I},2}\in\mathbf{A}$, with size estimates

$$\begin{aligned}\max_{i=1,2}\mathbf{t}(r_{\mathcal{I},i})&\leq nc^9\Big(h+(n+1)^3B\big(\vartheta[(\gamma_0D_1)^{2n}]+\gamma_0c'\log(2n+1)\big)+2c'\log M\Big)\\&\quad+C_0\varpi^4n^{15}c^{16}B^4D_1^2\Big(h+\vartheta[(\gamma_oD_1)^{2n}]+c'\log M+a+c'D_1\log(2n+2)\Big)\\&\leq C_0\varpi^4n^{17}c^{16}B^4D_1^2\Big(h+\vartheta[(\gamma_0D_1)^{2n}]+c'\log M+c'D_1\log(2n+2)\Big)\end{aligned}\tag{6.13}$$

Note that the denominator $r_{\mathcal{I},2}$ does not depend on the index $s\in\{0,\ldots,n\}$ chosen earlier. We take now as r_0 the product of all $r_{\mathcal{I},2}$, $\mathcal{I}\subset\{1,\ldots,n\}$ and obtain our final estimates thanks to (5.1). ■

Corollary 6.1. *Let p be a prime number and q a positive integer. Let $p_1,\ldots,p_M$ be polynomials in $\mathbf{F}_p[\tau_1,\ldots,\tau_q][x_1,\ldots,x_n]$ such that the corresponding degrees D_j in the x variables are in decreasing order, assume that the ideal they define in $\mathbf{F}_p[\tau,x]$ contains a non-zero element in $\mathbf{F}_p[\tau]$. Let δ be an upper bound for the degrees of the p_j in the τ variables, then one can find $a_0\in\mathbf{F}_p[\tau]$ and polynomials $a_1,\ldots,a_M$ in $\mathbf{F}_p[\tau,x]$ such that*

$$a_0=\sum_{j=1}^M a_jp_j$$

and, for $1\leq j\leq M$,

$$\begin{cases}\deg_x(a_jp_j)\leq n(n+1)^3B(D_1,\ldots,D_n)+n(D_1-1)\\ \deg_\tau(a_jp_j)\leq C_0n^{17}(n+q)^4B(D_1,\ldots,D_n)^4D_1^2(\delta+(n\log D_1)^{1/q})\end{cases}$$

where C_0 is an absolute constant.

Corollary 6.2. *Let $p_1,\ldots,p_M\in\mathbf{Z}[x_1,\ldots,x_n]$ such that their degrees D_j are in decreasing order and their maximum Mahler sizes $h(p_j)$ are bounded by h. Assume they have no common zeros in $\mathbf{C}^n$. Then, there are polynomials $q_j\in\mathbf{Z}[x]$ and a positive integer q_0 such that*

$$q_0=\sum_{j=1}^M q_jp_j$$

and, for $1\leq j\leq M$,

$$\begin{cases}\deg(q_jp_j)\leq n(n+1)^3B(D_1,\ldots,D_n)+n(D_1-1)\\ \max(\log q_0,h(q_j))\leq\kappa(n)B(D_1,\ldots,D_n)^4D_1^2(h+n\log D_1+D_1\log n+\log M)\end{cases}$$

where

$$\kappa(n) = C_0 n^{21} 2^n (8 \log(n+1))^{4n+8}$$

and C_0 is an absolute constant.

Remark 6.2. The estimates for the Mahler size in Corollary 6.2 are an improvement on the results from [BY1]. One can ameliorate the degree estimates in this corollary, while keeping the same size estimates, by taking into account (3.15) (valid in the characteristic 0 case) instead of Proposition 3.4 in order to improve Proposition 3.5 and thus sharpen the choice of N in Proposition 4.2. The degree estimates for the q_j will then be the same as in (1.4).

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