EFFECTIVE NULLSTELLENSATZ
AND GEOMETRIC DEGREE
FOR ZERO-DIMENSIONAL IDEALS

A. Fabiano, G. Pucci*

A. Yger**

Contents

1. Introduction
2. Newton sums and total sums of residues
3. A first approach to the Nullstellensatz
4. Some consequences of Briançon-Skoda theorem

1. Introduction.

Let $F := (F_1, \ldots, F_k) : \mathbb{C}^n \mapsto \mathbb{C}^k$ be a polynomial map with

$$\deg F_2 \geq \deg F_3 \geq \ldots \geq \deg F_k \geq \deg F_1 > 0$$

and G a polynomial in n variables such that $G|F^{-1}(0) \equiv 0$. The Hilbert Nullstellensatz guarantees the existence of polynomials $A_1, \ldots, A_k$ such that

$$(1.1) \qquad\qquad G^s = A_1 F_1 + \cdots + A_k F_k$$

where s is an integer ≥ 0. In the usual proofs of this result, one is not concerned with estimates about the degree of the A_j's and the exponent s. This question was considered by Brownawell [Br1] and later by N. Fitchas [Fi] and Kollár [K]. Kollár got the best estimates under the technical hypothesis $\deg F_j \neq 2$ for $j > 1$. Namely it is possible to solve (1.1) with estimates

$$(1.2) \qquad \begin{cases} \max(\deg A_j F_j) \leq (1 + \deg G) \prod_{j=1}^{\min(k,n)} \deg F_j \\ s \leq \prod_{j=1}^{\min(k,n)} \deg F_j \, . \end{cases}$$

The result of Brownawell-Kollár is in fact a result about homogeneous ideals which has also an interpretation in algebraic geometry. This was pointed out by Brownawell [Br2] (see also [T]) as follows:

---

* Università della Calabria. Dipartimento di Matematica, Arcavacata di Rende (Cs), Italy.

** Université Bordeaux I, Département de Mathématiques, 33405, Talence, France.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be the isolated prime components of the homogeneous ideal $\mathcal{U} := (^h F_1, \ldots, {}^h F_k)$ where $^h F$ is the homogeneized version of F. Then, if $\deg F_i \neq 2$ for $i > 1$, there exist integers $e_1, \ldots, e_m$ (depending on the decomposition of $\mathcal{U}$) such that

$$(1.3) \qquad \begin{cases} \displaystyle\sum_1^m e_i \leq \prod_{j=1}^{\min(k,n)} \deg F_j \\ \mathcal{P}_1^{e_1} \cap \ldots \cap \mathcal{P}_m^{e_m} \subset \mathcal{U} . \end{cases}$$

If $G \equiv 0$ on $F^{-1}(0)$, then $^h G \in \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_q$, where $\mathcal{P}_1, \ldots, \mathcal{P}_q$ are the isolated prime components which do not contain X_0; it follows from (1.3) that, if

$$s_1 := \sum_{i=1}^q e_i , \quad s_2 := \sum_{i=q+1}^m e_i ,$$

$$(^h G)^{s_1} X_0^{s_2} \in (\mathcal{P}_1^{e_1} \cap \ldots \cap \mathcal{P}_q^{e_q}) \cap (\mathcal{P}_{q+1}^{e_{q+1}} \cap \ldots \cap \mathcal{P}_m^{e_m}) \subset \mathcal{U} .$$

This implies, if one restricts the situation to $X_0 = 1$,

$$(1.4) \qquad \begin{cases} G^{s_1} = \displaystyle\sum_{j=1}^k A_j F_j \\ \deg A_j F_j \leq s_1 \deg G + s_2 \leq (s_1 + s_2) \max(\deg G, 1) \\ \qquad \leq \max(\deg G, 1) \displaystyle\prod_{j=1}^{\min(k,n)} \deg F_j \end{cases}$$

Note that the value of the exponent s in (1.1) (namely $s = s_1 + s_2$) which is provided that way depends on the decomposition of the homogeneous ideal.

On the other hand, in the particular case where $\# F^{-1}(0) < +\infty$, there is another version of the Nullstellensatz following the work of M. Noether. For any $\alpha \in F^{-1}(0)$, let

$$\nu_\alpha(F) := \min \left\{ p \in \mathbb{N}, \left(\sqrt{F_1 \mathcal{O}_\alpha + \cdots + F_k \mathcal{O}_\alpha} \right)^p \subset F_1 \mathcal{O}_\alpha + \cdots + F_k \mathcal{O}_\alpha \right\},$$

that is the local Noether exponent of the map F at α. It is well known [GH] that

$$\nu_\alpha(F) \leq \mu_\alpha(F) := \dim_\mathbb{C} \left(\frac{\mathcal{O}_\alpha}{F_1 \mathcal{O}_\alpha + \cdots + F_k \mathcal{O}_\alpha} \right) .$$

If $G \equiv 0$ on $F^{-1}(0)$, then

$$(1.5) \qquad G^\nu \in (F_1, \ldots, F_k)$$

where

$$\nu := \max_{\alpha \in F^{-1}(0)} \nu_\alpha(F) .$$

Nevertheless, this result, which provides an exponent

$$\nu = \max_{\alpha \in F^{-1}(0)} \nu_\alpha(F) \leq \max_{\alpha \in F^{-1}(0)} \mu_\alpha(F)$$

depending only on the ideal $(F_1, \ldots, F_k)$, does not give any information on the degree of the $A_j F_j$'s where

$$(1.6) \qquad\qquad G^\nu = \sum_{j=1}^k A_j F_j \, .$$

The main objective of this paper is, under the hypothesis $\#F^{-1}(0) < \infty$, to find some balance between good estimates for s (that is estimates depending on the affine situation and not on the projective one) as in (1.6) and good control on the degree of the $A_j F_j$'s such as in (1.4).

In order to do that, we consider two different approaches.

• The first one works only if $k = n$, $\#F^{-1}(0) < \infty$, and the map $(F_1, \ldots, F_n)$ is dominant (that is $[\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)] < \infty$, or also $F_1, \ldots, F_n$ are algebraically independent over $\mathbb{C}$). It uses essentially two ingredients.

Primo, some variants (which will be detailed in §2) of the fact that for any $G \in \mathbb{C}[X]$, the map

$$R_{F,G} \colon w \mapsto \left[\begin{array}{c} G dX_1 \wedge \ldots \wedge dX_n \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right]$$

(where [ ] denotes the total sum of residues in the sense of Grothendieck) is a rational map.

Secondo, some very classical combinatory argument due to Perron [Pe], and which is also a basic tool in the work of Jelonek, Ploski and Pi. Cassou-Noguès [Je], [CN], [CNPl], [Pl1], [Pl2]. This first approach allows us to deal with three particular situations:

(a) $(F_1, \ldots, F_n)$ is a proper map;
(b) $(F_1, \ldots, F_n)$ is a proper map over the origin;
(c) $(F_1, \ldots, F_n)$ satisfies some "separation condition" (as in [PlT]) over the origin.

(a) In case (a), $R_{F,G}$ is in fact a polynomial, with degree controled by the Lojasiewicz exponent $\delta > 0$ of the map $(F_1, \ldots, F_n)$, defined as

$$\delta := \min \left\{ r > 0, \liminf_{\|\zeta\| \to \infty} \frac{\|F(\zeta)\|}{\|\zeta\|^r} \right\} > 0.$$

There are in this subcase two different results; either one can profit from the knowledge of δ and get, using Cauchy-Weil formula, that if $G|F^{-1}(0) \equiv 0$, then, if $D := \max_{1 \leq j \leq n} (\deg F_j)$,

$\nu := \max_{\alpha \in F^{-1}(0)} \nu_\alpha(F)$,

$$(1.7) \qquad \begin{cases} G^\nu = \displaystyle\sum_1^k A_j F_j \\[2mm] \deg A_j F_j \leq D \left(\mathbf{E} \left(2/\delta \Big(\sum_{j=1}^n \deg F_j + \nu \deg G \Big) \right) - n \right) \end{cases}$$

3

(where $\mathbf{E}(\cdot)$ denotes the integral part); or one can essentially reinterpret some algebraic method (already introduced in [CN], [CNPl]), which gives, for some $s \le d$ (where d is the geometric *affine* degree $d := [\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)]$ of the map F),

(1.8)
$$\begin{cases} G^s = \sum_{j=1}^{n} A_j F_j \\ \deg A_j F_j \le \deg G \prod_{j=1}^{n} \deg F_j \end{cases}$$

Such estimates were obtained by Pi. Cassou-Noguès in [CN]. One can compare these estimates with (1.7) only if one has some estimate from below for the Lojasiewicz exponent.

(b) In case (b), the method that leads to (1.8) works exactly the same and provides exactly the same result (already obtained in [CNPl]). The main difference with [CNPl] is that here we can extend the method to some particular situation when the map is not proper at the origin, namely in case (c).

(c) In this case, if $G|F^{-1}(0) \equiv 0$, then, for some $s \le d$ (d being again the geometric degree of F),

(1.9)
$$\begin{cases} G^s = \sum_{1}^{n} A_j F_j \\ \deg A_j F_j \le n(\deg G + 1)(\deg F_1 \cdot \ldots \cdot \deg F_n + d) \end{cases}$$

• The second approach works in the general case where $\#F^{-1}(0) < \infty$, but it profits from the knowledge of the Lojasiewicz exponent q at infinity of the map $(F_1, \ldots, F_k)$; since the case of proper maps has already been treated, this approach is interesting in the case when $q \le 0$. Note that, from Brownawell [Br1], we know that

$$q \ge 1 - (n-1) \prod_{j=1}^{\min(k,n)} \deg F_j.$$

One just interprets the condition

(1.10)
$$\|F(\zeta)\| \ge \kappa \|\zeta\|^q, \qquad \|\zeta\| \gg 1$$

as a condition on the map $({}^h F_1, \ldots, {}^h F_k)$ on the unit sphere S^{2n+1}. The key algebraic ingredient is Briançon-Skoda theorem about integral closures of ideals ([BS], [LT]); if $G|F^{-1}(0) \equiv 0$, then

(1.11)
$$\begin{cases} G^{\gamma\nu} = \sum_{j=1}^{k} A_j F_j \\ \deg A_j F_j \le \gamma(\nu \deg G + D + \max(-q, 0)) \end{cases}$$

4

with

$$\gamma := \min(n+1, k) \ , \ \nu := \max_{\alpha \in F^{-1}(0)} \nu_\alpha(F) \ , \ D := \max_{1 \le j \le k} (\deg F_j) \,.$$

Moreover, this result can be improved if one assumes that the maximal ideal $(X_0, \ldots, X_n)$ is not an embedded prime component in the decomposition of $(^h F_1, \ldots, {}^h F_k)$ (or the depth of $^h F_{1 \, n+1} \mathcal{O}_0 + \cdots + {}^h F_{k \, n+1} \mathcal{O}_0$ is ≥ 2); in this case, if $G|F^{-1}(0) \equiv 0$, then

$$(1.12) \qquad \begin{cases} G^\nu = \displaystyle\sum_1^k A_j F_j \\ \deg A_j F_j \le \nu \deg G + n[D + \max(-q, 0)] \end{cases}$$

Note that such results englobe the result obtained in [CNPl] for "stably inconsistant maps" in the sense of [Fi] ($\#\{F^{-1}(w)\} = 0$ for $\|w\| \ll 1$.)

In order to be able to compare the different versions of the Nullstellensatz which we propose here, it is of course necessary to remark that estimates (1.7), (1.11), (1.12) have the advantage to be much more precise than (1.8) or (1.9), but need some *à priori* information about the Lojasiewicz exponent (δ in the proper case, q in the general case.) Note that, if the maps $(F_1, \ldots, F_n)$ is proper at 0, and if $G|F^{-1}(0) \equiv 0$, then, one has from (1.11)

$$(1.13) \qquad \begin{cases} G^{(n+1)\nu} = \displaystyle\sum_1^n A_j F_j \\ \deg A_j F_j \le (n+1)(\nu \deg G + D) \end{cases}$$

which seems clearly in general better than estimates (1.8); this shows the real power of Briançon-Skoda theorem. We will see that, though the first approach does not seem to lead to such estimates, it has the advantage to be completely constructive (in terms of computations of total sums of residues.) Another interesting remark is that the methods in section 3 provide some effective Nullstellensätze independently of the results of Brownawell. Such methods could be a starting point to get some complete analytic proof of Brownawell-Kollár results. Note also that all the results here have been obtained in the case when the zero set of the entries is *discrete*. This is a rather *good* situation for effectivity problems, since one knows in this particular case that the Buchberger algorithm (to get a standard basis for the ideal) runs in subexponential time (see [CoS]). The search for estimates depending on the *affine degree* instead of the *projective* one in such effectivity problems, dealing with the non discrete situation, has been recently investigated in [HGi].

2. Newton sums and total sums of residues.

5

Let $f = (f_1, \ldots, f_n)$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that $\#f^{-1}(0) < \infty$. For any $r \in \mathbb{C}(X_1, \ldots, X_n)$, with no poles on $V = f^{-1}(0)$, one can define the total sum of residues of r with respect to f as

(2.1)
$$
\left[\begin{array}{c} r dX_1 \wedge \ldots \wedge dX_n \\ f_1, \ldots, f_n \end{array} \right] :=
$$
$$
\lim_{\epsilon \to 0} \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \sum_{\alpha \in V} \int_{\|\zeta - \alpha\| = \epsilon} \frac{\sum_1^m (-1)^{j-1}\overline{f_j(\zeta)} \bigwedge_{l \neq j} \overline{df_l(\zeta)} \wedge r(\zeta)d\zeta}{\|f(\zeta)\|^{2n}} .
$$

If $r, f_1, \ldots, f_n$ belong respectively to $\mathbf{F}(X_1, \ldots, X_n)$, $\mathbf{F}[X_1, \ldots, X_n]$, where $\mathbf{F}$ is a subfield of $\mathbb{C}$, then the total sum of residues of r with respect to f is also in $\mathbf{F}$. If $J(f)$ denotes the Jacobian of the map $(f_1, \ldots, f_n)$,

(2.2)
$$
\left[\begin{array}{c} r J(f) dX_1 \wedge \ldots \wedge dX_n \\ f_1, \ldots, f_n \end{array} \right] = \sum_{\alpha \in V} \mu_f(\alpha) r(\alpha)
$$

where $\mu_f(\alpha)$ is the local multiplicity of the map f at α. In particular

$$
\left[\begin{array}{c} r J(f) dX_1 \wedge \ldots \wedge dX_n \\ f_1, \ldots, f_n \end{array} \right]
$$

can be interpreted as the trace of the multiplication operator

$$
\bar{r} : \ \mathbb{C}[X]/(f_1, \ldots, f_n) \ \mapsto \ \mathbb{C}[X]/(f_1, \ldots, f_n)
$$

defined (if $r = r_1/r_2$), as $\bar{r} = \overline{r_1} \cdot \overline{r_2}^{-1}$, where

$$
\overline{r_1} : \ \bar{g} \ \mapsto \ \overline{gr_1}
$$

$$
\overline{r_2} : \ \bar{g} \ \mapsto \ \overline{gr_2} .
$$

Moreover, if

$$
d := \dim_{\mathbb{C}} \frac{\mathbb{C}[X]}{(f_1, \ldots, f_n)} ,
$$

the symmetric functions of the collection $\{r(\alpha), \alpha \in V\}$, $\sigma_1, \ldots, \sigma_d$, can be computed in terms of the Newton sums $S_1, \ldots, S_d$ of these numbers, namely

(2.3)
$$
S_j = \left[\begin{array}{c} r^j J(f) dX_1 \wedge \ldots \wedge dX_n \\ f_1, \ldots, f_n \end{array} \right]
$$

Then, the characteristic polynomial of the multiplication operator $\bar{r}$,

$$
X^d - \sigma_1 X^{d-1} + \cdots + (-1)^d \sigma_d
$$

can be expressed in terms of total sums of residues modulo the Newton relations.

Let now $F = (F_1, \ldots, F_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$, i.e. a polynomial map such that $[\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)] < +\infty$. We have the following:

6

Lemma 2.1. *There exists some algebraic hypersurface Σ in $\mathbb{C}^n$ such that, for any $w \in \mathbb{C}^n \setminus \Sigma$, $\#\{\zeta, F_j(\zeta) = w_j \ j = 1, \ldots, n\} < +\infty$ and, for any $G \in \mathbb{C}[X_1, \ldots, X_n]$, $\exists R_{F,G} \in \mathbb{C}(X_1, \ldots, X_n)$, with poles on Σ, such that, for any $w \in \mathbb{C}^n \setminus \Sigma$,*

$$\left[\begin{array}{c} G dX_1 \wedge \ldots \wedge dX_n \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right] = R_{F,G}(w) \,.$$

PROOF. For any $j \in \{1, \ldots, n\}$, $\exists N_j \in \mathbb{N}^*$, $\exists A_{j,0,\ldots}, A_{j,N_j} \in \mathbb{C}[X_1, \ldots, X_n]$, such that

$$(2.4) \qquad A_{j,0}(F)X_j^{N_j} + A_{j,1}(F)X_j^{N_j-1} + \cdots + A_{j,N_j}(F) \equiv 0$$

(since $[\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)] < +\infty$.) For such j, one can rewrite (2.4) as

$$(2.5) \qquad \sum_{k=0}^{N_j} A_{j,k}(w) X_j^{N_j-k} = \sum_{k=1}^{n} (F_k - w_k) Q_{j,k}(F, w)$$

where $Q_{j,k}$, $k = 1, \ldots, n$, is in $\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. This can be done for each $j \in \{1, \ldots, n\}$. For any

$$w \in \mathbb{C}^n \setminus \left\{ \prod_{j=1}^{n} A_{j,0}(w) = 0 \right\},$$

one has $\#\{F_j = w_j, j = 1, \ldots, n\} < +\infty$. For such w,

$$(2.6) \qquad \left[\begin{array}{c} G dX_1 \wedge \ldots \wedge dX_n \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right] = \left[\begin{array}{c} G \det[Q_{j,k}]_{\substack{1 \le j \le n \\ 1 \le k \le n}} (F, w) dX_1 \wedge \ldots \wedge dX_n \\ \sum_{k=0}^{N_1} A_{1,k}(w) X_1^{N_1-k}, \ldots, \sum_{k=0}^{N_n} A_{n,k}(w) X_n^{N_n-k} \end{array} \right].$$

This follows from the transformation law for total sums of residues (see [Aiz]). On the other hand, for any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$,

$$(2.7) \qquad \left[\begin{array}{c} X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n} dX_1 \wedge \ldots \wedge dX_n \\ \sum_{k=0}^{N_1} A_{1,k}(w) X_1^{N_1-k}, \ldots, \sum_{k=0}^{N_n} A_{n,k}(w) X_n^{N_n-k} \end{array} \right] = \\ = \prod_{j=1}^{n} \left[\begin{array}{c} X^{\alpha_j} dX \\ \sum_{k=0}^{N_j} A_{j,k}(w) X^{N_j-k} \end{array} \right]$$

It is well known that the expression (2.7) is a rational function in w, with denominator

$$\prod_{j=1}^{n} (A_{j,0}(w))^{1+\max(\alpha_j - N_j + 1, 0)}$$

This implies that (2.6) equals $R_{F,G}$, where $R_{F,G}$ is a rational function of w with denominator

$$\left(\prod_{j=1}^{n} A_{j,0}(w)\right)^{n+\deg G}.$$

This completes the proof of lemma 2.1. Note that such a lemma was proved in [Bie1] using a completely different method. ◇

• When $F = (F_1, \ldots, F_n)$ is a proper map then, for any $G \in \mathbb{C}[X_1, \ldots, X_n]$, $R_{F,G}$ is a polynomial (since the $A_{j,0}$ in (2.4) can be taken as equal to 1.) We have in fact in this case the following more precise result:

Lemma 2.2. *Let $F = (F_1, \ldots, F_n)$ a polynomial proper map from $\mathbb{C}^n$ to $\mathbb{C}^n$, with Lojasiewicz exponent $\delta > 0$. Then, for any $G \in \mathbb{C}[X_1, \ldots, X_n]$ the map*

$$w \mapsto \left[\begin{array}{c} GdX_1 \wedge \ldots \wedge dX_n \\ F_1 - w_1, \ldots, F_n - w_n \end{array}\right]$$

is a polynomial map from $\mathbb{C}^n \to \mathbb{C}^n$, with degree at most:

$$\mathbf{E}\left(1/\delta\left(\deg G + \max_k \left(\sum_{j \neq k} \deg F_j\right) + n\right)\right) - 2n + 1.$$

PROOF. When $F = (F_1, \ldots, F_n)$ is a proper map, one has $\|F(\zeta)\| \geq \eta > 0$ for $\|\zeta\| \geq R$. Therefore, it follows from Bochner-Martinelli formula in its general form (see [Aiz] or [BGVY]) that, for $\|w\| \ll 1$,

$$\left[\begin{array}{c} GdX_1 \wedge \ldots \wedge dX_n \\ F_1 - w_1, \ldots, F_n - w_n \end{array}\right] =$$

(2.8)
$$= \frac{(-1)^{\frac{n(n-1)}{2}}(n-1)!}{(2i\pi)^n} \int_{\|\zeta\|=R} \frac{G(\zeta)\left(\sum_{j=1}^{n}(-1)^{j-1}\overline{F_j(\zeta)}\bigwedge_{k \neq j}\overline{dF_k(\zeta)}\right) \wedge d\zeta}{\left(\sum_{j=1}^{n}\overline{F_j(\zeta)}(F_j(\zeta) - w_j)\right)^n}$$

$$= \sum_{m \in \mathbf{N}^n} \tau_{n,m}(R) \prod_{j=1}^{n} w_j^{m_j}$$

where, for any $m = (m_1, \ldots, m_n) \in \mathbf{N}^n$, $|m| = m_1 + \cdots + m_n$ and

$$\tau_{n,m}(R) = \frac{(-1)^{\frac{n(n-1)}{2}}(n+|m|-1)!}{(2i\pi)^n} \int_{\|\zeta\|=R} \frac{G\prod_{j=1}^{n}\overline{F_j}^{m_j}\left(\sum_{j=1}^{n}(-1)^{j-1}\overline{F_j}\bigwedge_{k \neq j}\overline{dF_k}\right) \wedge d\zeta}{\|F\|^{2(n+|m|)}}.$$

8

From the fact that $\|F(\zeta)\| \geq \gamma \|\zeta\|^{\delta}$ for $\|\zeta\| \gg 1$, the Stokes formula and standard estimates show that

$$\int_{\|\zeta\|=R} \frac{G \prod_{j=1}^{n} \overline{F_j}^{m_j} \left(\sum_{j=1}^{m} (-1)^{j-1} \overline{F_j} \bigwedge_{k \neq j} \overline{dF_k} \right) \wedge d\zeta}{\|F\|^{2(n+|m|)}} = \lim_{R \to \infty} \int_{\|\zeta\|=R} \left(\; \cdot \; \right) = 0$$

if

$$\delta(n+|m|) > \deg G + \max_{k} \left(\sum_{j \neq k} \deg F_j \right) - (n-1)\delta + n \,.$$

This implies that (2.8) is a polynomial identity which is valid everywhere. $\diamondsuit$

REMARK. Note that this result (which already appeared in [BY1], [BY2] or [BGVY], section 4) was also obtained by G. Biernat in [Bie1] using a different argument (in particular the estimate for $\deg(F_G)$ we give here does not follow from such an argument.) This was pointed out to us by M. Elkadi. $\diamondsuit$

• When $F = (F_1, \ldots, F_n)$ is a polynomial map proper at the origin (that is $A_{0,1}, \ldots, A_{0,n}$ in (2.4) can be taken such that $A_{0,1}(0) \cdot \ldots \cdot A_{0,n}(0) \neq 0$), one has:

Lemma 2.3. *Let* $F = (F_1, \ldots, F_n)$ *be a proper map at the origin; then there exists an algebraic hypersurface* Σ *in* $\mathbb{C}^n$ *with* $0 \notin \Sigma$, *such that for any* $w \in \mathbb{C}^n \setminus \Sigma$, $\#\{\zeta, F_j(\zeta) = w_j, j = 1, \ldots, n\}$ *is finite and that, for any* $G \in \mathbb{C}[X_1, \ldots, X_n]$, *there exists a rational function* $R_{F,G}$ *in* n *variables with no poles at the origin such that, for* $w \notin \Sigma$,

$$\left[\frac{G \, dX_1 \wedge \ldots \wedge dX_n}{F_1 - w_1, \ldots, F_n - w_n} \right] = R_{F,G}(w) \,.$$

PROOF. It follows from formulas (2.6) and (2.7) and from the fact that a denominator for $R_{F,G}$ in lemma 2.1 is

$$\left(\prod_{j=1}^{n} A_{j,0}(w) \right)^{n + \deg G} \qquad\qquad \diamondsuit$$

• When F is just dominant, few things can be said about the rational function $R_{F,G}$. Nevertheless, there are two interesting situations that will be discussed later.

DEFINITION 2.1. Let $F = (F_1, \ldots, F_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$. The map F satisfies the *separation condition over the origin* if and only if there are constants $c > 0$, $0 < K_{1,j} < K_{2,j} < +\infty$, $j = 1, \ldots, n$ such that for any $j \in \{1, \ldots, n\}$

(2.9) $\|w\| \leq c$ and $F(\zeta) = w \Rightarrow |\zeta_j| \leq K_{1,j}$ or $|\zeta_j| \geq K_{2,j} \,.$

EXAMPLE. Suppose the map $F = (F_1, \ldots, F_n)$ is *commode* in the sense of Ploski (see for example [PlT]), that is $\exists q > 0$, $\exists \kappa > 0$, $\exists K > 0$ such that for any $j \in \{1, \ldots, n\}$,

either $|\zeta_j| \leq K$, or $\|F(\zeta)\| \geq \kappa|\zeta_j|^{-q}$. Then, the map F satisfies the separation condition over the origin. This is immediate to check: if $F(\zeta) = w$, $w \in (\mathbb{C}^n)^*$, one has, for any $j \in \{1, \ldots, n\}$, either $|\zeta_j| \leq K$, or $|\zeta_j| \geq (\kappa/\|w\|)^{\frac{1}{q}}$; for $w \in (\mathbb{C}^n)^*$, $\|w\| \ll 1$, we have, for any $j \in \{1, \ldots, n\}$

$$(2.10) \qquad\qquad F(\zeta) = w \Rightarrow |\zeta_j| \leq K \text{ or } |\zeta_j| \geq K + 1$$

one can take K large enough so that (2.10) remains valid when $w = 0$. So that (2.9) is fulfilled with $K_{1,j} = K$, $j = 1, \ldots, n$, $K_{2,j} = K + 1$, $j = 1, \ldots, n$.

REMARK. Note that the separation condition over the origin implies that the fiber $F^{-1}(w)$ remains discrete when $\|w\| \ll 1$. This follows from the maximum principle: let Γ be an irreducible branch with positive dimension of the fiber $F^{-1}(w)$, where $\|w\| \ll c$. Then, on Γ, the function

$$\zeta \mapsto \frac{1}{\frac{K_{1,j}+K_{2,j}}{2} - \zeta_j}$$

is a bounded holomorphic function, therefore a constant, which is a contradiction (since this is true for any $j \in \{1, \ldots, n\}$.)

Under such separation condition, we have the following lemma.

Lemma 2.4. *Let $F = (F_1, \ldots, F_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ which satisfies the separation condition over the origin (with constants c, $K_{1,j}$, $K_{2,j}$, $j = 1, \ldots, n$.) Let J be the Jacobian of F, there exists some algebraic hypersurface Σ in $\mathbb{C}^n$, such that for any $w \notin \Sigma$, the variety $F^{-1}(w)$ is discrete, the polynomial $\prod_{j=1}^n ((K_{1,j} + K_{2,j})/2) - X_j)$ does not vanish on it, and for any $G \in \mathbb{C}[X_1, \ldots, X_n]$, for any w outside Σ,*

$$\left[\frac{\left(GJ/\prod_{j=1}^n \left(\frac{K_{1,j} + K_{2,j}}{2} - X_j\right)^{\deg G}\right) dX_1 \wedge \ldots \wedge dX_n}{F_1 - w_1, \ldots, F_n - w_n} \right] = \widetilde{R}_{F,G}(w),$$

where $\widetilde{R}_{F,G}$ is a rational function with no poles at the origin.

PROOF. Let

$$d = \deg(F_1, \ldots, F_n) = \dim_{\mathbb{C}(w)}\left(\frac{\mathbb{C}(w)[X_1, \ldots, X_n]}{F_1 - w_1, \ldots, F_n - w_n}\right)$$

If

$$H(X) := \prod_{j=1}^n \left(\frac{K_{1,j} + K_{2,j}}{2} - X_j\right),$$

then for $\|w\| \leq c$, H does not vanish on the set $F^{-1}(w)$. For w outside some algebraic hypersurface, the fiber $F^{-1}(w)$ is discrete and the polynomial H does not vanish on it (this is just classical elimination theory.) We can in fact be more precise: there are polynomials

10

$\theta_1, \ldots, \theta_d$ in w, with $\theta_d(0) \neq 0$ such that, as linear operators in $\mathbb{C}(w)[X_1, \ldots, X_n]/(F_1 - w_1, \ldots, F_n - w_n)$, one has:

$$(2.11) \qquad \overline{H}^{-1} \equiv \frac{1}{\theta_d(w)} \left(\sum_{k=0}^{d-1} \theta_k(w) \overline{H}^{d-1-k} \right).$$

For w outside the hypersurface

$$\Sigma = \{A_{1,0} \cdots A_{n,0} \, \theta_d(w) = 0\}$$

the fiber $F^{-1}(w)$ is discrete and the polynomial H does not vanish on it. Moreover, for such w,

$$\left[\begin{array}{c} (GJ/H^{\deg G})dX_1 \wedge \ldots \wedge dX_n \\[2mm] F_1 - w_1, \ldots, F_n - w_n \end{array} \right]$$

equals exactly the trace of the operator

$$\frac{1}{(\theta_d(w))^{\deg G}} \overline{G} \cdot \left(\sum_{k=0}^{d-1} \theta_k(w) \overline{H}^{d-1-k} \right)^{\deg G}$$

as an operator in $\mathbb{C}[X_1, \ldots, X_n]/(F_1 - w_1, \ldots, F_n - w_n)$; then from the computations in lemma 2.1, it is a rational expression in w with denominator

$$(\theta_d(w))^{\deg G} \left(\prod_{j=1}^{n} A_{j,0}(w) \right)^{n(1+(d-1)\deg G)}.$$

It remains to see that, in fact, this rational function has no poles at the origin. Let $w \in \mathbb{C}^n \setminus \Sigma$, $\|w\| \leq c$; then

$$\left[\begin{array}{c} (GJ/H^{\deg G})dX_1 \wedge \ldots \wedge dX_n \\[2mm] F_1 - w_1, \ldots, F_n - w_n \end{array} \right] = \sum_{\alpha \in F^{-1}(w)} \frac{G(\alpha)}{H^{\deg G}(\alpha)}.$$

It is immediate to see that, if $k_1, \ldots, k_n$ are positive integers with $k_1 + \cdots + k_n \leq \deg G$ and if $\alpha \in F^{-1}(w)$,

$$(2.12) \qquad \frac{|\alpha_1^{N_1} \cdots \alpha_n^{N_n}|}{\left| \left(\dfrac{K_{1,1} + K_{1,2}}{2} - \alpha_1 \right)^{\deg G} \cdots \left(\dfrac{K_{1,n} + K_{2,n}}{2} - \alpha_n \right)^{\deg G} \right|} =$$

$$= \prod_{j=1}^{n} \frac{|\alpha_j|^{k_j}}{\left| \dfrac{K_{1,j} + K_{2,j}}{2} - \alpha_j \right|^{\deg G}} \leq C_G(k_1, \ldots, k_n)$$

for some constant $C_G(k_1, \ldots, k_n)$ independent of α. This follows from the separation condition over the origin (in fact $\zeta \to \zeta_j / ((K_{1,j} + K_{2,j})/2 - \zeta_j)$ is bounded on $\bigcup_{\|w\| \leq c} F^{-1}(w)$, for any $j \in \{1, \ldots, n\}$.) Therefore, since $\#F^{-1}(w) \leq d(F_1, \ldots, F_n)$ for $\|w\| \leq c$, one has, for $\|w\| \leq c$, $w \notin \Sigma$,

$$\left\| \left[\begin{array}{c} \left(GJ/H^{\deg G} \right) dX_1 \wedge \ldots \wedge dX_n \\ \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right] \right\| \leq C$$

for some constant C independent of w. Then $\widetilde{R}_{F,G}$ has in fact no poles at the origin and the lemma is proved. $\quad \diamond$

REMARK 1. Suppose that $(F_1, \ldots, F_n)$ is separated over the origin and *not proper* over the origin. There are some constants $K, q, \kappa > 0$ such that

$$\|\zeta\| \geq K \quad \Rightarrow \quad \|F(\zeta)\| \geq \kappa \|\zeta\|^{-q}.$$

This implies that if $\zeta \in F^{-1}(w)$, $w \neq 0$, either $\|\zeta\| \leq K$, or $\|\zeta\| \geq (\kappa/\|w\|)^{\frac{1}{q}}$. Following exactly the proof of lemma 2.4, one can see then that

$$\left[\begin{array}{c} \dfrac{GJ}{H^{\deg G+1}} dX_1 \wedge \ldots dX_n \\ \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right] = \sum_{\substack{\alpha \in F^{-1}(w) \\ \|\alpha\| \leq K}} \dfrac{G(\alpha)}{H^{\deg G+1}(\alpha)} + \epsilon(w)$$

where $\epsilon(w) \to 0$ when $\|w\| \to 0$.

REMARK 2. In fact, what we really used in the proof of Lemma 2.4 or in Remark 1 above was the fact that for any w, $\|w\| \leq c$, for any $\zeta \in F^{-1}(w)$

$$|H(\zeta)| \geq \kappa (1 + |\zeta_1|), \ldots, (1 + |\zeta_n|)$$

for some positive constant κ.

The above remark leads us to introduce the following weaker separation condition:

DEFINITION 2.2. Let $\delta > 0$ and H be a polynomial in n variables. A dominant map $(F_1, \ldots, F_n)$ satisfies the $(H - \delta)$ *separation condition* over the origin if and only if these are positive constants κ, c, such that

$$\left. \begin{array}{c} \|w\| \leq c \\ F(\zeta) = w \end{array} \right\} \Rightarrow |H(\zeta)| \geq \kappa (1 + \|\zeta\|)^\delta.$$

According to the above definition, we have the following:

Lemma 2.5. *Let $F = (F_1, \ldots, F_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ which satisfies the $(H-\delta)$ separation condition over the origin. Let J be the Jacobian of F. There exists some algebraic hypersurface Σ in $\mathbb{C}^n$ such that for any $w \notin \Sigma$, the fiber $F^{-1}(w)$ is discrete, H does not vanish on it, and for any $G \in \mathbb{C}[X_1, \ldots, X_n]$, for any w outside Σ,*

$$\left[\begin{array}{c} \dfrac{GJ}{H^{\mathbf{E}(\deg G/\delta)+1}} dX_1 \wedge \ldots \wedge dX_n \\ \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right] = \widetilde{R}_{F,G}(w) \, ,$$

where $\widetilde{R}_{F,G}$ is a rational function with no poles at the origin.

PROOF. It is exactly the same than the proof of lemma 2.4, except that estimate (2.12) has to be replaced by the following: if $\|w\| \leq c$ and $F(\alpha) = w$, then, for $k \in \mathbb{N}^n$,

$$\frac{|\alpha_1^{k_1}, \ldots, \alpha_n^{k_n}|}{|H(\alpha)|^{\mathbf{E}(\deg G/\delta)+1}} \leq \tilde{\kappa}_G \frac{\|\alpha\|^{\deg G}}{(1 + \|\alpha\|)^{\delta(1+\mathbf{E}(\deg G/\delta))}} \leq C_G$$

for some positive constants $\tilde{\kappa}_G$ and C_G since

$$\delta\left(1 + \mathbf{E}(\deg G/\delta)\right) \geq \deg G.$$

Therefore the function $\zeta \to G(\zeta)/H(\zeta)^{\mathbf{E}(\deg G/\delta)+1}$ is bounded on $\bigcup_{\|w\| \leq c} F^{-1}(w)$, and the lemma follows exactly as in the proof of lemma 2.4. There is also in this case a remark analogous to Remark 1 above. $\diamondsuit$

3. A first approach to the Nullstellensatz.

The first result we will prove in this section is some explicit version of the algebraic Nullstellensatz for proper polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ with Lojasiewicz exponent $\delta > 0$. The methods here are inspired of those used to provide a solution with economic bounds (taking into account the projective degree, that is in general the product of the degrees of the entries) for the algebraic Bezout identity (see for example [BY], [BGVY], section 5, [ElY], [Y].)

Theorem 3.1. Let $F = (F_1, \ldots, F_n)$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that

$$\|Z\| \geq K \Rightarrow \|P(Z)\| \geq \kappa \|Z\|^{\delta}$$

for some $K \geq 0$, $\kappa > 0$, $\delta > 0$. Let G be some element in $\mathbb{C}[X_1, \ldots, X_n]$ such that $G|F^{-1}(0) \equiv 0$. Then, if $D := \max_{1 \leq j \leq n}(\deg F_j)$ and $\nu := \max_{\alpha \in F^{-1}(0)} \nu_{\alpha}(F)$

(3.1)
$$\begin{cases} G^{\nu} = \sum_1^n A_j F_j \\ \\ \deg A_j F_j \leq D\left(\mathbf{E}\left(2/\delta\left(\sum_{j=1}^n \deg F_j + \nu \deg G\right)\right) - n\right) \end{cases}$$

PROOF. Let $Q_{1,1}, \ldots, Q_{n,n}$ be polynomials in $2n$ variables, such that $\deg(Q_{j,k}) \leq \deg F_j - 1$ for $j = 1, \ldots, n$ $k = 1, \ldots, n$ and

(3.2)
$$\forall Z \in \mathbb{C}^n, \ \forall \zeta \in \mathbb{C}^n, \ F_j(Z) - F_j(\zeta) = \sum_{k=1}^n Q_{j,k}(Z, \zeta)(z_k - \zeta_k);$$

let $\Delta(Z, \zeta)$ be the determinant of the matrix $[Q_{j,k}]_{1 \leq j,k \leq n}$ (a *Bézoutian* for the map $(F_1, \ldots, F_n)$.) For any $Z \in \mathbb{C}^n$, the set $F^{-1}(F(Z))$ is finite (since F is a proper map.)

13

Then one can apply the global transformation law for multidimensional residues [Aiz] and get, for any G such that $G|F^{-1}(0) \equiv 0$, for any Z in $\mathbb{C}^n$

$$(3.3) \qquad G^\nu(Z) = \left[\begin{array}{c} G(X)^\nu \Delta(Z,X) dX_1 \wedge \cdots \wedge dX_n \\ \\ F_1 - F_1(Z), \ldots, F_n - F_n(Z) \end{array} \right].$$

Let

$$\Delta(Z,X) = \sum_{|\alpha|+|\beta| \le \sum_1^n \deg F_j - n} c_{\alpha,\beta} z_1^{\alpha_1} \cdots z_n^{\alpha_n} X_1^{\beta_1} \cdots X_n^{\beta_n}.$$

One has, from (3.3), for any $Z \in \mathbb{C}^n$,

$$(3.4) \qquad G^\nu(Z) = \sum_{|\alpha|+|\beta| \le \sum_1^n \deg F_j - n} c_{\alpha,\beta} \left[\begin{array}{c} G(X)^\nu X^\beta dX_1 \wedge \cdots \wedge dX_n \\ \\ F_1 - F_1(X), \ldots, F_n - F_n(X) \end{array} \right]$$

where we used the abridged notation $X^\beta := X_1^{\beta_1} \cdots X_n^{\beta_n}$. From lemma 2.2, we know that

$$\left[\begin{array}{c} G(X)^\nu X^\beta dX_1 \wedge \cdots \wedge dX_n \\ \\ F_1 - w_1, \ldots, F_n - w_n \end{array} \right] = \mathcal{Q}_\beta[G](w),$$

where $\mathcal{Q}_\beta[G]$ is a polynomial with total degree at most

$$\mathbf{E}\left(1/\delta\left(\nu \deg G + |\beta| + \max_k(\sum_{j \ne k} \deg F_j) + n\right)\right) - 2n + 1.$$

Since G^ν is in the ideal $(F_1, \ldots, F_n)$ because of Noether's theorem, one has

$$\mathcal{Q}_\beta[G](0) = 0.$$

Therefore

$$\mathcal{Q}_\beta[G](w) = \sum_{j=1}^n w_j \mathcal{Q}_\beta^{[j]}[G](w).$$

for some polynomials $\mathcal{Q}_\beta^{[j]}$. Formula (3.4) can be written as

$$(3.5) \qquad G^\nu(Z) = \sum_{|\alpha|+|\beta| \le \sum_1^n \deg F_j - n} \sum_{k=1}^n c_{\alpha,\beta} F_k(Z) Z^\alpha \mathcal{Q}_\beta^{[k]}[G](F(Z)).$$

It is clear that (3.5) is a formula of the form

$$G^\nu = \sum_{j=1}^n A_j F_j$$

14

where

$$\deg A_j F_j \leq \max_{|\alpha|+|\beta|\leq\sum_1^n \deg F_j - n} [|\alpha| + D \deg(\mathcal{Q}_\beta[G])]$$

$$\leq \max_{\alpha,\beta} \left(|\alpha| + D \left(\mathbf{E}\left(1/\delta\left(\nu \deg G + |\beta| + \max_k \sum_{j\neq k} \deg F_j + n \right) \right) - 2n + 1 \right) \right)$$

$$\leq D \left(\mathbf{E}\left(2/\delta\left(\sum_{j=1}^n \deg F_j + \nu \deg G \right) \right) - n \right). \qquad\qquad \diamondsuit$$

REMARK. When $F = (F_1, \ldots, F_n)$ is proper over the origin, and $G|F^{-1}(0) \equiv 0$, then one knows from lemma 2.3 that

$$\mathcal{Q}_\beta[G](w) = \frac{\mathcal{N}_\beta[G](w)}{\left(\prod_{j=1}^n A_{j,0}(w) \right)^{n+|\beta|+\nu \deg G}}$$

where $\mathcal{N}_\beta[G]$ is a polynomial in w vanishing at 0 and the $A_{j,0}$ come from (2.4). The same method as above can be applied for $\|w\| \ll 1$ and we obtain for $\|Z\|$ close to 0,

$$(3.6) \qquad G^\nu(Z) = \sum_{|\alpha|+|\beta|\leq\sum_1^n \deg F_j - n} c_{\alpha,\beta} F_k(Z) Z^\alpha \frac{\mathcal{N}_\beta^{[k]}[G](F(Z))}{\left(\prod_{j=1}^n A_{j,0} F(Z) \right)^{n+|\beta|+\nu \deg G}}$$

where

$$\mathcal{N}_\beta[G](w) = \sum_1^n w_j \mathcal{N}_\beta^{[j]}[G](w).$$

Raising denominators in (3.6) provides also some effective Nullstellensatz since $A_{j,0}(0) \neq 0$. The problem one has doing that is that the bounds for the degrees of the $A_{j,0}$ and the $\mathcal{N}_\beta^{[j]}[G]$ can only be estimated using for example Perron's theorem ([Pe], Satz 57) and the transformation law.

In fact, with respect to the Nullstellensatz, nice estimates (for the case of maps proper over the origin) can be obtained following a method due to A. Ploski and Pi. Cassou-Noguès. In order to be complete, we will recall here their result and its proof.

Theorem 3.2 [CNPl]. *Let $F = (F_1, \ldots, F_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ proper over the origin, and $G \in \mathbb{C}[X]$ such that $G|F^{-1}(0) \equiv 0$, then for some exponent $s \leq d = \deg(F)$,*

$$\begin{cases} G^s = \sum_1^n A_j F_j \\ \deg A_j F_j \leq \deg G \deg F_1 \cdots \deg F_n \end{cases}$$

15

PROOF. For any w close to 0,

$$\#F^{-1}(w) = d = [\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)]$$

if the points are counted with multiplicities. Let $S_1(w), \ldots, S_d(w)$ be the Newton symmetric polynomials in the values $(G(\zeta_1(w)), \ldots, G(\zeta_d(w)))$, where $\{\zeta_1(w), \ldots, \zeta_d(w)\} = F^{-1}(w)$. Since for $1 \leq j \leq d$

$$S_j(w) = \left[\frac{G^j J dX_1 \wedge \ldots \wedge dX_n}{F_1 - w_1, \ldots, F_n - w_n} \right]$$

(J as usual being the jacobian of F), it follows from lemma 2.3 that S_j is a rational function with no poles at 0. The same is true for the symetric functions $\sigma_1(w), \ldots, \sigma_d(w)$ of $\{\zeta_1(w), \ldots, \zeta_d(w)\}$. For any w close to zero and such that $F^{-1}(w)$ contains only simple points,

$$G^d - \sigma_1(w) G^{d-1} + \cdots + (-1)^d \sigma_d(w) \in (F_1 - w_1, \ldots, F_n - w_n).$$

If

$$\sigma_j(w) = \frac{A_j(w)}{A_0(w)},$$

with $A_j \in \mathbb{C}[w]$, then one has the polynomial identity

$$(3.7) \qquad A_0(F) G^d - A_1(F) G^{d-1} + \cdots + (-1)^d A_d(F) \equiv 0$$

where $A_0(0) \neq 0$, $A_1(0) = \ldots = A_d(0) = 0$ (all $G(\zeta_j(0))$ are zero since $G|F^{-1}(0) = 0$.) One has also an equation for G

$$(3.8) \qquad \widetilde{A}_0(F) G^s + \widetilde{A}_1(F) G^{s-1} + \cdots + \widetilde{A}_s(F) \equiv 0$$

($s \leq d$) which is irreducible over $\mathbb{C}[F_1, \ldots, F_n]$ and such that $\widetilde{A}_0(0) \neq 0$, $\widetilde{A}_1(0) = \ldots = \widetilde{A}_s(0) = 0$. Let

$$\mathcal{Q}(X_1, \ldots, X_n, Y) = \widetilde{A}_0(X) Y^s + \cdots + \widetilde{A}_s(X).$$

Since the polynomials $(F_1, \ldots, F_n, G)$ are algebraically dependent, there exists from Perron's theorem ([Pe], Satz 56 & 57, pages 125-126) some polynomial $\widetilde{\mathcal{Q}}$ in $n+1$ variables such that

$$\begin{cases} \widetilde{\mathcal{Q}}(F_1, \ldots, F_n, G) \equiv 0 \\ \deg_{w_{n+1}} \widetilde{\mathcal{Q}} > 0 \\ \deg_w \widetilde{\mathcal{Q}}(w_1^{\deg F_1}, \ldots, w_n^{\deg F_n}, w_{n+1}^{\deg G}) \leq \deg G \deg F_1 \ldots \deg F_n. \end{cases}$$

We now use an argument of Jelonek (see the proof of proposition 12 in [Je].) Since $\mathcal{Q}$ is irreducible, $\mathcal{Q}$ divides $\widetilde{\mathcal{Q}}$, so that we can compare the weighted degrees:

$$\deg_w \mathcal{Q}(w_1^{\deg F_1}, \ldots, w_n^{\deg F_n}, w_{n+1}^{\deg G}) \leq \deg_w \widetilde{\mathcal{Q}}(w_1^{\deg F_1}, \ldots, w_n^{\deg F_n}, w_{n+1}^{\deg G}).$$

Now, one can rewrite (3.8) as

$$\widetilde{A}_0(0)G^s = -G^s \left(\sum_{k=1}^n F_k \widetilde{B}_{0,k}(F) \right) + \sum_{l=1}^s G^{s-l} \left(\sum_{k=1}^n F_k \widetilde{B}_{j,k}(F) \right)$$

where

$$\widetilde{A}_j(w) = \widetilde{A}_j(0) + \sum_{k=1}^n w_k \widetilde{B}_{j,k}(w) \,.$$

This provides the Nullstellensatz with the convenient estimates. $\diamondsuit$

When $(F_1, \ldots, F_n)$ is dominant and separated over the origin (definition 2.1) one has:

Theorem 3.3. *Let* $F = (F_1, \ldots, F_n)$ *be a dominant polynomial map from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$ *which is separated over the origin; let* $G \in \mathbb{C}[X]$, $G|F^{-1}(0) \equiv 0$; *then, for some* $s \leq d = [\mathbb{C}(X_1, \ldots, X_n) : \mathbb{C}(F_1, \ldots, F_n)]$,

$$\begin{cases} G^s = \sum_1^n A_j F_j \\ \deg A_j F_j \leq n(\deg G + 1)[\deg F_1 \cdots \deg F_n + d] \end{cases}$$

PROOF. One can suppose that F is not proper over the origin. Let $G \in \mathbb{C}[X_1, \ldots, X_n]$, $G|F^{-1}(0) \equiv 0$. Let $K_{1,j}, K_{2,j}, j = 1, \ldots, n$ be the constants attached to the fact that the map F is separated at the origin (see definition 2.1) and

$$H(X) := \prod_1^n \left(\frac{K_{j,1} + K_{j,2}}{2} - X_j \right) \,.$$

Consider, for $\|w\| \ll 1$, and such that $\#F^{-1}(w) = d$, the Newton sums $S_1^{(G)}, \ldots, S_d^{(G)}$ of the values

$$\left\{ \frac{G(\zeta_1(w))}{H^{\deg G + 1}(\zeta_1(w))}, \ldots, \frac{G(\zeta_d(w))}{H^{\deg G + 1}(\zeta_d(w))} \right\}$$

where $F^{-1}(w) = \{\zeta_1(w), \ldots, \zeta_d(w)\}$. Following remark 1 after the proof of lemma 2.4, the functions S_j^G can be written near the origin as

$$S_j^{(G)}(w) = \sum_{\substack{\alpha \in F^{-1}(w) \\ \|\alpha\| \leq K}} \left(\frac{G(\alpha)}{H^{\deg G + 1}(\alpha)} \right)^j + \epsilon_j(w)$$

where K is a constant and $\lim_{w \mapsto 0} \epsilon_j(w) = 0$. Therefore, since $G|F^{-1}(0) \equiv 0$, one has

$$S_j^{(G)}(0) = 0 \,, \qquad j = 1, \ldots, d \,.$$

17

The same is true for the symmetric polynomials $\sigma_1^{(G)}, \dots, \sigma_d^{(G)}$ of the values

$$\left\{ \frac{G(\zeta_1(w))}{H^{(\deg G+1)}(\zeta_1(w))}, \dots, \frac{G(\zeta_d(w))}{H^{(\deg G+1)}(\zeta_d(w))} \right\}$$

where $F^{-1}(w) = \{\zeta_1(w), \dots, \zeta_d(w)\}$. As in the proof of theorem 3.2, for any w close to zero such that $\#F^{-1}(w) = d$ and $F^{-1}(w)$ contains only simple points, one has:

$$\left(\frac{G}{H^{\deg G+1}} \right)^d - \sigma_1^{(G)}(w) \left(\frac{G}{H^{\deg G+1}} \right)^{d-1} + \cdots + \sigma_d^{(G)}(w) \in (F_1 - w_1, \dots, F_n - w_n).$$

If $\widetilde{G}$ is the rational function $\dfrac{G}{H^{\deg G+1}}$, then one has the identity:

$$A_0^{(G)}(F)\widetilde{G}^d - A_1^{(G)}(F)\widetilde{G}^{d-1} + \cdots + (-1)^d A_d^{(G)}(F) \equiv 0,$$

where

$$\sigma_j^{(G)}(w) = \frac{A_j^{(G)}(w)}{A_0^{(G)}(w)},$$

with $A_1^{(G)}(0) \neq 0$, $A_1^{(G)}(0) = \dots = A_d^{(G)}(0) = 0$. There is some equation for $\widetilde{G}$ of the form

(3.9) $$\widetilde{A}_0^{(G)}(F)\widetilde{G}^s + \widetilde{A}_1^{(G)}(F)\widetilde{G}^{s-1} + \cdots + \widetilde{A}_s^{(G)}(F) = 0$$

which is irreducible over $\mathbb{C}[F_1, \dots, F_n]$ and such that $\widetilde{A}_0^{(G)}(0) \neq 0$, $\widetilde{A}_1^{(G)}(0) = \dots = \widetilde{A}_d^{(G)}(0) = 0$. Since $\widetilde{G}$ is now a rational function $\widetilde{G} = G/H^{\deg G+1}$, we cannot apply Perron's theorem immediately as in the proof of the theorem 3.2. In order to find a substitute argument, we use some variant of the famous Rabinowicz trick. Let us add some additional variable X_0; the polynomials

$$(F_1(X), \dots, F_n(X), H^{\deg G+1}(X) - X_0 G(X), X_0)$$

are algebraically dependent; there exists, by Perron's theorem ([Pe], Satz 56 and 57), some polynomial $\mathcal{Q}^G$ in $n + 2$ variables such that

(3.10)
$$\begin{cases} \mathcal{Q}^G(F_1, \dots, F_n, X_0, H^{\deg G+1} - X_0 G) \equiv 0 \\ \deg_w \mathcal{Q}^G(w_1^{\deg F_1}, \dots, w_n^{\deg F_n}, w_{n+1}, w_{n+2}^{n(\deg G+1)}) \leq n \deg F_1 \cdots \deg F_n(\deg G + 1). \end{cases}$$

Then, substituting $X_0 = H^{\deg G+1}/G$ in (3.10), one gets:

(3.11) $$\mathcal{Q}_0^G\left(F_1, \dots, F_n, \frac{1}{\widetilde{G}} \right) \equiv 0$$

18

where

$$\mathcal{Q}_0^G(X_1, \ldots, X_n, X_{n+1}) := \mathcal{Q}^G(X_1, \ldots, X_n, X_{n+1}, 0).$$

One has

(3.12) $\qquad \deg_w \mathcal{Q}^G(w_1^{\deg F_1}, \ldots, w_n^{\deg F_n}, w_{n+1}) \le n \deg F_1 \cdots \deg F_n (\deg G + 1).$

Since $(F_1, \ldots, F_n)$ is dominant, the relation (3.11), multiplied by $\widetilde{G}^{\deg w_{n+1}} \mathcal{Q}_0^G$ provides an equation for $\widetilde{G}$ (which is such that (3.9) divides it, so that we have estimates on the weighted degrees of the $\widetilde{A}_k^{(G)}$ thanks to estimates (3.10).) The relation (3.9) can be rewritten as

(3.13) $$\sum_{k=0}^{s} \widetilde{A}_k^{(G)}(F) G^{s-k} H^{k(\deg G+1)} \equiv 0.$$

Since $\widetilde{A}_0^{(G)}(0) H(0) \ne 0$, the equation (3.13) solves the Nullstellensatz. One can rewrite (3.13) as

$$G^s = \sum_1^n A_j F_j$$

$$\deg_j F_j \le n(\deg G + 1) \deg F_1 \cdots \deg F_n + sn(\deg G + 1)$$

$$\le n(\deg G + 1)[\deg F_1 \cdots \deg F_n + d].$$

This concludes the proof of Theorem 3.3.　$\diamondsuit$

When $F = (F_1, \ldots, F_n)$ is dominant and satisfies the $(H - \delta)$ separation condition in definition 2.2, one has a similar result.

Theorem 3.4. *Let $F = (F_1, \ldots, F_n)$ be a dominant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ which satisfies the $(H - \delta)$ condition for some $H \in \mathbb{C}[X]$ and $\delta > 0$. Then, for $G \in \mathbb{C}[X]$ such that $G|F^{-1}(0) \equiv 0$, one has for some $s \le d(F_1, \ldots, F_n)$,*

$$\begin{cases} G^s = \sum_1^n A_j F_j \\ \deg A_j F_j \le \deg H (\mathbf{E}(\deg G / \delta) + 1)(\deg F_1 \cdots \deg F_n + d). \end{cases}$$

PROOF. Let $G \in \mathbb{C}[X]$ such that $G|F^{-1}(0) = 0$. We let now:

$$\widetilde{G} = \frac{G}{H^{\mathbf{E}(\deg G/\delta)+1}}.$$

We use lemma 2.5 instead of lemma 2.4. We obtain as before a relation of the form (3.9), with the same conditions on $\widetilde{A}_0^{(G)}, \ldots, \widetilde{A}_s^{(G)}$. On the other hand, by the same argument than in the proof of Theorem 3.3, there is a relation

(3.14) $\qquad \mathcal{Q}^{(G)}(F_1, \ldots, F_n, \widetilde{G}) \equiv 0$

19

with

$$\deg_w \mathcal{Q}^{(G)}(w_1^{\deg F_1}, \ldots, w_n^{\deg F_n}, w_{n+1}) \le \deg F_1 \cdots \deg F_n \deg H \left(\mathbf{E}(\deg G/\delta) + 1\right).$$

Since (3.9) divides (3.14), one has

$$G^s = \sum_1^n A_j F_j$$

$$\deg A_j F_j \le \deg F_1 \cdots \deg F_n \deg H (\mathbf{E}(\deg G/\delta) + 1) + s(\mathbf{E}(\deg G/\delta) + 1) \deg H$$
$$\le \deg H (\mathbf{E}(\deg G/\delta) + 1)(\deg F_1 \cdots \deg F_n + d). \qquad \Diamond$$

4. Some consequences of Briançon-Skoda theorem.

In this section, we prove that for maps with a negative Lojasiewicz exponent, the Nullstellensatz can be solved with estimates (1.11) or (1.12).

Theorem 4.1. *Let $F = (F_1, \ldots, F_k)$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^k$ such that*

(4.1) $$\|X\| > K \Rightarrow \|F(X)\| \ge \kappa \|X\|^q$$

for some $K \ge 0$, $q \le 0$ and $\kappa > 0$. Let $G \in \mathbb{C}[X_1, \ldots, X_n]$, $G \equiv 0$ on $F^{-1}(0)$; then, if $\gamma := \min(n+1, k)$, $\nu := \max\limits_{\alpha \in F^{-1}(0)} (\nu_\alpha)$, $D := \max\limits_{1 \le j \le k} \deg F_j$,

$$\begin{cases} G^{\gamma \nu} = \sum_1^k A_j F_j \\ \deg A_j F_j \le \gamma(\nu \deg G + D + |q|). \end{cases}$$

PROOF. From (4.1), for any $\widetilde{X} = (X_0, X) = (X_0, \ldots, X_n) \in \mathbb{C}^{n+1}$, $X_0 \ne 0$, $\|X\| \ge k|X_0|$,

(4.2) $$\left\| F\left(\frac{X}{X_0}\right) \right\| \ge \kappa \left(\frac{\|X\|}{|X_0|}\right)^q,$$

that is

(4.3) $$|X_0|^{|q|} \le \frac{1}{\kappa} \|X\|^{|q|} \left\| F\left(\frac{X}{X_0}\right) \right\|,$$

so

(4.4) $$|X_0|^{|q|+D} \le \frac{1}{\kappa} \|X\|^{|q|} \left(\sum_1^k \left|{}^h F_j(\widetilde{X})\right|^2 |X_0|^{2(D-\deg F_j)}\right)^{\frac{1}{2}}.$$

Let $G \in \mathbb{C}[X_1, \ldots, X_n]$, $G \equiv 0$ on $F^{-1}(0)$; then $G^\nu \in (F_1, \ldots, F_k)$ by Noether's theorem. There exists some constant $C > 0$ such that, for any $\widetilde{X} \in S^{2n+1} := \{\widetilde{X} \in \mathbb{C}^{n+1}, \|\widetilde{X}\| = 1\}$, with $|X_0| \ge \frac{1}{K}$

$$\left|{}^h G^\nu(\widetilde{X})\right| \le C \|{}^h F(\widetilde{X})\|.$$

On the other hand, because of (4.4), $\exists C' > 0$ such that for any $\widetilde{X} \in S^{2n+1}$, with $|X_0| \leq 2/K$,

$$(4.5) \qquad\qquad |X_0|^{|q|+D} \leq C' \|{}^h F(\widetilde{X})\| .$$

On the unit sphere S^{2n+1},

$$\left| {}^h G^\nu(\widetilde{X}) |X_0|^{|q|+D} \right| \leq C'' \|{}^h F(\widetilde{X})\|$$

for some $C'' > 0$. So for any $\widetilde{X} \in \mathbb{C}^{n+1} \setminus \{0\}$

$$\left| {}^h G^\nu \left(\widetilde{X}/\|\widetilde{X}\| \right) \left(X_0/\|\widetilde{X}\| \right)^{|q|+D} \right| \leq C'' \left\| {}^h F \left(\widetilde{X}/\|\widetilde{X}\| \right) \right\| .$$

So for any $\widetilde{X} \in \mathbb{C}^{n+1} \setminus \{0\}$

$$\left| {}^h G^\nu(\widetilde{X}) \right| |X_0|^{|q|+D} \leq C'' \|X_0\|^{|q|+D+\nu} \left\| {}^h F \left(\widetilde{X}/\|\widetilde{X}\| \right) \right\| .$$

Then for $\widetilde{X} \in \mathbb{C}^{n+1} \setminus \{0\}$,

$$(4.6) \qquad \left| {}^h G^\nu(\widetilde{X}) \right| |X_0|^{|q|+D} \leq C'' \|X_0\|^{|q|+\nu \deg G} \left(\sum_1^k \left| {}^h F_j(\widetilde{X}) \right|^2 \|X_0\|^{2(D - \deg F_j)} \right)^{\frac{1}{2}} .$$

The inequality (4.6) remains valid in $\mathbb{C}^{n+1}$. At any point $\widetilde{Z}$ in $\mathbb{C}^{n+1}$, the germ at $\widetilde{Z}$ of ${}^h G^\nu X_0^{|q|+D}$ is in the integral closure of the ideal generated by the germs of ${}^h F_1, \ldots, {}^h F_k$ at this point. Then, by Briançon-Skoda theorem ([BS], or [LT] for an algebraic version)

$$\left({}^h G^\nu X_0^{|q|+D} \right)^\gamma \in ({}^h F_1, \ldots, {}^h F_k)_{\mathrm{loc}} .$$

So, there are polynomials $\widetilde{A}_1, \ldots, \widetilde{A}_k \in \mathbb{C}[X_0, \ldots, X_n]$ such that

$$(4.7) \qquad \left({}^h G^\nu X_0^{|q|+D} \right)^\gamma = \sum_1^k \widetilde{A}_j(\widetilde{X}) {}^h F_j(\widetilde{X}) .$$

Taking $X_0 = 1$ in (4.6), we get the result. $\quad \Diamond$

Under some additional hypothesis, we have

Theorem 4.2. *Let $F = (F_1, \ldots, F_k)$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^k$ such that*

$$\|X\| \geq K \Rightarrow \|F(X)\| \geq \kappa \|X\|^q$$

for some constants $K \geq 0$, $q \leq 0$, $\kappa > 0$. Assume that the depth of the ideal

$$^h F_{1\ n+1} \mathcal{O}_0 + \cdots + {^h F_{k\ n+1}} \mathcal{O}_0$$

in $\mathcal{O}_0$ is bigger than 2. Let $G \in \mathbb{C}[X_1, \ldots, X_n]$, $G \equiv 0$ on $F^{-1}(0)$; then, if $\nu := \max\limits_{\alpha \in F^{-1}(0)} (\nu_\alpha)$, $D := \max\limits_{1 \leq j \leq k} \deg(F_j)$,

$$G^\nu = \sum_1^k A_j F_j$$

with the estimates

$$\deg(A_j F_j) \leq \nu \deg G + n(D + |q|).$$

PROOF. It is exactly similar to the proof of theorem 4.1. At each point $\widetilde{Z} \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $|Z_0| \leq \frac{2}{K}|\widetilde{Z}|$, it follows from (4.5) and Briançon-Skoda theorem that

$$\left(X_0^{n(|q|+D)} \right)_{\widetilde{Z}} \in {^h F_{1\ n+1}} \mathcal{O}_{\widetilde{Z}} + \cdots + {^h F_{k\ n+1}} \mathcal{O}_{\widetilde{Z}}.$$

At each point $\widetilde{Z} \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $|Z_0| \geq \frac{1}{K}\|\widetilde{Z}\|$,

$$\left({^h G^\nu} \right)_{\widetilde{Z}} \in {^h F_{1\ n+1}} \mathcal{O}_{\widetilde{Z}} + \cdots + {^h F_{k\ n+1}} \mathcal{O}_{\widetilde{Z}}.$$

So, at any point $\widetilde{Z} \in \mathbb{C}^{n+1} \setminus \{0\}$,

$$(4.8) \qquad \left(X_0^{n(|q|+D)}\ {^h G^\nu} \right)_{\widetilde{Z}} \in {^h F_{1\ n+1}} \mathcal{O}_{\widetilde{Z}} + \cdots + {^h F_{k\ n+1}} \mathcal{O}_{\widetilde{Z}}.$$

Because of the hypothesis, it follows (as kindly pointed to us by Roger Gay) from [Ban], corollary 4.3 page 42 (which is in fact a variant of Hartogs theorem), that (4.8) is also valid at $\widetilde{Z} = 0$. So, once again

$$X_0^{n(|q|+D)}\ {^h G^\nu} = \sum_1^k \widetilde{A}_j(\widetilde{X}) {^h F_j}(\widetilde{X}).$$

Taking $X_0 = 1$ provides the result. $\quad \diamond$

References

[Aiz] L. A. Aizenberg, The multiple logarithmic residue and its applications, *Encyclopaedia of Math. Sci*, vol. 8 (*Several Complex Varibles* II), Springer-Verlag.

[Ban] C. Banica, O. Stănăşila, *Algebraic methods in the global theory of complex spaces*, John Wiley & sons, 1976, New York.

[BGVY] C. A. Berenstein, R. Gay, A. Vidras, A. Yger, *Residue currents and Bézout identities*, Progress in Mathematics 114, Birkhäuser, Basel, 1993.

[Bie1] G. Biernat, On the sum of residues for a polynomial mapping, *Bulletin de la Société des Sciences et Lettres de Lodz*, XL, 18 (1990), 73-83.

[Bie2] G. Biernat, On the Jacobi-Kronecker formula for a polynomial mapping having zeroes at infinity, *Bulletin de la Société des Sciences et Lettres de Lodz*, XLII, 29, 103-111.

[Br1] D. W. Brownawell, Bounds for the degrees in the Nullstellensatz, *Annals of Math.* 126 (1987) 577-591.

[Br2] D. W. Brownawell, A prime power version of the Nullstellensatz, *Preprint*, 1989.

[BS] J. Briançon, H. Skoda, Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbb{C}^n$, *Comptes Rendus Acad. Sci. Paris, sér. A, Math.* 278 (1974), 949-951.

[BY1] C. A. Berenstein, A. Yger, Effective Bézout identities in $Q[z_1, \ldots, z_n]$, *Acta Mathematica* 166 (1991), 69-120.

[BY2] C. A. Berenstein, A. Yger, Une formule de Jacobi et ses conséquences, it Ann. Sci. Ec. Norm. Sup. Paris 24 (1991), 363-377.

[CN] Pi. Cassou-Noguès, Quelques remarques sur les applications polynomiales propres, *manuscript*, 1994.

[CNPl] Pi. Cassou-Noguès, A. Ploski, Un théorème des zéros effectif, *Bulletin of the Polish Academy of Sciences*, 1996.

[CoS] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties and Algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, 1991.

[GH] P. Griffiths, J. Harris, *Principles of Algebraic geometry*, Wiley-Interscience, New York, 1978.

[ElY] M. Elkadi, A. Yger, Residue currents and complexity problems, in *Topics in Complex Analysis*, Banach Center Publications, 31, Institute of Mathematics, Polish Academy of Sciences, Warszawa 1995, 173-186.

[Fi] N. Fitchas, A. Galligo, Nullstellensatz effectif et conjecture de Serre (théorème de Quillen-Suslin) pour le Calcul Formel, *Math. Nachr.* 149 (1990), 231-253.

[HGi] M. Giusti, J. Heintz, J. E. Morais, L. M. Pardo, When polynomial equation systems can be "solved" fast, *Preprint, Laboratoire GAGE, Ecole Polytechnique*, 1995.

[Je] Z. Jelonek, The set of points at wich a polynomial map is not proper, *Ann. Polon. Math.* XLVIII, 3 (1993), 259-266.

[K] J. Kollár, Sharp effective Nullstellensätz, *Journal of Amer. Math. Soc.* 1, 963-975.

[LT] J. Lipman, B. Teissier, Pseudo-Rational local rings and a theorem of Briançon-Skoda, *Michigan Math. J.* 28 (1991), 97-116.

[Pe] O. Perron, *Algebra* I (Die Grundlagen) Göschens Lehrbücherei, Berlin und Leipzig, 1932.

[Pl1] A. Ploski, On the growth of proper polynomial mappings, *Ann. Polon. Math.* XLV (1985), 287-309.

[Pl2] A. Ploski, Sur l'exposant d'une application analytique I, *Bulletin of the Polish Academy of Sciences* 32 (1984), 669-673.

[PlT] A. Ploski, P. Tworzewski, A separation condition for polynomial mappings, *Preprint Instytut Matematyki Uniwersytet Jagiellónski, Krakow*, 1995/10.

[T] B. Teissier, Résultats récents d'Algèbre commutative effective, *Séminaire Bourbaki*, 1989/1990, *exposé* 718, *Astérisque* 189-190, pages 107-131.

[Y] A. Yger, *Courants résidus et applications*, Publications de l'Ecole Doctorale, Bordeaux, 1994.