Semestre 9, Année 2011-2012 SESSION 1

UE : N1MA0114 (Master 2) - Géométrie Approfondie 2

Date : May 3rd, 8h30-11h00. Duration : 3 h

Notes of the course authorized.

Notations used in the problem are those used all through the notes.

1. Consider a polynomial $F \in \mathbb{C}[X_1, X_2]$ with support the set

$$\{(0,0),(1,0),(2,0),(3,0),(0,3),(1,1),(2,2)\}.$$

Draw its Newton diagram Δ (on a figure called from now on figure 1) and compute its 2-dimensional euclidean volume.

Let $\mathcal{A}_{V_{\mathbb{T}}(F)}$ be the archimedean amœba of the polynomial F.

2. Show that the number of <u>unbounded</u> connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ cannot exceed 8.

3. How many (exactly) unbounded components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ among those (eventual) 8 mentionned in **2** do have a 2-dimensional cone as recession cone ? Draw on a separate figure (figure 2) the 2-dimensional cones corresponding to such components and picture roughly on the same figure the corresponding unbounded connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$.

4. What are the possible 1-recession cones for the other (eventual) unbounded components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$? How many such components at most do share each of these 1-dimensional recession cones ?

5. Why does the number of <u>bounded</u> connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ does not exceed 3 ? Assuming that $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ has exactly the maximal number of connected components, that is 11, complete figure 2 as a new figure (figure 3) where you sketch the drawing of the 11 connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$.

6. Let

$$R_F : (x_1, x_2) \in \mathbb{R}^2 \longmapsto \frac{1}{4\pi^2} \iint_{[0,2\pi]^2} \log |F(e^{x_1+i\theta_1}, e^{x_2+i\theta_2})| \, d\theta_1 d\theta_2$$

be the Ronkin fonction R_F and

$$p_{R_F} : \mathbb{R}^2 = (\mathrm{Trop} \setminus \{-\infty\})^2 \mapsto \mathbb{R}$$

be the evaluation of the tropical polynomial $\mathfrak{p}_{R_F}$ (recall how such $\mathfrak{p}_{R_F}$ is deduced from R_F). Explain why the following inequality holds :

$\forall (x_1, x_2) \in \mathbb{R}^2,\ p_{R_F}(x_1, x_2) \leq R_F(x_1, x_2)$? **PTO**

7. In the particular case where $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ has exactly 4 connected components, compute the Monge-Ampère real measure $\mu[p_{R_F}, ..., p_{R_F}]$ attached to the convex function p_{R_F}. In the general case (where the number of connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ lies between 4 and 11), compute $\iint_{\mathbb{R}^2} d\mu[p_{R_F}, ..., p_{R_F}]$ and compare it to $\iint_{\mathbb{R}^2} d\mu[R_F, ..., R_F]$; what is the support of $\mu[R_F, ..., R_F]$? that of $\mu[p_{R_F}, ..., p_{R_F}]$?

8. What is the relation between the number of bounded 2-dimensional faces in the roof of $\mathfrak{p}_{R_F}$ and the number of nodes in the tropical deformation $V_{\rm trop}(\mathfrak{p}_{R_F})$ of the Ronkin function R_F ? On which condition on the roof of $\mathfrak{p}_{R_F}$ does $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ have exactly 4 connected components ? Why does the number of edges in the tropical deformation $V_{\rm trop}(\mathfrak{p}_{R_F})$ equal the number of bounded edges of the roof of $\mathfrak{p}_{R_F}$?

9. When the coefficient of F are generic, what is the topological degree of the logarithmic Gauß map γ_F, considered as a rational map from the Zariski closure of $V_{\mathbb{T}}(F)$ (in the toric variety $\mathcal{X}(\Delta)$) into $\mathbb{P}^1(\mathbb{C})$? Is the toric variety $\mathcal{X}(\Delta)$ simplicial ? Is it a 2-dimensional complex manifold ?

10. Suppose that

$$F(X_1, X_2) = 27 + 4X_1^3 - 4X_2^3 + 18X_1X_2 - X_1^2X_2^2,$$

that is the Sylvester resultant of the polynomial $X^3 + X_1X^2 + X_2X - 1$, considered as a polynomial in X, and its derivative with respect to X, namely $3X^2 + 2X_1X + X_2$. What is the degree of the logarithmic Gauß map γ_F, from the Zariski closure of $V_{\mathbb{T}}(F)$ (in the toric variety $\mathcal{X}(\Delta)$) into $\mathbb{P}^1(\mathbb{C})$? How can the contour of the amœba $\mathcal{A}_{V_{\mathbb{T}}(F)}$ be parametrized in that case ?

11. We admit that, under the hypothesis in **10**, the number of connected components of $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ equals exactly 4. Starting from figure 1 (featuring Δ), draw a picture of the compactified amœba of F in that particular case.

12. Take F as in **1**. Consider the toric variety $\mathcal{X}(\Delta)$ and its algebraic moment map $\mu : \mathcal{X}(\Delta) \to \Delta$. Suppose that the Zariski closure of $V_{\mathbb{T}}(F)$ in $\mathcal{X}(\Delta)$ hits transversally each of the 4 toric curves corresponding to the 4 rays in the rational fan $\Sigma(\Delta)$ (dual to Δ), and that the images of such intersection points by μ are all distinct. How many connected components does $\mathbb{R}^2 \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ has in that case ? Starting from figure 1 again (featuring Δ), draw a picture of the compactified amœba of F in that situation.

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