NOTE ON TORSION CONJECTURE

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Abstract. In this note, we give an elementary and effective proof of the fact that the torsion conjecture for jacobian varieties implies the torsion conjecture for abelian varieties.

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1. Introduction

The classical torsion conjecture for abelian varieties over number fields can be stated as follows.

Conjecture 1.1. *Let $d \geq 0$ be an integer then:*

- *(Weak form): Given a number field k, there exists an integer $N := N(k, d) \geq 0$ such that for any d-dimensional abelian variety A over k, one has:*
$$A(k)_{tors} \subset A[N].$$
- *(Strong form): Given an integer $\delta \geq 1$, there exists an integer $N := N(\delta, d) \geq 0$ such that for any number field k of degree $\leq \delta$ and any d-dimensional abelian variety A over k, one has:*
$$A(k)_{tors} \subset A[N].$$

Completing a body of works initiated by B. Mazur in the mid-1970's [Ma77], L. Merel achieved a proof of the $d = 1$ case of the strong torsion conjecture in the mid-1990's [Me96]. But the $d > 1$ case remains widely open though recent results of the authors show that the strong torsion conjecture for the p-primary part of the torsion holds for d-dimensional abelian varieties parametrized by curves [CT09].

The aim of this note is to give a proof of the following statement, which, in particular, shows that the torsion conjecture for abelian varieties is equivalent to the torsion conjecture for jacobian varieties.

Theorem 1.2. *Let $d > 0$ be an integer. Then there exists an integer $g(d) > 0$ satisfying the following property: For any infinite field k and any d-dimensional principally polarized abelian variety A over k there exists a smooth, geometrically connected curve $C \hookrightarrow A$ of genus $g_C \leq g(d)$ that induces a smooth surjective homomorphism with connected kernel $J_{C|k} \to A$ and a closed immersion $A \to J_{C|k}$ of abelian varieties.*

More precisely, one may take $C \hookrightarrow A$ of genus $g_C = g(d)$ with:
$$g(d) = 1 + 6^d (d-1)! \frac{d(d-1)}{2}.$$

Note that if A is an arbitrary (i.e., a not necessarily principally polarized) d-dimensional abelian variety over k then, by Zarhin's trick, $(A \times A^\vee)^4$ is an $8d$-dimensional principally polarized abelian variety over k. In particular, to prove the torsion conjecture for d-dimensional abelian varieties, it is enough to prove it for $g(8d)$-dimensional jacobian varieties.

Roughly speaking, the curve C in the statement of theorem 1.2 is constructed by cutting $d-1$ times A by "nice" hyperplanes. For the proof of the crucial fact that C can be chosen to have genus bounded only in terms of d, we give two different approaches in section 2 and section 3 respectively. The first approach given in section 2 is very elementary and relies on a certain explicit genus computation. More precisely, in subsection 2.1, we show that given a smooth, geometrically connected projective variety X over an infinite field k and a fixed embedding $X \hookrightarrow \mathbb{P}^n_k$, the curves obtained by cutting X by "nice" hyperplanes all have the same Hilbert polynomial, which depends only on the Hilbert polynomial of X. We then compute in subsection 2.2 effectively the Hilbert polynomial of the resulting curves. The second approach given in section 3 is more conceptual and relies on the theory of Hilbert schemes. More precisely, in subsection 3.1 we show that given a scheme S and a closed subscheme $X \hookrightarrow \mathbb{P}^n_S$ smooth, geometrically connected and (purely) of relative dimension $d > 0$ over S there exists a surjective smooth morphism $\pi : U \to S$ of finite type and a universal curve $q : C \hookrightarrow X \times_S U \to U$ such that any curve constructed in X_s by cutting $d-1$ times X_s by "nice" hyperplanes arises as the fiber of q at some $h \in U_s$ (see proposition 3.1 for the precise statement). In subsection 3.2, we discuss the representability of the Hilbert functor. In subsection 3.3, we combine these results to recover the desired boundedness of the genus. Eventually, in section 4, we conclude the proof of theorem 1.2 by resorting to a weak version of Lefschetz theorem and a duality argument.

Remark 1.3.
1. If k is a finite field, one may still show that for any (positive-dimensional) abelian variety A over k there exists a smooth, geometrically connected curve $C \hookrightarrow A$ for which the assertions of theorem 1.2 hold. Indeed, one has only to replace the classical hyperplane Bertini theorem by more recent hypersurface Bertini theorems due to O. Gabber [G01, Cor. 1.6] and B. Poonen [P04, Thm. 1.1] and note that lemma 4.1 also works for hypersurfaces. However, the genus of the curve constructed by this method is uncontrolled in Gabber's method and depends on the poles of the zeta functions of the successive sections in Poonen's method. So it is difficult to obtain a bound of the genus independent of the finite base field as in theorem 1.2.
2. If the characteristic of k is 0, our proof of theorem 1.2 is entirely elementary and classical. On the other hand, if the characteristic of k is positive, this argument only shows that the morphism $J_{C|k} \to A$ is surjective with connected kernel and the morphism $A \to J_{C|k}$ is finite with kernel having connected Cartier dual. To get the full statement in positive characteristic, we resort to [G01, Prop. 2.4], which may be less elementary.
3. The problem of how to realize abelian varieties as quotients of jacobian varieties is classical and the first proof of the fact that this can always be done (over an algebraically closed field) seems to go back to [M52]. Other references include [Mi86] and the already mentioned [G01].

We end this section by the following lemma, which is used in both of the two approaches.

Lemma 1.4. *Let k be a field and A a d-dimensional abelian variety over k equipped with a polarization $\lambda : A \to A^\vee$ of degree δ^2 ($\delta > 0$). Let $\mathcal{L}$ denote the invertible sheaf $(id_A, \lambda)^*(\mathcal{P}_A)$ on A, where $\mathcal{P}_A$ is the normalized Poincaré sheaf on $A \times_k A^\vee$, so that $\phi_{\mathcal{L}} = 2\lambda$* [MF82, Chap.6, §2, Prop. 6.10]*, and that $\mathcal{O}_A(1) := \mathcal{L}^{\otimes 3}$ is very ample relatively to $A \to k$* [Mu70, III, §17] *and induces a closed immersion $A \hookrightarrow \mathbb{P}^n_k$. Then the Hilbert polynomial $P(T)$ of A with respect to this embedding is given by:*

$$P(T) = 6^d \delta T^d.$$

In particular, $P(T)$ depends only on (d, δ).

Proof. This is stated in [MF82, Chap.7, §2] with a hint of proof. More explicitly, from the Riemann-Roch theorem [Mu70, III, §16], one has:

$$\chi(\mathcal{O}_A(n))^2 = \deg(\phi_{\mathcal{L}^{\otimes 3n}}) = \deg(3n\phi_{\mathcal{L}}) = \deg(6n\lambda) = (6n)^{2d}\deg(\lambda),$$

whence $\chi(\mathcal{O}_A(n)) = 6^d \delta n^d$ and $P(T) = 6^d \delta T^d$. $\square$

2. First approach — Genus computation

2.1. **General case.** Let k be an infinite field and let $X_0 \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective and geometrically connected variety of dimension $d > 0$ over k. If $d - 1 > 0$, it follows from Bertini's theorem [J83, I, Th. 6.10 and Th. 7.1] that there exists a hyperplane $H_1 \hookrightarrow \mathbb{P}^n_k$ such that $X_1 := X_0 \times_{\mathbb{P}^n_k} H_1$ is a smooth, geometrically connected k-variety of dimension $d-1$. Iterating the process, one can construct a smooth geometrically connected k-variety X_i of dimension $d - i$ inductively for $0 \leq i \leq d-1$. More precisely, one set $X_i := X_{i-1} \times_{\mathbb{P}^n_k} H_i = X_0 \times_{\mathbb{P}^n_k} H_1 \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} H_i$. Also, let $\mathcal{O}_{X_i}(1)$ denote the very ample invertible sheaf relatively to k induced by the projective embedding $X_i \hookrightarrow \mathbb{P}^n_k$. Then, one has the short exact sequences of $\mathcal{O}_{X_i}$-modules:

$$0 \to \mathcal{O}_{X_i}(-1) \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i+1}} \to 0.$$

Hence, tensoring by $\mathcal{O}_{X_i}(n)$:

$$0 \to \mathcal{O}_{X_i}(n-1) \to \mathcal{O}_{X_i}(n) \to \mathcal{O}_{X_{i+1}}(n) \to 0,$$

from which it follows that $P_{i+1}(T) = P_i(T) - P_i(T-1)$, where $P_i(T)$ denotes the Hilbert polynomial of X_i. A straightforward inductive computation then yields:

$$P_i(T) = \sum_{0 \le k \le i} \binom{i}{k} (-1)^k P_0(T-k).$$

And, in particular, one can compute the genus $g(X_{d-1})$ of X_{d-1}:

$$g(X_{d-1}) = \dim_k(H^1(X_{d-1}, \mathcal{O}_{X_{d-1}})) = 1 - \sum_{0 \le k \le d-1} \binom{d-1}{k} (-1)^k P_0(-k).$$

Remark 2.1. (Comparison with Castelnuovo's bound) By construction, the curves X_{d-1} obtained by cutting out X by $d-1$ hyperplanes all have the same degree as X - say a. From Castelnuovo's bound [ACGH85, p. 116], this implies that the genus of X_{d-1} is bounded from above by a constant

$$\pi(a,n) = \frac{q(q-1)}{2}(n-1) + qr,$$

where q and r denote the quotient and remainder of the euclidean division of $a-1$ by $n-1$ respectively.

2.2. **The case of polarized abelian varieties.** We would like to apply the preceding computation to a d-dimensional abelian variety $X_0 = A$ over k equipped with a degree δ^2 polarization $\lambda : A \to A^\vee$ ($\delta > 0$). So, as in lemma 1.4, let $\mathcal{L}$ be the invertible sheaf on A such that $\phi_{\mathcal{L}} = 2\lambda$, hence $\mathcal{O}_A(1) := \mathcal{L}^{\otimes 3}$ is very ample relatively to $A \to k$ and induces a closed immersion $A \hookrightarrow \mathbb{P}^n_k$. Now, by lemma 1.4, the Hilbert polynomial $P_0(T)$ with respect to this embedding is given by: $P_0(T) = 6^d \delta T^d$. As a result:

$$g(X_{d-1}) = 1 + 6^d \delta (-1)^{d-1} \sum_{0 \le k \le d-1} \binom{d-1}{k} (-1)^k k^d.$$

It remains to compute:

$$u(d) := \sum_{0 \le k \le d-1} \binom{d-1}{k} (-1)^k k^d.$$

For this, set $u_0(x) = (1-x)^{d-1}$ and $u_i(x) = x u'_{i-1}(x)$, $i \ge 1$. Then straightforward inductive computations show that, on the one hand:

$$u_i(x) = \sum_{1 \le k \le d-1} \binom{d-1}{k} (-1)^k k^i x^k, \ i \ge 1,$$

and that, on the other hand:

$$u_i(x) = \sum_{1 \le k \le i} a_{i,k} x^k u_0^{(k)}(x), \ i \ge 1,$$

with $a_{1,1} = 1$ and

$$a_{i,k} = \begin{cases} a_{i-1,i-1}, & k = i, \\ k a_{i-1,k} + a_{i-1,k-1}, & 2 \le k \le i-1, \\ a_{i-1,1}, & k = 1. \end{cases}$$

In particular, $a_{i,i} = a_{i-1,i-1} = \cdots = a_{1,1} = 1$, $a_{i,1} = a_{i-1,1} = \cdots = a_{1,1} = 1$, and

$$a_{i,i-1} = (i-1) + a_{i-1,i-2} = \cdots = (i-1) + (i-2) + \cdots + 2 + a_{2,1} = \frac{i(i-1)}{2}.$$

So, as $u_0^{(k)}(x) = (-1)^k \frac{(d-1)!}{(d-k-1)!}(1-x)^{d-k-1}$, one eventually gets:

$$u(d) = u_d(1) = a_{d,d-1}u_0^{(d-1)}(1) = (-1)^{d-1}(d-1)!\frac{d(d-1)}{2}$$

and:

$$g(X_{d-1}) = 1 + 6^d\delta(d-1)!\frac{d(d-1)}{2}.$$

Remark 2.2. (Comparison with Castelnuovo's bound - continued) In our situation, $X_0 = A$ is embedded with degree $d!(6^d\delta)$ into $\mathbb{P}^n$, where $n = 6^d\delta - 1$. As X_{d-1} is obtained by cutting X_0 out by generic hyperplane sections, the degree of X_{d-1} is, again, $a = d!(6^d\delta)$. In particular, for $d \to +\infty$, Castelnuovo's bound is asymptotically equivalent to

$$\frac{1}{2}(d!)^2 6^d\delta$$

whereas our bound is asymptotically equivalent to

$$\frac{1}{2}d!d6^d\delta.$$

(The authors are grateful to the referee for mentioning the connection of their work with Castelnuovo's bound.)

3. Second approach — Universal curve

3.1. **Existence.** Let S be a scheme and let $X \hookrightarrow \mathbb{P}^n_S$ be a closed subscheme smooth, geometrically connected and (purely) of relative dimension $d > 0$ over S. Let $G_S(n) \simeq \mathbb{P}^n_S \to S$ denote the Grassmannian of projective linear subschemes of codimension 1 in $\mathbb{P}^n_S$ and $\mathcal{H} \hookrightarrow \mathbb{P}^n_{G_S(n)}$ the universal hyperplane. Set

$$G_S^{d-1}(n) := \underbrace{G_S(n) \times_S \cdots \times_S G_S(n)}_{d-1} \xrightarrow{\pi} S,$$

and write $p_i : G_S^{d-1}(n) \to G_S(n)$ for the natural ith projection ($i = 1, \dots, d-1$).

Proposition 3.1. (Universal curve) *There exist an open subscheme $U \subset G_S^{d-1}(n)$ such that $\pi : U \to S$ remains surjective and a smooth, geometrically connected projective curve $C \subset X \times_S U \subset \mathbb{P}^n_U$ over U with the following 'universal property': for any $h = (H_1, \dots, H_{d-1}) \in G_S^{d-1}(n)$, the scheme $I_h := (X_s)_{k(h)} \times_{\mathbb{P}^n_{k(h)}} H_1 \times_{\mathbb{P}^n_{k(h)}} \cdots \times_{\mathbb{P}^n_{k(h)}} H_{d-1}$, where $s := \pi(h) \in S$, is a smooth, geometrically connected curve over $k(h)$ if and only if $h \in U$. Moreover, one has $I_h = C_h \subset \mathbb{P}^n_{k(h)}$ for any $h \in U$.*

Proof. Since the problem is local on S, one may reduce to the case where S is affine. Then, since X is of finite presentation over S, it follows from the standard argument (cf. [EGA4-3, Prop. (8.9.1)(i)]) that one may reduce to the case where S is noetherian. Consider the closed subscheme

$$C_0 := \pi^*X \times_{\mathbb{P}^n_{G_S^{d-1}(n)}} p_1^*\mathcal{H} \times_{\mathbb{P}^n_{G_S^{d-1}(n)}} \cdots \times_{\mathbb{P}^n_{G_S^{d-1}(n)}} p_{d-1}^*\mathcal{H} \subset \mathbb{P}^n_{G_S^{d-1}(n)}$$

obtained as the (scheme-theoretic) intersection of $\pi^*X = X \times_S G_S^{d-1}(n)$ and $d-1$ hyperplanes $p_1^*\mathcal{H}, \dots, p_{d-1}^*\mathcal{H}$ in $\mathbb{P}^n_{G_S^{d-1}(n)}$. Note that one may also write:

$$\mathbb{P}^n_{G_S^{d-1}(n)} = \underbrace{\mathbb{P}^n_{G_S(n)} \times_{\mathbb{P}^n_S} \cdots \times_{\mathbb{P}^n_S} \mathbb{P}^n_{G_S(n)}}_{d-1}$$

and correspondingly:

$$p_i^*\mathcal{H} = \mathbb{P}^n_{G_S(n)} \times_{\mathbb{P}^n_S} \cdots \times_{\mathbb{P}^n_S} \underset{i}{\mathcal{H}} \times_{\mathbb{P}^n_S} \cdots \times_{\mathbb{P}^n_S} \mathbb{P}^n_{G_S(n)},$$

hence one gets:

$$C_0 = X \times_{\mathbb{P}^n_S} \underbrace{\mathcal{H} \times_{\mathbb{P}^n_S} \cdots \times_{\mathbb{P}^n_S} \mathcal{H}}_{d-1}$$

(that is, for any $(x, h = (H_1, \dots, H_{d-1})) \in X \times_S G_S^{d-1}(n)$, $(x,h) \in C_0$ if and only if $x \in H_1 \times_{\mathbb{P}^n_{k(h)}} \cdots \times_{\mathbb{P}^n_{k(h)}} H_{d-1}$). We will use the following notation:

$$\begin{tikzcd}
C_0 \arrow[r, hook] \arrow[rr, bend left, "q_0"] \arrow[dr] & X \times_S G_S^{d-1}(n) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & G_S^{d-1}(n) \arrow[d, "\pi"] \\
& X \arrow[r] & S
\end{tikzcd}$$

By construction, $q_0 : C_0 \to G_S^{d-1}(n)$ is projective (as a closed immersion composed with the base change of a projective morphism) with geometrically connected fibers of dimension ≥ 1 ([J83, I, Th. 7.1]).

From the semicontinuity of dimension, the subset:

$$U_1 := \{h \in G_S^{d-1}(n) \mid \dim(q_0^{-1}(h)) = 1\}$$

is open in $G_S^{d-1}(n)$. Write $q_1 : C_1 \to U_1$ for the base change of $q_0 : C_0 \to G_S^{d-1}(n)$ *via* the open immersion $U_1 \hookrightarrow G_S^{d-1}(n)$. Then $q_1 : C_1 \to U_1$ remains projective over U_1 hence closed. Define the following loci:

$$C_1^{smooth} := \{c \in C_1 \mid q_1 : C_1 \to U_1 \text{ is smooth at } c\}$$

$$C_1^{reg} := \{c \in C_1 \mid C_{1,q_1(c)} \to \mathrm{spec}(k(q_1(c))) \text{ is smooth (that is, } C_{1,\overline{q_1(c)}} \text{ is regular)}\}$$

and set $U := U_1 \setminus q_1(C_1 \setminus C_1^{smooth})$, $U' := U_1 \setminus q_1(C_1 \setminus C_1^{reg})$. Then

- U is open in U_1 (since C_1^{smooth} is open in C_1 and $q_1 : C_1 \to U_1$ is closed);
- $U \subset U'$;
- $\pi(U') = S$ ([J83, I, Th. 6.10]).

Write $q : C \to U$ for the base change of $q_1 : C_1 \to U_1$ *via* the open immersion $U \hookrightarrow U_1$. It only remains to prove that $\pi : U \to S$ is surjective and that $C \hookrightarrow \mathbb{P}^n_U$ satisfies the announced universal property. This will follow from the claim below which, in particular, shows that $U = U'$:

<u>Claim.</u> $C_1^{smooth} = C_1^{reg}$.

Indeed, this follows from the following claim:

<u>Claim.</u> $q_1 : C_1 \to U_1$ is flat.

To prove the latter claim, let $x \in C_1$ and set $h = q_1(x)$. Let A (resp. B, resp. $\overline{B}$) be the local ring of U_1 at h (resp. of X_{U_1} at x, resp. of C_1 at x), and k the residue field of A (i.e., $k = k(h)$). It follows from the reduction step at the beginning of the proof that A, B and $\overline{B}$ are noetherian. Let $f_i \in B$ be the image in B of a local defining equation of the hyperplane $(p_i^*\mathcal{H})_{U_1} \subset \mathbb{P}^n_{U_1}$. Then one has $\overline{B} = B/(f_1, \dots, f_{d-1})$. Now, one has to prove that $\overline{B}$ is flat as an A-module. For this, it suffices to show that $g_1, \dots, g_{d-1}$ is a regular sequence of $B \otimes_A k$, where g_i is the image of f_i in $B \otimes_A k$, by [EGA4-1, Chap. 0_{IV}, Prop. (15.1.16), c)$\Rightarrow$b)]. But this latter fact follows from [EGA4-1, Chap. 0_{IV}, Cor. (16.5.6), b)$\Rightarrow$a)]. (The second author would like to thank S. Yasuda very much for the useful discussion on this proof.) $\square$

3.2. **Representability of the Hilbert functor.** Let *Sets* and Sch^{op} denote the category of sets and the opposite category of locally noetherian schemes, respectively. From [FGA] that for any $N \geq 0$ and $P \in \mathbb{Q}[T]$ the Hilbert functor $\mathcal{H}ilb_{N,P} : Sch^{op} \to Sets$ defined by

$$\mathcal{H}ilb_{N,P}(X) = \left\{Y \subset \mathbb{P}^N_X \ \textit{closed subscheme} \mid Y \to X \textit{ is flat and has Hilbert polynomial } P\right\}$$

is representable by a projective $\mathbb{Z}$-scheme $Hilb_{N,P} \to \mathbb{Z}$; we will denote by $Y_{N,P} \subset \mathbb{P}^N_{Hilb_{N,P}}$ the universal object over it.

Now, set $P_{d,\delta}(T) := 6^d\delta T^d \in \mathbb{Q}[T]$ and $m := 6^d\delta - 1$. Define the functor $\mathcal{A}_{d,\delta} : Sch^{op} \to Sets$ by

$$\mathcal{A}_{d,\delta}(X) = \left\{ (A \xrightarrow{\pi} X, \lambda, \phi) \mid \begin{array}{l} A \xrightarrow{\pi} X \text{ is an abelian scheme of dimension } d; \\ \lambda : A \to A^\vee \text{ is a polarization of degree } \delta^2; \\ \phi : \mathbb{P}(\pi_*(\mathcal{L}^\Delta(\lambda)^{\otimes 3})) \tilde{\to} \mathbb{P}^m_X \text{ is a linearization.} \end{array} \right\}$$

Here $\mathcal{L}^\Delta(\lambda)$ stands for the invertible sheaf $(id_A, \lambda)^*(\mathcal{P}_A)$ on A, where $\mathcal{P}_A$ is the normalized Poincaré sheaf on $A \times_X A^\vee$. Set $Y^{(1)}_{m,P_{d,\delta}} := Y_{m,P_{d,\delta}} \times_{Hilb_{m,P_{d,\delta}}} Y_{m,P_{d,\delta}} \xrightarrow{pr_2} Y_{m,P_{d,\delta}}$, which is naturally equipped with the diagonal section $\epsilon : Y_{m,P_{d,\delta}} \to Y^{(1)}_{m,P_{d,\delta}}$. Then $(Y^{(1)}_{m,P_{d,\delta}} \hookrightarrow \mathbb{P}^m_{Y_{m,P_{d,\delta}}}, \epsilon) \to Y^{(1)}_{m,P_{d,\delta}}$ is the universal family for closed subschemes of $\mathbb{P}^m_X$ flat over X with Hilbert polynomial $P_{d,\delta}$ and 1 section. Thus, by definition of $\mathcal{A}_{d,\delta} : Sch^{op} \to Sets$, one has a natural functor morphism

$$\Phi : \mathcal{A}_{d,\delta} \to \mathrm{Hom}(-, Y^{(1)}_{m,P_{d,\delta}}).$$

Then, applying steps (I), (II), (V) and (VI) of the proof of [MF82, Prop. 7.3] (where level structures are also considered), one can show:

<u>Claim.</u> *There exists a locally closed subscheme $A_{d,\delta} \subset Y^{(1)}_{m,P_{d,\delta}}$ such that Φ induces a functor isomorphism*

$$\mathcal{A}_{d,\delta} \tilde{\to} \mathrm{Hom}(-, A_{d,\delta}) \subset \mathrm{Hom}(-, Y^{(1)}_{m,P_{d,\delta}}).$$

In other words, $A_{d,\delta}$ represents $\mathcal{A}_{d,\delta}$.

3.3. **Boundedness of the genus.** We combine the results of subsections 3.1 and 3.2 to prove that there exists an integer $g(d) > 0$ such that for any d-dimensional principally polarized abelian variety A over an infinite field, any curve $C \hookrightarrow A$ constructed as in subsection 2.2 has genus $g_C \le g(d)$.

As $Y^{(1)}_{m,P_{d,1}}$ is projective over $\mathbb{Z}$, it is noetherian and, consequently, $A_{d,1}$ is as well. Let $(\mathcal{A} \xrightarrow{\pi} A_{d,1}, \lambda, \phi)$ denote the universal family over $A_{d,1}$ and $\mathcal{A} \hookrightarrow \mathbb{P}(\pi_*(\mathcal{L}^\Delta(\lambda)^{\otimes 3})) \xrightarrow[\sim]{\phi} \mathbb{P}^m_{A_{d,1}}$ the closed immersion induced by λ and ϕ. Now, proposition 3.1 applied to $\mathcal{A} \subset \mathbb{P}^m_{A_{d,1}}$ yields an open subscheme $U \subset G^{d-1}_{A_{d,1}}(m)$ and a universal (projective, smooth, geometrically connected) curve $q : \mathcal{C} \to U$. As U is noetherian, it has only finitely many connected components. Thus, only finitely many values appear as the genus of a fiber of $q : \mathcal{C} \to U$. Let $g(d)$ be the maximum of such values.

Any d-dimensional abelian variety $A \xrightarrow{\pi} \mathrm{spec}(k)$ over a field k with a principal polarization $\lambda : A \tilde{\to} A^\vee$ can be equipped with a linearization $\phi : \mathbb{P}(\pi_*(\mathcal{L}^\Delta(\lambda)^{\otimes 3})) \tilde{\to} \mathbb{P}^m_k$ defined over k, which corresponds to a k-rational point a of $A_{d,1}$. Further, a choice of $d-1$ "nice" hyperplanes $H_1, \dots, H_{d-1}$ corresponds to a k-rational point b of the fiber U_a (which is a nonempty open subscheme of $G^{d-1}_k(m)$) and the resulting curve $C \hookrightarrow A$ is identified with the fiber $\mathcal{C}_b$ at b of $q : \mathcal{C} \to U$. Thus, $g_C \le g(d)$.

4. End of the proof

Recall first the following classical result [SGA1, Chap. X, Lem. 2.10]:

Lemma 4.1. *Let k be an algebraically closed field, and X a normal, irreducible scheme proper over k. Let $g : X \to \mathbb{P}^n_k$ be a morphism such that $g(X)$ has dimension ≥ 2. Then, for any hyperplane $H \subset \mathbb{P}^n_k$, the scheme $X \times_{\mathbb{P}^n_k} H$ is connected and the natural homomorphism*

$$\pi_1(X \times_{\mathbb{P}^n_k} H) \to \pi_1(X)$$

of étale fundamental groups induced by the first projection $X \times_{\mathbb{P}^n_k} H \to X$ is surjective.

From now on, fix an integer $d \geq 2$ (as the $d = 1$ case is trivial), an infinite field k and a d-dimensional abelian variety A over k equipped with a principal polarization $\lambda : A \tilde{\to} A^\vee$. As in subsection 2.2 let $\mathcal{L}$ be the invertible sheaf on A defining 2λ and $A \hookrightarrow \mathbb{P}^n_k$ the closed immersion induced by $\mathcal{L}^{\otimes 3}$.

Then, for any hyperplane $H_1 \hookrightarrow \mathbb{P}^n_k$ such that $A \cap H_1$ is a smooth, geometrically connected k-variety of dimension $d-1$ it follows from lemma 4.1, that the closed immersion $A \cap H_1 \hookrightarrow A$ induces a surjective homomorphism $\pi_1(A \cap H_1) \twoheadrightarrow \pi_1(A)$ over Γ_k. Iterating $d-1$ times the process, one can construct a smooth geometrically connected k-curve $C := A \times_{\mathbb{P}^n_k} H_1 \times_{\mathbb{P}^n_k} \cdots \times_{\mathbb{P}^n_k} H_{d-1}$ such that the closed immersion $C \hookrightarrow A$ induces a surjective homomorphism of fundamental groups $\pi_1(C) \twoheadrightarrow \pi_1(A)$ over the absolute Galois group Γ_k of k or, in other words, a surjective homomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(C_{\overline{k}}) & \longrightarrow & \pi_1(C) & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(A_{\overline{k}}) & \longrightarrow & \pi_1(A) & \longrightarrow & \Gamma_k & \longrightarrow & 1 \end{array} \tag{1}$$

Since $\pi_1(A_{\overline{k}}) = T(A)$ (the full Tate module) is abelian, the homomorphism $\pi_1(C_{\overline{k}}) \to \pi_1(A_{\overline{k}})$ kills the commutator subgroup $\pi_1(C_{\overline{k}})' \subset \pi_1(C_{\overline{k}})$ and one gets:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(C_{\overline{k}})^{ab} & \longrightarrow & \pi_1(C)^{(ab)} & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(A_{\overline{k}}) & \longrightarrow & \pi_1(A) & \longrightarrow & \Gamma_k & \longrightarrow & 1, \end{array} \tag{2}$$

where $\pi_1(C)^{(ab)}$ denotes the quotient $\pi_1(C)/\pi_1(C_{\overline{k}})'$. Further, the homomorphism $\pi_1(C_{\overline{k}})^{ab} \to \pi_1(A_{\overline{k}})$ is identified with the natural homomorphism $T(J) \to T(A)$ of full Tate modules induced by the albanese morphism $J \to A$ associated with $C \hookrightarrow A$, where $J = J_{C|k}$ denotes the jacobian variety of C over k.

Since the morphism $J \to A$ induces a surjection $T(J) \twoheadrightarrow T(A)$ of full Tate modules, it is surjective with connected kernel. Indeed, let $\mathcal{K}$, $\mathcal{I}$ and $\mathcal{C}$ denote the kernel, the image and the cokernel of $J \to A$, respectively. Note that $\mathcal{I}$ and $\mathcal{C}$ are abelian varieties, while $\mathcal{K}$ may not in general. The exact sequence

$$0 \to \mathcal{I} \to A \to \mathcal{C} \to 0$$

of abelian varieties induces an exact sequence

$$0 \to T(\mathcal{I}) \to T(A) \to T(\mathcal{C}) \to 0$$

of full Tate modules. Since the image of $T(J)$ in $T(A)$ is contained in $T(\mathcal{I}) \subset T(A)$, one must have $T(\mathcal{C}) = 0$, hence $\mathcal{C} = 0$, i.e., $J \to A$ is surjective. Next, let $\mathcal{K}^0$ denote the connected component at 0 of $\mathcal{K}$, and $\mathcal{K}^{0,red}$ the reduced closed subscheme associated with $\mathcal{K}^0$. Then $\mathcal{K}^{0,red} \subset \mathcal{K}^0 \subset \mathcal{K} \subset J$ are closed subgroup schemes; $\mathcal{K}^{et} := \mathcal{K}/\mathcal{K}^0$ is finite and étale; $K^{loc} := \mathcal{K}^0/\mathcal{K}^{0,red}$ is finite and connected; and $\mathcal{K}^{0,red}$ is an abelian variety. Now, by using the snake lemma, one gets the exact sequence

$$0 \to T(\mathcal{K}^{0,red}) \to T(J) \to T(A) \to \mathcal{K}^{et}(\overline{k}) \to 0,$$

from which one gets $\mathcal{K}^{et}(\overline{k}) = 0$, hence $\mathcal{K}^{et} = 0$, and $\mathcal{K} = \mathcal{K}^0$ is connected.

Next, the dual morphism $A^\vee (= A) \to J^\vee (= J)$ is finite with kernel having connected Cartier dual. Indeed, set $\tilde{A} := J/\mathcal{K}^{0,red}$. Then one gets the following two exact sequences:

$$0 \to \mathcal{K}^{0,red} \to J \to \tilde{A} \to 0,$$

$$0 \to \mathcal{K}^{loc} \to \tilde{A} \to A \to 0.$$

The dual morphism $A^\vee \to J^\vee$ factors as $A^\vee \to \tilde{A}^\vee \to J^\vee$. Here, the kernel of the dual isogeny $A^\vee \to \tilde{A}^\vee$ is identified with the Cartier dual of the kernel $\mathcal{K}^{loc}$ of the morphism $\tilde{A} \to A$, and the dual morphism $\tilde{A}^\vee \to J^\vee$ is a closed immersion. Indeed, for the former fact, see, e.g., [Mu70, III, §15, Th. 1]. For the latter fact, write $\mathcal{K}_1$ and $\mathcal{I}_1$ for the kernel and the image of the morphism $\tilde{A}^\vee \to J^\vee$, respectively. From Poincaré's complete reducibility theorem, there exists an abelian subvariety $A' \subset J$

such that the composite of $A' \hookrightarrow J \twoheadrightarrow \tilde{A}$ is an isogeny. Thus, the kernel $\mathcal{K}_1$ is contained in the kernel of the dual isogeny $\tilde{A}^\vee \to (A')^\vee$, hence is finite. Since the dual morphism $\tilde{A}^\vee \to J^\vee$ factors as $\tilde{A}^\vee \to \mathcal{I}_1 \to J^\vee$, the original morphism $J \to \tilde{A}$ factors as $J(= J^{\vee\vee}) \to (\mathcal{I}_1)^\vee \to \tilde{A}(= \tilde{A}^{\vee\vee})$. Here, the morphism $(\mathcal{I}_1)^\vee \to \tilde{A}$ is an isogeny whose kernel is identified with the Cartier dual $(\mathcal{K}_1)^D$ of $\mathcal{K}_1$. This, together with the surjectivity of $J \to \tilde{A}$, implies that the morphism $J \to (\mathcal{I}_1)^\vee$ must be also surjective. Now, the surjections $J \twoheadrightarrow (\mathcal{I}_1)^\vee \twoheadrightarrow \tilde{A}$ yield an epimorphism from the kernel $\mathcal{K}^{0,red}$ of $J \twoheadrightarrow \tilde{A}$ to the kernel $(\mathcal{K}_1)^D$ of $(\mathcal{I}_1)^\vee \twoheadrightarrow \tilde{A}$. As $\mathcal{K}^{0,red}$ is an abelian variety (hence divisible) and $(\mathcal{K}_1)^D$ is finite, this implies that $(\mathcal{K}_1)^D = 0$, hence $\mathcal{K}_1 = 0$, as desired. Combining these facts, one sees that the kernel of $A^\vee \to J^\vee$ coincides with the Cartier dual of the (finite, connected) group scheme $\mathcal{K}^{loc}$.

Now, if the characteristic of k is 0, the surjectivity of the morphism $J \to A$ automatically implies the smoothness (cf. [Co02, Lem. 2.1]), and the connectedness of the Cartier dual of the kernel of the morphism $A \to J$ automatically implies the triviality. Thus, the proof is completed in characteristic 0.

On the other hand, if the characteristic of k is positive, the assertion that the morphism $J \to A$ is smooth with connected kernel follows from [G01, Prop. 2.4(iii)]. Namely, notations being as above, one has $\mathcal{K} = \mathcal{K}^{0,red}$ and $A = \tilde{A}$. Thus, it follows from the above argument that the morphism $A \to J$ is a closed immersion. (See also [G01, Cor. 2.5].)

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