Université de Bordeaux
Master 2 MIMSE 14/12/2016

Final exam
Jump Markov processes

3h00

Exercise 1

We consider a machine with two identical electronic components. The machine works as soon as at least one of the electronic component works. We assume that:

- Each component has a failure rate λ (i.e. the failure times are i.i.d. exponential random variables with parameter λ).
- The repair time of one component while the other works has an exponential distribution with parameter μ.
- When one component fails, the other one can also fail instantly with probability c.
- When the two electronic components fail, the machine is out of order and cannot be repaired. In this case, electronic components are not repaired.

We denote X_t the number of electronic components in working conditions at time t. $(X_t)_{t\in\mathbb{R}^+}$ is an homogeneous jump Markov process with state space $E = \{0, 1, 2\}$ and with initial law $X_0 = 2$.

1. Let $R \sim \mathcal{E}(\alpha)$ and $S \sim \mathcal{E}(\beta)$ two independent random variables. Show that $\min(R, S) \sim \mathcal{E}(\alpha + \beta)$. We denote $M = \min(R, S)$. Show that
$$\mathbb{P}(M = R) = \mathbb{P}(R \leqslant S) = \frac{\alpha}{\alpha + \beta}.$$
2. Give the transition matrix of the embedded Markov chain $(Z_n)_{n\in\mathbb{N}}$.
3. Give the infinitesimal generator of the process $(X_t)_{t\in\mathbb{R}^+}$.
4. Give communicating classes and their properties.
5. Calculate the expected hitting time of $\{0\}$.

We assume now that the machine can be repaired when the two components fail. In this case the repair time for each electronic component has an exponential distribution with parameter μ and these times are independent.

1. Give the new transition matrix of the embedded Markov chain $(Z_n)_{n\in\mathbb{N}}$.

2. give the new infinitesimal generator of the process $(X_t)_{t\in\mathbb{R}^+}$.

3. Give communicating classes and their properties.

4. Does there exist an invariant probability measure? If so, calculate it.

5. We assume that the company earns 100 euros per unit of time when the machine works and loses 50 euros per unit of time when the machine does not work. What is the average gain per unit of time of the company?

Exercise 2

We consider a population of bacteria. We denote X_t the number of bacteria at time $t \geqslant 0$ and we assume that $(X_t)_{t\geqslant 0}$ is an homogeneous jump Markov process with state space $\mathbb{N}$, initial law $X_0 = x_0$ and infinitesimal generator

$$Q_{xy} = \begin{cases} \lambda x & \text{if } y = x+1 \\ \nu x & \text{if } y = x-1 \\ \alpha & \text{if } y = x \\ 0 & \text{otherwise,} \end{cases}$$

for $x > 0$, with $\lambda > 0$ and $\nu > 0$ two given parameters. We define also $Q_{01} = \lambda$ and $Q_{00} = -\lambda$.

1. What is the value of α?

2. Show that the non explosion condition is satisfied.

3. Give communicating classes.

4. We recall that non-symetric random walks $(Y_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}$ are transient and $\lim_{n\to+\infty} Y_n = +\infty$ (respectively $-\infty$) if $\mathbb{P}(Y_{n+1} - Y_n = 1) > \mathbb{P}(Y_{n+1} - Y_n = -1)$ (respectively $<$).

   (a) If $\nu < \lambda$, show that the process $(X_t)_{t\geqslant 0}$ is transient.

   (b) If $\nu > \lambda$, show that the process $(X_t)_{t\geqslant 0}$ is recurrent.

Exercise 3

Let us consider $(W_t)_{t\in\mathbb{R}^+}$ a real valued Brownian motion, $(N_t)_{t\in\mathbb{R}^+}$ a Poisson process with intensity λ, $(Z_i)_{i\in\mathbb{N}^*}$ a sequence of real random variables i.i.d. with a given distribution ν and a, b some constants. We assume that $(W_t)_{t\in\mathbb{R}}$, $(N_t)_{t\in\mathbb{R}}$ and $(Z_i)_{i\in\mathbb{N}^*}$ are independent. We define the process $(X_t)_{t\in\mathbb{R}^+}$ by setting, for all $t \geqslant 0$,

$$X_t = at + bW_t + \sum_{n=1}^{N_t} Z_n.$$

We denote by $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ the filtration generated by the process $(X_t)_{t\in\mathbb{R}^+}$.

1. Prove that $(X_t)_{t\in\mathbb{R}^+}$ has stationnary and independent increments.

2. We assume that $a = b = 0$ and ν is a discrete distribution. Give the state space E of $(X_t)_{t\in\mathbb{R}^+}$ and prove that $(X_t)_{t\in\mathbb{R}^+}$ is a jump Markov process.

   (a) We assume that $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-\pi}$, i.e. $\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = -\pi) = 1/2$. Give E, communicating classes and their properties. Calculate the infinitesimal generator of $(X_t)_{t\in\mathbb{R}^+}$.

   (b) We assume that $\nu = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_{-1}$, i.e. $\mathbb{P}(Z_i = 2) = \mathbb{P}(Z_i = -1) = 1/2$. Give E and communicating classes. Show that $\frac{X_t}{t}$ converges almost surely and give its limit. Deduce properties of communicating classes.

3. We assume that $(Z_i)_{i\in\mathbb{N}^*}$ are integrable random variables and ν is a discrete distribution.

   (a) Calculate $\mathbb{E}[\sum_{n=1}^{N_t} Z_n | N_t = k]$ for all $k \in \mathbb{N}$ and $\mathbb{E}[\sum_{n=1}^{N_t} Z_n]$.

   (b) Show that $(X_t)_{t\in\mathbb{R}^+}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ if we impose a relation between a, λ and $\mathbb{E}[Z_1]$. We recall that a martingale $(M_t)_{t\in\mathbb{R}^+}$ (with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$) is an integrable process such that $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ when $t > s$.

   (c) We assume that $Z_i = c$. Show that $(X_t^2 - dt)_{t\in\mathbb{R}^+}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ if we impose a relation between a, b, c and d.

4. We assume that $a = b = 0$ and ν is a finite discrete distribution: $\nu = \sum_{k=1}^{n} p_k \delta_{x_k}$, i.e. $\mathbb{P}(Z_i = x_k) = p_k$. Show that $(X_t)_{t\in\mathbb{R}^+}$ can be rewritten as

$$X_t = \sum_{k=1}^{n} x_k N_t^k$$

   where N^1,...,N^n are n independent Poisson processes whose intensity has to be determined.