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Dessins from a geometric point of view

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Abstract

In this paper we study the topological aspects of dessins (via analytic description) with two distinct goals. Firstly we are interested in fields of definition and fields of moduli. We give a topological proof that there exist some dessins with no model defined over their field of moduli. This answers explicitly a question asked in [Har87]. Our second motivation is to collect practical and theoretical data for the explicit computation of covers given by some topological description, following ideas of Atkin [ASD71] Oesterlé and ourselves. This leads to a method for the computation of the linear space associated to a divisor on a given dessin.

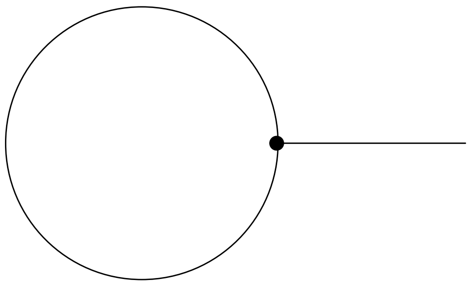
§1. Introduction

This paper develops some practical applications of the archimedean analytic description of coverings through Puiseux series. In the second section, we recall a classical result due to Klein concerning the classification of genus zero Galois coverings, and related to the classification of regular polytopes. In the third section we give a review of many possible definitions of what a moduli field is. We do not claim to exhaust the list of various contradictory notions denoted by these words, but simply to avoid the frequent confusion about it. The fourth section is an illustration of what knowledge can be provided by local considerations at infinity. We show that such a study leads to interesting examples of coverings with strange rationality properties, which we can state by mere combinatorial considerations. In the fifth section we recall quite classical results related to the Legendre form of elliptic curves, which are useful in the next section. The sixth section consists in the analytic description of the linear systems associated with some divisors on the curve corresponding to a given dessin. It leads to a method for the explicit computation of an algebraic model. This provides us with an algorithmic correspondance between abstract dessins and explicit Belyi functions. We give quite general techniques. In the case where the genus of the dessin is small, the equations have a simple general form which helps make the method more beautiful. In view of that, it is convenient to have an a priori description of the equation of the curve and Belyi function, which is given by classical curve theory for low genera. We detail that in the seventh section.

The authors wish to thank Leila Schneps for many useful discussions and for the organization of the Luminy conference in April 1993, where we found the motivation for this work (especially the four talks given by Joseph Oesterlé).

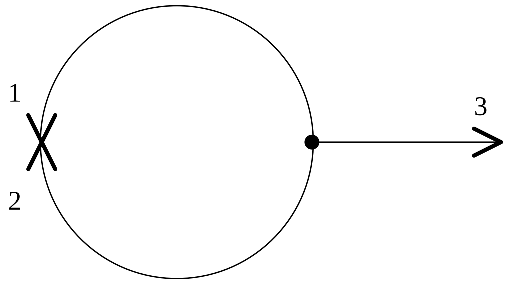
Throughout this paper we represent the coverings as dessins. For definitions and motivations, the reader should read the article by Leila Schneps in this volume, and of course the introduction to this book. There are many possible combinatorial descriptions with such dessins. Let us illustrate this on a small example which we will consider throughout this paper every time we need to be more explicit. In our drawings, the points over 0 are denoted by a black bullet and the points over 1 correspond to the middles of the segments and to the extremities without bullets (unramified points over 1). This corresponds to Grothendieck's normalization. We ask that the ramification above 1 be equal to 1 or 2.

Let us consider the following genus zero and degree 3 dessin:

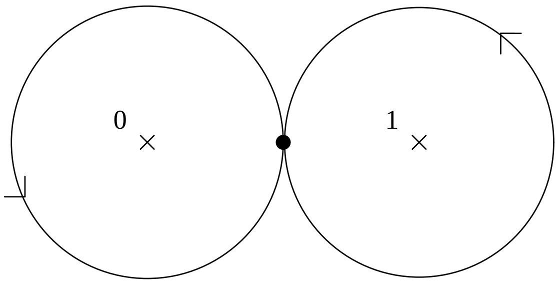


It has one vertex of multiplicity 3, corresponding to the totally ramified point over 0. There is one circular edge, corresponding to a point over 1 with ramification degree equal to 2, and one half-edge the extremity of which is an unramified point over 1. To finish, there are two faces. The inner one is unramified and the outer one is ramified of order 2.

Since the dessin is of degree 3, there are 3 flags which are drawn in the following picture

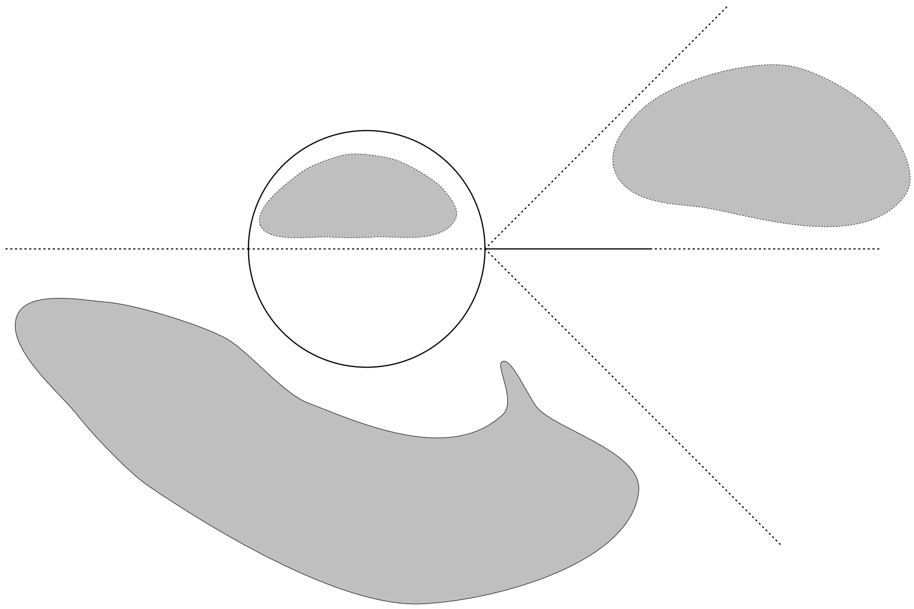


The monodromy of the dessin is given by the following three permutations of the flags which correspond to the elementary loops around 0, 1 and ∞.



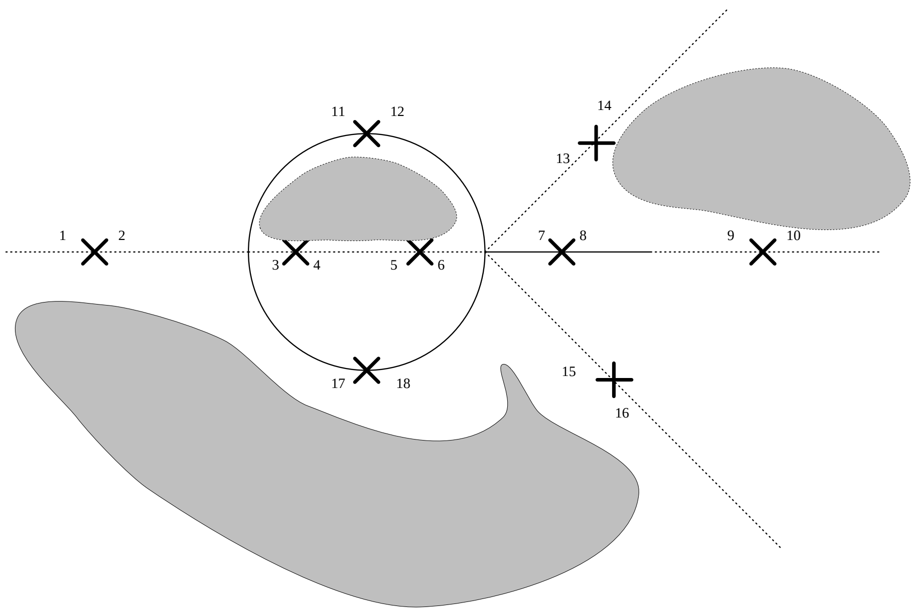
$$\sigma_0 = (1,2,3), \quad \sigma_1 = (1,2), \quad \sigma_\infty = (2,3).$$

The dessin itself is the preimage of the segment $[0,1]$ under the Belyi function. If we consider rather the preimage of the full real axis, we get a coloured triangulation of the sphere, consisting of three (grey) triangles oriented in the positive direction and sent by the Belyi function onto the upper half plane, and three (white) triangles, oriented in the inverse direction and lying above the lower half plane. This way, the dessin can be considered as a combinatorial covering of coloured triangulations.



We now consider the elementary triangle $0, 1, \infty$ on the Riemann sphere. The middles of the three edges are -1, $1/2$ and 2. This splits the real axis into six open segments plus three points. The six open segments are called *standards* and we give each of them a name which will become clearer later. The segment $(0, 1/2)$ is denoted by $\vec{01}$; the segment $(1/2, 1)$ is denoted by $\vec{10}$, the segment $(1, 2)$ is denoted by $\vec{1\infty}$, the segment $(2, \infty)$ is denoted by $\overrightarrow{\infty 1}$, the segment $(\infty, -1)$ is denoted by $\overrightarrow{\infty 0}$ and the segment $(-1, 0)$ is denoted by $\overrightarrow{0\infty}$.

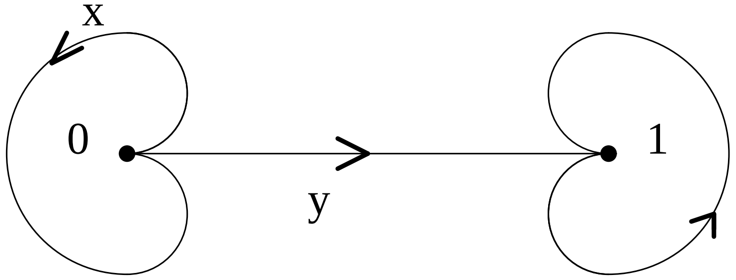
The preimages of these six standards under the Belyi function give 3×6 standards on the dessin. We draw these standards as little arrows and give an arbitrary number to each of them.



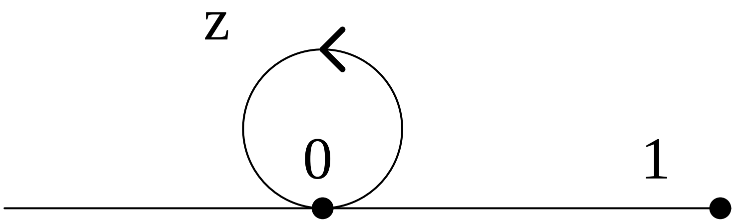
The standards above $\overrightarrow{01}$ are $\{7, 12, 18\}$, the standards above $\overrightarrow{10}$ are $\{11, 17, 8\}$ and so on. There is, on those standards, an action of the fundamental groupoid with "tangential base points"

$$\mathcal{B} = \{\overrightarrow{01}, \overrightarrow{10}, \overrightarrow{1\infty}, \overrightarrow{\infty 1}, \overrightarrow{\infty 0}, \overrightarrow{0\infty}\}$$

as defined by Deligne in [Del89] (see the appendix by Lochak and Emsalem to Ihara's article in this volume). This groupoid is generated by the paths $x_{\vec{v}}$ and $y_{\vec{v}}$ and $z_{\vec{v}}$ where $\vec{v}$ runs through the six standards. The paths $x_{\overrightarrow{01}} = x$, $y_{\overrightarrow{01}} = y$ and $x_{\overrightarrow{10}}$ are shown in the following drawing (we let the reader imagine what the other ones could be.)



We also show $z_{\vec{01}} = z$ below:



This action is given by the following maps:

$$x_{\vec{01}} = (12, 18, 7),\ y_{\vec{01}} = (7 \mapsto 8, 12 \mapsto 11, 18 \mapsto 17),\ y_{\vec{10}} = y_{\vec{01}}^{-1} ...$$

§2. Topological classification of genus zero covers

In this section, we recall a quite classical result first stated by Klein in its modern formulation [Kle13]. We need to introduce a certain number of Galois genus zero coverings of the sphere, corresponding to well-known dessins.

The first family corresponds to the dessins consisting of a star with e rays where e is a positive integer. A corresponding Belyi function is

$$y = f(x) = x^e$$

where e is the degree of the covering, totally ramified over 0 and ∞ and unramified elsewhere. We call these dessins C_e. Their topological Galois group is the cyclic group with e elements, C_e.

The second family corresponds to the polygon with $2e$ edges and admits the following Belyi function

$$-4y = x^e + x^{-e} - 2.$$

We call these dessins D_{2e}. Their topological Galois group is the dihedral group with $2e$ elements, D_{2e}.

We then have three coverings consisting of

- The tetrahedron which we call T of degree 12 and with Galois group the alternating permutation group on 4 letters A_4. A corresponding Belyi function is given by

$$yx^3(x^3 + 8)^3 = 2^6(x^3 - 1)^3.$$

- The octahedron O, of degree 24 with Galois group the full symmetric group on 4 letters S_4. A corresponding Belyi function is given by

$$y(x^8+14x^4+1)^3 = 2^2.3^3.x^4(x^4-1)^4.$$

- The icosahedron I, of degree 60 with Galois group the alternating permutation group on 5 letters A_5. A corresponding Belyi function is given by

$$y(x^{20}+228x^{15}+494x^{10}-228x^5+1)^3 = x^5(x^{10}-11x^5-1)^5.$$

We can now state Klein's theorem ([Kle13]), in which two coverings $\chi : \mathcal{C} \to \mathcal{D}$ and $\chi' : \mathcal{C}' \to \mathcal{D}'$ are said to be weakly isomorphic if there exist two isomorphisms c and d such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "c"] \arrow[d, "\chi"] & \mathcal{C}' \arrow[d, "\chi"] \\
\mathcal{D} \arrow[r, "d"] & \mathcal{D}'
\end{tikzcd}$$

They are *strongly* isomorphic if $\mathcal{D} = \mathcal{D}'$ and d can be chosen to be the identity.

Theorem 1: *Any algebraic Galois covering of the sphere is weakly isomorphic to one of the following: C_e or D_{2e} with $e \geq 1$, or T, O, or I.*

Proof: We first note that the Galois group G of such a covering G is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$. Such subgroups are known to be isomorphic to one of the following: C_e, D_{2e} with $e \geq 2$, A_4, S_4, or A_5. The proof is quite elementary and uses the fact that a non-trivial element of finite order in $\mathrm{PGL}_2(\mathbb{C})$ has two fixed points ([Arm88, p 104]). If we call X the set of such fixed points, then G acts on X. One of the consequences of the proof is that there are 2 orbits if G is cyclic and 3 otherwise. The order of the stabilizer of a point in X just depends on its orbit. For each orbit $\mathcal{O}$ we denote by $(o_{\mathcal{O}}, s_{\mathcal{O}})$ the couple consisting of its cardinality and the order of the stabilizer of some element in $\mathcal{O}$. We list the values we obtain in each case:

- $(1,e)$, $(1,e)$ for C_e.
- $(e,2)$, $(e,2)$, $(2,e)$ for D_{2e}.
- $(4,3)$, $(6,2)$, $(4,3)$ for A_4.

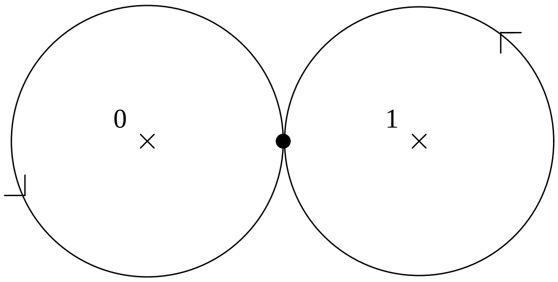
- $(6,4)$, $(12,2)$, $(8,3)$ for S_4.
- $(12,5)$, $(30,2)$, $(20,3)$ for A_5.

It is clear that these fixed points are the ramification points of the covering with orders of ramification the orders of their stabilizer. This proves that either the covering is cyclic or there are exactly three singular values.

If the covering is cyclic, one can suppose that it is totally ramified over 0 and ∞ and that the single point above 0 is 0 and the single point above ∞ is ∞. We then get a function of the form $y = Ax^e$ which is clearly equivalent to the one we gave.

If there are three ramification values we can send them on 0, 1 and ∞ using the 3-transitivity of $\mathrm{PGL}_2(\mathbb{C})$. Note that we have put those three ramification values in some definite order in the above table. We respect this order in that we send the first one to 0, the second one to 1 and the third one to ∞.

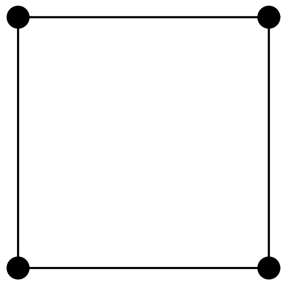
Then, a strong isomorphism class of finite coverings is given by a subgroup of finite index of $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{0,1,\infty\}, b)$ where $b = 1/2$ is the base point. We choose the following basis of $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{0,1,\infty\}, b)$ that induces an isomorphism to the free group with two generators (σ_0, σ_1):



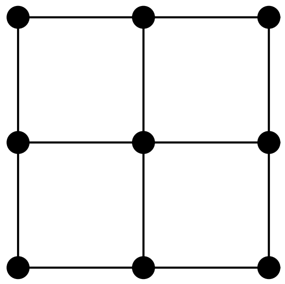
We write $\sigma_\infty^{-1} = \sigma_0\sigma_1$. Now we can associate to G a subgroup g of π_1. Let us write G_0 for the covering in the list given above which has G as Galois group, and let g_0 be the corresponding subgroup. We prove that $g = g_0$.

Suppose for example that $G = A_4$. Both g_0 and g have index 12. They both contain σ_0^3, σ_1^2 and σ_∞^3 because of the ramification orders. The point now is that the subgroup generated by σ_0^3, σ_1^2 and σ_∞^3 is of finite index 12 (trivial from the classical presentation of A_4) so that $g = g_0 = \langle \sigma_0^3, \sigma_1^2, \sigma_\infty^3 \rangle$. The remaining cases are similar and reduce to the classical presentations of rotation groups.

Remark: Such a method no longer works for arbitrary genus. For example the following genus one dessin (where the opposite sides are identified) is Galois and the corresponding subgroup contains σ_0^4, σ_1^2 and σ_∞^4.



But the following dessin has the same property, and yet it is different. Indeed the subgroup generated by σ_0^4, σ_1^2 and σ_∞^4 is not of finite index.



§3. Fields of definition, fields of moduli

In this section we simply recall a certain number of definitions in order to clarify our terminology for the rest of the paper. Let $\mathcal{D}$ be a dessin, that is, an isomorphism class over $\overline{\mathbb{Q}}$ of Belyi pairs. We recall that a Belyi pair is a made of a curve $\mathcal{C}$ defined over $\bar{\mathbb{Q}}$ and a function $\chi : \mathcal{C} \to \mathbb{P}_1(\bar{\mathbb{Q}})$ defined over $\bar{\mathbb{Q}}$ and unramified outside $\{0, 1, \infty\}$. Two Belyi pairs are said to be equivalent if the corresponding coverings are strongly isomorphic.

Let $\mathbb{K}$ be a number field and $\mathcal{C}$ a projective curve and ϕ a function on $\mathcal{C}$, defined over $\mathbb{K}$. If the Belyi pair $(\mathcal{C}, \phi)$ belongs to $\mathcal{D}$, we say that $\mathbb{K}$ is *a field of definition* of $\mathcal{D}$.

There is an action of $\mathbf{\Gamma}$ on the set of dessins. This action can be seen as the naive action on the coefficients of the equations of any Belyi pair. We call $\mathbf{\Gamma}_{\mathcal{D}}$ the stabilizer of $\mathcal{D}$ and $\mathbb{K}_{\mathcal{D}}$ its fixed field. We call $\mathbb{K}_{\mathcal{D}}$ the moduli field of $\mathcal{D}$.

The moduli field is contained in any field of definition and is actually the intersection of all the possible fields of definition ([CH85]). It need not be a field of definition itself as we will show in section 4.

Note that we do not ask that the automorphisms of the covering (if any) be defined over the field of definition, which could have the effect of augmenting it. On the other hand, the field of moduli of $\mathcal{C}$ itself might be strictly smaller than the one of the dessin. For genus zero dessins, $\mathbb{P}_1(\mathbb{C})$ has $\mathbb{Q}$ as field of moduli but Lenstra proved that there exist genus zero dessins with arbitrary field of moduli (see the article by L. Schneps in this volume). To finish with the *distinguo* we should warn the reader that people studying modular forms over possibly non-congruence subgroups, usually consider structures that are somewhat richer than dessins. Following Birch (see his contribution to this volume), we define marked dessins to be dessins plus a fixed marked point over infinity. In this case, of course, the field of moduli might become bigger but it is more likely to be a field of definition (for example, it will always be one in genus zero).

In the case where the dessin has no automorphisms, it must admit a model over its field of moduli $\mathbb{K}_{\mathcal{D}}$ by Weil's criterion ([Wei56]). In this case we note that the corresponding $\mathbb{K}_{\mathcal{D}}$-isomorphism class of curves is characteristic of the dessin. We will see an example of this in the next section.

§4. Galois action. Descending from $\mathbb{C}$ to $\mathbb{R}$

In this section we illustrate the problems of fields of definition and descent on the toy example of descending from $\mathbb{C}$ to $\mathbb{R}$. This is particularly interesting because we can give topological criteria for the descent. Further results on this subject can be found in [FD90]. Here, we are only interested in descent with extensions ramified over three points which thus can be chosen to be real, and which we take to be our favourite ones.

Let us denote by $\mathbf{S_3}$ the sphere minus three points $\mathbb{P}_1(\mathbb{C}) - \{0, 1, \infty\}$ with base point $b = 1/2$ and the same basis as above for the π_1. A covering is thus given by two permutations a_0 and a_1 of the fibre over b, corresponding to the paths σ_0 and σ_1.

We write $\mathbb{M}_{0,1,\infty}$ for the maximal extension of $\mathbb{R}(t)$ unramified outside $\{0, 1, \infty\}$. We consider the following tower of extensions

$$\begin{array}{c} \mathbb{M}_{0,1,\infty} \\ \Big| \hat{\pi}_1 \\ \mathbb{C}(t) \\ \Big| \mathbb{Z}/2\mathbb{Z} \\ \mathbb{R}(t) \end{array}$$

and the corresponding exact sequence of groups

$$1 \to \hat{\pi}_1 \to \mathcal{G} \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

We recall that there exist two $\mathbb{R}$-isomorphism classes of genus zero curves, the class of the straight line $\mathbb{P}_1(\mathbb{R})$ and the class of the plane curve given by the equation

$$x^2 + y^2 + z^2 = 0,$$

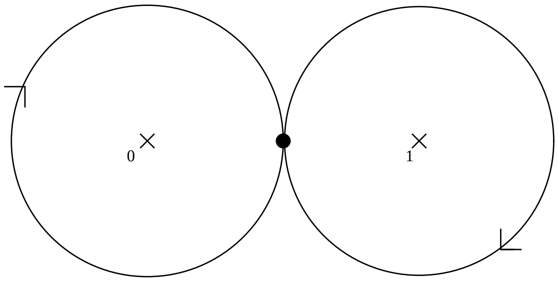
which we call $\tilde{\mathbb{P}}_1(\mathbb{R})$.

Given a dessin $\mathcal{D}$ by its monodromy (a_0, a_1), or equivalently, a triangulation of a surface, we ask three questions:

- Is the moduli field of $\mathcal{D}$ equal to $\mathbb{C}$ or $\mathbb{R}$?
- If the moduli field is $\mathbb{R}$, does the dessin admit a model over $\mathbb{R}$?
- If a real genus zero dessin has no automorphisms, it admits a real model. Then can we say whether the underlying curve is $\mathbb{P}_1$ or $\tilde{\mathbb{P}}_1$?

We will give examples of all the possible situations and finish with an example of a real dessin (i.e. a dessin with real moduli field) with no real model.

To answer the first question we note that the outer action of $\mathbb{Z}/2\mathbb{Z}$ on $\hat{\pi}_1$ comes from an action on π_1 itself. Let τ denote the reflection of the plane induced by the unique non-trivial element $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. This reflection is continuous and thus induces an involution of π_1. The images of σ_0, σ_1, σ_∞ are given by ${}^\tau\sigma_0 = \sigma_0^{-1}$, ${}^\tau\sigma_1 = \sigma_1^{-1}$, and ${}^\tau\sigma_\infty = \sigma_1\sigma_0$.



Now let $\chi : \mathcal{C} \to \mathbf{S_3}$ be an algebraic covering of degree d and ${}^\tau\chi : {}^\tau\mathcal{C} \to \mathbf{S_3}$ its conjugate under τ. There is a bijection induced by τ between the fibre of χ above b and the fibre of ${}^\tau\chi$ above b. Let $\{b_1, b_2, ..., b_d\}$ denote the points above b and $\{{}^\tau b_1, {}^\tau b_2, ..., {}^\tau b_d\}$ their images under τ. If σ is a closed path in π_1 and b_i a point above b on $\mathcal{C}$, then $\sigma(b_i)$ denotes the extremity of the lifted path on $\mathcal{C}$, with origin b_i. On the other hand, we can lift ${}^\tau\sigma$ onto ${}^\tau\mathcal{C}$, with origin ${}^\tau b_i$, so ${}^\tau\sigma({}^\tau b_i)$ is the extremity of the lifted path. Then ${}^\tau\sigma({}^\tau b_i) = {}^\tau(\sigma(b_i))$.

This means that the action of σ on the fibre $\chi^{-1}(b)$ is conjugated by τ to the action of ${}^\tau\sigma$ on the fibre ${}^\tau\chi^{-1}(b)$. Therefore, if χ was given by its monodromy (a_0, a_1), the monodromy of ${}^\tau\chi$ is (a_0^{-1}, a_1^{-1}) where a_0^{-1} and a_1^{-1} can be seen as permutations of $\{{}^\tau b_1, {}^\tau b_2, ..., {}^\tau b_d\}$ through the bijection with

$\{b_1, b_2, ..., b_d\}$ induced by τ. This gives an explicit description of the outer action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ in the above exact sequence.

Now, a dessin of degree d will be said to be real if and only if its field of moduli is $\mathbb{R}$. If (a_0, a_1) is its monodromy, this is just saying that there exists a permutation $\omega \in \mathcal{S}_{\{b_1, b_2, \ldots, b_d\}}$ such that

$$^{\tau}(a_0, a_1) = (a_0^{-1}, a_1^{-1}) = {}^{\omega}(a_0, a_1) = (\omega^{-1} a_0 \omega, \omega^{-1} a_1 \omega)$$

If this is the case, we note that ω belongs to the normalizer of $G = \langle a_0, a_1 \rangle$ in $\mathcal{S}_{\{b_1, b_2, \ldots, b_d\}}$, and is defined up to an automorphism of $\mathcal{D}$ (we recall that the automorphism group of $\mathcal{D}$ is $a = \mathcal{Z}_{\mathcal{S}_{\{b_1, b_2, \ldots, b_d\}}}(G)$, the centralizer of G in the full permutation group, see [Cou94]). Furthermore, since τ is an involution, we have $\omega^2 \in a$. Now, the dessin $\mathcal{D}$ admits a model over $\mathbb{R}$ if and only if ω can be chosen to satisfy Weil's cocycle condition

$$\omega^2 = 1.$$

Indeed, associated to ω, there is a morphism $H : \mathcal{C} \to {}^{\tau}\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "H"] \arrow[dr, "\chi"] & & {}^{\tau}\mathcal{C} \arrow[dl, "{}^{\tau}\chi"'] \\
& \mathbb{P}_1(\mathbb{C}) &
\end{tikzcd}$$

and H and ω are linked by the following identity

$$H(b_i) = {}^{\tau}(\omega(b_i)).$$

The cocycle condition on H for the existence of a real model is ${}^{\tau}HH = I$ which is immediately translated on ω as $\omega^2 = 1$.

We now come to the situation where the dessin $\mathcal{D}$ has no automorphisms. In this case ω is unique and ω^2 can only be equal to 1 and we have a model over $\mathbb{R}$ (here H is nothing but the identity)

$$\begin{tikzcd}
\mathcal{C} \arrow[d, dash, "\chi"] \\
\mathbb{P}_1(\mathbb{R})
\end{tikzcd}$$

and the action of τ extends to the real curve $\mathcal{C}$ in a way that makes the following diagram commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tau"] \arrow[d, "\chi"] & \mathcal{C} \arrow[d, "\chi"'] \\ \mathbb{P}_1(\mathbb{C}) \arrow[r, "\tau"] & \mathbb{P}_1(\mathbb{C}) \end{tikzcd}$$

The action of τ on the fibre above b is thus given by the formula

$$^{\tau}b_i = \omega(b_i)$$

and since ω conjugates a_0 and a_0^{-1}, it induces a permutation of the cycles of a_0 which gives the action of τ on the fibre $\chi^{-1}(0)$. In the same way we describe the Galois action on $\chi^{-1}(1)$ and $\chi^{-1}(\infty)$.

Suppose that among the cycles of σ_0 and σ_1 there is one which is fixed under the action of ω. Then, the corresponding point on $\mathcal{C}$ is real and thus $\mathcal{C}$ is isomorphic over $\mathbb{R}$ to the projective line $\mathbb{P}_1(\mathbb{C})$.

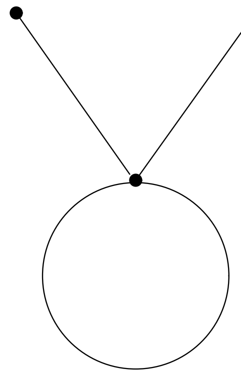
To state the reciprocal assertion, we need to work a bit more. Suppose that χ is a real rational fraction: $\chi : \mathbb{P}_1(\mathbb{R}) \to \mathbb{P}_1(\mathbb{R})$ associated to the dessin $\mathcal{D}$. Let c be some connected component of the preimage of the open segment $(0,1)$. Because χ is real and unramified over $(0,1)$, c is either contained in $\mathbb{R}$, or does not intersect it. If there exists such a c contained in $\mathbb{R}$ then its extremities are real thus proving the assertion that at least one point over $\{0,1\}$ is real, and so the corresponding cycle must be fixed by ω. On the other hand, suppose that $\chi^{-1}((0,1)) \cap \mathbb{R}$ is empty. We note that $\chi^{-1}([0,1]) \cap \mathbb{R}$ cannot be empty because $\chi^{-1}([0,1])$ is a connected non-empty subset of the plane which is invariant under the reflection τ. This again proves the desired statement.

We finish by stating

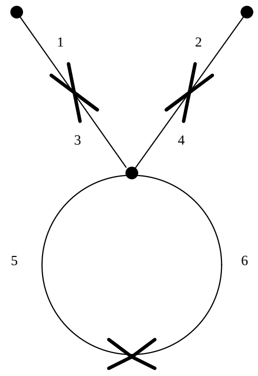
Theorem 2: *Let $\mathcal{D}$ be a dessin, given by its monodromy (a_0, a_1, a_∞). Then the field of moduli of $\mathcal{D}$ is $\mathbb{R}$ if and only if there exists some ω such that $a_0^{-1} = {}^{\omega}a_0$ and $a_1^{-1} = {}^{\omega}a_1$. In the latter case, the dessin admits a real model if and only if ω can be chosen so that $\omega^2 = 1$. If $\mathcal{D}$ is of genus zero and its automorphism group (the centralizer of $\langle a_0, a_1\rangle$) is trivial, then the dessin admits a rational model over some real genus 0 curve. This curve is isomorphic to $\mathbb{P}_1(\mathbb{R})$ if and only if the action of ω over the cycles of a_0 and a_1 has at least one fixed point.*

Examples

The rabbit is a real dessin with no automorphisms and admits a real model on the projective line.



To see this, we give numbers to the flags and compute the monodromy.



$$a_0 = (3,5,6,4), \ \ a_1 = (1,3)(2,4)(5,6), \ \ a_\infty = (4,2,6,3,1).$$

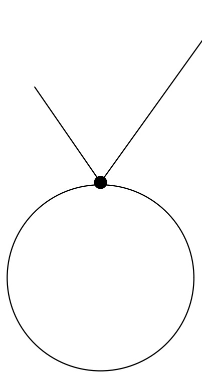
It is clear that there are no automorphisms. If a is a permutation which commutes with a_0, it must fix 1. But since it commutes with a_1 as well, it must fix 3 as well. Now, coming back to a_0 we see that a must be the identity. Furthermore we have

$$\omega = (3,4)(1,2)(5,6),$$

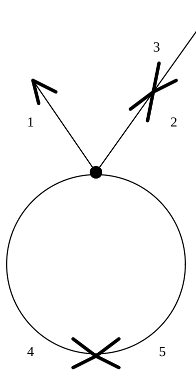
and check that the dessin is real.

The action of ω on the cycles of a_0 and a_1 fixes the cycle $(3,5,6,4)$ in a_0 and the cycle $(5,6)$ in a_1. This is more than enough to prove that the dessin has a real model on the projective line.

The rabbit with a lopped off left ear is a non-real dessin.



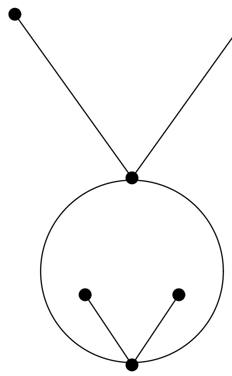
The monodromy is given by:



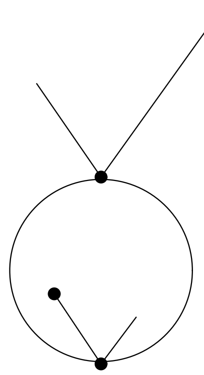
$$a_0 = (1,4,5,2),\ a_1 = (2,3)(4,5),\ a_\infty = (5,1,2,3).$$

Here there is no hope of finding an ω since such a permutation should fix 3 (from a_0) and 4 (from a_∞) and thus 2 and 5 as well (from a_1). This does not work.

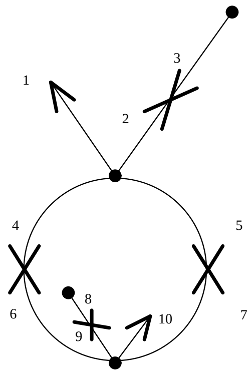
The smiling rabbit obviously has an automorphism group of order 2.



The rabbit with a lopped off left ear and a sidelong smirk on the right hand side is a real dessin with no non-trivial automorphisms and real model on the real curve $\tilde{\mathbb{P}}_1$ with equation $x^2 + y^2 + z^2 = 0$.



Its monodromy is:



$$a_0 = (1,4,5,2)(9,6,7,10),\ a_1 = (2,3)(4,6)(8,9)(7,5),$$

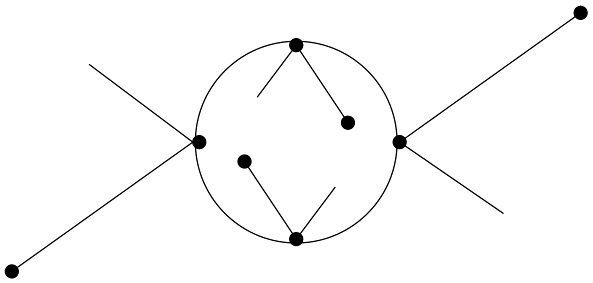
$$a_\infty = (4,9,8,10,7)(3,5,6,1,2).$$

There are no non-trivial automorphisms (exercise) and there is a unique ω defined as

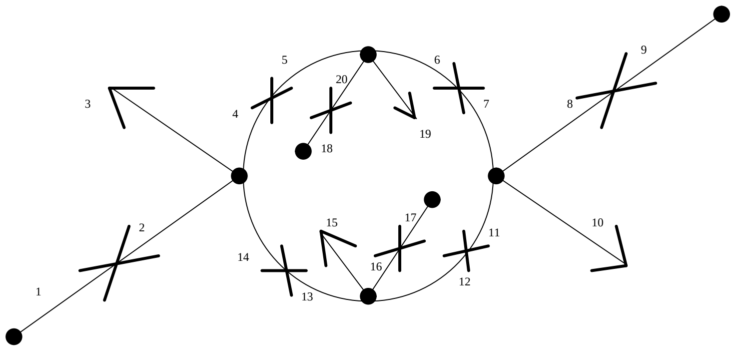
$$\omega = (1,10)(3,8)(9,2)(6,5)(4,7),$$

and none of the cycles of a_0 and a_1 are fixed by ω.

The double rabbit is a real dessin with no real model.



Its monodromy is:



$$a_0 = (3,2,14,4)(5,20,19,6)(7,11,10,8)(15,13,12,16),$$

$$a_1 = (1,2)(4,5)(6,7)(11,12)(13,14)(16,17)(8,9)(18,20),$$

$$a_\infty = (18,5,14,15,16,17,12,7,19,20)(4,6,8,9,10,11,13,2,1,3).$$

There is an automorphism group of order 2 generated by

$$a = (3,10)(8,2)(9,1)(14,7)(6,13)(5,12)(11,4)(20,16)(15,19)(17,18).$$

The dessin is real for we can choose ω to be

$$\omega = (15,3,19,10)(16,2,20,8)(17,1,18,9)(13,4,6,11)(12,14,5,7).$$

We could have chosen $a\omega$ instead. But $(a\omega)^2 = \omega^2$ is *not* the identity. This proves that our dessin although real, has no real model.

§5. Spheres minus four points

In this section we recall the basics about the Legendre form for elliptic curves. We are interested in building moduli spaces for spheres minus four points. To begin with, we define two different kinds of spheres minus four points. A non-coloured sphere minus four points is defined as a set of four distinct points $\{a,b,c,d\}$ on the complex projective line. A coloured sphere is a quadruplet of distinct points in $\mathbb{P}_1(\mathbb{C})$.

There are actions of $\mathrm{PGL}_2(\mathbb{C})$ on both sets, defined by

$$H\{a,b,c,d\} = \{Ha,Hb,Hc,Hd\},\ H(a,b,c,d) = (Ha,Hb,Hc,Hd).$$

The set of coloured spheres is $\mathcal{C} = \mathbb{P}_1^4 - \mathcal{D}$ where $\mathcal{D}$ is the discriminant variety defined as $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) = 0$. The group $\mathcal{S}_4$ acts naturally on $\mathcal{C}$. The set of non-coloured spheres is the quotient $\mathcal{N}$ of $\mathcal{C}$ by $\mathcal{S}_4$.

We thus have a decolouration covering s_4 which is Galois with Galois group $\mathcal{S}_4$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "s_4"] \\ \mathcal{D} \end{tikzcd}$$

We define the classical cross-ratio function on $\mathcal{C}$

$$[a,b,c,d] = \lambda(a,b,c,d) = \frac{b-c}{b-a} \cdot \frac{d-a}{d-c}.$$

It is well known that two elements in $\mathcal{C}$ belong to the same $\mathrm{PGL}_2(\mathbb{C})$-orbit if and only if λ takes the same value at those points.

We note that λ is invariant under the Klein group, seen as the subgroup V of $\mathcal{S}_4$ generated by the permutations of type $(2,2)$. This subgroup is normal so that the covering splits in two. We note $\mathcal{H} = \mathcal{C}/V$, v the corresponding V-covering, and s_3 the $\mathcal{S}_3$-covering of the lower part:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "v"] & \\ \mathcal{H} \arrow[r, "\lambda"] \arrow[d, "s_3"] & \mathbb{P}_1 \\ \mathcal{D} & \end{tikzcd}$$

and we have the exact sequence

$$1 \to V \to \mathcal{S}_4 \to \mathcal{S}_3 \to 1.$$

It is tempting (although not particularly original...) to look at the action of $\mathcal{S}_3$ on λ. It is given in the following list:

$$\begin{aligned} [[1,2]] \quad & \lambda \mapsto 1-\lambda \\ [[1,3]] \quad & \lambda \mapsto 1/\lambda \\ [[2,3]] \quad & \lambda \mapsto \lambda/(\lambda-1) \\ [[1,2,3]] \quad & \lambda \mapsto (\lambda-1)/\lambda \\ [[1,3,2]] \quad & \lambda \mapsto 1/(1-\lambda) \end{aligned}$$

This action is killed by the function

$$J(\lambda) \stackrel{def}{=} 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

which defines a Galois covering with (strong) automorphism group $\mathcal{S}_3$. Note the following amusing fact: J also admits a weak automorphism, namely

$$J(\frac{\lambda-2}{2\lambda-1}) = \frac{1728J}{J-1728} \tag{1}$$

The linear fraction

$$\delta(\lambda) = \frac{\lambda-2}{2\lambda-1}$$

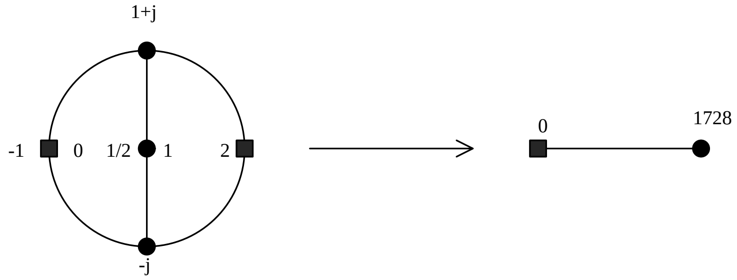
is the one which sends the triangle $(0,1,\infty)$ to the triangle $(2,-1,1/2)$. It is of order six. The linear fraction

$$\rho(J) = \frac{1728J}{J-1728}$$

is of order two and permutes the ramification locus of J. We have

$$\rho J = J\delta.$$

This will appear later on. We can draw the reciprocal image of $[0,1728]$ under J and find the dessin below:



We set $J(a,b,c,d) = J(\lambda(a,b,c,d))$ and get a symmetric function of (a,b,c,d) defined over $\mathcal{D}$:

$$J(a,b,c,d) = 2^8 \cdot \frac{E_4^3}{\mathrm{disc}(a,b,c,d)}$$

and

$$J-1728 = 2^6 \cdot \frac{E_6^2}{\mathrm{disc}(a,b,c,d)}, \tag{2}$$

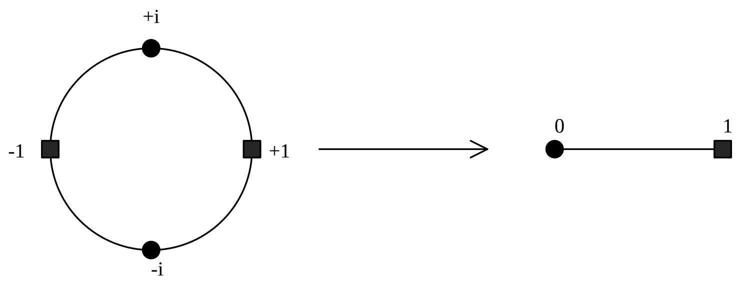
where $E_4 = 12\sigma_4 + \sigma_2^2 - 3\sigma_1\sigma_3$ and $E_6 = 72\sigma_2\sigma_4 - 2\sigma_2^3 + 9\sigma_1\sigma_2\sigma_3 - 27\sigma_3^2 - 27\sigma_4\sigma_1^2$, and $\mathrm{disc}(a,b,c,d) = \Delta$ denotes the discriminant. Those polynomials satisfy the expected invariance relations. For example, if $H(z) = (uz+v)/(wz+k)$ then we have

$$\frac{\Delta(H(a),H(b),H(c),H(d))}{\Delta(a,b,c,d)} = \frac{(uk-vw)^{12}}{(-u^4+\sigma_1 u^3 w - \sigma_2 u^2 w^2 + \sigma_3 u w^3 - \sigma_4 w^4)^6}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "v"] & \\
\mathcal{H} \arrow[r, "\lambda"] \arrow[d, "s_3"] & \mathbb{P}_1 \arrow[d, "J"] \\
\mathcal{D} \arrow[r, "J"] & \mathbb{P}_1
\end{tikzcd}$$

We note that the above commutative diagram is compatible with the Galois actions of $\mathcal{S}_3$ on each side. It seems as well that the right hand side of this is incomplete (one level is lacking). In the sequel we try to see what can be done to complete this construction. We first remember of the existence of a Galois genus 0 extension of the sphere with group $\mathcal{S}_4$. We build such an extension in the following way. Let $\mathcal{B}(x) = 1/4(x+1/x)^2 = 1+1/4(x-1/x)^2$ be the Galois function with automorphism group V, ramified over 0, 1, ∞.

We draw the corresponding dessin:



The composition $J \circ \mathcal{B}$ is a function ramified over 0, 1728, ∞ which defines the only genus zero $\mathcal{S}_4$-extension of $\mathbb{P}_1$ ramified at those places (in *that* order).

To each value of x we associate the quadruplet $Q(x) = (x, -x, 1/x, -1/x)$ such that $\lambda(Q(x)) = [x, -x, 1/x, -1/x] = \mathcal{B}(x)$ and get the following commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "v"] & \mathbb{P}_1 \arrow[l, "Q"'] \arrow[d, "\mathcal{B}"] \\ \mathcal{H} \arrow[r, "\lambda"] \arrow[d, "s_3"] & \mathbb{P}_1 \arrow[d, "J"] \\ \mathcal{D} \arrow[r, "J"] & \mathbb{P}_1. \end{tikzcd}$$

Note also that we can define $q(\lambda) = v(Q(x)) = v(x, -x, 1/x, -1/x)$ where x is any point such that $\mathcal{B}(x) = \lambda$. Such a point $q(\lambda)$ on $\mathcal{H}$ can be defined by its V-symmetric functions $\sigma_1 = 0$, $\sigma_2 = 2 - 4\lambda$, $\sigma_3 = 0$, $\sigma_4 = 1$, and $[x, -x, 1/x, -1/x] = \lambda$. This way we get the following commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d, "v"] & \mathbb{P}_1 \arrow[l, "Q"'] \arrow[d, "\mathcal{B}"] \\ \mathcal{H} \arrow[d, "s_3"] & \mathbb{P}_1 \arrow[l, "q"'] \arrow[d, "J"] \\ \mathcal{D} & \mathbb{P}_1. \end{tikzcd}$$

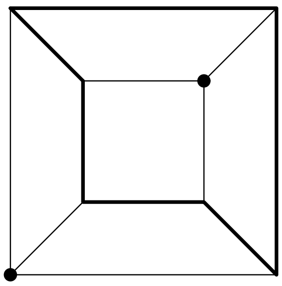
We would like to build four algebraic functions $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ with the following properties:

- The set $\{f_1, f_2, f_3, f_4\}$ is invariant under the automorphism group of $J \circ \mathcal{B}$, and this group $\mathcal{G}$ acts on $\{f_1, f_2, f_3, f_4\}$ like $\mathcal{S}_4$.
- The cross-ratio $[f_1, f_2, f_3, f_4]$ is (something like) $\lambda = [x, -x, 1/x, -1/x] = \mathcal{B}(x)$.

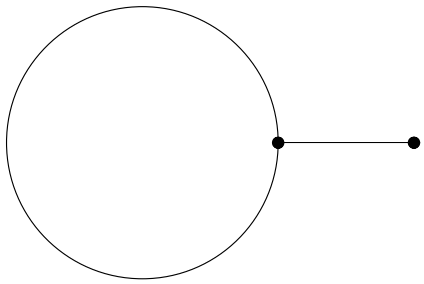
To do this, we write $\mathcal{S}_3^i$ for the stabilizer of i in $\mathcal{S}_4$ for $i \in \{1, 2, 3, 4\}$. The corresponding subextensions of $\mathbb{C}(x)/\mathbb{C}(j)$ are genus zero fields. We choose f_1 to be a generator of $\mathbb{C}(x)^{\mathcal{S}_3^1}$. We then can choose $f_2(x) = f_1(-x)$, $f_3(x) = f_1(1/x)$ and $f_4(x) = f_1(-1/x)$. Note that the f_i are defined up to a linear transform on the left.

$$\begin{tikzcd}
 & & \mathbb{C}(x) \arrow[dll, dash, "\mathcal{S}_3^1"'] \arrow[dl, dash, "\mathcal{S}_3^2"] \arrow[dr, dash, "\mathcal{S}_3^3"'] \arrow[drr, dash, "\mathcal{S}_3^4"] & & \\
\mathbb{C}(f_1) \arrow[drr, dash] & \mathbb{C}(f_2) \arrow[dr, dash] & & \mathbb{C}(f_3) \arrow[dl, dash] & \mathbb{C}(f_4) \arrow[dll, dash] \\
 & & \mathbb{C}(j) & &
\end{tikzcd}$$

We can be more precise if we look for the minimal polynomial of f_1 with coefficients in $\mathbb{C}(j)$. To compute it, we just quotient the dessin corresponding to a cube by the group $\mathcal{S}_3^1$ which can be seen as the stabilizer of one of the four diagonals of the cube.



We thus get the following dessin.



If we send the vertex of order three to zero and the one of order one to one, the corresponding Belyi function will be $X \mapsto Y$ such that

$$Y + 2^{14} \cdot X^3(X-1) = 0.$$

In other words, we choose for f_i the four roots of the equation

$$j + 2^{14} \cdot f^3(f-1) = 0. \tag{3}$$

On the other hand, the map $x \mapsto f_1$ is a Galois covering with group $\mathcal{S}_3$. As we saw in the second section, such a covering must be equal to the classical J covering up to linear transforms L and R on both sides:

$$f_1(x) = L(J(R(x))).$$

We don't worry too much about L since it does not change the cross-ratio $[f_1, f_2, f_3, f_4]$. As for R, it can be defined as follows. Let r be the primitive 8-th root of unity given by

$$r = \sqrt{2} \cdot \frac{1+i}{2}$$

and let R be the linear transform defined by the matrix

$$R = \begin{pmatrix} 3 - r - 2r^2 + 4r^3 & 1 - r + 2r^2 \\ 3 - 2r - r^2 + 2r^3 & 2 + r - 2r^2 + 3r^3 \end{pmatrix}.$$

We set $f_1(x) = L(J(R(x)))$ and $f_2(x) = f_1(-x)$, $f_3(x) = f_1(1/x)$ and $f_4(x) = f_1(-1/x)$. Then it can be easily shown that the cross-ratio $[f_1, f_2, f_3, f_4]$ satisfies

$$[f_1(x), f_2(x), f_3(x), f_4(x)] = \delta(\mathcal{B}(x))$$

We also get the j-invariant thanks to (2)

$$J(f_1(x), f_2(x), f_3(x), f_4(x)) = J(f_1(x), f_1(-x), f_1(1/x), f_1(-1/x)) =$$

$$\rho(J(x, -x, 1/x, -1/x)) = \rho(j) = \frac{1728j}{j - 1728}.$$

The symmetric functions of the f_i are given by (3):

$$\sigma_1 = 1,\ \sigma_2 = 0,\ \sigma_3 = 0,\ \sigma_4 = 2^{-14}j.$$

It is important not to confuse j, the invariant of $[x, -x, 1/x, -1/x]$, with $\rho(j)$, the invariant of the f_i.

We now define three maps. The first one, called D, from the j-space $\mathbb{P}_1(\mathbb{C})$ to the non-coloured space $\mathcal{D}$, is such that $D(j)$ is the point of $\mathcal{D}$ defined by its symmetric functions

$$\sigma_1 = 1, \sigma_2 = 0, \sigma_3 = 0, \sigma_4 = 2^{-14}j.$$

The second map, called H, from the λ-space $\mathbb{P}_1(\mathbb{C})$ to the half-coloured space $\mathcal{H}$, is such that $H(\lambda)$ is defined by its V-symmetric functions

$$\sigma_1 = 1, \sigma_2 = 0, \sigma_3 = 0, \sigma_4 = 2^{-14}J(\lambda) = 2^{-6}(\lambda^2 - \lambda + 1)^3/\lambda^2/(\lambda - 1)^2,$$

and the cross-ratio defined as

$$\delta(\lambda) = (\lambda - 2)/(2\lambda - 1).$$

The third map, called C, from the x-space $\mathbb{P}_1(\mathbb{C})$ to the coloured space $\mathcal{C}$, is such that $C(x)$ is defined by the quadruplet $(f_1(x), f_2(x), f_3(x), f_4(x))$ as above.

We then get the following commutative diagram in which the actions of $\mathcal{S}_4$ as a Galois group on both sides are compatible with the arrows.

$$\begin{array}{ccc}
\mathcal{C} & \xleftarrow{C} & \mathbb{P}_1 \\
\downarrow v & & \downarrow \mathcal{B} \\
\mathcal{H} & \xleftarrow{H} & \mathbb{P}_1 \\
\downarrow s_3 & & \downarrow J \\
\mathcal{D} & \xleftarrow{D} & \mathbb{P}_1 .
\end{array}$$

We have thus realized the covering of moduli spaces as a restriction of the covering of naive spaces. We finish by noting that $\lambda H = \delta$ and $JD = \rho$ which stresses the importance of (1). The functions (H, D) define something which is almost but not quite a section of (λ, J).

§6. Approximating dessins from Puiseux series

In this section we now come to the problem of computing explicitly some algebraic model for a given abstract dessin. In fact, we will do better: we will compute the linear space associated with any given divisor on the dessin. The result is given as Puiseux series. Of course, we must truncate the series and consider floating point coefficients if we want to work with finite memory and time. We show in the next section how to obtain some exact solution from such approximations. We consider the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ consisting of six linear transforms permuting 0, 1, and ∞. We describe it explicitly as follows:

$$H_{0\vec{1}}(\lambda) = \lambda = \lambda_{0\vec{1}}, \quad H_{0\vec{\infty}}(\lambda) = \frac{\lambda}{\lambda - 1} = \lambda_{0\vec{\infty}},$$

$$H_{1\vec{0}}(\lambda) = 1 - \lambda = \lambda_{1\vec{0}}, \quad H_{1\vec{\infty}}(\lambda) = \frac{\lambda - 1}{\lambda} = \lambda_{1\vec{\infty}},$$

$$H_{\infty\vec{0}}(\lambda) = \frac{1}{1-\lambda} = \lambda_{\infty\vec{0}}, \quad H_{\infty\vec{1}}(\lambda) = \frac{1}{\lambda} = \lambda_{\infty\vec{1}}.$$

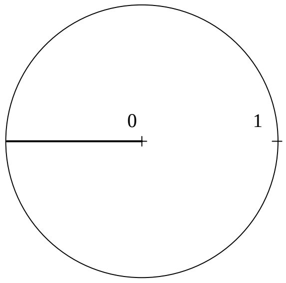
We note that for any standard $\vec{v}$ we have $H_{\vec{v}}(\vec{v}) = \vec{01} = (0, 1/2)$. Now let e be a positive integer. We build an e-th root of $\lambda_{\vec{v}}$ as follows. First let $\Lambda_{\vec{01},e}$ be defined for $\lambda_{\vec{01}} \in \mathbb{C} - \{\infty, 0\}$ as

$$\Lambda_{\vec{01},e}(\lambda_{\vec{01}}) = \lambda_{\vec{01}}^{1/e} = exp(2i\pi \mathrm{Log}(\lambda_{\vec{01}})e^{-1})$$

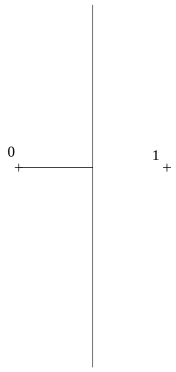
where Log is the principal determination of the logarithm. We then define the $\Lambda_{\vec{v},e}$ as

$$\Lambda_{\vec{v},e}(\lambda_{\vec{01}}) = \Lambda_{\vec{01},e}(H_{\vec{v}}(\lambda_{\vec{01}})) = \Lambda_{\vec{01},e}(\lambda_{\vec{v}}).$$

Now we define the domain $\mathcal{K}_{\vec{01}}$ to be the open circle of center 0 and radius 1 minus the segment $(-1, 0)$,



and similarly, $\mathcal{K}_{\vec{v}}$ is such that $H_{\vec{v}}(\mathcal{K}_{\vec{v}}) = \mathcal{K}_{\vec{01}}$. For example $\mathcal{K}_{0\vec{\infty}}$ is the half-plane $\Re(z) < 1/2$ minus the segment $(0, 1/2)$.



Note that there are two uniformizing parameters at any given point. For example, $\Lambda_{\vec{01},e}$ will be useful for analytic continuation from 0 to 1 and $\Lambda_{0\vec{\infty},e}$ will be useful for analytic continuation from 0 to ∞. The six domains of convergence form a covering of $\mathbb{P}_1$.

We consider a dessin $\mathcal{D}$ together with a Belyi function $\chi : \mathcal{C} \to \mathbb{P}_1 - \{0, 1, \infty\}$ for some algebraic curve $\mathcal{C}$, and a divisor D over $\mathcal{D}$, i.e. a divisor over the underlying curve $\mathcal{C}$ whose points all lie over $\{0, 1, \infty\}$. We write

$$D = \sum_i o_i P_i$$

and we write $\mathcal{L}(D)$ for the corresponding linear space. We will characterize it as the kernel of a certain operator built from some universal hermitian blocks. Let f be some function in $\mathcal{L}(D)$. Associated to each standard $\vec{v}$ above $\vec{01}$ there is a connected component of $\chi^{-1}(\mathcal{K}_{\vec{01}})$, and also an expansion of f as a series in $\Lambda_{\vec{01},e_i}$, where e_i is the ramification at the associated point P_i over 0.

$$f = \sum_{k \geq o_i} a_{\vec{v},k} \Lambda^k_{\vec{01},e_i},$$

where o_i is the valence of f at P_i.

Similarly, we define uniformizing parameters and expansions of f at any standard of the dessin. We call $\mathbf{S}$ the set of all standards in the dessin. To a function $f \in \mathcal{L}(D)$ we associate the list of sequences of coefficients of its expansions at all standards

$$((a_{\vec{v},k})_k)_{\vec{v}\in\mathbf{S}}.$$

The sequence $(a_{\vec{v},k})_k$ is such that the associated entire series

$$\sum_k a_{\vec{v},k} X^k$$

is convergent on the open disk of radius one and is bounded outside any neighbourhood of $\{0, 1\}$ in the disk. Such sequences form a linear space which we call $\mathbf{J}$. To each function $f \in \mathcal{L}(D)$ we associate a vector in $\mathbf{J}^{\mathbf{S}}$. This clearly induces an injection of linear spaces. We want to characterize its image as the kernel of a certain linear operator.

We now study the relations between the various expansions. The relations will be of three types. The first two types involve expansions at various standards related to the same point. The third one relates the expansions at two standards facing each other.

Let P_i be a point above 0 with ramification order e_i and $\vec{v}$ a standard at P_i. Let's say that $\vec{v}$ is over $\vec{01}$. We call $\vec{w} = x_{\vec{01}}(\vec{v})$ the next standard over $\vec{01}$ at P_i reached when turning counterclockwise.

A typical situation of that is in our example, the standards 7 and 12. We write the two corresponding expansions

$$f = \sum_{k \geq o_i} a_{\vec{v},k} \Lambda^k_{\vec{01},e_i},$$

$$f = \sum_{k \geq o_i} a_{\vec{w},k} \Lambda^k_{\vec{01},e_i},$$

where the coefficients are related by the obvious relations

$$a_{\vec{w},k} = \zeta^k_{e_i} \cdot a_{\vec{v},k} \tag{4}$$

where $\zeta_{e_i} = exp(2i\pi e_i^{-1})$ is the smallest primitive e_i-th root of unity. This relation simply expresses the monodromy of the logarithm.

We may think now of relating the expansion at $\vec{v}$ and the expansion at $\vec{u} = z_{\vec{01}}(\vec{v})$, which is the first flag over $\vec{0\infty}$ met when turning counterclockwise.

$$f = \sum_{k \geq o_i} a_{\vec{v},k} \Lambda^k_{\vec{01},e_i},$$

$$f = \sum_{k \geq o_i} a_{\vec{u},k} \Lambda^k_{\vec{0\infty},e_i}.$$

This requires no more than expressing $\Lambda_{\vec{0\infty},e_i}$ from $\Lambda_{\vec{01},e_i}$. Let $\xi_{e_i} = exp(i\pi e_i^{-1})$ be the smallest e_i-th root of -1; we find that

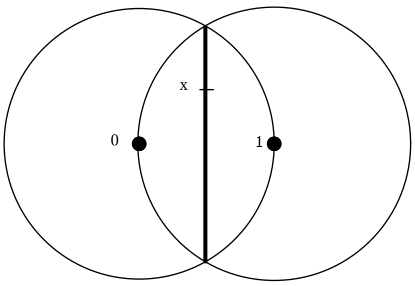
$$\Lambda_{\vec{0\infty},e_i} = \xi_{e_i} \cdot \frac{\Lambda_{\vec{01},e_i}}{(1-\lambda_{\vec{01}})^{1/e_i}} = \xi_{e_i} \cdot \Lambda_{\vec{01},e_i} \sum_{k \geq 0} \binom{{e_i}^{-1}}{k} \lambda^k_{\vec{01}} \tag{5}$$

Now comes the only non-trivial type of relation, the one concerning for example the flags $\vec{v}$ and $\vec{t} = y_{\vec{01}}(\vec{v})$. This time the two expansions are not over the same point since when $\vec{v}$ is over $\vec{01}$ and concerns a point P_i over 0, on the contrary $\vec{t}$ is over $\vec{10}$ and is attached to some point P_j above 1. We have the two corresponding expansions

$$f = \sum_{k \geq o_i} a_{\vec{v},k} \Lambda^k_{\vec{01},e_i},$$

$$f = \sum_{k \geq o_j} a_{\vec{t},k} \Lambda^k_{\vec{10},e_j}.$$

Following Atkin [ASD71], we now equate these two expansions at some point x on the open segment $(\rho, \bar{\rho})$ where $\rho = exp(2i\pi/6)$ is a sixth root of unity. It is to be noted that for such an x, $1 - x = \bar{x}$.



For convenience we adopt the following notation. Let $|a_{\vec{v},k}\rangle$ denote the infinite column vector of all coefficients in the expansion at $\vec{v}$, namely

$$|a_{\vec{v},k}\rangle = (a_{\vec{v},o_i}, a_{\vec{v},o_i+1}, a_{\vec{v},o_i+2}, \ldots)^{\tilde{}}$$

where the tilde stands for transposition.

We also write $\langle x, o_i, e_i|$ for the infinite line vector

$$\langle x, o_i, e_i| = (\Lambda^{o_i}_{\vec{01},e_i}(x), \Lambda^{o_i+1}_{\vec{01},e_i}(x), \Lambda^{o_i+2}_{\vec{01},e_i}(x), ...)$$

Then the value taken by f at x is given by

$$f(x) = \langle x, o_i, e_i||a_{\vec{v},k}\rangle$$

and the relation between the two expansions can be expressed for all x in the segment $(\rho, \bar{\rho})$ as

$$\langle x, o_i, e_i||a_{\vec{v},k}\rangle = \langle 1-x, o_j, e_j||a_{\vec{t},k}\rangle = \langle \bar{x}, o_j, e_j||a_{\vec{t},k}\rangle \tag{6}$$

We write $|\bar{x}, o_i, e_i\rangle$ for the adjoint of $\langle x, o_i, e_i|$ and similarly, $|\bar{x}, o_j, e_j\rangle$ for the adjoint of $\langle x, o_j, e_j|$. We also define the operators v_{x,o_i,e_i}, v_{x,o_j,e_j} and c_{x,o_i,e_i,o_j,e_j} by

$$\begin{aligned}
v_{x,o_i,e_i} &= |\bar{x}, o_i, e_i\rangle\langle x, o_i, e_i| \\
v_{x,o_j,e_j} &= |\bar{x}, o_j, e_j\rangle\langle x, o_j, e_j| \\
c_{x,o_i,e_i,o_j,e_j} &= |\bar{x}, o_i, e_i\rangle\langle \bar{x}, o_j, e_j|.
\end{aligned}$$

We deduce from (6) that

$$\begin{pmatrix} v_{x,o_i,e_i} & -c_{x,o_i,e_i,o_j,e_j} \\ -c^*_{x,o_i,e_i,o_j,e_j} & \bar{v}_{x,o_j,e_j} \end{pmatrix} |a_{\vec{v},k}\rangle \oplus |a_{\vec{t},k}\rangle = 0 \tag{7}$$

where $\bar{v}_{x,o_j,e_j}$ is the conjugate of v_{x,o_j,e_j}. This proves that the direct sum $|a_{\vec{v},k}\rangle \oplus |a_{\vec{t},k}\rangle$, obtained as the concatenation of the two column-vectors, is in the kernel of a given hermitian positive operator which we call

$$j_{x,o_i,e_i,o_j,e_j} = \begin{pmatrix} v_{x,o_i,e_i} & -c_{x,o_i,e_i,o_j,e_j} \\ -c^*_{x,o_i,e_i,o_j,e_j} & \bar{v}_{x,o_j,e_j} \end{pmatrix}.$$

We now choose a positive measure μ with non-finite support on $(\rho, \bar{\rho})$, such that μ is small enough around ρ and $\bar{\rho}$. For example we can choose it with non-finite compact support in $(\rho, \bar{\rho})$. This is safe enough but it may be even better to take $\mu(x)dx$, where $\mu(x)$ is a suitable power of $1-|x|^2 = 1-x+x^2$ or equivalently $J(x) = (1-x+x^2)^3 x^{-2}(x-1)^{-2}$. Then we integrate (7) over $(\rho, \bar{\rho})$. We define the integrals of the above operators:

$$\begin{aligned}
V_{o_i,e_i} &= \int v_{x,o_i,e_i} d\mu \\
V_{o_j,e_j} &= \int v_{x,o_j,e_j} d\mu \\
C_{o_i,e_i,o_j,e_j} &= \int c_{x,o_i,e_i,o_j,e_j} d\mu \\
J_{o_i,e_i,o_j,e_j} &= \int j_{x,o_i,e_i,o_j,e_j} d\mu,
\end{aligned}$$

and we obtain

$$J_{o_i,e_i,o_j,e_j}|a_{\vec{v},k}\rangle \oplus |a_{\vec{t},k}\rangle = 0. \tag{8}$$

This finishes the characterization of $\mathcal{L}(D)$. We can collect all the relations in a blockwise matrix. The blocks are universal junction matrices, and the disposition of all the blocks reflects the topology of the dessin since it comes from the action of the fundamental groupoid on the standards. We note that all the entries of the junction operators are of the form

$$\int x^a(1-x)^b d\mu$$

where a and b are rationals. We can think of expressing them with the beta function plus some hypergeometric functions.

Now, for the actual computation of a dessin we first choose a divisor on the dessin. For example, the Riemann-Hurwitz formula gives us a divisor which is in the canonical class, made up of ramification points. Indeed, if $\mathcal{D}$ is our dessin, we call P_i, Q_j, R_k the points above 0, 1, ∞ respectively, and p_i, q_j, r_k their multiplicities. If the dessin is clean, $q_j = 2$. We call $\mathcal{K}$ the following divisor, which is in the canonical class by the Riemann-Hurwitz formula:

$$\mathcal{K} = -\sum_i (P_i) + \sum_j (q_j - 1)(Q_j) - \sum_k (R_k).$$

If the genus is greater than or equal to 2 then the divisor $2\mathcal{K}$ is very ample. We compute the associated linear space $\mathcal{L}(\in\mathcal{K})$ with enough accuracy as the kernel of the operator introduced above. To achieve this, we choose a given precision P and write down the junction matrices, truncated at rank P. We then build a blockwise matrix corresponding to the above divisor, from the truncated junction matrices, and compute its kernel. Actually, this matrix, being no more than an approximation, is not very likely to have a kernel. We just look for vectors with small images under this matrix, using the least-squares method. This provides us with an explicit, though approximate, description of the linear space $\mathcal{L}(\in\mathcal{K})$ given by some base $(f_1, .., f_{\ell(2\mathcal{K})})$ where the f_i are Puiseux series with P terms.This linear space defines an embedding of the curve in a projective space. By looking for algebraic dependencies between the the f_i, we build an algebraic regular model $\mathcal{C}$ for the curve provide the genus is greater than or equal to 2 (if the genus is 0, a model of the curve is $\mathbb{P}_1$; if the genus is 1, we have some elliptic curve which can be determinated by looking at any ample divisor, although there exist simpler techniques).

Now it remains to compute the Belyi function $\varphi(f_1, .., f_{\ell(2\mathcal{K})})$ from $\mathcal{C}$ to $\mathbb{P}_1$. We first compute the linear space associated to the divisor $-(\varphi) =$

$\sum_k r_k(R_k) - \sum_i p_i(P_i)$ in just the same way as above. It is of dimension 1, and we take a generator which we normalize with the conditions $\varphi(Q_j) = 1$. From all the Puiseux series expansions we have (both for φ and the f_i), we can express φ as an algebraic function of $(f_1, .., f_{\ell(2\mathcal{K})})$.

Of course, all this is quite tedious, and we must develop sharper techniques depending on the genus of the curve as we will see in the next section for genus 0. Thanks to some *a priori* description, we can refine the approximation with an iterative method such as the one detailed below, which leads us to an exact algebraic solution over $\bar{\mathbb{Q}}$.

§7. Iterative ad hoc methods

In this section we describe iterative methods to compute genus zero dessins as rational functions from $\mathbb{P}_1(\mathbb{C})$ to $\mathbb{P}_1(\mathbb{C})$. We compute the positions α_i, β_i and γ_i of the points over 0, 1 and ∞. Algebraic methods are feasible in the case of relatively small dessins of low degree; numerous examples are given in the articles by Shabat, Malle, Birch in this volume. However it is possible to do the calculations much more efficiently via approximation and iteration.

Our method consists of three stages:

- Computing approximations of the positions of the points with ad hoc methods.
- Use an iterative algorithm to obtain the numeric convergence and compute the positions with hundreds of digits. We obtain a very good approximation of the Belyi function.
- Find the number field where the coefficients of the Belyi function are. We use a lattice reduction.

We work the whole time with the geometrical definition of the dessins: we try to compute some *canonical* positions of the vertices of a coloured triangulation of the curve. Thanks to this intuitive approach, we can occasionally help the computer with human intervention, if part of the solution seems obvious.

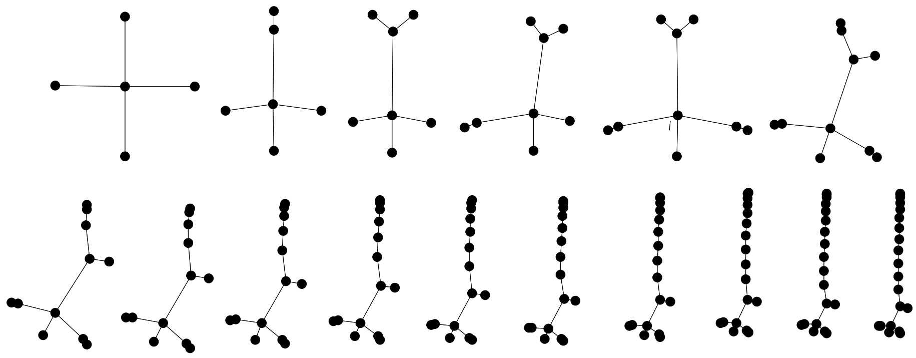
Finding the first approximation

The more general method uses Puiseux series, as stated in section 6, but this method needs memory and time, and cannot be used to analyse families of dessins.

In genus 0, we can use visual intuition and very quickly build an approximation of the Belyi function.

We say that two dessins are close if their combinatorial structure is close. For example, if we add one vertex to a dessin, the resulting dessin has a similar shape. This leads to a method of *growing families of dessins* where we add branches to an initial seed.

Here is for example a family of dessins, with the vertices in their correct positions. We see that the deformation caused by adding a point is small, so each time we can build an approximation of the positions of the vertices from the previous dessin.



This stage needs our intuition, so we can place the new point near its future position, with visual considerations such as the regularity of the graph $\phi^{-1}([0,1])$ and the local symmetry around a vertex.

Numeric convergence

The second stage consists in solving some equations to find the position of the vertices with arbitrary precision.

We want to have a good system of equations, so as to really be able to compute the solutions with our computer. This system must be as simple and as small as possible and it must be stable, so we can converge to the exact solution even with rough approximations.

The case of trees

A dessin is called a "tree" if it is clean, of genus 0 and totally ramified over ∞. If the dessin is a tree, there is a system of polynomial equations in the coordinates of the vertices, that is easy to compute and easy to solve.

We follow the ideas of [Cou94].

If we write ϕ a Belyi function of degree d, it is a rational function with ramification points α_i of degree ν_i over 0 ($i \in A$; we denote by $N = |A|$ the number of vertices of the tree), ramification points β_i of degree 2 over 1

$(i \in B)$ and totally ramified over ∞. Hence we can write, if we put ∞ over ∞:

$$\phi = \frac{-1}{\lambda} \prod_{i \in A} (X - \alpha_i)^{\nu_i} = \frac{-1}{\lambda} \left(\prod_{i \in B} (X - \beta_i) \right)^2 + 1.$$

Since we fixed ∞ over ∞, the positions of the vertices are defined up to an affine transformation: we have 2 degrees of liberty. Now we have an equation in the positions of the vertices (α_i) and the segments (β_i):

$$\prod_{i \in A} (X - \alpha_i)^{\nu_i} = \left(\prod_{i \in B} (X - \beta_i) \right)^2 - \lambda \quad \text{which we write as:} \quad \Pi = Q^2 - \lambda. \tag{9}$$

This is the equation used by Atkin in [ASD71] and Shabat and Birch in this volume. This system could be solved with a Gröbner basis reduction algorithm, but the dessin should not be too large.

We want to build a better system of equations. We want to reduce the number of unknowns and obtain a system of equations independent of the (β_i).

We can factor the right-hand side of equation (9): $Q^2 - \lambda = (Q - \sqrt{\lambda})(Q + \sqrt{\lambda})$, so we can split the set of vertices A in two subsets A^+ and A^- – the blue and the red vertices – such that $i \in A^+$ if and only if $Q(\alpha_i) = +\sqrt{\lambda}$. We factor the left-hand part of equation (9):

$$\Pi^+ = \prod_{i \in A^+} (X - \alpha_i)^{\nu_i} = \left(Q + \sqrt{\lambda} \right)$$

$$\Pi^- = \prod_{i \in A^-} (X - \alpha_i)^{\nu_i} = \left(Q - \sqrt{\lambda} \right)$$

We use the notation:

$$\Theta = \prod_{i \in A} (X - \alpha_i)$$

$$\Sigma = \sum_{i \in A} \frac{\nu_i}{X - \alpha_i} \qquad \Sigma^+ = \sum_{i \in A^+} \frac{\nu_i}{X - \alpha_i} \qquad \Sigma^- = \sum_{i \in A^-} \frac{\nu_i}{X - \alpha_i}$$

$$\sigma = \Theta\Sigma \qquad \sigma^+ = \Theta\Sigma^+ \qquad \sigma^- = \Theta\Sigma^-$$

to differentiate equation (9):

$$\Pi\Sigma = 2QQ' \quad \text{i.e.} \quad \frac{\Pi}{\Theta}\sigma = 2QQ',$$

but since Q is prime to Π and to $\frac{\Pi}{\Theta}$, we deduce

$$\sigma = dQ \quad \text{i.e.} \quad \sigma^+ + \sigma^- = dQ \tag{10}$$

Now, we can compute $\sigma^+ - \sigma^-$, and we obtain an equation with the (α_i) and λ but without the (β_i).

$$\sigma^+ - \sigma^- = d\sqrt{\lambda} \tag{11}$$

because $(\sigma^+ - \sigma^-) - d\sqrt{\lambda}$ is a polynomial of degree less than $N-1$ with N distinct roots:

for i in A^+, $(\sigma^+ - \sigma^-)(\alpha_i) = \sigma^+(\alpha_i) = dQ(\alpha_i) = d\sqrt{\lambda}$

for i in A^-, $(\sigma^+ - \sigma^-)(\alpha_i) = -\sigma^-(\alpha_i) = -dQ(\alpha_i) = d\sqrt{\lambda}$.

If we add and substract equations (10) and (11), we obtain a system in the (α_i) that respects the coloration of the vertices:

$$2\sigma^+ = d\Pi^- \quad \text{and} \quad 2\sigma^- = d\Pi^+.$$

Let us define $\bar{\nu}_i = \nu_i$ for $i \in A^+$ and $\bar{\nu}_i = -\nu_i$ for $i \in A^-$. Equation (11) divided by Θ and with $U = 1/X$ is

$$\frac{d\sqrt{\lambda}U^{N-1}}{\prod_{i\in A}(1-U\alpha_i)} = \sum_{i\in A^+} \frac{\nu_i}{1-U\alpha_i} - \sum_{i\in A^-} \frac{\nu_i}{1-U\alpha_i} = \sum_{i\in A} \frac{\bar{\nu}_i}{1-U\alpha_i}.$$

Now, λ does not interfere with the terms of degree less than N; to eliminate λ, we write down the $N-1$ first terms of the Taylor expansion of this equation:

$$\forall\, 0 \le k \le N-2, \qquad \sum_{i\in A} \bar{\nu}_i \alpha_i^k = 0. \tag{12}$$

The equation for $k = 0$ is trivial, so we have $N-2$ equations for N indeterminates. The set of solutions is invariant under affine transformations $(\alpha_i)_i \mapsto (A\alpha_i + B)_i$.

We add a few inequalities to the system, namely $\alpha_i \neq \alpha_j$ if $i \neq j$. This defines a smooth variety of dimension 2 in the space of dimension N. If we quotient this by the action of the group of affine transformations, we get a variety of dimension 0 in $\mathbb{P}_{N-2}(\mathbb{C})$. It is not necessarily a single point, not even necessarily irreducible over $\mathbb{Q}$, but one point must correspond to our Belyi function.

This proves that our system has a unique solution near our first approximation: the dessin we want to compute.

The case of dessins with all ramification orders even

If all the ramification orders of the dessin are even, we obtain equations similar to (12).

Let α_i, of ramification $2\nu_i$, denote the vertices and γ_i of ramification $2\mu_i$ the faces. Since all ramification indices are even, we can colour the vertices and the faces. We denote $\bar{\nu}_i$ and $\bar{\mu}_i$ the algebraic ramifications.

Then we have the system:

$$\forall\, i, \quad \forall\, 0 \le k \le \nu_i - 1, \qquad \sum_j \frac{\bar{\mu}_j}{(\gamma_j - \alpha_i)^k} = 0$$

$$\forall\, j, \quad \forall\, 0 \le k \le \mu_j - 1, \qquad \sum_i \frac{\bar{\nu}_i}{(\alpha_i - \gamma_j)^k} = 0.$$

Solving the system

The system (12) is a Vandermonde-like system. Let $\mathcal{A} = (\alpha_\infty, ..., \alpha_\mathcal{N})$ and the function $\mathcal{F}(\mathcal{A}) = (\sum_{\rangle\in\mathcal{A}} \not{\nearrow}_\rangle \alpha_\rangle^{\|})_{\|=\infty\ldots\mathcal{N}-\in}$ be such that our system is $\mathcal{F}(\mathcal{A}) = \prime$. Newton's algorithm for solving such equations begins with an approximation of the solution $\mathcal{A}_\prime$ and iterates the formula: $\mathcal{A}_{\setminus+\infty} = \mathcal{A}_\setminus - {\mathcal{F}'_{\mathcal{A}_\setminus}}^{-\infty}\mathcal{F}(\mathcal{A}_\setminus)$.

But $\mathcal{F}'_{\mathcal{A}_\setminus}$ is not invertible and ${\mathcal{F}'_{\mathcal{A}_\setminus}}^{-1}$ is defined up to an element of the kernel. We choose an ${\mathcal{F}'_{\mathcal{A}_\setminus}}^{-1}$ orthogonal to the kernel.

Now we have a method to solve $\mathcal{F}(\mathcal{A}) = \prime$, but we must be careful: the numeric representation forces us to work with $\mathcal{A}_\setminus \in \mathbb{C}^\mathcal{N}$, but we must be aware that $\mathcal{A}_\setminus$ is defined up to affine transformation.

We must normalize the $\mathcal{A}_\setminus$ to avoid a shift to infinity. We fix the center of the dessin at 0, and the scale of the dessin to a diameter of 1. The sum of the coordinates of the vertices is 0 and the maximal distance between two vertices is 1.

To handle large numbers, we use the PARI library.

Back to the algebraic point of view

The third stage uses the powerful lattice reduction tools to go from the geometrical point of view to the algebraic description: given very precise complex approximations of algebraic complex numbers, we build a lattice such that the shortest vector of this lattice gives the minimal polynomial. We use the method described in [LLL82].

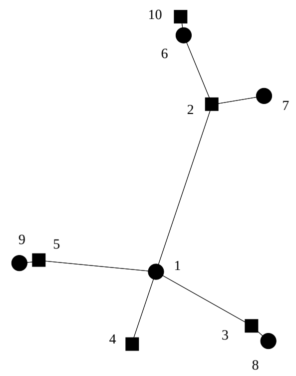
The problem of finding a short vector in an integer lattice is hard, but efficient algorithms have been published in [LLL82], [Sch87] and [Sch88].

The previous stage allows us to compute these numbers to any desired precision. However, a priori we do not know what precision is necessary to find the exact solution with the lattice reduction method. We compute upper bounds for the degree and we guess the size of the coefficients of the polynomial.

The degree is lower than the number of "combinatorial" conjugates of the dessin i.e. the cardinal of the variety of dimension 0 in $\mathbb{P}_{N-2}(\mathbb{C})$, solution of our system. This number can be approximated by character formulae, see [Ser92].

Application to an example

Consider the dessin given by this graph, a tree with 10 vertices:



It is the 6th of the family of dessins shown above. We make the tree grow step by step, and then we compute the solution to 2000 digits.

The minimal polynomial of α_1/α_2 is the polynomial given here, of degree 24 and of discriminant $-1.2^{799}.3^{270}.5^{90}.7^{54}.N^2$ (N is a large number with no smaller factor than 127):

$$
\begin{aligned}
&1216396531470080000\,x^{24} + 15167128532892096000\,x^{23}\\
&+ 88567164003405619200\,x^{22} + 320465331330548463040\,x^{21}\\
&+ 801926461469806116168\,x^{20} + 1468854325860309911334\,x^{19}\\
&+ 2037128673503852027315\,x^{18} + 2189254042743982149456\,x^{17}\\
&+ 1858352449953325455855\,x^{16} + 1271096908385844699688\,x^{15}\\
&+ 717291487653207390204\,x^{14} + 342482003051130999024\,x^{13}\\
&+ 140622333198259937516\,x^{12} + 49205805780202178532\,x^{11}\\
&+ 13997991682162739850\,x^{10} + 2897517763455570160\,x^{9}\\
&+ 284441186456050050\,x^{8} - 67794459856593624\,x^{7}\\
&- 41017353384312340\,x^{6} - 10862737575891504\,x^{5}\\
&- 1796582490031788\,x^{4} - 178029020920154\,x^{3}\\
&- 7529198821413\,x^{2} + 14589968448\,x - 34245281017
\end{aligned}
$$

Note the factorization of the leading coefficient

$$1216396531470080000 = 2^{11}.5^{4}.950309790211.$$

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