Global descent obstructions for varieties

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Abstract

We show how to transport descent obstructions from the category of covers to the category of varieties. We deduce examples of curves having $\mathbf{Q}$ as field of moduli, that admit models over every completion of $\mathbf{Q}$, but have no model over $\mathbf{Q}$.

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Contents

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1 Introduction

If k is a field, a k-variety is by definition a separated scheme of finite type over $\mathrm{Spec}(k)$. A k-curve is a variety of dimension 1 over k. A k-surface is a variety of dimension 2 over k.

1.1 The main results

In this article, we construct descent obstructions in the category of varieties. For example, we show the following theorem:

Theorem 1.1 *There exists a projective, integral and smooth curve over $\overline{\mathbf{Q}}$, having $\mathbf{Q}$ as field of moduli, which is defined over all the completions of $\mathbf{Q}$ but not over $\mathbf{Q}$ itself.*

The main idea is to start from a descent obstruction in the category of covers of curves, and to transport it into various other categories: the category of quasi-projective surfaces, the category of proper surfaces, and finally the category of smooth curves. This process is summarized by the following theorem:

Theorem 1.2 *Let k be a field of characteristic zero, k^a an algebraic closure of k. Let X_k be a smooth, projective, geometrically integral curve over k and let X denote the base change to k^a of X_k. Let Y be a smooth, projective, integral curve over k^a and let $\varphi : Y \to X$ be a (possibly ramified) cover over k^a, having k as field of moduli. There exists a smooth, projective, integral curve over k^a having k as field of moduli and having exactly the same fields of definition as the initial cover φ.*

Examples of obstructions to descent have been mostly constructed in the categories of G-covers and covers [CH85, DF94, CG94] and in the category of dynamical systems [Sil95]. A key technical point is that, in many cases, one can measure these obstructions in terms of the Galois cohomology of a finite abelian group. As far as we know, no example of global obstructions was known for varieties. Mestre gave some examples of local descent obstructions for hyperelliptic curves in [Mes91]. Dèbes and Emsalem [DE99] give a criterion for a curve to be defined over its field of moduli. This criterion involves a particular model for the quotient of the curve by its automorphism group. Dèbes and Emsalem prove that the local-global principle applies to the descent problem for a curve *together with its automorphisms*. However they leave open the question of the local-global principle for a curve (and a variety in general).

Global descent obstructions for covers have been constructed by Ros and Couveignes:

Theorem 1.3 (cf.[CR04], Corollaire 2) *There exists a connected ramified $\overline{\mathbf{Q}}$-cover of $\mathbf{P}^1_{\mathbf{Q}}$ having $\mathbf{Q}$ as field of moduli, which is defined over all the completions of $\mathbf{Q}$ but which does not admit any model over $\mathbf{Q}$.*

If we apply theorem 1.2 to these obstructions, we prove theorem 1.1.

1.2 Overview of the paper

Let k be a field with characteristic zero. Let X_k be a smooth, projective and geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. The starting point of all the constructions, in the sequel, is a smooth projective and integral curve Y over k^a which covers X, i.e. there exists a non-constant morphism $\varphi : Y \to X$ of k^a-curves. We would like to construct a variety having the same moduli and definition properties (the same field of moduli and the same fields of definition) as φ. A first natural idea is to consider the complement $X \times Y - G(\varphi)$ of the graph $G(\varphi)$ of φ in the product $X \times Y$. We call it the *mark* of φ. We hope this surface will have the expected property: same field of moduli and same fields of definition as φ. This will be true in many cases. In order to prove it, we shall associate to φ the stack of all its models. We shall similarly associate to the mark of φ the stack of all its models. We then try to construct a morphism between these two stacks. If this morphism happens to be an equivalence, then we are done.

In section 2, we recall the definition of the stack and gerbe of "models" of an algebraic variety over k^a (or of a cover of curves over k^a). We then explain how a morphism between the two gerbes of models associated with two objects relates the definition and module properties of either objects. In the sequel we shall make use of these functorial properties to transport descent obstructions from a category to another one. It turns out that the key point is to control the group of automorphisms of the involved objects.

To make this task easier, in section 3, we prove that we can suppose that the base curve X of our starting cover φ, does not have any non trivial k^a-automorphism. In other words, we construct another k-curve X'_k without any non-trivial k^a-automorphism and a k^a-cover $Y' \to X'_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ having the same field of moduli and the same fields of definition as φ.

In section 4, we do suppose that X does not have any non-trivial automorphism and we prove that the *mark* of φ has the same field of moduli and the same fields of definition as φ.

In section 5, we assume that the field of moduli of the cover φ is k and we construct a proper normal k^a-surface having k as field of moduli and the same fields of definition as φ. This proper surface is constructed as a cover of $X \times Y$ which is strongly ramified along the graph of φ.

Finally, in section 6, we construct a projective k^a-curve, having k as field of moduli, and having the same fields of definition as the initial cover φ. This curve is drawn on the previous surface. It is obtained by deformation of a stable curve chosen to have the same automorphism group as the surface.

Notations. If k is a field, we denote by k^a its algebraic closure. Let l be a k-extension and let X_l be a l-variety. We denote by $\mathrm{Aut}_l(X_l)$ or simply $\mathrm{Aut}(X_l)$, the group of automorphisms of the l-variety X_l (i.e. automorphisms over $\mathrm{Spec}(l)$). On the other hand, we denote by $\mathrm{Aut}_k(X_l)$ the group of automorphisms of the k-*scheme* X_l (i.e. automorphisms over $\mathrm{Spec}(k)$). For $f \in l(X_l)$, $(f)_0$ denote the divisor of zeros of the function f while $(f)_\infty$ denote the divisor of poles of the function f.

2 Stack of "models"

In this section k is a field of characteristic zero and k^a is an algebraic closure of k.

2.1 The conjugate of a variety

Let X be a k^a-variety. We denote by $\pi : X \to \mathrm{Spec}(k^a)$ the structural morphism. Let $\sigma : k^a \to k^a$ be a k-isomorphism. We denote ${}^\sigma X$ the k^a-variety defined to be X itself with the structural morphism ${}^\sigma\pi = \mathrm{Spec}(\sigma)^{(-1)} \circ \pi$. It is clear that the square below is cartesian. So ${}^\sigma\pi$ is the pullback of π along $\mathrm{Spec}(\sigma)$.

$$\begin{tikzcd}
{}^\sigma X \arrow[r, "\pi"] \arrow[d, "\mathrm{Id}"] & \mathrm{Spec}(k^a) \arrow[r, "\mathrm{Spec}(\sigma)^{(-1)}"] & \mathrm{Spec}(k^a) \arrow[d, "\mathrm{Spec}(\sigma)"] \\
X \arrow[rr, "\pi"] & & \mathrm{Spec}(k^a)
\end{tikzcd}$$

With this (slightly abusive) notation one has ${}^\tau({}^\sigma(\pi)) = {}^{\tau\sigma}\pi$ and ${}^\tau({}^\sigma(X)) = {}^{\tau\sigma}X$. If X is an affine variety, then ${}^\sigma X$ is obtained from X by letting σ act on the coefficients in the defining equations of X. One may prefer to write $X^{\mathrm{Spec}(\sigma)}$ rather than ${}^\sigma X$. This is fine also and we do have $(X^{\mathrm{Spec}(\sigma)})^{\mathrm{Spec}(\tau)} = X^{\mathrm{Spec}(\sigma)\circ\mathrm{Spec}(\tau)}$.

2.2 The field of moduli

It is natural to ask if X and ${}^\sigma X$ are isomorphic. They are certainly isomorphic as schemes (and even equal by definition). But as varieties over k^a, they are isomorphic if and only if there exists an isomorphism ϕ_σ that makes the following diagram commute

$$\begin{tikzcd}
X \arrow[rr, "\phi_\sigma"] \arrow[ddr, "\pi"'] & & X \arrow[d, "\pi"] \\
& & \mathrm{Spec}(k^a) \arrow[dl, "\mathrm{Spec}(\sigma)^{(-1)}"] \\
& \mathrm{Spec}(k^a) &
\end{tikzcd}$$

The above triangle gives rise to a commutative square

$$\begin{tikzcd}
X \arrow[r, "\phi_\sigma"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\
\mathrm{Spec}(k^a) \arrow[r, "\mathrm{Spec}(\sigma)"] & \mathrm{Spec}(k^a)
\end{tikzcd} \tag{1}$$

The existence of such a square means that the isomorphism $\mathrm{Spec}(\sigma)$ of $\mathrm{Spec}(k^a)$ lifts to an isomorphism ϕ_σ of X. If there exists such a lift ϕ_σ for every σ in the absolute Galois group of k, then we say that the condition *field of moduli* is met, or that X has k as field of moduli.

2.3 Fields of definition

Another natural question is, given $l \subset k^a$ an algebraic extension of k, does there exist an l-variety $\pi_l : X_l \to \mathrm{Spec}(l)$ and a cartesian square

$$\begin{tikzcd}
X_l \arrow[d, "\pi_l"'] & X \arrow[l] \arrow[d, "\pi"] \\
\mathrm{Spec}(l) & \mathrm{Spec}(k^a) \arrow[l, "\mathrm{Spec}(\subset)"']
\end{tikzcd} \tag{2}$$

where the line downstairs is the spectrum of the inclusion. If such a square exists we say that l is a *field of definition* of X. We say that $\pi_l : X_l \to \mathrm{Spec}(l)$ is a *model* of $\pi : X \to \mathrm{Spec}(k^a)$ over l. One may wonder if it is important to impose the arrow downstairs in the definition above. The answer is yes in general. The existence of such a cartesian square may depend on the chosen arrow downstairs. However, if the condition *field of moduli* is met, then we may compose the cartesian squares in 1 and 2

$$\begin{tikzcd}
X_l \arrow[d, "\pi_l"'] & X \arrow[l] \arrow[d, "\pi"] & X \arrow[l, "\phi_\sigma"'] \arrow[d, "\pi"] \\
\mathrm{Spec}(l) & \mathrm{Spec}(k^a) \arrow[l, "\mathrm{Spec}(\subset)"'] & \mathrm{Spec}(k^a) \arrow[l, "\mathrm{Spec}(\sigma)"']
\end{tikzcd}$$

and choose the arrow downstairs we prefer.

Another simple observation: if X has a model $\pi_k : X_k \to \mathrm{Spec}(k)$ over k then the condition *field of moduli* is met. Indeed, we write X as a fiber product $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ and we take for ϕ_σ the fiber product $\mathrm{Id}_{X_k} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\sigma)$ where $\mathrm{Id}_{X_k} : X_k \to X_k$ is the identity on X_k.

One may ask if the converse is true.

2.4 Descent obstructions

Assume the condition *field of moduli* holds true. Does there exist a model over k ? If the answer is no, we say that there is a *descent obstruction*. In case k is a number field, we say that the obstruction is *global* if

1. the condition *field of moduli* holds true,
2. there is no model over k,
3. but for every place v of k there exists a model of X over some extension $l \subset k^a$ such that l can be embedded in the completion k_v of k at v.

2.5 The fibered category of "models" of a variety

We denote by $\mathrm{Et} / \mathrm{Spec}(k)$ the category of (finite) étale morphisms over $\mathrm{Spec}(k)$. An object U in this category is a structural morphism $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ where $k \hookrightarrow l$ is a finite étale k-algebra. We define a *covering* of U to be a surjective family $(U_i \to U)_i$ of morphisms in $\mathrm{Et} / \mathrm{Spec}(k)$. This turns $\mathrm{Et} / \mathrm{Spec}(k)$ into a site called the étale site on $\mathrm{Spec}(k)$. It indeed satisfies the three axioms of site : the pullback of a covering exists and is a covering; a covering of a covering is a covering; and the identity is a covering.

Note that in this paper, we use the word *covering* in the context of sites. We keep the word *cover* for a non-constant (separable) morphism between two smooth projective and geometrically integral curves.

Now given a k^a-variety X, we define the fibered category over $\mathrm{Et} / \mathrm{Spec}(k)$ of its "models". So, for any k-algebra l we must say what we mean by a "model" of X over $\mathrm{Spec}(l)$. We first assume that l is a field. We say that an l-variety $\pi_l : X_l \to \mathrm{Spec}(l)$ is a "model" of X over $\mathrm{Spec}(l)$ if and only if there exists an embedding $e : l \hookrightarrow k^a$ of k-algebras and a cartesian square:

$$\begin{tikzcd} X_l \arrow[d, "\pi_l"'] & X \arrow[l] \arrow[d, "\pi"] \\ \mathrm{Spec}(l) & \mathrm{Spec}(k^a) \arrow[l, "\mathrm{Spec}(e)"'] \end{tikzcd}$$

We insist that this time we do not fix an embedding of l into k^a. In particular, if l is a subfield of k^a containing k, we accept models of X but also models of all its conjugates. So the word model here is less restrictive than in section 2.3. This is why we write the word model between quotation marks in that case. Of course, as already noticed, the two notions are equivalent when the condition *field of moduli* holds true.

If l is any finite étale algebra over k, then it is a direct product of finitely many finite field extensions of k. We define a "model" of X over $\mathrm{Spec}(l)$ to be a disjoint union of "models" of X over every connected component of $\mathrm{Spec}(l)$.

Definition 2.1 (The category $\mathbb{M}_X$ of "models" of X) *Let X be a k^a-variety. The category of "models" of X, denoted $\mathbb{M}_X$, is the category:*

- *whose objects are all "models" of X over all finite étale k-algebras,*
- *and whose morphisms are the cartesian squares*

$$\begin{tikzcd} X_l \arrow[rr] \arrow[d, "\pi_l"'] & & X_m \arrow[d, "\pi_m"] \\ \mathrm{Spec}(l) \arrow[rr] \arrow[dr] & & \mathrm{Spec}(m) \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

The functor that associates $\mathrm{Spec}(l)$ to every "model" over $\mathrm{Spec}(l)$ turns $\mathbb{M}_X$ into a fibered category over $\mathrm{Et}\,/\,\mathrm{Spec}(k)$; we denote by $\mathbb{M}_X(l)$ or $\mathbb{M}_X(\mathrm{Spec}(l))$ the fiber over $\mathrm{Spec}(l)$.

In particular, we can pullback a "model" $X_l \to \mathrm{Spec}(l)$ along any morphism $\mathrm{Spec}(m) \to \mathrm{Spec}(l)$ over $\mathrm{Spec}(k)$. Note that pulling back is not quite innocent since it can turn a model into its conjugates so to say.

2.6 Descent data

In fact, under mild conditions, the fibered category $\mathbb{M}_X$ happens to be a stack. In order to see this, we need to recall a few definitions and elementary results about descent data (see Giraud [Gir64] or the more accessible notes [Vis04] by Vistoli).

Let $\mathbb{S}$ be a site and let $\mathbb{X}$ be a fibered category over $\mathbb{S}$. Let U be an object in $\mathbb{S}$ and let $\mathcal{U} = (U_i \to U)_i$ be a covering of U. A *descent datum* from $\mathcal{U}$ to U is a collection of objects $X_i \to U_i$. For every i and every j we also want a morphism $\phi_{ij} : \pi_2^*(X_j) \to \pi_1^*(X_i)$ where π_1 and π_2 are the two "projections" in the cartesian diagram

$$\begin{tikzcd} & U_i \times_U U_j \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ U_i \arrow[dr] & & U_j \arrow[dl] \\ & U & \end{tikzcd}$$

We also require that the following compatibility relation holds true for any i, j, and k

$$\pi_{12}^*(\phi_{ij}) \circ \pi_{23}^*(\phi_{jk}) = \pi_{13}^*(\phi_{ik}) \tag{3}$$

where the π_{12}, π_{23}, π_{31} are the partial "projections" in the cube below

$$\begin{tikzcd}
& U_{ijk} \arrow[rr, "\pi_{23}"] \arrow[dl, "\pi_{12}"'] \arrow[dd, "\pi_{13}"' near end] & & U_{jk} \arrow[dl] \arrow[dd] \\
U_{ij} \arrow[rr, crossing over] \arrow[dd] & & U_j & \\
& U_{ik} \arrow[rr] \arrow[dl] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U \arrow[from=uu, crossing over] &
\end{tikzcd}$$

and $U_{ij} = U_i \times_U U_j$, $U_{ijk} = U_i \times_U U_j \times_U U_k$.

A morphism of descent data is a collection of local morphisms that are compatible with the glueing morphisms on either sides. We thus obtain a category $\mathrm{Desc}_{\mathbb{X}}(\mathcal{U}, U)$ for every covering $\mathcal{U}$ of U. We denote by $\mathbb{X}(U)$ the fiber of $\mathbb{X}$ above U. There is a functor $\mathbb{X}(U) \to \mathrm{Desc}_{\mathbb{X}}(\mathcal{U}, U)$ that associates to any object over U the collection of its restrictions over the U_i. These constructions are functorial. For example, if $\mathbb{Y}$ is another fibered category and $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ a cartesian functor, then $\mathbb{F}$ induces a functor from $\mathbb{X}(U)$ to $\mathbb{Y}(U)$ and a functor from $\mathrm{Desc}_{\mathbb{X}}(\mathcal{U}, U)$ to $\mathrm{Desc}_{\mathbb{Y}}(\mathcal{U}, U)$. Further, the composite functors $\mathbb{X}(U) \to \mathbb{Y}(U) \to \mathrm{Desc}_{\mathbb{Y}}(\mathcal{U}, U)$ and $\mathbb{X}(U) \to \mathrm{Desc}_{\mathbb{X}}(\mathcal{U}, U) \to \mathrm{Desc}_{\mathbb{Y}}(\mathcal{U}, U)$ are isomorphic.

A fibered category $\mathbb{X}$ over a site $\mathbb{S}$ is a *stack* if and only if all the functors $\mathbb{X}(U) \to \mathrm{Desc}_{\mathbb{X}}(\mathcal{U}, U)$ are equivalences of categories.

2.7 When $\mathbb{M}_X$ is a stack, next a gerbe

If X is a k^a-variety then $\mathbb{M}_X$ is a fibered category over $\mathrm{Et} / \mathrm{Spec}(k)$ and it makes sense to ask if it is a stack.

We first notice that if l and m are two finite field extensions of k and if $l \subset m$, then $\mathrm{Spec}(m) \to \mathrm{Spec}(l)$ is a covering of $\mathrm{Spec}(l)$. If further m is a Galois extension of l, then a descent datum from $\mathrm{Spec}(m)$ to $\mathrm{Spec}(l)$ is a model $\pi_m : X_m \to \mathrm{Spec}(m)$ of X over $\mathrm{Spec}(m)$ and, for every σ in $\mathrm{Gal}(m/l)$, an automorphism $\phi_\sigma : X_m \to X_m$ of l-*scheme*, such that the following diagram commutes

$$\begin{tikzcd}
X_m \arrow[r, "\phi_\sigma"] \arrow[d, "\pi_m"'] & X_m \arrow[d, "\pi_m"] \\
\mathrm{Spec}(m) \arrow[r, "\mathrm{Spec}(\sigma)"] & \mathrm{Spec}(m)
\end{tikzcd}$$

We emphasize the fact that each ϕ_σ need not be an automorphism of the m-variety X_m but only an automorphism of the l-scheme X_m. Let $\mathrm{Aut}_l(X_m)$ denote the set of automorphisms of the

l-scheme X_m. The compatibility condition (3) states that the map $\mathrm{Spec}(\sigma) \mapsto \phi_\sigma$ must be a group homomorphism from $\mathrm{Aut}_{\mathrm{Spec}(l)}(\mathrm{Spec}(m))$ into $\mathrm{Aut}_l(X_m)$.

Proposition 2.2 *Let X be a variety over k^a. If X is affine or projective or if every finite subset of $X(k^a)$ is contained in an affine subvariety, then the fibered category $\mathbb{M}_X$ is a stack over* $\mathrm{Et}\,/\,\mathrm{Spec}(k)$.

Proof — This is a consequence of Weil's descent theory. See the initial article of Weil [Wei56] or Serre's book [Ser59, Chap V,§4]. □

Recall that a locally non-empty and locally connected stack is called a *gerbe*. More precisely a stack $\mathbb{X}$ over a site $\mathbb{S}$ is a gerbe if and only if

1. For every object U in $\mathbb{S}$ there exists a covering $(U_i \to U)_i$ of U such that the fibers over the U_i are non-empty,
2. Given two objects $X \mapsto U$ and $Y \mapsto U$ above U there exists a covering $(U_i \to U)_i$ such that for every i the pullbacks $X \times_U U_i$ and $Y \times_U U_i$ are isomorphic over U_i,
3. For every object U in $\mathbb{S}$ the fiber $\mathbb{X}(U)$ is a groupoid.

The stack $\mathbb{M}_X$ of "models" of a variety X always satisfies conditions one and three, while the second one holds true if and only if k is the field of moduli of X.

2.8 The stack, next the gerbe, of "models" of a cover of a curves

Since the starting point of our construction is a cover of curves, we need to define the stack of "model" of a cover of curves. So we adapt the notions presented in the preceding subsections to this context.

Let X_k be a smooth, projective, geometrically integral curve over k. We set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth projective and integral curve over k^a and let $\varphi : Y \to X$ be a non-constant morphism of k^a varieties. Since k has zero characteristic, the morphism φ is separable. We say that φ is a cover of X. Note that we allow branch points. An isomorphism between two covers $\varphi : Y \to X$ and $\psi : Z \to X$ is an isomorphism of k^a-varieties $i : Y \to Z$ such that $\psi \circ i = \varphi$.

The conjugate of a cover — If σ is a k-automorphism of k^a, the conjugate variety ${}^\sigma X$ is obtained from X by composing the structural morphism on the left with $\mathrm{Spec}(\sigma)^{(-1)}$. The same is true for Y. So any k^a-morphism φ between X and Y can be seen as a k^a morphism ${}^\sigma\varphi$ between ${}^\sigma X$ and ${}^\sigma Y$. Since X is the fiber product of X_k and $\mathrm{Spec}(k^a)$ over $\mathrm{Spec}(k)$, we have a canonical isomorphism $\phi_\sigma = \mathrm{Id}_{X_k} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\sigma)$ between X and ${}^\sigma X$. The composite map $\phi_\sigma^{(-1)} \circ {}^\sigma\varphi$ is a morphism of k^a-varieties between ${}^\sigma Y$ and X. We call it the conjugate of φ by σ. We may denote it ${}^\sigma\varphi$ also by abuse of notation.

The field of moduli — We say that the condition *field of moduli* holds true for φ, or that k is the field of moduli of φ, if ${}^\sigma\varphi$ is isomorphic to φ for every k-automorphism σ of k^a.

Fields of definition, models — If $l \subset k^a$ is an algebraic extension of k we set $X_l = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$. Let Y_l be a smooth projective and geometrically connected l-curve. Let $\varphi_l : Y_l \to X_l$ be a non-constant (separable) map. If we lift φ_l along the spectrum of the inclusion $l \subset k^a$ we obtain a morphism from $Y_l \times_{\mathrm{Spec}(l)} \mathrm{Spec}(k^a)$ onto $X = X_l \times_{\mathrm{Spec}(l)} \mathrm{Spec}(k^a)$. If this cover is isomorphic to $\varphi : Y \to X$ then we say that φ_l is a model of φ over l.

So it makes sense to ask if there exist (global) obstructions to descent for covers of curves. It it proven in [CR04] that this is indeed the case.

The fibered category of "models" of a cover — Given a finite étale k-algebra l we explain what we mean by a "model" of φ over l.

Assume first that l is a finite field extension of k. Set $X_l = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$. Let Y_l be a smooth projective and geometrically integral curve over $\mathrm{Spec}(l)$ and $\varphi_l : Y_l \to X_l$ be a non-constant morphism over $\mathrm{Spec}(l)$. We pick any embedding $e : l \to k^a$ of k-algebras. The pullback of X_l along $\mathrm{Spec}(e)$ is X (up to unique isomorphism) and we have the following diagram

$$\begin{tikzcd}
 & & Y_l \arrow[dl, "\varphi_l"'] \arrow[ddl] & & Y_l \times_{\mathrm{Spec}(l)} \mathrm{Spec}(k^a) \arrow[ll] \arrow[dl] \arrow[ddl] \\
X_k \arrow[d] & X_l \arrow[l] \arrow[d] & & X \arrow[ll] \arrow[d] & \\
\mathrm{Spec}(k) & \mathrm{Spec}(l) \arrow[l] & & \mathrm{Spec}(k^a) \arrow[ll, "\mathrm{Spec}(e)"'] &
\end{tikzcd}$$

We say that φ_l is a "model" of φ if the cover

$$\varphi_l \times_{\mathrm{Spec}(l)} \mathrm{Spec}(k^a) : Y_l \times_{\mathrm{Spec}(l)} \mathrm{Spec}(k^a) \to X$$

is isomorphic to φ.

Again we don't care about the choice of the embedding e. We just ask that such an embedding exists.

If l is any finite étale algebra over k, we define a "model" of φ over $\mathrm{Spec}(l)$ to be a disjoint union of "models" of φ over every connected component of $\mathrm{Spec}(l)$. We write $\mathbb{M}_\varphi$ for the category of all models of φ. This is a fibered category over $\mathrm{Et}\,/\,\mathrm{Spec}(k)$.

The following proposition is a consequence of Weil's descent theorem:

Proposition 2.3 *Let X_k be a smooth, projective, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth projective and integral curve over k^a and let $\varphi : Y \to X$ be a non-constant morphism of k^a curves. Then the fibered category $\mathbb{M}_\varphi$ is a stack over* $\mathrm{Et}\,/\,\mathrm{Spec}(k)$.

Like for varieties, the stack $\mathbb{M}_f$ is a gerbe if and only if k is the field of moduli of f.

2.9 Transporting obstructions

In this subsection by *cover of curves*, we mean a cover $f : Y \to X$ satisfying the hypotheses of section 2.8.

Let us emphasize the following easy facts from the preceding sections.

Proposition 2.4 *Let X be a k^a-variety (or a cover a curves) then:*

1. *the field k is the field of moduli of X if and only if the stack $\mathbb{M}_X$ is a Gerbe;*
2. *the field l is a field of definition of X if and only if the fiber $\mathbb{M}_X(l)$ is not empty.*

Let X and Y be two k^a-varieties. Using a (cartesian) morphism of stacks from $\mathbb{M}_X$ to $\mathbb{M}_Y$, we are now able to transport descent obstruction for X to descent obstruction for Y. Recall a cartesian morphism is a functor of fibered categories that transforms cartesian square into cartesian squares. So a cartesian morphism $\mathbb{F} : \mathbb{M}_X \to \mathbb{M}_Y$, associates an l-model $\mathbb{F}(X_l)$ of Y to every l-model X_l of X, and commutes with base change.

Proposition 2.5 *Let X and Y be either k^a-varieties or covers of curves. Suppose that there exists a morphism $\mathbb{F} : \mathbb{M}_X \longrightarrow \mathbb{M}_Y$ of stacks.*

1. *If X has k as field of moduli then Y has k as field of moduli;*
2. *If l is a field of definition of X then l is also a field of definition of Y.*

Moreover, if the first condition holds true and if $\mathbb{F}$ is fully faithful then:

3. *l is a field of definition of X if and only if l is a field of definition of Y.*

In that case, there is a descent obstruction for X if and only if there is one for Y.

Proof — The two first conditions are easy consequences of the preceding result. The third one can be deduced form the following more general lemma. □

Lemma 2.6 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two gerbes over a site $\mathbb{S}$ and let $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ be a cartesian morphism. If $\mathbb{F}$ is fully faithful then $\mathbb{F}$ is essentially surjective.*

Proof — Let U be an object in $\mathbb{S}$ and $Y \to U$ an object in the fiber $\mathbb{Y}(U)$. Locally $\mathbb{X}(U)$ is not empty: there exists a covering $(U_i \to U)_i$ of U and objects $X_i \in \mathbb{X}(U_i)$ for all i. Set $Y_i = X \times_U U_i$. Locally, Y_i and $\mathbb{F}(X_i)$ are isomorphic: there exists a covering $(U_{ij} \to U_i)_j$ such that $Y_i \times_{U_i} U_{ij}$ and $\mathbb{F}(X_i \times_{U_i} U_{ij})$ are isomorphic. Set $X_{ij} = X_i \times_{U_i} U_{ij}$ and $Y_{ij} = Y_i \times_{U_i} U_{ij}$.

Note that the collection of objects $(Y_{ij} \to U_{ij})_{ij}$ defines a descent datum from $(U_{ij} \to U)_{ij}$ to U; indeed for every i, j, i', j', pulling back identity gives rise to isomorphisms:

$$\Phi_{iji'j'} : Y_{ij} \times_{U_{ij}} U_{iji'j'} \longrightarrow Y_{i'j'} \times_{U_{i'j'}} U_{iji'j'}$$

which clearly satisfy the compatibility conditions (3) of §2.6.

Since $\mathbb{F}$ is fully faithful, there exist isomorphisms

$$\Psi_{iji'j'} : X_{ij} \times_{U_{ij}} U_{iji'j'} \longrightarrow X_{i'j'} \times_{U_{i'j'}} U_{iji'j'}$$

which, in turn, satisfy the compatibility conditions (3) of §2.6. We deduce that there exists $X \to U$ in $\mathbb{X}(U)$ such that $\mathbb{F}(X) = Y$. □

We end this section by an example of morphism between two stacks of "models" of a variety.

Proposition 2.7 *Let X be a integral variety over k^a having k as field of moduli and let G be a finite subgroup of $\mathrm{Aut}_{k^a}(X)$ which is normal in the group $\mathrm{Aut}_k(X)$. Assume that every orbit of G is contained in an affine open subset of X. Then there is a morphism from $\mathbb{M}_X$ to $\mathbb{M}_{X/G}$, where X/G denotes the quotient variety of X by G.*

Let $Y \subset X/G$ be the complement of the branch locus of $X \to X/G$. This an open sub-variety of X/G and there is a morphism of stacks from $\mathbb{M}_X$ to $\mathbb{M}_Y$.

Proof — We first need to define the image of an object. Let l be an extension of k and let X_l be a "model" of X in $\mathbb{M}_X(l)$. All the elements of G may not be defined over l, but there exists a finite Galois extension m of l over which they are. Put $X_m = X_l \times_{\mathrm{Spec}(l)} \mathrm{Spec}(m)$. One can now consider the quotient of X_m by the group G: let $p_m : X_m \to X_m/G$ be the canonical projection.

Of course X_l is a model of X_m over l; by section 2.7, there exists a group homomorphism $\sigma \mapsto \phi_\sigma$ from $\mathrm{Gal}(m/l)$ to $\mathrm{Aut}_l(X_m)$.

Since G is a normal subgroup of $\mathrm{Aut}_k(X)$, it is, a fortiori, a normal subgroup of $\mathrm{Aut}_l(X_m)$ and thus, for every $g \in G$ and every $\sigma \in \mathrm{Gal}(m/l)$, one has $\phi_\sigma \circ g \circ \phi_\sigma^{-1} \in G$. We deduce that $p_m \circ \phi_\sigma \circ g = p_m \circ \phi_\sigma$ for every $g \in G$. This implies that ϕ_σ factorizes into $\psi_\sigma : X_m/G \to X_m/G$. By uniqueness of this factorization, the correspondence $\sigma \mapsto \psi_\sigma$ is necessarily a group homomorphism from $\mathrm{Gal}(m/l)$ to $\mathrm{Aut}_l(X_m/G)$; therefore the quotient X_m/G descents to l.

Next, we need to define the image of a morphism. Let $X_i \to \mathrm{Spec}(l_i)$, $i = 1, 2$ two "models" of X. One can complete a cartesian square involving the X_i's in the following way:

$$\begin{tikzcd}
X_1 \arrow[d] & X_2 \arrow[l] \arrow[d] & X_m \arrow[l] \arrow[d] \\
\mathrm{Spec}(l_1) & \mathrm{Spec}(l_2) \arrow[l] & \mathrm{Spec}(m) \arrow[l]
\end{tikzcd}$$

where m is a finite Galois extension of k such that all elements of G are defined over m. We know that there exist isomorphisms Φ_1, Φ_2 making the following diagrams commute:

$$\begin{tikzcd}
Y_1 \arrow[d] & X_m/G \arrow[l, "\Phi_1"'] \arrow[d] \\
\mathrm{Spec}(l_1) & \mathrm{Spec}(m) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_2 \arrow[d] & X_m/G \arrow[l, "\Phi_2"'] \arrow[d] \\
\mathrm{Spec}(l_2) & \mathrm{Spec}(m) \arrow[l]
\end{tikzcd}$$

The image of the starting cartesian square is nothing else than:

$$\begin{tikzcd}
Y_1 \arrow[d] & Y_2 \arrow[l, "\Phi_1 \circ \Phi_2^{-1}"'] \arrow[d] \\
\mathrm{Spec}(l_1) & \mathrm{Spec}(l_2) \arrow[l]
\end{tikzcd}$$

This completes the proof of the first statement.

The second statement is true because taking the branch locus commutes with base changes.
□

3 Cancellation of the automorphism group of the base curve

In this section, k is a field of characteristic zero, k^a an algebraic closure of it, $l \subset k^a$ an algebraic extension of k. Let X_k be a projective, smooth, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. We assume we are given a smooth projective and integral curve Y over k^a and a cover $\varphi : Y \to X$ having k as field of moduli.

We want to construct other covers having the same field of moduli and the same fields of definition as φ but satisfying additional properties. In particular, we want to show that one can assume that the base curve X has no non-trivial k^a-automorphism.

We first prove that the degree of the cover can be multiplied by any prime integer not dividing the initial degree.

Proposition 3.1 *Let X_k be a smooth, projective, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth projective and integral curve over k^a and let $\varphi : Y \to X$ be a cover over k^a of degree d. For every prime p not dividing d, there exists a smooth projective curve Y' over k^a and a cover $\psi : Y' \to X$ of degree pd, having the same field of moduli and the same fields of definition as φ.*

Proof — Let $f \in k(X_k)$ be a non-constant function whose divisor is simple and does not meet the ramification locus of φ. The equation $h^p = f$ defines a degree p extension of $k(X_k)$. We denote by X'_k the smooth, projective, geometrically integral curve corresponding to this function field and we set $X' = X'_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. The morphism $\nu : X' \to X$ is a cyclic Galois cover of degree p. We fix an algebraic closure Ω of $k^a(X)$ and embeddings of $k^a(X')$ and $k^a(Y)$ in Ω. Let Y' be the smooth projective k^a-curve corresponding to the compositum of $k^a(Y)$ and $k^a(X')$. Since the field extensions $k^a(Y)$ and $k^a(X')$ are linearly disjoint over $k^a(X)$, the cover $\psi : Y' \to X$ has degree pd:

$$\begin{tikzcd} & Y' \arrow[dl, "\varphi'"'] \arrow[dr, "\nu'"] \arrow[dd, "\psi"] & \\ X' \arrow[dr, "\nu"'] & & Y \arrow[dl, "\varphi"] \\ & X & \end{tikzcd}$$

Let us prove that this construction yields a morphism of stacks $\mathbb{F} : \mathbb{M}_\varphi \to \mathbb{M}_\psi$.

Let $l \subset k^a$ be a finite extension of k. Set $X_l = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$, $X'_l = X'_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$ and consider $\varphi_l : Y_l \to X_l$ an l-model of φ. In the construction above, one can replace X, X', Y by X_l, X'_l, Y_l. The l-curve Y'_l corresponding to the compositum of the two function fields $l(X'_l)$ and $l(Y_l)$ is smooth, projective, geometrically integral (because l is algebraically closed in the compositum) curve and the l-cover $\psi_l : Y'_l \to X_l$ is an l-model of ψ. We define the morphism $\mathbb{F}$ by putting $\mathbb{F}(\varphi_l) = \psi_l$. Since the function f has been chosen in $k(X)$, the functor $\mathbb{F}$ maps cartesian squares to cartesian squares. Thus $\mathbb{F}$ is a morphism of stacks. By proposition 2.5, if l is a field of definition of φ then l is a field of definition of ψ and if φ has k as field of moduli then ψ has k as field of moduli.

To prove the converse, we use proposition 2.7 in order to construct a morphism the other way around. Let ν' denote the Galois cover $Y' \to Y$. We need to show that the group $\mathrm{Aut}(\nu')$ is normal in $\mathrm{Aut}_k(\psi)$. Let $\Phi' \in \mathrm{Aut}_k(\psi)$. It induces maps $\Phi : Y \to Y$ and $\Psi : X \to X$ such that the following diagram commute:

$$\begin{tikzcd}
Y' \arrow[rr, "\Phi'"] \arrow[d, "\nu'"'] \arrow[dd, bend right=60, "\psi"'] & & Y' \arrow[d, "\nu'"] \arrow[dd, bend left=60, "\psi"] \\
Y \arrow[rr, "\Phi"] \arrow[d, "\varphi"'] & & Y \arrow[d, "\varphi"] \\
X \arrow[rr, "\Psi"] \arrow[dr] & & X \arrow[dl] \\
& \mathrm{Spec}(k) &
\end{tikzcd} \tag{4}$$

(horizontal arrows are morphisms of k-schemes). The existence of Ψ is a consequence of the fact that X is defined over k. The morphism Φ exists because $Y \xrightarrow{\varphi} X$ is the maximal sub-cover of $Y' \xrightarrow{\psi} X$ unramified at the support of f. And f is k-rational. Now if $\Lambda \in \mathrm{Aut}_{k^a}(\nu')$, i.e. $\nu' \circ \Lambda = \nu'$, then:

$$\nu' \circ \Phi' \circ \Lambda = \Phi \circ \nu' \circ \Lambda = \Phi \circ \nu' = \nu' \circ \Phi'$$

so $\Phi' \circ \Lambda \circ \Phi'^{-1} \in \mathrm{Aut}_{k^a}(\nu')$, which was to be proven. In conclusion, we do have a morphism $\mathbb{G} : \mathbb{M}_\psi \to \mathbb{M}_\varphi$ of stacks and the lemma follows. $\square$

> **Remark –** The functor $\mathbb{F} : \mathbb{M}_\varphi \to \mathbb{M}_\psi$ is not fully faithful because ψ has more automorphisms than φ. This is why, we do not apply point (3) in proposition 2.5 here. We instead construct another functor $\mathbb{G} : \mathbb{M}_\psi \to \mathbb{M}_\varphi$ and we apply points (1) and (2) in proposition 2.5 to either functors $\mathbb{F}$ and $\mathbb{G}$ successively. We notice that $\mathbb{G}$ is a left inverse of $\mathbb{F}$.

Next, we show that the base curve can be assumed to have genus greater than 2.

Proposition 3.2 *Let X_k be a smooth, projective, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth projective and integral curve over k^a and let $\varphi : Y \to X$ be a cover over k^a of degree d. There exists a smooth, projective, geometrically integral curve X'_k over k of genus greater than 2 and a cover $\varphi' : Y' \to X'_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ having the same field of moduli and the same fields of definition as φ.*

Proof — We use the construction and notation of diagram 4 above. We further assume that the chosen function has degree at least f has degree at least 3. By Hurwitz genus formula, the curve X' has a genus greater than or equal to 2.

This construction yields a morphism of stacks $\mathbb{F} : \mathbb{M}_\varphi \to \mathbb{M}_{\varphi'}$. The cover $\varphi : Y \to X$ is the maximal sub-cover of $\psi : Y' \to X$ unramified at the support of f. Therefore, there exists a morphism from $\mathrm{Aut}_{k^a}(\varphi') \to \mathrm{Aut}_{k^a}(\varphi)$. This morphism is bijective because $k^a(X')$ and $k^a(Y)$ are linearly disjoint over $k^a(X)$. So the morphism $\mathbb{F}$ is fully faithful. We conclude, this time, using proposition 2.5. $\square$

Last, we prove that one can assume the base curve to have no non-trivial k^a-automorphism.

Proposition 3.3 *Let X_k be a smooth, projective, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth projective and integral curve over k^a and let $\varphi : Y \to X$ be a cover over k^a. There exists a smooth, projective, geometrically integral curve X'_k over k, of genus greater that 2, such that $X' = X'_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ does not have any non-trivial automorphism and there exists a cover $\varphi' : Y' \to X'$ over k^a having the same field of moduli and the same fields of definition as φ.*

Proof — Thanks to proposition 3.2, one can assume that the genus $g(X)$ of X is greater then 2. Consequently, the group $\mathrm{Aut}(X)$ of k^a-automorphisms is finite.

Let $p \geq 3$ be a prime integer. To begin with, we show that there exists a non-constant function $f \in k(X)$ which is non-singular above 2, -2 and ∞, of degree greater than $2 + 4p(g(X) - 1) + 2p^2$, such that the set $f^{-1}(\{-2, 2\})$ is not invariant by any non-trivial automorphism of X, and such that the set of singular values of φ does not meet the set $f^{-1}(\{2, -2, \infty\})$.

Let D be a simple effective divisor on X with degree greater than $2 + 4p(g(X) - 1) + 2p^2$. We also assume that D is disjoint from the set of singular values of φ and the linear space $L(D)$ associated with D generates $k^a(X)$ over k^a. In particular, for every $\theta \in \mathrm{Aut}(X)$, this linear space is not contained in the Kernel of $\theta - \mathrm{Id}$. It is not contained in the kernel of $\theta + \mathrm{Id}$ either because it contains k^a. If D has been chosen with a large enough degree, the functions in $L(D)$ having degree less than the degree of D are contained in a finite union of strict vector subspaces. Therefore there exists a non constant function $f \in L(D)$ such that $\deg(f) = \deg(D)$ and $\theta(f) \neq \pm f$ for all $\theta \in \mathrm{Aut}(X) \setminus \{\mathrm{Id}\}$. By construction, this function is not singular above ∞ and $f^{-1}(\infty)$ does not meet the singular values of φ. We can also assume that $f \in k(X_k)$.

By construction, the function f^2 does not have any non-trivial automorphism (in short one has $\mathrm{Aut}_{k^a(f^2)}(k^a(X)) = \{\mathrm{Id}\}$). Using lemma 7.3, we deduce that almost all the fibers of f^2 are non singular and not fixed by any non trivial automorphism of $\mathrm{Aut}(X)$. In particular, there exists $\lambda \in k^*$ such that the fiber of f^2 above λ^2 is non singular, not fixed by any non-trivial automorphism in $\mathrm{Aut}(X)$ and does not meet the singular values of φ. The function $2f/\lambda$ satisfies all the properties we want. Let us denote it by f.

Now the equation $h^p + h^{-p} - f = 0$ defines a regular extension of $k(X_k)$. Let X''_k be the smooth, projective, geometrically integral curve associated to this function field. We denote by w the automorphism of X''_k given by $w(h) = h^{-1}$ and by X'_k the quotient $X''_k/\langle w \rangle$; this is a smooth, projective, geometrically integral k-curve, covering X_k by a k-cover $\nu_k : X'_k \to X_k$ of degree p. Extending scalars to k^a, we obtain a Galois cover $X'' \to X$ of k^a-curves, with Galois group D_p, and whose singular values are exactly $f^{-1}(\{2, -2, \infty\})$. Since the subgroup $\langle w \rangle$ is self-normalized in D_p, the quotient by this subgroup is a sub-cover $\nu : X' \to X$ of k^a-curves of degree p which does not have any non-trivial automorphism.

Because the ramification loci do not meet, the function fields $k^a(X'')$ and $k^a(Y)$ are linear disjoint over $k^a(X)$. Let Y' (resp. Y'') be the smooth, projective, integral curve corresponding to the compositum of $k^a(Y)$ with $k^a(X')$ (resp. $k^a(X'')$). We have the following diagram:

$$
\begin{tikzcd}
 & & Y'' \arrow[dl] \arrow[dr] & & \\
 & X'' \arrow[dl, "h"'] \arrow[dr] & & Y' \arrow[dl, "\varphi'"] \arrow[dr] & \\
\mathbf{P}^1 \arrow[dr] & & X' \arrow[dl, "h+\frac{1}{h}"'] \arrow[dr, "\nu"'] & & Y \arrow[dl, "\varphi"] \\
 & \mathbf{P}^1 \arrow[dr] & & X \arrow[dl, "f"'] \arrow[d, phantom, "\text{k-rational towers}"] & \\
 & & \mathbf{P}^1 & {} &
\end{tikzcd}
$$

The cover $Y'' \to Y$ is again a D_p-Galois cover and the cover $Y'' \to Y'$ has degree 2.

Let us show that the cover $\varphi' : Y' \to X'$ has the expected properties.

First of all, it is clear that the construction above yields a morphism of stacks $\mathbb{F} : \mathbb{M}_\varphi \to \mathbb{M}_{\varphi'}$. The Galois equivariance is a direct consequence of the fact that the middle diagonal tower is defined over k. This morphism is in fact fully faithful because the sub-cover $\varphi : Y \to X$ of $\nu \circ \varphi' : Y' \to X$ is the maximal sub-cover unramified at $f^{-1}(\{2, -2, \infty\})$.

Last we have to prove that the curve X' does not have any non-trivial automorphism. Let θ' be an automorphism of X'. Call Z the image Z of $\nu \times (\nu \circ \theta') : X' \to (X \times X)$. Let $\pi_1 : X \times X \to X$ be the projection to the first factor. The map ν factors as:

$$\nu : X' \longrightarrow Z \xrightarrow{\pi_1} X$$

and it has prime degree p. So Z is either isomorphic to X or birationaly equivalent to X'. In the latter case, the geometric genus of Z would be $> \frac{1}{4}\deg(f)p \geq 1 + 2p(g(X) - 1) + p^2$ by Hurwitz genus formula. But the bi-degree of Z is $\leq (p, p)$; so, by lemma 7.1, its virtual arithmetic genus is less than $1 + 2p(g(X) - 1) + p^2$. A contradiction. Therefore Z is a correspondence of bi-degree $(1, 1)$ which defines an automorphism θ of X such that $\theta \circ \nu = \nu \circ \theta'$. Such an automorphism preserves the ramification data of ν, the one of its Galois closure $X'' \to X$ and also the one of the unique subcover of degree 2 of the cover $X'' \to X$. Since this last cover is exactly ramified above $f^{-1}(\{-2, 2\})$, we deduce that $\theta = \mathrm{Id}$ and then that θ' is a k^a-automorphism of the cover ν. Since ν does not have any non-trivial automorphism, necessarily $\theta' = \mathrm{Id}$. $\square$

4 Quasi-projective surfaces

Let k be a field of characteristic zero. In this section, we give a general process which associates to each k^a-cover of curves, a smooth quasi-projective integral k^a-surface with the same field of moduli and fields of definition.

Theorem 4.1 *Let X_k be a smooth, projective, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth, projective, integral curve over k^a and let $\varphi : Y \to X$ be a non-constant morphism of k^a curves. Then there exists a smooth quasi-projective integral k^a-surface having the same field of moduli and the same fields of definition as φ.*

First of all, by propositions 3.2 and 3.3, one can assume that the base curve X has genus greater than 2 and has no non-trivial k^a-automorphism.

We consider the product $X \times Y$ of the two curves and we denote by $G(\varphi)$ the graph of φ inside this product. Let U be the open complementary set of $G(\varphi)$ in $X \times Y$.

The surface we are looking for is nothing else than the open set U. We call it the *mark* of the cover $\varphi : Y \to X$ and we now prove that is has the same field of moduli and the same fields of definition as φ.

We need two lemmas.

Lemma 4.2 *Let l/k be a finite extension of k inside k^a. Let X_k be a smooth, projective, geometrically integral k-curve. Set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ and assume that the genus of X is greater than 2 and that X has no non-trivial k^a-automorphism. Let U_l and V_l be the marks of two non-trivial geometrically integral l-covers $\varphi_l : Y_l \to X_l$ and $\psi_l : Z_l \to X_l$, where $X_l = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$.*

Then every morphism of covers between $\varphi_l : Y_l \to X_l$ and $\psi_l : Z_l \to X_l$ induces a morphism between the corresponding marks U_l and V_l. Conversely, every surjective l-morphism from U_l to V_l is equal to $\mathrm{Id} \times \gamma_l$ where $\gamma_l : Y_l \to Z_l$ is a l-morphism between the covers $\varphi_l : Y_l \to X_l$ and $\psi_l : Z_l \to X_l$.

Proof — Recall that a l-morphism between the covers $Y_l \xrightarrow{\varphi_l} X_l$ and $Z_l \xrightarrow{\psi_l} X_l$ is a l-morphism of l-curves $\gamma_l : Y_l \to Z_l$ such that $\psi_l \circ \gamma_l = \varphi_l$. The product $\mathrm{Id} \times \gamma_l : X_l \times Y_l \to X_l \times Z_l$ maps the graph of φ_l to the graph of ψ_l and also the mark U_l to the mark V_l.

Conversely, let υ_l be a surjective l-morphism form U_l to V_l. We denote by $\upsilon : U \to V$, $\varphi : Y \to X$, $\psi : Z \to X$ the base change to k^a of υ_l, φ_l, ψ_l respectively.

Let y be a closed k^a-point of Y. Let $\pi_2 : X \times Z \to Z$ be the projection on the second factor. The restriction of $\pi_2 \circ \upsilon$ to $(X \times \{y\}) \cap U$ is a constant function because the genus of X is less than the one of Z. We denote by $\gamma(y)$ this constant; this defines a morphism $\gamma : Y \to Z$ which cannot be constant since υ is surjective. Let $\pi_1 : X \times Z \to X$ be the projection on the first factor. The restriction of $\pi_1 \circ \upsilon$ to $(X \times \{y\}) \cap U$ is a morphism β_y with values in X. Let $F \subset Y$ the set of closed k^a-points of Y such that the morphism β_y is constant. This is a closed set; and a finite one because υ is surjective. For a closed k^a-point $y \notin F$ the morphism β_y induces an automorphism of X, which is trivial since X does not have any non-trivial automorphism. Thus we have $\upsilon(x, y) = (x, \gamma(y))$ for every closed k^a-point x on X and y on Y with $y \notin F$ and $(x, y) \in U$. Let x be a closed k^a-point of X. The restriction of $\pi_1 \circ \upsilon$ to $(\{x\} \times Y) \cap U$ is constant and equal to x on the non-empty open set $(\{x\} \times (Y - F)) \cap U$. So it is a constant function. So F is empty and υ is the restriction of $\mathrm{Id} \times \gamma$ to U. Thus $\mathrm{Id} \times \gamma$ maps U to V and therefore $\psi \circ \gamma = \varphi$. Moreover γ must be defined over l since υ, U, V are defined over l. $\square$

Lemma 4.3 *Let X_k be a smooth, projective, geometrically integral k-curve. Set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ and assume that the genus of X is greater than 2 and that X has no non-trivial k^a-automorphism. Let U be the mark of a non-constant k^a-cover $\varphi : Y \to X$, where Y is a smooth, projective, integral k^a-curve. Then:*

1. *k is the field of moduli of U (in the category of quasi-projective varieties) if and only if it is the field of moduli of the cover $\varphi : Y \to X$;*
2. *an algebraic extension of k is a field of definition of U if and only if it is the field of definition of the cover $\varphi : Y \to X$.*

Proof — It is clear that the construction of the mark from the cover commutes with base change. This yields a morphism of stacks $\mathbb{F} : \mathbb{M}_\varphi \to \mathbb{M}_U$ which is fully faithful according to lemma 4.2. The result follows by proposition 2.5. In particular, $\mathbb{F}$ has an inverse functor $\mathbb{G} : \mathbb{M}_U \to \mathbb{M}_\varphi$. $\square$

5 Proper normal surfaces

In this section k is a field of characteristic zero. We start from a cover of curves, having k as field of moduli, and we construct a proper normal integral surface over k^a, having the same field of moduli and the same fields of definition as the original cover.

Theorem 5.1 *Let X_k be a smooth, projective, geometrically integral curve over k and set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$. Let Y be a smooth projective, integral curve over k^a and let $\varphi : Y \to X$ be a cover. Assume that k is the field of moduli of φ. Then, there exists a proper, normal and integral surface S over k^a, having k as field of moduli, and having the same fields of definition as φ.*

The proof of this theorem is given in the rest of this section. The surface is constructed as the cover of a surface $X \times Z$, strongly ramified along the graph of ψ, where $\psi : Z \to X$ is a suitably chosen cover of curves deduced from φ.

5.1 Construction of the surface S

The construction of the surface is divided in several steps.

Step 0. Starting point.

We keep notation and assumptions of theorem 5.1. We denote by $g(X)$ the genus of X and by d the degree of the cover φ. According to proposition 3.3, we may assume that $g(X)$ is at least 2 and that X has no non-trivial automorphism over k^a.

Step 1. A system of generators $f_1, \ldots, f_I$ of the function field $k(X_k)$.

We need to exhibit some k-**rational** functions on X.

Lemma 5.2 (The functions f_i on X and the primes p_i) *There exist $I \in \mathbf{N}^*$, some prime integers $p_1, \ldots, p_I > d$, and functions $f_1, \ldots, f_I \in k(X_k)$ satisfying the following conditions:*

1. *the functions $(f_i)_{1 \leq i \leq I}$ generate the field $k(X_k)$ over k;*
2. *for every $1 \leq i \leq I$ and every $\lambda \in k^a$, none of the functions $f_i - \lambda$ is a p_i-th power in $k^a(X)$;*
3. *let $\Pi = \prod_{i=1}^{I} p_i$ and let M (resp. m) be the maximum (resp. minimum) among the degrees of the f_i, then:*

$$\forall 1 \leq i \leq I, \qquad 1 + 2(g(X) - 1)\Pi + \Pi^2 < m \leq \deg(f_i) \leq M.$$

Proof — We first choose a finite generating system $(h_j)_{1 \leq j \leq J}$ of $k(X_k)$ over k. We assume that none of the h_j is a power in $k^a(X)$. We set $I = 2J$ and let $\Pi = \prod_{i=1}^{I} p_i$ be the product of the first I prime integers greater than the degree d of φ. We choose two distinct prime integers a and b, both bigger than $1 + 2(g(X) - 1)\Pi + \Pi^2$. For every $1 \leq j \leq J$, we set:

$$f_j = h_j^a, \qquad \text{and} \qquad f_{j+J} = h_j^b.$$

We can choose a and b in such a way that none of the functions $f_i - \lambda$ is a p_i-th power in $k^a(X)$ for $\lambda \in k^a$ and $1 \leq i \leq I$: this is evident for $\lambda = 0$. If $\lambda \neq 0$ and if $h_i^a - \lambda = \prod_{0 \leq k \leq a-1}(h_i - \zeta_a^k \lambda^{\frac{1}{a}})$ is a power, then h_i has at least a distinct singular values. This is impossible if we choose an a bigger than the number of singular values of h_i.

We note also that the $(f_i)_{1 \leq i \leq I}$ generate $k(X_k)$ over k and that they all have a degree greater than $1 + 2(g(X) - 1)\Pi + \Pi^2$, as expected. □

Step 2. A cover $\psi : Z \to X$ of large enough degree.

Let p be a prime integer bigger than $(g(X) + IM)\Pi$. We call Z the curve and $\psi : Z \to X$ the degree pd cover given by proposition 3.1. The genus of Z is bigger than $dp > (g(X) + IM)\Pi$ and the covers φ and ψ have the same field of moduli and the same fields of definition.

Step 3. A system of functions $g_1, \ldots, g_I$ on $X \times Z$.

Using the previous functions f_i, we define functions on $X \times Z$.

Lemma 5.3 (The functions g_i on $X \times Z$) *For every $1 \leq i \leq I$, let g_i be the function on $X \times Z$ defined by:*

$$g_i(P, Q) = f_i(\psi(Q)) - f_i(P).$$

Then:

1. *the negative part $(g_i)_\infty$ of the divisor of g_i is $(f_i)_\infty \times Z + X \times (f_i \circ \psi)_\infty$;*
2. *the positive parts $(g_i)_0$ are such that $\mathrm{gcd}_i((g_i)_0) = G(\psi)$, where $G(\psi)$ is the graph of ψ;*
3. *for every point $P \in X$ the function $Q \mapsto g_i(P, Q)$ on $P \times Z$ is not a p_i-th power.*

Proof — The first two points are easy. To prove the third one, recall that each function f_i is such that none of the $f_i - \lambda$ for $\lambda \in k^a$ is a p_i-th power (lemma 5.2). Since the degree pd of ψ is prime to p_i, none of the function $f_i \circ \psi - \lambda$ is a p_i-th power in $k^a(Z)$. Condition 3 follows. $\square$

Let us note that, if ψ is defined over a field l, then so are the functions g_i.

Step 4. At last, the surface S.

Let $k^a(X \times Z)$ be the field of functions of the surface $X \times Z$. We define a regular radicial extension of $k^a(X \times Z)$ by setting

$$y_i^{p_i} = g_i.$$

We denote by S the normalization of $X \times Z$ in the latter radicial extension. It is a normal surface by construction and there is a ramified cover:

$$\chi : S \to X \times Z$$

which is a Galois cover of surfaces over k^a with Galois group $\prod_{i=1}^{I} \mathbf{Z}/p_i\mathbf{Z}$.

5.2 The group of automorphisms of S

We denote by A the group of k^a-automorphisms of ψ. An element in A induces a k^a-automorphism of the surface $X \times Z$, and this latter automorphism can be lifted uniquely to an automorphism of $k^a(S)/k^a$ that fixes all y_i and stabilizes $k^a(X \times Z)$. In the sequel we shall use the same notation for an automorphism of ψ, the induced automorphism of $X \times Z$ and its lift to S. In other words, A can be identified with a subgroup of $\mathrm{Aut}_{k^a}(S)$, the group of k^a-automorphisms of S.

We know another subgroup of $\mathrm{Aut}_{k^a}(S)$, namely the Galois group $B = \prod_{i=1}^{I} \mathbf{Z}/p_i\mathbf{Z}$ of the extension $k^a(S)/k^a(X \times Z)$.

To summarize, A is the set of α such that the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\chi"'] & S \arrow[d, "\chi"] \\ X \times Z \arrow[r, "\alpha"] & X \times Z \end{tikzcd} \tag{5}$$

while B is the set of β such that the following diagram commute:

$$\begin{tikzcd} S \arrow[rr, "\beta"] \arrow[dr, "\chi"'] & & S \arrow[dl, "\chi"] \\ & X \times Z & \end{tikzcd} \tag{6}$$

It is clear that $A \times B \subset \mathrm{Aut}_{k^a}(S)$. We now prove that this inclusion is an equality. To this end, we introduce a family of curves on S.

Lemma 5.4 (The curves E_Q) *For any point Q on Z, we call E_Q the inverse image of $X \times Q$ by χ and we denote by $\chi_Q : E_Q \to X \times Q$ the restriction of χ to E_Q. The geometric genus of E_Q can be bounded from above:*

$$g(E_Q) \leq (g(X) + IM)\Pi < g(Z), \tag{7}$$

and the genus of any non-trivial subcover of χ_Q can be bounded from below:

$$1 + 2(g(X) - 1)\Pi + \Pi^2 < m \leq g(\textit{non-trivial subcover of } \chi_Q : E_Q \to X). \tag{8}$$

Proof — If Q is the generic point on Z, then E_Q is a geometrically integral curve and χ_Q is a degree Π, geometrically connected cover. The degree of the ramification divisor of this cover is bounded from above by the product $2IM$ (where I is the number of functions in the family $(f_i)_i$ and M is the maximum of the degrees of these functions). The upper bound follows.

For the lower bound, let us consider a non-trivial subcover of χ_Q. Such a cover has degree at least $p_1 \geq 3$ and its ramification divisor has degree at least m (where m is the minimum among the degrees of the functions f_i). So its genus is greater than m and the lower bound follows. □

Lemma 5.5 *The group* $\mathrm{Aut}(S)$ *of k^a-automorphisms of S is $A \times B$.*

Proof — Let θ be a k^a-automorphism of S.

First of all, let Q be the generic point of Z. We know from inequality (7) of lemma 5.4 that $g(E_Q) < g(Z)$. We deduce that $\theta(E_Q) = E_{\alpha(Q)}$ where α is a k^a-automorphism of Z.

We now prove that the isomorphism between E_Q and $E_{\alpha(Q)}$ induced by θ makes the following diagram commute:

$$\begin{tikzcd} E_Q \arrow[r, "\theta"] \arrow[d, "\chi_Q"'] & E_{\alpha(Q)} \arrow[d, "\chi_{\alpha(Q)}"] \\ X \times Q \arrow[r, "\mathrm{Id}\times\alpha"] & X \times \alpha(Q) \end{tikzcd} \tag{9}$$

Indeed, the cartesian product of the maps χ_Q and $\chi_{\alpha(Q)} \circ \theta$ defines a morphism:

$$E_Q \xrightarrow{\chi_Q \times \left(\chi_{\alpha(Q)} \circ \theta\right)} X \times X,$$

whose image W is a divisor with bidegree $\leq (\Pi, \Pi)$. Using lemma 7.1 we deduce that the arithmetic genus of W is smaller than or equal to $1 + 2(g(X) - 1)\Pi + \Pi^2$. Let $\pi_1 : X \times X \to X$ be the projection on the first factor. The morphism χ_Q factors as:

$$\chi_Q : E_Q \longrightarrow W \xrightarrow{\pi_1} X.$$

The map $W \xrightarrow{\pi_1} X$ is a birational isomorphism. Otherwise, it would define a non-trivial subcover of $\chi_Q : E_Q \to X$. But we know from inequality (8) of lemma 5.4 that such a subcover has geometric genus greater than or equal to $m > 1+2(g(X)-1)\Pi+\Pi^2$. A contradiction. We deduce that W is a correspondence of bidegree $(1,1)$. Since X has no non-trivial k^a-automorphism we deduce that diagram (9) commutes.

We now prove that $\alpha \in A$. We just showed that θ induces an isomorphism between the covers $\chi_Q : E_Q \to X$ and $\chi_{\alpha(Q)} : E_{\alpha(Q)} \to X$. Therefore these two covers have the same ramification data: for every $1 \leq i \leq I$ the points P such that $f_i(P) = f_i(\psi(Q))$ and those such that $f_i(P) = f_i(\psi(\alpha(Q)))$ are the same. Thus:

$$\forall i, \quad f_i(\psi(Q)) = f_i(\psi(\alpha(Q)))$$

therefore $\psi(Q) = \psi(\alpha(Q))$, because the f_i generate $k^a(X)$ over k^a (lemma 5.2). So $\psi = \psi \circ \alpha$, and $\alpha \in A$.

Diagram 9 implies that the map $\chi_{\alpha(Q)} \circ \theta : E_Q \to E_{\alpha(Q)}$ is equal to $(\mathrm{Id} \times \alpha) \circ \chi_Q$. And this is $\chi_{\alpha(Q)} \circ \alpha$ according to diagram (5). We set $\beta = \theta \circ \alpha^{-1}$ and we check that $\chi_{\alpha(Q)} \circ \beta = \chi_{\alpha(Q)}$. Since Q is generic and α surjective we deduce that $\chi \circ \beta = \chi$ so $\beta \in B$. We conclude that $\theta = \beta\alpha \in A \times B$ as was to be shown. $\square$

Remark – We have proven something slightly stronger than lemma 5.5: the group of birational k^a-automorphisms of S is $A \times B$. We shall not need this stronger result.

5.3 Field of moduli and fields of definition of S

To prove theorem 5.1, we have to show that the cover φ and the surface S share the same field of moduli and the same fields of definition. In fact, by construction, one can replace the cover φ by the cover ψ, since those two covers have same field of moduli and fields of definition.

The construction of section 5.1 yields a morphism of stacks $\mathbb{F} : \mathbb{M}_\psi \to \mathbb{M}_S$. To see this, let us consider l/k extension inside k^a and let $\psi_l : Z_l \to X_l$ be an l-model of ψ. We just follow the line of the construction, replacing ψ by ψ_l. Since the functions f_i are k-rational, the functions g_i lie in $l(X_l \times Z_l)$. Then the radical extension defined by the equations $y_i^{p_i} = g_i$ is a regular extension of $l(X_l \times Z_l)$. The normalization of $X_l \times Z_l$ in this extension is a surface S_l which is defined over l. Of course, this surface S_l is an l-model of S and the morphism $\mathbb{F}$ is defined on objects by $\mathbb{F}(\psi_l) = S_l$. Because functions f_i are k-rational, $\mathbb{F}$ is a morphism of stacks. By proposition 2.5, k is the field of moduli of S and every field of definition of ψ (or φ) is a field of definition of S.

Unfortunately, $\mathbb{F}$ is not fully faithful. As in proposition 3.1, we use proposition 2.7 to construct a morphism the other way around. The group $\mathrm{Aut}_{k^a}(S)$ is a normal subgroup of $\mathrm{Aut}_k(S)$. Conjugation by an element of $\mathrm{Aut}_k(S)$ induces an automorphism of $\mathrm{Aut}_{k^a}(S)$. This automorphism must stabilize the unique sub-group of order Π of $\mathrm{Aut}_{k^a}(S)$, which is nothing but $B = \mathrm{Aut}_{k^a}(\chi)$. Let U be the mark of the cover ψ. This is the complementary set of the branch locus of the quotient map $\chi : S \to X \times Z$. According to proposition 2.7, taking the complementary set of the branch locus of a quotient map defines a morphism of stacks $\mathbb{G} : \mathbb{M}_S \to \mathbb{M}_U$. Therefore, every field of definition of S is a field of definition of the mark of ψ and then also a field of definition of ψ by lemma 4.3. Indeed the proof of this lemma provides a morphism from $\mathbb{M}_U$ to $\mathbb{M}_\psi$ and the proof of proposition 3.1 provides a morphism from $\mathbb{M}_\psi$ to $\mathbb{M}_\varphi$.

6 Curves

In this section k is still a field of characteristic zero. We start from a cover of curves, having k as field of moduli, and we construct, a projective normal integral curve over k^a, having the same field of moduli and the same fields of definition as the original cover. This will prove theorem 1.2.

We shall make use of the surface S constructed in section 5. So we keep the notation of section 5. We know that S has field of moduli k and the same fields of definition as the initial cover $\varphi : Y \to X$ (or equivalently $\psi : Z \to X$).

The main idea is to draw on S a singular (but stable) curve inheriting the field of moduli and fields of definition of S; then to deform it to obtain a smooth projective curve.

6.1 Two stable curves

In section 5.1, we have constructed a cover $\chi : S \to X \times Z$ strongly ramified along the graph of $\psi : Z \to X$. For any point P on X, we call F_P the inverse image of $P \times Z$ by χ and $\chi_P : F_P \to P \times Z$ the correstriction of χ to $P \times Z$. We call Γ the union of the supports of all divisors of the functions g_i of lemma 5.3. It contains the ramification locus of the cover χ.

Lemma 6.1 *There exist two non-constant k-rational functions $f, g \in k^a(X)$ such that:*

1. *the divisor $((f)_0 + (f)_\infty) \times Z$ crosses transversally Γ;*
2. *the divisor $X \times ((g \circ \psi)_0 + (g \circ \psi)_\infty)$ crosses transversally $\Gamma \cup [((f)_0 + (f)_\infty) \times Z]$;*
3. *any k^a-automorphism of Z that stabilizes the fiber $(g \circ \psi)_0$ is an automorphism of the cover ψ (note that the preceding condition implies that this fiber is simple);*
4. *for any zero P of f, the cover $g \circ \psi \circ \chi_P : F_P \to \mathbf{P}^1$ has no automorphism other than the elements of $A \times B$:*

$$\mathrm{Aut}_{k^a}(g \circ \psi \circ \chi_P) = \mathrm{Aut}_{k^a}(\psi \circ \chi_P) = A \times B.$$

Proof — Let $f \in k^a(X)$ be a k-rational non-constant function. We apply lemma 7.2 to k, X, Z, Γ and f. We deduce that there exist two distinct scalars x and y in k^a such that $(f)_x \times Z$ and $(f)_y \times Z$ cross transversally Γ. We even can choose x and y in k and such that for every point P in $f^{-1}(x)$ or $f^{-1}(y)$, the fiber of every function $f_i \in k(X)$ above $f_i(P)$ does no meet the singular values of ψ, that is:

$$\forall P \in f^{-1}(x) \cup f^{-1}(y), \qquad f_i^{-1}(f_i(P)) \cap \{\text{singular values of } \psi\} = \emptyset. \tag{10}$$

We replace f by $(f-x)/(f-y)$ and the first condition is fulfilled.

Now, for every zero P of f, we see that F_P is smooth and geometrically integral, because $(f)_0 \times Z$ crosses transversally the ramification locus Γ of χ. We now prove that

$$\mathrm{Aut}_{k^a}(\psi \circ \chi_P) = A \times B.$$

Indeed the function field $k^a(F_P)$ is the compositum:

$$\begin{tikzcd}
 & k^a(F_P) \arrow[dl, dash] \arrow[dd, dash] \arrow[dr, dash] & \\
k^a(Z) \arrow[dr, dash, "pd"'] & & k^a(X') \overset{\text{def.}}{=} k^a\left((f_i - f_i(P))^{\frac{1}{p_i}}, 1 \le i \le I\right) \\
 & k^a(X) \arrow[ur, dash, "\prod_i p_i"'] &
\end{tikzcd}$$

where $X' \to X$ is an abelian cover with Galois group $B = \prod_{i=1}^{I} \mathbf{Z}/p_i\mathbf{Z}$. The $k^a(X)$-extensions $k^a(Z)$ and $k^a(X')$ are linearly disjoint (their degrees are coprime and one of them is Galois) and condition (10) implies that the extension $k^a(Z)/k^a(X)$ is not ramified above the zeros of the functions $f_i - f_i(P)$.

Now, any subcover of $X' \to X$ is ramified above the zeros of at least one of the functions $f_i - f_i(P)$. The same is true for any subcover of $F_P \to Z$. We deduce that $Z \to X$ is the maximal subcover of $F_P \to X$ that is not ramified above the zeros of the functions $f_i - f_i(P)$. Therefore any $k^a(X)$-automorphism of $k^a(F_P)$ stabilizes $k^a(Z)$. Thus:

$$\mathrm{Aut}_{k^a(X)}(k^a(F_P)) = \mathrm{Aut}_{k^a(X)}(k^a(Z)) \times \mathrm{Aut}_{k^a(X)}(k^a(X')),$$

as was to be shown.

Next we look for a function g in $k(X)$ such that $g \circ \psi$ has no k^a-automorphism but elements of A and, for every zero P of f, the cover $g \circ \psi \circ \chi_P$ has no k^a-automorphism but elements of $\mathrm{Aut}_{k^a}(\psi \circ \chi_P) = A \times B$. According to lemma 7.4, the functions in $k(X)$ that do not fulfill all these conditions lie in a finite union of strict sub-k-algebras. Therefore there exists such a function g.

According to lemma 7.2, the scalars x in k such that $(g \circ \psi)_x$ does not cross $\Gamma \cup [((f)_0 + (f)_\infty) \times Z]$ transversally, are finitely many.

According to lemma 7.3, the x in k such that $(g \circ \psi)_x$ has a k^a-automorphism not in A, are finitely many.

Therefore there exist two distinct scalars x and y in k such that $(g \circ \psi)_x$ and $(g \circ \psi)_y$ cross $\Gamma \cup [((f)_0 + (f)_\infty) \times Z]$ transversally and $(g \circ \psi)_x$ has no automorphism but those in A. We replace g by $(g - x)/(g - y)$ and the last three conditions are satisfied. $\square$

The curves C_0 and D_0.

Let C_0 be the curve on $X \times Z$ with equation:

$$f(P) \times g \circ \psi(Q) = 0.$$

Let D_0 be the inverse image of C_0 by χ. These are singular curves over k^a. The two following lemmas are concerned with the stability and the automorphism groups of these two curves.

Lemma 6.2 *The curve C_0 is stable and $\mathrm{Aut}_{k^a}(C_0) \simeq A$.*

Proof — The curve C_0 is geometrically reduced because the zeros of f and $g \circ \psi$ are simple. The singular points on C_0 are the couples (P, Q) on $X \times Z$ such that $f(P) = g \circ \psi(Q) = 0$. These are ordinary double points. Therefore C_0 is semistable. It is geometrically connected also. Its irreducible components are isomorphic to X or Z. So they all have genus ≥ 2. Therefore C_0 is a stable curve.

We now prove that $\mathrm{Aut}_{k^a}(C_0) \simeq A$, i.e. that the group of k^a-automorphisms of C_0 is the group A of k^a-automorphisms of ψ. It is clear that A is included in $\mathrm{Aut}_{k^a}(C_0)$. Conversely, let θ be a k^a-automorphism of C_0. Then θ permutes the irreducible components of C_0. Some of these components are isomorphic to X, and the others are isomorphic to Z. Since X and Z are not k^a-isomorphic, θ stabilizes the two subsets of components. If we restrict θ to a component isomorphic to X then compose with the projection on X, we obtain a non-constant k^a-morphism from X to itself. This morphism must be the identity because X has no non-trivial k^a-automorphism. Therefore θ stabilizes each component isomorphic to Z. The singular points on such a component are the zeros of $g \circ \psi$. The set of these zeroes is stabilized by no k^a-automorphism of Z but those of ψ by (3) of lemma 6.1. So the restriction of θ to any component isomorphic to Z is an element in A. If we compose θ with some well chosen element in A, we may assume that θ is trivial on one component isomorphic to Z. Therefore θ stabilizes every component isomorphic to X. Since these components have no non-trivial automorphism, θ acts trivially on them. Now let $P \times Z$ be a component of C_0 isomorphic to Z. The restriction of θ to it is an automorphism that fixes the singular points. These points are the zeros of $g \circ \psi$. So θ restricted to $P \times Z$ is an element of A. But A acts faithfully on the set of zeros of $g \circ \psi$. We deduce that θ acts trivially on every component isomorphic to Z. □

Controlling the full group of k^a-automorphisms of D_0 seems difficult to us. So we shall be interested in the subgroup consisting of *admissible* automorphisms. This subgroup is denoted $\mathrm{Aut}^{\text{adm.}}_{k^a}(D_0)$. We now explain what we mean by an admissible automorphism.

We first notice that the components of D_0 are of two different kinds. Some of them are covers of some $X \times Q$ where Q is a zero of $g \circ \psi$. We denote such a component by E_Q. The other components are covers of some $P \times Z$ where P is a k^a-zero of f. Such a component is denoted by F_P. We call $\chi_P : F_P \to P \times Z$ and $\chi_Q : E_Q \to X \times Q$ the restrictions of χ to components of D_0. Now let T be a singular point on D_0 such that $\chi(T) = (P, Q)$. So T lies in the intersection of E_Q and F_P. The point on E_Q corresponding to T is denoted U. The point on F_P corresponding to T is denoted V. So $\chi_Q(U) = P$ and $\chi_P(V) = Q$. And $f \circ \chi_Q$ is a uniformizing parameter for E_Q at U, while $g \circ \psi \circ \chi_P$ is a uniformizing parameter for F_P at V. Let θ be an automorphism of D_0 and let $T' = (U', V')$ be the image of $T = (U, V)$ by θ. We write $\chi(T') = (P', Q')$. We observe that $f \circ \chi_{Q'} \circ \theta$ is a uniformizing parameter for E_Q at U and $g \circ \psi \circ \chi_{P'} \circ \theta$ is a uniformizing parameter for F_P at V.

We say that θ is an *admissible automorphism* of D_0 if for every singular point T of D_0 we have

$$\frac{f \circ \chi_{Q'} \circ \theta}{f \circ \chi_Q}(U) \times \frac{g \circ \psi \circ \chi_{P'} \circ \theta}{g \circ \psi \circ \chi_P}(V) = 1 \tag{11}$$

where $\chi(T) = (P, Q)$ and $\chi(\theta(T)) = (P', Q')$.

The justification for this definition is given at paragraph 6.2. Admissible automorphisms form a subgroup of the group of k^a-automorphisms of D_0.

Lemma 6.3 *The curve D_0 is stable and* $\mathrm{Aut}_{k^a}^{adm.}(D_0) \simeq A \times B$.

Proof — First, it is clear that $A \times B$ acts faithfully on D_0, and the corresponding automorphisms are admissible.

The curve D_0 is drawn on S. Let us prove that D_0 is a stable curve. Points (1) and (2) of lemma 6.1 imply that the ramification locus Γ of χ does not contain any singular points of C_0. Therefore every singular point on C_0 gives rise to $\deg(\chi)$ singular points on D_0; and all these singular points are ordinary double points. To prove that D_0 is connected, we observe that the function g_i restricted to any irreducible component of C_0 is not a p_i-th power because none of the functions $f_i - \lambda$, $\lambda \in k^a$ is a p_i-th power (and the $f_i \circ \psi - \lambda$ are not either) as shown in lemma 5.2. Also the irreducible components of D_0 correspond bijectively to those of C_0.

Now let us prove that $\mathrm{Aut}_{k^a}^{\mathrm{adm.}}(D_0) \simeq A \times B$. The components F_P and E_Q have different genera. Therefore no F_P is k^a-isomorphic to some E_Q. Thus any k^a-automorphism θ of D_0 stabilizes the set of all components F_P (and also the set of all E_Q).

Let Q and Q' be two k^a-zeros of $g \circ \psi$ such that $\theta(E_Q) = E_{Q'}$. As in the proof of lemma 5.5, we notice that the image of E_Q in the product $X \times X$, by the morphism $\chi_Q \times \chi_{Q'} \circ \theta$, has an arithmetic genus smaller than or equal to $1 + 2(g(X) - 1)\Pi + \Pi^2$. Again, this implies that this image is k^a-isomorphic to X (otherwise, this image would have geometric genus bigger than $1+2(g(X)-1)\Pi+\Pi^2$ by Hurwitz formula). Since X has no k^a-automorphism, we deduce as before that θ induces an isomorphism of covers between the restrictions $\chi_Q : E_Q \to X$ and $\chi_{Q'} : E_{Q'} \to X$ of χ. Thus

$$\chi_Q = \chi_{Q'} \circ \theta. \tag{12}$$

This implies that θ stabilizes every component F_P where P is any k^a-zero of f. Indeed, let us start from a singular point $T = (U, V) \in E_Q \cap F_P$, where P is a k^a-zero of f and Q is a k^a-zero of $g \circ \psi$. We thus have $\chi(T) = (P, Q) \in X \times Z$. We know there exists $P' \in X(k^a)$ and $Q' \in Z(k^a)$ such that $\theta(T) \in F_{P'} \cap E_{Q'}$. We deduce from Equation (12) that:

$$P' = \chi_{Q'} \circ \theta(T) = \chi_Q(T) = P.$$

We conclude that $P = P'$ and $\theta(F_P) = F_P$.

Now, we deduce from formulae (11) and (12) that:

$$\frac{g \circ \psi \circ \chi_P \circ \theta}{g \circ \psi \circ \chi_P}(V) = 1. \tag{13}$$

Call θ_P the restriction of θ to F_P. This is an automorphism of F_P. We prove that θ_P is the restriction to F_P of an element of $A \times B$. To this end, we introduce the function $h_P = g \circ \psi \circ \chi_P \in k^a(F_P)$. The degree of h_P is $\deg(g) \times pd \times \Pi$ and its zeros are all simple. These zeros are the intersection points between F_P and the other components of D_0. Since θ_P permutes these zeros, the functions $h_P \circ \theta_P$ and h_P have the same divisor of zeros. Therefore the only possible poles of the function $\frac{h_P}{h_P \circ \theta_P} - 1$ are the poles of h_P. Thus the degree of $\frac{h_P}{h_P \circ \theta_P} - 1$ is smaller than or equal to the degree of h_P . But according to (13), the zeros of h_P are also zeros of $\frac{h_P}{h_P\circ\theta_P}-1$. So we just proved that if the function $\frac{h_P}{h_P\circ\theta_P}-1$ is non-zero, it has the same divisor as h_P. Therefore there exists a constant $c\in k^a$ such that:

$$\frac{h_P}{h_P\circ\theta_P}-1=ch_P \qquad \text{or equivalently:} \qquad \frac{1}{h_P\circ\theta_P}=\frac{1}{h_P}+c.$$

Since θ_P has finite order e and k^a has characteristic zero, we deduce that $ce=0$, then $c=0$, then $h_P\circ\theta_P=h_P$. In other words, θ_P is an automorphism of the cover $h_P=g\circ\psi\circ\chi_P:F_P\to\mathbf{P}^1$. According to point (4) of lemma 6.1, we deduce that θ_P is the restriction to F_P of an element in $A\times B$. We replace θ by θ composed with the inverse of this element. So we now assume that θ acts trivially on F_P for some P. In particular θ fixes every singular point on F_P. So θ stabilizes every component E_Q. The restriction θ_Q of θ to E_Q is an automorphism of χ_Q according to (12). Further θ_Q fixes one point (and even every point) in the unramified fiber above P of the Galois cover $\chi_Q:E_Q\to X$. Therefore θ_Q is the identity. We have proved that θ is trivial on every component E_Q.

To finish with, we now prove that θ is also trivial on the components $F_{P'}$ for every zeros P' of f. Remind we have already assumed this to be true for one of these zeros. We call $\theta_{P'}$ the restriction of θ to $F_{P'}$. We already proved that $\theta_{P'}$ is the restriction of an element in $A\times B$. Further $\theta_{P'}$ fixes all the singular points of D_0 lying on $F_{P'}$. These points are the zeros of $g\circ\psi\circ\chi_{P'}$. So we just need to prove that the action of $A\times B$ on the set of zeros of $g\circ\psi\circ\chi_{P'}$ is free. This is certainly the case for elements in B because the zeros of $g\circ\psi$ are, by hypothesis, unramified in the Galois cover $\chi_{P'}:F_{P'}\to Z$. This is true also for elements in $A\times B$ because the action of A on the set of zeros of $g\circ\psi$ is free. $\square$

6.2 Deformations

In this paragraph we deform the two stable curves C_0 and D_0. If $t\in k^a$ is a scalar, it is natural to consider the curve C_t drawn on the surface $W=X\times Z$ and defined by the equation $f(P)\times g(\psi(Q))=t$. We call D_t the inverse image of C_t by χ. In this paragraph and in the next one, we shall prove that for almost all scalars t in k, the curve D_t is smooth, geometrically integral, with k^a-automorphism group equal to $A\times B$, and with the same field of moduli and the same fields of definition as the original cover φ. To this end, we would like to consider the families $(C_t)_t$ and $(D_t)_t$ as fibrations above $\mathbf{P}^1$. We should be careful however : the family $(C_t)_t$ has base points. So we first have to blow up $W=X\times Z$ along

$$\Delta=((f)_\infty\times(g\circ\psi)_0)\cup((f)_0\times(g\circ\psi)_\infty).$$

Note that Δ is the union of $2\times\deg(f)\times\deg(g\circ\psi)$ simple geometric points. We denote by $W_{\infty,\infty}\subset W=X\times Z$ the complementary open set of $((f)_\infty\times Z)\cup(X\times(g\circ\psi)_\infty)$ in $X\times Z$. We similarly define $W_{0,0}$, $W_{0,\infty}$, $W_{\infty,0}$. These four open sets cover $X\times Z$.

Let $\mathbf{P}^1=\mathrm{Proj}(k^a[T_0,T_1])$ be the projective line over k^a. We set $F=1/f$ and $G=1/g$. Let $C_{\infty,0}\subset W_{\infty,0}\times\mathbf{P}^1$ be the set of $(P,Q,[T_0:T_1])$ such that $f(P)T_0=G(\psi(Q))T_1$. Let $C_{0,\infty}\subset W_{0,\infty}\times\mathbf{P}^1$ be the set of $(P,Q,[T_0:T_1])$ such that $g(\psi(Q))T_0=F(P)T_1$. Let $C_{\infty,\infty}\subset$

$W_{\infty,\infty} \times \mathbf{P}^1$ be the set of $(P, Q, [T_0 : T_1])$ such that $f(P)g(\psi(Q))T_0 = T_1$. Let $C_{0,0} \subset W_{0,0} \times \mathbf{P}^1$ be the set of $(P, Q, [T_0 : T_1])$ such that $T_0 = F(P)G(\psi(Q))T_1$. We glue together these four algebraic varieties and obtain a variety $C \subset W \times \mathbf{P}^1$. Let $\pi_W : C \to W$ be the projection on the first factor and let $\pi_C : C \to \mathbf{P}^1$ be the projection on $\mathbf{P}^1$. This is a flat, projective, surjective morphism.

Let $D \subset S \times \mathbf{P}^1$ be the inverse image of C by $\chi \times \mathrm{Id}$ where $\mathrm{Id} : \mathbf{P}^1 \to \mathbf{P}^1$ is the identity. This is the blow up of S along $\chi^{-1}(\Delta)$. Note that $\chi^{-1}(\Delta)$ is the union of $\deg(\chi) \times \deg(f) \times \deg(g \circ \psi)$ simple geometrical points because χ is unramified above Δ. Actually, D is the normalization of C in $k^a(S \times \mathbf{P}^1)$. We denote by $\chi : D \to C$ the corresponding morphism. We call $\pi_S : D \to S$ the projection on the first factor. We call $\pi_D : D \to \mathbf{P}^1$ the projection on the second factor. This is the composed morphism $\pi_D = \pi_C \circ \chi$. This is a flat, proper and surjective morphism.

Let $\mathbf{A}^1 \subset \mathbf{P}^1$ be the spectrum of $k^a[T]$ where $T = \frac{T_1}{T_0}$. Using the function T we identify $\mathbf{P}^1(k^a)$ and $k^a \cup \{\infty\}$. If t is a point on $\mathbf{P}^1(k^a)$ we denote by C_t the fiber of π_C above t and D_t the fiber of π_D above t. The restriction of π_W to C_t is a closed immersion. So we can see C_t as a curve drawn on $W = X \times Z$. Similarly, the restriction of π_S to D_t is a closed immersion. So we can see D_t as a curve drawn on S. In particular, the fiber of π_C at 0 is isomorphic by π_W to the stable curve C_0 introduced in paragraph 6.1. Similarly, the fiber of π_D at 0 is isomorphic by π_S to the stable curve D_0 introduced in paragraph 6.1.

We call C_η the generic fiber of π_C and D_η the generic fiber of π_D.

We now show that the curve C_η over $k^a(\mathbf{P}^1)$ is geometrically connected and that for almost every $t \in \mathbf{P}^1(k^a)$ the curve C_t over k^a is connected. According to Stein's factorization theorem [Liu02, Chapter 5, Exercise 3.11], we can factor $\pi_C : C \to \mathbf{P}^1$ as $\pi_f \circ \pi_c$ where π_c has geometrically connected fibers and π_f is finite and dominant. The fiber of π_f above 0 is trivial because C_0 is connected and reduced. Therefore the degree of π_f is 1 according to [Liu02, Chapter 5, Exercise 1.25]. Therefore π_f is an isomorphism above a non-empty open set of $\mathbf{P}^1$. The generic fiber C_η is geometrically connected over $k^a(\mathbf{P}^1)$ and for almost all $t \in \mathbf{P}^1(k^a)$ the curve C_t over k^a is connected.

We now show that C_η is smooth (and therefore geometrically integral). Indeed, it is smooth outside the points $(P, Q) \in C_\eta \subset X \times Z$ where $df(P) = 0$ and $d(g \circ \psi)(Q) = 0$. Such points are defined over k^a. Therefore the function $f(P) \times g(\psi(Q))$ cannot take the transcendental value T at these points.

The ramification locus $\Gamma \subset W$ of χ cuts the fiber C_0 transversally. Therefore it cuts the generic fiber C_η transversally. So D_η is smooth and geometrically integral. Thus for almost every $t \in k^a$ the fibers C_t and D_t are smooth and integral.

Finally, our knowledge of $\mathrm{Aut}^{\mathrm{adm.}}_{k^a}(D_0)$ enables us to show that $\mathrm{Aut}_{k(\mathbf{P}^1)^s}(D_\eta) \simeq A \times B$. Indeed, set $R = k^a[[T]]$ the completed local ring at the point $T = 0$ of $\mathbf{P}^1$. The curve $\hat{D} = D \times_{\mathbf{P}^1} \mathrm{Spec}(R)$ is stable over the spectrum of R. According to [Liu02, Chapter 10, Proposition 3.38, Remark 3.39] the functor "automorphism group" $t \mapsto \mathrm{Aut}_t(\hat{D}_t)$ is representable by a finite unramified scheme over $\mathrm{Spec}\, R$ and the specialization morphism $\mathrm{Aut}_{k^a((T))}(\hat{D}_\eta) \to \mathrm{Aut}_{k^a}(D_0)$ is injective. According to lemma 7.6, the image of this morphism is included in the subgroup of admissible k^a-automorphisms of D_0. Since $\mathrm{Spec}\, R$ has no unramified cover, we deduce the following estimate for the automorphism group of the generic fiber

$$A \times B \subset \mathrm{Aut}_{k^a(\mathbf{P}^1)^s}(D_\eta) \subset \mathrm{Aut}_{k^a((T))}(\hat{D}_\eta) \subset \mathrm{Aut}_{k^a}^{\text{adm.}}(D_0).$$

We have already seen that the rightmost group is equal to $A \times B$. So $\mathrm{Aut}_{k^a(\mathbf{P}^1)^s}(D_\eta) = A \times B$ as was to be proved.

6.3 Fields of moduli and fields of definition of fibers

As we have seen in paragraph 6.2, for almost all $t \in \mathbf{A}^1(k)$, the fiber D_t is smooth and geometrically integral. Using lemma 7.7 on the specialization of the automorphism group in a family of curves, we deduce that for almost all $t \in \mathbf{A}^1(k)$, the group of k^a-automorphisms of the fiber D_t is isomorphic to the group of $k(\mathbf{A}^1)^s$-automorphisms of the generic fiber. Since the latter group is isomorphic to the automorphism group $A \times B$ of the surface S, we deduce that, for almost all t, the restriction map above is an isomorphism:

$$\mathrm{Aut}_{k^a}(S) \xrightarrow{\simeq} \mathrm{Aut}_{k^a}(D_t). \tag{14}$$

Now let $t \in k$ be such that D_t is smooth and geometrically integral and such that $\mathrm{Aut}_{k^a}(D_t) = A \times B$. We call $\pi_t : D_t \to S$ the corresponding embedding.

We construct a functor $\mathbb{F}_t : \mathbb{M}_S \to \mathbb{M}_{\pi_t}$. We first define the image of an object by $\mathbb{F}_t$. Let l be a finite extension of k inside k^a and S_l an l-model of S. Using the functor $\mathbb{M}_S \to \mathbb{M}_U$ given in section 5.3 followed by the functor $\mathbb{M}_U \to \mathbb{M}_\psi$ of the proof of lemma 4.3, we obtain an l model $\psi_l : Z_l \to X_l$ of the cover ψ, where $X_l = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$ and Z_l is a l-model of Z. There exists also an abelian cover $\chi_l : S_l \to X_l \times Z_l$. It is uniquely defined up to an automorphism of S_l. We denote by $C_{t,l}$ the curve on $X_l \times Z_l$ defined by the equation $f \cdot g \circ \psi_l - t = 0$. Let $D_{t,l}$ be the inverse image of $C_{t,l}$ by χ_l. Let $\pi_{t,l} : D_{t,l} \hookrightarrow S_l$ be the inclusion map. The image of the object S_l by the functor $\mathbb{F}_t$ is defined to be $\pi_{t,l}$. We still need to define the image of a morphism by the functor $\mathbb{F}_t$. Let l' be another finite extension of k and let $\sigma : l \to l'$ be a k-homomorphism. Let $S'_{l'}$ be an l'-model of S and let $\alpha : S_l \to S'_{l'}$ be a morphism above $\mathrm{Spec}(\sigma)$. We call $\pi'_{t,l'} : D'_{t,l'} \hookrightarrow S'_{l'}$ the image by $\mathbb{F}_t$ of $S'_{l'}$. Then α maps $D_{t,l}$ to $D'_{t,l'}$. We denote by β the restriction of α to $D_{t,l}$. The image of α by $\mathbb{F}_t$ is defined to be the morphism (α, β) from $\pi_{t,l}$ to $\pi'_{t,l'}$. If we compose $\mathbb{F}_t : \mathbb{M}_S \to \mathbb{M}\pi_t$ with the forgetful functor $\mathbb{M}_{\pi_t} \to \mathbb{M}_{D_t}$, we obtain a cartesian functor $\mathbb{G}_t : \mathbb{M}_S \to \mathbb{M}_{D_t}$. Further, identity (14) implies that the functor $\mathbb{G}_t$ is fully faithful. Therefore, by proposition 2.5, both S and D_t have k as field of moduli and a k-extension is a field of definition of S if and only if it is a field of definition of D_t. In view of section 5.3, D_t, ψ and φ also share the same fields of definition. Theorem 1.2 is proved.

7 Six lemmas about curves and surfaces

In this section we state and prove seven lemmas that are needed in the proof of theorem 1.2.

7.1 About curves and products of two curves

Lemma 7.1 *Let k be a algebraically closed field. Let X and Y be two projective, smooth and integral curves over k. Let β be the genus of X and let γ be the genus of Y. We fix a geometric point P on X and a geometric point Q on Y. We identify the curves X and $X \times Q$ and the curves Y and $P \times Y$. Let Γ be a divisor on $X \times Y$ of bidegree (b, c), i.e. $b = X \cdot \Gamma$ and $c = Y \cdot \Gamma$. The virtual arithmetic genus π of Γ is at most $1 + bc + c(\beta - 1) + b(\gamma - 1)$. When $b = c$ this bound reads $1 + 2b(\beta - 1) + b^2$.*

Proof — We follow the lines of Weil's proof of the Riemann hypothesis for curves (cf. [Har77, Exercise V-1.10]).

The algebraic equivalence class of the canonical divisor on $X \times Y$ is $K = 2(\beta - 1)Y + 2(\gamma - 1)X$. We recall that the virtual arithmetic genus π, as defined in [Har77, Exercise V-1.3], is such that $\pi = \frac{D\cdot(D+K)}{2} + 1$ for every divisor D. Thus:

$$\pi = \frac{D \cdot (D + 2(\beta - 1)Y + 2(\gamma - 1)X)}{2} + 1 = \frac{D \cdot D + 2c(\beta - 1) + 2b(\gamma - 1)}{2} + 1,$$

and we just need to bound the self intersection $D \cdot D$. We deduce from Castelnuovo's and Severi's inequality (cf.[Har77, Exercise V-1.9]) that $D \cdot D \leq 2bc$. This finishes the proof of the lemma. $\square$

Lemma 7.2 *Let k be an algebraically closed field. Let X and Y be two projective, smooth, integral curves over k. Let Γ be an effective divisor without multiplicity on the surface $X \times Y$. Let $f \in k(X)$ be a non-constant function. For all but finitely many scalars x in k, the divisor $(f)_x \times Y$ crosses transversally Γ, where $(f)_x$ is the positive part of the divisor of $f - x$.*

Proof — We call $p_X : X \times Y \to X$ the projection on the first factor. Let E be the set of points in $X(k)$ such that at least one of the following condition holds: $p_X^{-1}(P)$ contains a singular point on Γ, or $p_X^{-1}(P)$ contains a ramified point of the morphism $p_X : \Gamma \to X$, or the fiber $p_X^{-1}(P)$ is contained in Γ. The set E is finite. For all $x \in k$ but finitely many, the fiber $f^{-1}(x)$ avoids E and it is simple. $\square$

Lemma 7.3 *Let k be an algebraically closed field. Let X be a projective, smooth, integral curve over k. Assume the genus of X is at least 2. Let $f \in k(X)$ be a non-constant function. We note G the group of k-automorphisms of f. This is the set of all k-automorphisms θ of X such that $f \circ \theta = f$. For any $x \in \mathbf{P}^1(k)$, we note $(f)_x = f^{-1}(x)$ the fiber above x and G_x the group of k-automorphisms of X that stabilize the set of k-points of $(f)_x$.*

For all x in $\mathbf{P}^1(k)$ but finitely many we have $G_x = G$.

Proof — The group $H = \mathrm{Aut}_k(X)$ of k-automorphisms of X is finite because the genus of X is at least two. Let θ be an automorphism in $H \setminus G$ and let $x \in \mathbf{P}^1(k)$. Assume that the k-points in $(f)_x$ are permuted by θ. Let P be one of them. Then $f \circ \theta(P) = f(P) = x$. So P is a zero of the non-zero function $f \circ \theta - f$. For each θ there are finitely many such zeros. And the θ are finitely many. So the images by f of such P's are finitely many also. $\square$

Lemma 7.4 *Let k be a field. Let X_k be a projective, smooth, geometrically integral curve over k. Set $X = X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^a)$ and assume that X has genus at least 2. Let Y be a projective, smooth, integral curve over k^a and let $\varphi : Y \to X$ be a non-constant k^a-cover. If f is any non-constant function in $k^a(X)$ then $\mathrm{Aut}(\varphi) \subset \mathrm{Aut}(f \circ \varphi)$. Let $V \subset k(X_k)$ be the set of functions $f \in k(X_k)$ such that $\mathrm{Aut}(\varphi) \neq \mathrm{Aut}(f \circ \varphi)$. This set V is contained in a finite union of strict k-subalgebras of $k(X_k)$.*

Proof — The statement to be proven concerns the three function fields $k^a(f) \subset k^a(X) \subset k^a(Y)$ and the groups involved are the following ones:

$$\begin{cases} \mathrm{Aut}(\varphi) = \mathrm{Aut}_{k^a(X)}(k^a(Y)), \\ \mathrm{Aut}(f \circ \varphi) = \mathrm{Aut}_{k^a(f)}(k^a(Y)), \\ \mathrm{Aut}(Y) = \mathrm{Aut}_{k^a}(k^a(Y)), \end{cases} \quad \Rightarrow \quad \mathrm{Aut}(\varphi) \subset \mathrm{Aut}(f \circ \varphi) \subset \mathrm{Aut}(Y).$$

Now, the set V can be described as follows:

$$V = \left(\bigcup_{\theta \in \mathrm{Aut}(Y) \setminus \mathrm{Aut}(\varphi)} k^a(Y)^\theta \cap k^a(X) \right) \cap k(X_k) = \bigcup_{\theta \in \mathrm{Aut}(Y) \setminus \mathrm{Aut}(\varphi)} k^a(Y)^\theta \cap k(X_k).$$

This is a union of sets indexed by elements in the finite set $\mathrm{Aut}(Y) \setminus \mathrm{Aut}(\varphi)$ (remind $\mathrm{Aut}(Y)$ is finite because the genus of Y is at least 2). Since $\theta \notin \mathrm{Aut}(\varphi)$, each $k^a(Y)^\theta \cap k^a(X)$ is a strict subfield of $k^a(X)$ containing k^a. Therefore $k^a(Y)^\theta \cap k(X_k) \subsetneq k(X_k)$. □

7.2 Deformation of an automorphism of a nodal curve

In this paragraph we give a *necessary* condition for extending an automorphism of a nodal curve to a given deformation of this curve.

Let R be a complete discrete valuation ring. Let π be a uniformizing parameter and let k be the residue field. We assume k is algebraically closed. Let D be a semi-stable curve over $\mathrm{Spec}(R)$. We note D_η the generic fiber and D_0 the special fiber. We assume D_η is smooth over the fraction field of R. Let T be a singular point of D_0. According to [Liu02, Chapter 10, Corollary 3.22], the completion of the local ring of D at T takes the form:

$$\hat{\mathcal{O}}_{D,T} = R[[f,g]]/\langle fg - \pi^e \rangle$$

where e is a positive integer. This integer is called the *thickness* of D at T. We also say that f and g form a coordinate system for D at T. If we reduce modulo π, we obtain the completion of the local ring of D_0 at T:

$$\hat{\mathcal{O}}_{D_0,T} = \hat{\mathcal{O}}_{D,T}/\langle \pi \rangle = k[[\overline{f}, \overline{g}]]/\langle \overline{f}\overline{g} \rangle,$$

where $\overline{f} = f \bmod \pi$ and $\overline{g} = g \bmod \pi$. Because T is an ordinary double point, D_0 has two branches F and G at T. These correspond to the two irreducible components of the completion at T. Be careful that these two branches may lie on the same irreducible component of D_0. Anyway, the functions $\overline{f}$ and $\overline{g}$ are the uniformizing parameters of either branches. We call P and Q the points of F and G above T.

Now let T' be another singular point of D_0, and let f', g', e', F', and G' the corresponding data.

Let θ be an automorphism of D over R such that $\theta(T) = T'$ and $\theta(F) = F'$, $\theta(G) = G'$. One easily checks that the functions $f' \circ \theta$ and $g' \circ \theta$ form a coordinate system for D at T. We deduce that $e' = e$ and that both $f' \circ \theta / f$ and $g' \circ \theta / g$ are units in $\hat{\mathcal{O}}_{D,T}$ (indeed, in either fraction, the numerator and denominator have the same Weil divisor). Since $f \times g = \pi^e = f' \circ \theta \times g' \circ \theta$, we have $\frac{f' \circ \theta}{f}(T) \times \frac{g' \circ \theta}{g}(T) = 1$. We reduce this identity modulo π and obtain the following identity where the first factor is a function on F evaluated at P while the second factor is a function on G evaluated at Q:

$$\frac{\overline{f}' \circ \overline{\theta}}{\overline{f}}(P) \times \frac{\overline{g}' \circ \overline{\theta}}{\overline{g}}(Q) = 1. \tag{15}$$

This leads us to the following definition.

Definition 7.5 *Let R be a complete discrete valuation ring. Assume that the residue field k is algebraically closed. Let D be a semi-stable curve over $\operatorname{Spec}(R)$. The generic fiber of D is assumed to be smooth. Assume we are given a coordinate system at each singular point of the special fiber D_0. Let $\bar{\theta}$ be an automorphism of the special fiber D_0. We say that $\bar{\theta}$ is admissible in $D/\operatorname{Spec}(R)$ if for every singular point T of D_0, the image $\bar{\theta}(T)$ has the same thickness as T in D, and if equality (15) holds true.*

We have just proved the following lemma.

Lemma 7.6 *With the notation of definition 7.5, the set of automorphisms of D_0 that are admissible in $D/\operatorname{Spec}(R)$ form a subgroup of $\operatorname{Aut}_k(D_0)$. If θ is an automorphism of D over $\operatorname{Spec}(R)$, its reduction $\bar{\theta} = \theta \bmod \pi$ is an automorphism of D_0 and is admissible in $D/\operatorname{Spec}(R)$.*

One may compare this statement with [Wew99, Theorem 3.1.1] where the deformation of morphisms between two distinct curves is studied.

> **Remark –** It must be pointed out that the converse of lemma 7.6 is not true. For example, consider the elliptic curve E with modular invariant $j = 0$ (or 1728). Every automorphism of E is admissible because there are no singular points on the curve (the condition in definition 7.5 is empty). However, the only automorphisms that can be extended to the generic elliptic curve are the identity and the involution.

7.3 Automorphisms of curves in a family

In this section we state and prove a lemma of specialization of the automorphism group of curves in a family.

Lemma 7.7 *Let k be a field and let U be a smooth, geometrically integral curve over k. Let X be a quasi-projective, smooth, geometrically integral surface over k. Let $\pi : X \to U$ be a surjective, projective, smooth morphism of relative dimension 1. Assume that for any point x of U, the fiber X_x at x is geometrically integral. We call η the generic point of U and $\bar{X}_\eta = X_\eta \times_{\mathrm{Spec}(k(U))} \mathrm{Spec}(k(U)^a)$ the generic fiber, seen as a curve over the algebraic closure of the function field of the basis U. We assume the genus of X_η is at least 2.*

There exists a non-empty open subset V of U over k such that for any geometric point $x \in V(k^a)$ the group of k^a-automorphisms of the fiber at x is equal to the group $\mathrm{Aut}_{k(U)^a}(\bar{X}_\eta)$ of automorphisms of $\bar{X}_\eta$.

The following proof was communicated to us by Qing Liu.

Proof — This is a consequence of a general result by Deligne-Mumford. Let $X \to S$ be a flat projective morphism over a noetherian scheme S. The functor $T \to \mathrm{Aut}_T(X_T)$ from the category of S-schemes to the category of groups is representable by a group scheme $\mathrm{Aut}_{X/S}$ over S. See [Kol96, Exercise 1.10.2] for example.

When $X \to S$ is a stable curve with genus at least 2, Deligne and Mumford [DM69, Thm 1.11] prove that the scheme $\mathrm{Aut}_{X/S}$ is finite and unramified over S.

In our lemma, S is a the smooth, geometrically integral curve U over k. Replacing S by a non-empty open subset, we may assume that $\mathrm{Aut}_{X/S}$ is finite étale over S. At the expense of a finite surjective base change $T \to S$, we may assume that the generic fiber of $\mathrm{Aut}_{X/S} \to S$ consists of rational points. So $\mathrm{Aut}_{X/S} \to S$ is now a disjoint union of étale sections and the fibers have constant degree. In particular, the fibers are constant and the specialization maps $\mathrm{Aut}_S(X) = \mathrm{Aut}_{X/S}(S) \to \mathrm{Aut}_s(X_s) = \mathrm{Aut}_{X/S}(k(s))$ are isomorphisms. □

References

[CG94] Jean-Marc Couveignes and Louis Granboulan. Dessins from a geometric point of view. In Leila Schneps, editor, *The Grothendieck theory of dessins d'enfants*. Cambridge University Press, 1994.

[CH85] Kevin Coombes and David Harbater. Hurwitz families and arithmetic galois groups. *Duke Mathematical Journal*, 52(4):821–839, 1985.

[CR04] Jean-Marc Couveignes and Nicolas Ros. Des obstructions globales à la descente des revêtements. *Acta Arithmetica*, 114(4):331–348, 2004.

[DE99] Pierre Dèbes and Michel Emsalem. On fields of moduli of curves. *Journal of Algebra*, 211:42–56, 1999.

[DF94] Pierre Dèbes and Michael Fried. Non-rigid constructions in galois theory. *Pacific Journal of Mathematics*, 163(1):81–122, 1994.

[DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l'I.H.E.S.*, 36:75–109, 1969.

[Gir64] Jean Giraud. *Méthode de la descente*, volume 2 of *Mémoires de la S.M.F.* S.M.F., 1964.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*, volume 32 of *A series of Modern Surveys in Mathematics*. Springer, 1996.

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford, 2002.

[Mes91] Jean-François Mestre. Construction de courbes de genre 2 à partir de leurs modules. In Teo Mora and Carlo Traverso, editors, *Effective Methods in Algebraic Geometry*, volume 94 of *Progress in Mathematics*, pages 313–334. Birkäuser, 1991.

[Ser59] Jean-Pierre Serre. *Groupes algébriques et corps de classes*. Hermann, deuxième edition, 1959.

[Sil95] Joseph H. Silverman. The field of definition for dynamical systems on $\mathbf{p}^1$. *Compositio Math.*, 98(3):269–304, 1995.

[Vis04] Angelo Vistoli. *Notes on Grothendieck topologies, fibered categories and descent theory*. *arXiv*, math/0412512, 2004.

[Wei56] André Weil. The field of definition of a variety. *Amer. J. of Math.*, 78:509–524, 1956.

[Wew99] Stefan Wewers. Deformation of tame admissible covers of curves. In Helmut VÖLKEIN, David HARBATER, Peter MÜLLER, and J.G. THOMPSON, editors, *Aspects of Galois theory*, volume 256 of *London Mathematical Society Lecture Note Series*, pages 239–282. Cambridge, 1999.