Tools for the computation of families of coverings

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Abstract

We list several techniques for efficient computation of families of coverings and we illutrate them on an example.

1 Introduction

The computation of algebraic models for coverings of the line is interesting both for theoretical reasons (e.g. the inverse Galois problem) and computational ones (as a test example for computer algebra tools). In many cases one reduces to solving a zero dimensional algebraic system (see [22, 2, 21] for many examples). This can be achieved using Buchberger algorithm. For many reasons, however, one would like to avoid using such an expensive algorithm from the point of view of complexity. In particular, the algebraic system one can associate to a covering does not provide a very sharp characterization and usually admits many solutions having nothing to do with the initial problem. Indeed, such a system may easily encode multiplicities but certainly not such discrete invariants as the monodromy group. On the other hand, famous work by Atkin and Swinnerton-Dyer achieves quite non-trivial computations using methods from numerical analysis [1] and similar methods were applied succesfully in different contexts. Some time ago Ralph Dentzer asked about how to compute an algebraic model for a covering of the sphere ramified above four points with monodromy group M_{24} given in [20]. This computation was achieved by Granboulan in [14] with a lot of numerical methods. At that time I collected several tricks and constructions in order to help with this computation, but this was not published because of the length of the result itself and also because the computational challenge appeared to be the most important. It seems that this information, however, may be of some use to

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other people performing similar computations. Also I have developed it a little bit further and I give in this work an illustration of it on a simple though non trivial example. It gave me an opportunity to consider these computations from a more conceptual point of view. In particular I realized the importance of explicit patching à la Harbater [15], and I tried to detail the algorithmic aspects of it for genus zero coverings. In this special case one can use the explicit description of the moduli spaces [12] to deal efficiently with *not necessarily Galois* coverings. The reason why patching is efficient is that it allows the computation of formal fibers without computing the extension of the base. The latter may really be huge in non-rigid cases (degree 144 in [14]). On the contrary, the extension of the basis is derived from the model for the fiber.

The paper is organized as follows. In section 2 we present down to earth techniques for computing an algebraic model for a covering. These techniques were known to Fricke. We illustrate them on simple examples. In section 3 we define the main family of coverings that will serve us as an example and start studying its combinatorial properties. The main point there is Hurwitz braid action. In section 4 and 5 we show how to compute an algebraic model for our family of coverings from the consideration of degenerate ones. Once a model has been computed for a degenerate cover, we first compute an analytic deformation of it which consists of a one parameter family of coverings, the parameter taking its values in a real interval. This is the purpose of section 4 and does not require more than linear algebra computations. From this analytic family we derive an algebraic one in section 5. This again reduces to linear algebra. In section 6 we give a more general and more conceptual description of our method. It uses the explicit description of the compactification of moduli spaces of curves given in [12] for the case of genus zero curves. We explain in particular how to compute in advance the degree of the coefficients that appear in the algebraic model we are looking for. Geometrically, these degrees are expressed in terms of the "thickness" of intersections in some formal curves.

It should be clear that the example we present is a toy that we chose for its simplicity and for such a small covering there exist simpler, faster methods.

The methods presented here apply to any genus zero covering of the sphere minus r points. Computations will be more difficult for higher genera, however, (except small values) because of the lack of a sufficiently explicit description for the corresponding moduli spaces.

We hope the reader will be convinced that the rich recent theory together with old computational methods make the computation of coverings much easier than it appears provided one does not rely too much on Buchberger's algorithm, as useful as it is.

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2 Two coverings ramified over three points

In this section we shall compute an algebraic model for two coverings of the sphere minus three points. This will be useful in the next sections. We take this opportunity to recall how people have been efficiently computing simple coverings since the last century.

Let $\mathbb{P}_1(\mathbb{C})$ be the projective line over the field of complex numbers and let R_1, R_2, R_3 be three distinct points on it. By a coordinate on $\mathbb{P}_1(\mathbb{C})$ we mean a generator of its function field. For P a point and z a coordinate we denote by $z(P)$ the value of z at P. There is a unique coordinate z such that $z(R_1) = 0$, $z(R_2) = 1$, $z(R_3) = \infty$. Let β be the point with z-coordinate equal to $z(\beta) = i + 1/2$ and let Σ_1, Σ_2 and Σ_3 be the three loops represented on figure 1 in the plane with coordinate z.

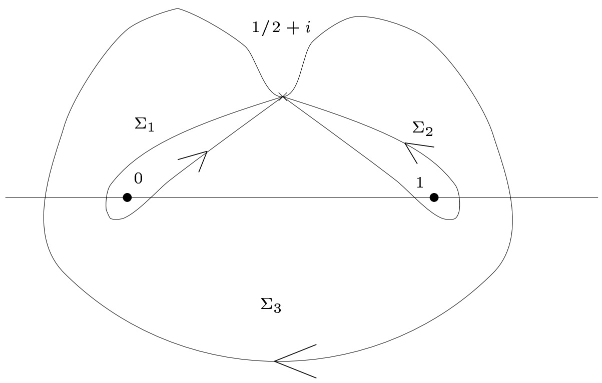

Figure 1: $\pi_1(\mathbb{P}_1 - \{R_1, R_2, R_3\}, b)$

Let ρ_1, ρ_2, ρ_3 be the three permutations below

$$\begin{aligned}
\rho_1 &= [1,2,3,4,5,6,7], \\
\rho_2 &= [1,2], \\
\rho_3 &= (\rho_2\rho_1)^{-1}.
\end{aligned}$$

We consider the covering of $\mathbb{P}_1(\mathbb{C}) - \{R_1, R_2, R_3\}$ with monodromy (ρ_1, ρ_2, ρ_3) in the basis $(\Sigma_1, \Sigma_2, \Sigma_3)$.

The Riemann-Hurwitz formula shows that this is a genus zero covering that is a map $f : \mathbb{P}_1 \to \mathbb{P}_1$ unramified outside $\{R_1, R_2, R_3\}$. This map f can be represented as a rational fraction F provided we pick a coordinate on the left and a coordinate on the right. Let S_1 be the unique point above R_1 and S_2 the unique point with multiplicity 1 above R_3 and S_3 the unique point with multiplicity 6 above R_3. Let x be the coordinate which takes the values 0, 1 and ∞ at S_1, S_2, S_3 respectively.

We now can represent the covering f as a rational fraction $F(x) = z$ which satisfies

$$F(x) = K_1 \frac{x^7}{x-1}$$

with K_1 a constant and $F(x) - 1$ has a double zero i.e. $K_1 x^7 - (x-1)$ has a double zero. We therefore solve the system $\{K_1 x^7 - x + 1 = 0, 7K_1 x^6 - 1 = 0\}$ and trivialy find the unique solution $x = 7/6$ and $K_1 = 6^6/7^7$. We thus get

$$F(x) = \frac{6^6}{7^7} \frac{x^7}{x-1} \tag{1}$$

and

$$F(x) - 1 = \frac{6^6}{7^7} \frac{(x - \frac{7}{6})^2 (x^5 + \frac{7}{3}x^4 + \frac{49}{12}x^3 + \frac{343}{54}x^2 + \frac{12005}{1296}x + \frac{16807}{1296})}{x-1}. \tag{2}$$

We observe that the coefficients in the expressions above are rational. This could have been deduced *a priori* using a rigidity criterion.

The reader who is familiar with the theory of Grothendieck's *dessins d'enfant* as presented in [22] may like to see the *dessin* corresponding to f. It is the preimage of $[0, 1]$ by F. In figure 2, bullets correspond to points above 0 and arrows to points above 1.

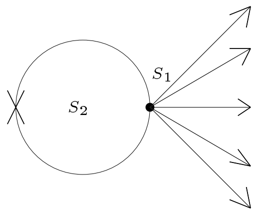

Figure 2: Dessin

We now consider a slightly more difficult example. Let τ_1, τ_2, τ_3 be the three permutations below

$$\begin{aligned} \tau_1 &= [1], [2,3,4], [5], [6,7], \\ \tau_2 &= [1,2,5,6], \\ \tau_3 &= (\tau_2 \tau_1)^{-1}. \end{aligned}$$

They define a genus zero covering $g : \mathbb{P}_1 \to \mathbb{P}_1$ unramified outside $\{R_1, R_2, R_3\}$ with monodromy (τ_1, τ_2, τ_3) in the basis $(\Sigma_1, \Sigma_2, \Sigma_3)$. We call T_1 the unique point with multiplicity 3 above R_1 and T_2 the unique point with multiplicity 4 above R_2 and T_3 the unique point above R_3. We take y to be the unique coordinate on

$\mathbb{P}_1$ which takes values 1, 0 and ∞ at T_1, T_2 and T_3 respectively. We also call T_4 the unique point with multiplicity 2 above R_1.

The covering g may be represented as a polynomial function $G(y) = z$ which satisfies

$$G(y) = K_2(y-1)^3(y-y(T_4))^2C(y) \text{ and } G(y) - 1 = K_2y^4A(y)$$

where $C(y)$ is a degree 2 monic polynomial and $A(y)$ is a degree 3 monic polynomial and K_2 is a constant. We set $C(y) = y^2 - ay + b$ and $A(y) = y^3 - cy^2 + dy - e$ and $y(T_4) = f$ and we write the identity

$$K_2(y-1)^3(y-f)^2(y^2-ay+b) - 1 = K_2y^4(y^3-cy^2+dy-e). \tag{3}$$

We differentiate the identity above with respect to the variable y and find

$$(y-f)(y-1)^2(7y^3+(-6a-4-5f)y^2+(4fa+2f+3a+5b)y-fa-2b-3fb)=y^3(7y^3-6cy^2+5dy-4e)$$

and since $f \neq 0$ we deduce that y^3 divides $7y^3 + (-6a-4-5f)y^2 + (4fa+2f+3a+5b)y - fa - 2b - 3fb$ and thus that

$$-6a-4-5f = 0 \text{ and } 4fa+2f+3a+5b = 0 \text{ and } fa+2b+3fb = 0.$$

We solve the system above and find that f is one of the three solutions of

$$10f^3 + 12f^2 + 9f + 4 = 0 \tag{4}$$

and

$$C(y) = y^2 - (-\frac{5}{6}f - \frac{2}{3})y + \frac{2}{3}f^2 + \frac{19}{30}f + \frac{2}{5}$$

and

$$A(y) = y^3 - (\frac{7}{6}f + \frac{7}{3})y^2 + (\frac{14}{5}f + \frac{7}{5})y - \frac{7}{4}f \tag{5}$$

and

$$K_2 = \frac{60}{7f-4}. \tag{6}$$

Simple combinatorial considerations (see [5] or [6, Theorem 2]) show that the field of moduli of our covering is real. There is a single real solution to equation 4. We therefore take f to be this real solution. This finishes the computation of a model for the covering g. We draw the corresponding dessin on figure 3.

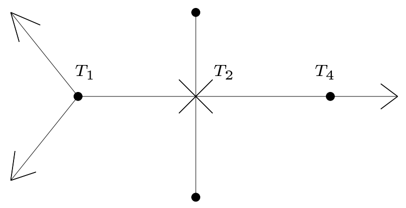

Figure 3: Dessin

3 Topological description

In this section we describe a simple family of coverings branched over 4 points and we start studying it from the point of view of combinatorics. We start with an integer $d = 7$ and four partitions of d, namely $\mathcal{P}_1 = \{1, 1, 2, 3\}$, $\mathcal{P}_2 = \{4, 1, 1, 1\}$, $\mathcal{P}_3 = \{2, 1, 1, 1, 1, 1\}$, and $\mathcal{P}_4 = \{6, 1\}$. Associated to these data we consider the set of isomorphism classes of connected coverings of the sphere ramified over four ordered points, of degree d and with ramification data given by the four above partitions in this order. The (not *a priori* connected) topological configuration space for such coverings is called a Hurwitz space. Its construction and finer ones are given in [16, 11, 10, 9, 23]. A nice introduction to these questions is [24, Chapter 10].

Our goal in this section is to obtain topological information on the Hurwitz space associated to the data above through a simple combinatorial study.

We say that two permutation vectors on d letters $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ and $(\nu_1, \nu_2, \nu_3, \nu_4)$ are conjugate if there is a $u \in \mathcal{S}_d$ such that $\nu_i = {}^u\zeta_i$ for $i \in \{1, 2, 3, 4\}$.

We call $\tilde{\mathcal{P}}_i$ the conjugacy class in $\mathcal{S}_d$ associated to the partition $\mathcal{P}_i$. We first collect all vectors of permutations on d letters $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ up to conjugacy, such that the following conditions hold

1. $\zeta_i \in \tilde{\mathcal{P}}_i$ for $i \in \{1, 2, 3, 4\}$
2. $\zeta_4\zeta_3\zeta_2\zeta_1 = 1$
3. the ζ_i generate a transitive subgroup G of $\mathcal{S}_d$.

Being transitive of prime degree, G is primitive. Since it contains a transposition it must be the full symmetric group on 7 letters [25, Theorem 13.3].

There exist various formulae for counting vectors of permutations ([3, 4, 17, 13] among many others). Unfortunately we also need to exhibit all these vectors and the known methods do not provide an elegant feature for this task.

In general we may just do an exhaustive search with a computer. For the example under study, however, we can find all solutions by hand. A useful though trivial tool is the following "thickening" lemma. We first give a few natural definitions.

Definition 1 *Let d be a positive integer and S a permutation in $\mathcal{S}_d$. Let $\mathcal{I}$ be a non empty subset of $\{1, 2, 3, ..., d\}$. We define a permutation $S_{|\mathcal{I}}$ in the following manner. For any $x \in \mathcal{I}$ we call K the smallest positive integer k such that $S^k(x)$ is in $\mathcal{I}$ and we set $S_{|\mathcal{I}}(x) = S^K(x)$. We call $[x, S(x), ..., S^{K-1}(x)]$ the tail of x. If x is not in $\mathcal{I}$ we set $S_{|\mathcal{I}}(x) = x$. We call $S_{|\mathcal{I}}$ the restriction of S to $\mathcal{I}$.*

Lemma 1 *Let ζ and S be two permutations in $\mathcal{S}_d$ and let $\mathcal{I} = Supp(\zeta)$ be the support of ζ and $S_{|\mathcal{I}}$ the restriction of S to it. Then the restriction $(\zeta S)_{|\mathcal{I}}$ of the product ζS is equal to the product $\zeta S_{|\mathcal{I}}$. This means that the product ζS is obtained from the product $\zeta S_{|\mathcal{I}}$ by replacing every x not fixed by ζ by its tail. Similarly $(S\zeta)_{|\mathcal{I}} = S_{|\mathcal{I}}\zeta$.*

This is useful when multiplying a fixed permutation by a permutation whose cycle lengths depend on a few parameters. For example

Corollary 1 *Let m, n, p be three positive integers. Then the product $[1, m+n+1, m+1] * [1, 2, ..., m+n][m+n+1, ..., m+n+p]$ is $[1, 2, ..., m][m+1, ..., m+n, m+n+1, ..., m+n+p]$.*

Note that the restriction of a product is not the product of restrictions. However lemma 1 is of some theoretical interest. Indeed, consider the set $\mathcal{A}$ of infinite sequences of permutations $(\sigma_n)_{n\in\mathbb{N}, n>0}$ such that $\sigma_n \in \mathcal{S}_n$ and the restriction of σ_{n+1} to $\{1, 2, ..., n\}$ is σ_n. This is the inverse limit of the *sets* $\mathcal{S}_n$ with respect to the restriction maps. This $\mathcal{A}$ is not a group by the above remark. However, lemma 1 implies that the group $\mathcal{S}_\infty$ of permutations of the positive integers with bounded support acts on $\mathcal{A}$ by action on coordinates. Indeed for any permutation τ of degree m and any $\sigma = (\sigma_n)_n$ in $\mathcal{A}$ we define $\tau.\sigma$ as follows. For any $n \geq m$ set $\mu_n = \tau\sigma_n$ and for any $n < m$ take μ_n to be the restriction of μ_m to $\{1, 2, ..., n\}$. Then $\tau.\sigma = (\mu_n)_n$ is in $\mathcal{A}$. One can also define a right action. These actions clearly have no fixed points.

Elements of $\mathcal{A}$ admit a more geometric description. Consider pairs of the form $(\mathcal{U}, \iota)$ where $\mathcal{U}$ is a finite or enumerable disjoint union of oriented circles and ι is an injection of the set of positive integers into $\mathcal{U}$. Two such pairs $(\mathcal{U}_1, \iota_1)$ and $(\mathcal{U}_2, \iota_2)$ are said to be equivalent if and only if there is an orientation preserving homeomorphism h from $\mathcal{U}_1$ to $\mathcal{U}_2$ such that $\iota_2 = h \circ \iota_1$. We call such an equivalence class a propermutation. The left and right actions of $\mathcal{S}_\infty$ on propermutations can be seen as cutting and glueing circles.

As elementary as they are, these considerations allow mental computation with permutations.

We proceed as in [7] and find that there are exactly 48 vectors satisfying conditions 1, 2, 3 above (up to conjugacy). We give these vectors in the following definition in which a residue class modulo a positive integer N is identified with its smallest *positive* element.

Definition 2 *For any* $k \bmod 7$ *a residue class modulo* 7 *we denote by* a_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2],[3,4],[5,6,7],\\
\zeta_2 &= [1,2,3,5],\\
\zeta_3 &= [k \bmod 7, k+1 \bmod 7],\\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 7 *such vectors form the* A *family.*

For any $k \bmod 7$ *a residue class modulo* 7 *we denote by* b_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2],[3,4,5],[6,7],\\
\zeta_2 &= [1,2,3,6],\\
\zeta_3 &= [k \bmod 7, k+1 \bmod 7],\\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 7 *such vectors form the* B *family.*

For any $k \bmod 7$ *a residue class modulo* 7 *we denote by* c_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2,3,4],[5],[6,7],\\
\zeta_2 &= [1,2,5,6],\\
\zeta_3 &= [k \bmod 7, k+1 \bmod 7],\\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 7 *such vectors form the* C *family.*

For any $k \bmod 3$ *a residue class modulo* 3 *we denote by* d_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2],[3,4],[5,6,7],\\
\zeta_2 &= [1,2,3,4],\\
\zeta_3 &= [k \bmod 3, 5],\\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 3 *such vectors form the* D *family.*

For any $k \bmod 3$ *a residue class modulo* 3 *we denote by* e_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2],[3,4,5],[6,7],\\
\zeta_2 &= [1,2,3,5],\\
\zeta_3 &= [k \bmod 3, 6],\\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 3 *such vectors form the* E *family.*

For any $k \bmod 5$ *a residue class modulo* 5 *we denote by* f_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2,3],[4,5,6],[7],\\
\zeta_2 &= [1,2,4,6],\\
\zeta_3 &= [k \bmod 5, 7],\\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 5 *such vectors form the* F *family.*

For any $k \bmod 5$ *a residue class modulo* 5 *we denote by* g_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1,2],[3],[4,5,6],[7], \\
\zeta_2 &= [1,3,4,6], \\
\zeta_3 &= [k \bmod 5, 7], \\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 5 *such vectors form the* G *family.*

For any $k \bmod 5$ *a residue class modulo* 5 *we denote by* h_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1],[2,3,4],[5,6],[7], \\
\zeta_2 &= [1,2,5,6], \\
\zeta_3 &= [k \bmod 5, 7], \\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 5 *such vectors form the* H *family.*

For any $k \bmod 5$ *a residue class modulo* 5 *we denote by* i_k *the vector* $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ *with*

$$\begin{aligned}
\zeta_1 &= [1,2,3],[4],[5,6],[7], \\
\zeta_2 &= [1,4,5,6], \\
\zeta_3 &= [k \bmod 5, 7], \\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

We say that the 5 *such vectors form the* I *family.*

We now compute the action of braids on these 48 vectors. To this end we consider the configuration space $X_{0,4} = {\mathbb{P}_1}^4 - \Delta$ of spheres minus four pairwise distinct points. A point $Q = (Q_1, Q_2, Q_3, Q_4)$ in $X_{0,4}$ corresponds to the sphere $\mathbb{P}_1 - (Q_1, Q_2, Q_3, Q_4)$. If Z is a coordinate on $\mathbb{P}_1$ we denote by $Z(Q)$ the vector $(Z(Q_1), Z(Q_2), Z(Q_3), Z(Q_4))$ and we call it the Z-coordinate of Q. We pick such a coordinate Z and choose as a base point for $X_{0,4}$ the point $P = (P_1, P_2, P_3, P_4)$ with Z-coordinate $Z(P) = (0, 1, 2, \infty)$. We also choose a base point b on the corresponding sphere $\mathbb{P}_1 - (P_1, P_2, P_3, P_4)$. We take for b the whole upper half plane in the Z-coordinate. This makes sense because the upper half plane is a contractible set. We also pick generators $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ for $\pi_1(\mathbb{P}_1 - \{P_1, P_2, P_3, P_4\}, b)$ as on figure 4.

The fundamental group $\pi_1(X_{0,4}, P)$ is generated by braids $t_{1,2}$, $t_{2,3}$, $t_{3,4}$ defined in the classical way. For example $t_{1,2}$ is represented by the map $t_{1,2}(u)$ with Z-coordinates $Z(t_{1,2}(u)) = (1/2 - 1/2e^{2i\pi u}, 1/2 + 1/2e^{2i\pi u}, 2, \infty)$ for $u \in [0, 1]$. The action on monodromy vectors in these basis is then given by $t_{1,2}((\zeta_1, \zeta_2, \zeta_3, \zeta_4)) = ({}^{\zeta_2\zeta_1}\zeta_1, {}^{\zeta_2\zeta_1}\zeta_2, \zeta_3, \zeta_4)$. See [16]. Straightforward calculation then gives the following fact.

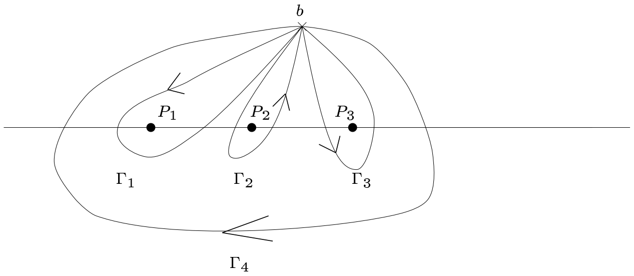

Figure 4: $\pi_1(\mathbb{P}_1 - \{P_1, P_2, P_3, P_4\}, b)$ in the Z-coordinate

Fact 1 *For any k a residue modulo 7*

$$\begin{aligned} t_{1,2}(a_k) &= a_{k-1} \\ t_{1,2}(b_k) &= b_{k-1} \\ t_{1,2}(c_k) &= c_{k-1}. \end{aligned}$$

For any k a residue modulo 3

$$t_{1,2}(d_k) = d_{k-1}.$$

For any k a residue modulo 4

$$t_{1,2}(e_k) = e_{k-1}.$$

For any k a residue modulo 5

$$\begin{aligned} t_{1,2}(f_k) &= f_{k-1} \\ t_{1,2}(g_k) &= g_{k-1} \\ t_{1,2}(h_k) &= h_{k-1} \\ t_{1,2}(i_k) &= i_{k-1}. \end{aligned}$$

The action of $t_{2,3}$ on our 48 vectors is given below (trivial cycles are omitted)

$$[B_1,B_2,C_5][C_1,A_1,A_2][C_7,I_1,D_2,H_5,C_6][A_3,A_4,D_1,I_4,I_5][A_7,F_1,F_2,E_3,A_5][B_7,H_1,H_2,D_3,B_6][B_3,B_5,E_1,G_3,G_4][C_2,C_4,G_1,E_2,F_4].$$

The above fact gives us the combinatorial description of the configuration space $\mathcal{M}$ parametrizing our family. Since the coverings we consider have no automorphisms we even have a covering of universal curves:

$$\begin{tikzcd} \mathcal{M} \arrow[d, "\Lambda"] & \mathcal{T} \arrow[l, "\Gamma"'] \arrow[d, "\Phi"] \\ X_{0,4} & X_{0,5} \arrow[l] \end{tikzcd}$$

Following [8] we embed the moduli space of spheres minus four points $M_{0,4} = \mathbb{P}_1 - \{R_1, R_2, R_3\}$ in the configuration space $X_{0,4}$ as the subvariety of points (R_1, R_2, Q, R_3) for $Q \in \mathbb{P}_1 - \{R_1, R_2, R_3\}$. The restriction of Λ to the curve $M_{0,4}$ is a covering of curves also called $\Lambda : \mathcal{H} \to \mathbb{P}_1 - \{R_1, R_2, R_3\}$. The curve $\mathcal{H}$ is the moduli space of our family of coverings and is often called a Hurwitz space. This Hurwitz space is mapped by Λ onto the moduli space $M_{0,4}$ of spheres minus four points.

From fact 1 and the Hurwitz formula we deduce that the curve $\mathcal{H}$ has genus zero. Recall that z is the coordinate such that $z(R_1) = 0$, $z(R_2) = 1$ and $z(R_3) = \infty$. The preimage by Λ of the segment consisting of points with z-coordinates in $[1, \infty]$ is a connected graph on the sphere which we represent below. The bullets correspond to points above ∞ and the other vertices to points over 1. The points above 0 are associated with faces. To any point above ∞ corresponds one of the nine families A, B, C, D, E, F, G, H, and I.

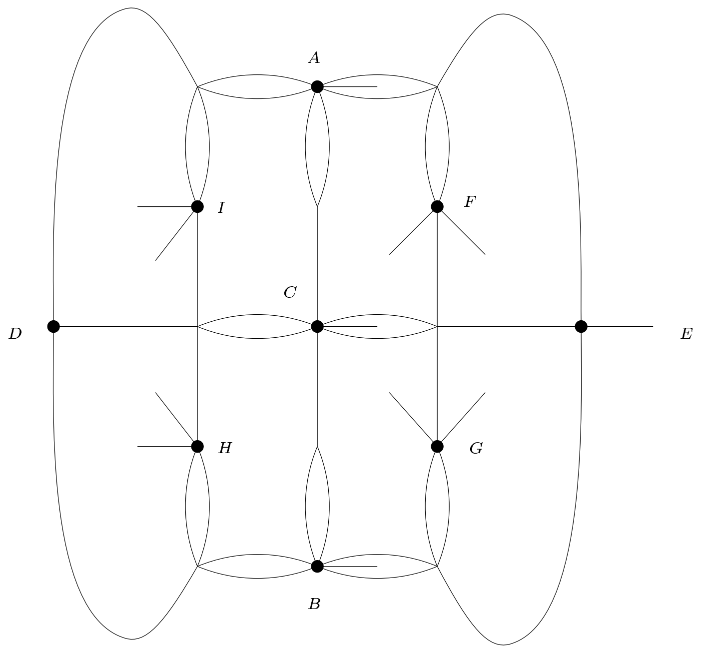

Figure 5: The Hurwitz space $\mathcal{H}$

4 Patching

In this section we compute an analytic model for the family of coverings presented in section 3.

Let again $\mathbb{P}_1(\mathbb{C})$ be the projective line over the field of complex numbers and let U_1, U_2, U_3, U_4 be four distinct points on it. We call Z the unique coordinate such that $Z(U_1) = 0$, $Z(U_2) = 1$, $Z(U_4) = \infty$. We also set $\lambda = Z(U_3)$ and $\chi = 1/\lambda$. For convenience we introduce another coordinate $W = Z/\lambda$ such that $W(U_1) = 0$, $W(U_2) = \chi$, $W(U_3) = 1$, and $W(U_4) = \infty$.

Assume first that λ is a real greater than 1 and let us choose as a base point b the whole upper half plane in the Z-coordinates (which is also the upper half plane in the W-coordinate since $Z = \lambda W$ and λ is real positive). We also pick generators $(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$ for $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{U_1, U_2, U_3, U_4\}, b)$ as represented on figure 6. Note that these data depend continuously on λ. For $\lambda = 2$ we find ourselves in the situation of figure 4.

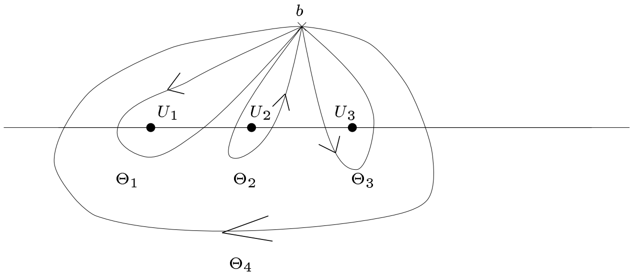

Figure 6: $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{U_1, U_2, U_3, U_4\}, b)$ in the Z-coordinate

We define $\mu = K_3 \chi^{1/7}$ where K_3 is a constant that will be chosen most conveniently latter and $\chi^{1/7}$ is the unique real seventh root of χ. The reason for introducing this μ is that we are going to study the Hurwitz space $\mathcal{H}$ locally at the point C represented on figure 5. This point corresponds to the C family and it is mapped onto $R_3 \in \mathbb{P}_1$ by the Hurwitz map Λ. The ramification index of Λ at C is 7. Therefore μ is a local parameter at C on the Hurwitz space and we expect all coordinates arising in the algebraic model we are looking for to be Laurent series in μ.

For any $\mu \in]0, 1[$ we call $\phi_\mu : \mathbb{P}_1 \to \mathbb{P}_1$ the covering with monodromy $c_5 = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$

$$\begin{aligned}
\zeta_1 &= [1], [2, 3, 4], [5], [6, 7], \\
\zeta_2 &= [1, 2, 5, 6], \\
\zeta_3 &= [1, 5], \\
\zeta_4 &= (\zeta_3 \zeta_2 \zeta_1)^{-1}.
\end{aligned}$$

in the basis of $\pi_1(\mathbb{P}_1 - \{U_1, U_2, U_3, U_4\}, b)$ given in figure 6. Again, one can show that this covering is defined over the field of real numbers (see [5, 6]). We call V_1 the unique point above U_1 with multiplicity 2 and V_2 the unique point above U_1 with multiplicity 3. We call V_3 the unique point above U_2 with multiplicity 4 and V_4 the unique point above U_3 with multiplicity 2 and V_5 the unique point above U_4 with multiplicity 1 and V_6 the unique point above U_4 with multiplicity 6.

We now try to understand what happens when μ tends to zero.

If we look at things from the point of view of W-coordinates we see that $W(U_2) = \chi$ tends to $0 = W(U_1)$ while $W(U_3) = 1$ and $W(U_4) = \infty$. In the limit we get a sphere minus three points $U_1 = U_2$, U_3 and U_4 and a fundamental group $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{U_1 = U_2, U_3, U_4\}, b)$ generated by $\Gamma_{1,2} = \Gamma_1 \Gamma_2, \Gamma_3, \Gamma_4$ as on figure

7. Indeed, when U_1 and U_2 coalesce, turning around the resulting point is just turning around U_1 then U_2. Topologically, this is equivalent to punching a big grey hole containing U_1 and U_2 or equivalently removing the segment $[U_1, U_2]$. Turning around this big hole is equivalent to turning around U_1 and then around U_2.

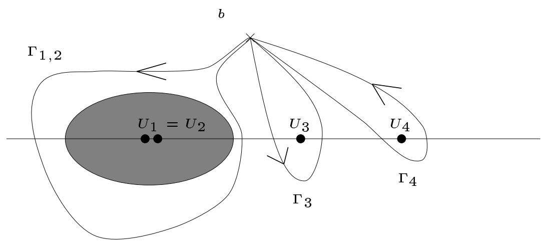

Figure 7: $\pi_1(\mathbb{P}_1 - \{U_1 = U_2, U_3, U_4)\}, b)$

We conclude that when μ tends to zero the covering ϕ_μ tends to a covering of the sphere minus three points with monodromy $(\zeta_2\zeta_1, \zeta_3, \zeta_4)$ in the basis $(\Gamma_{1,2}, \Gamma_3, \Gamma_4)$ of $\pi_1(\mathbb{P}_1 - \{U_1 = U_2, U_3, U_4\}, b)$ shown on figure 7. Coming back to section 2 we see that $\zeta_2\zeta_1 = \rho_1$ and $\zeta_3 = \rho_2$ and $\zeta_4 = \rho_3$. Therefore ϕ_μ tends to f when μ tends to zero.

In order to take advantage of this we shall write down a model for ϕ_μ. As usual, we must chose coordinates on each side. As for the right-hand side we shall of course consider W-coordinates. We note that when μ tends to zero W tends to the coordinate z of section 2 since it takes values 0, 1 and ∞ at the three ramification points of the limit covering.

On the left-hand side we must pin three points to 0, 1 and ∞. We must be careful to choosing three points that do remain distinct when μ tends to zero. For example, points above U_1 and U_2 may well coalesce since U_1 and U_2 coalesce. On the other hand, a point above U_1 and a point above U_3 will not coalesce. Two points above U_3 will not coalesce either. This corresponds *mutatis mutandis* to the notion of *admissible* families of points introduced in Définition 3 of [7]. The key mathematical idea underneath is to be found in sections 2 and 3 of [11].

In our situation we see that $\{V_3, V_5, V_6\}$ form an admissible family because V_5 and V_6 map to U_4 which does not coalesce to any other point. Indeed when μ tends to zero then V_3 tends to the point S_1 of section 2 because V_3 is above U_2 and U_2 tends to R_1 and S_1 is the unique point above R_1. Similarly V_5 tends to S_2 because V_5 is above U_4 and U_4 tends to R_3 and the multiplicity of V_5 is equal to 1 and is not affected as μ tends to zero because U_4 is simple (i.e. does not coalesce) in the W-coordinate, see [11]. Also V_6 tends to S_3. We call X the coordinate which takes values 0, 1 and ∞ at V_3, V_5 and V_6 respectively. Then X tends to the coordinate x of section 2.

The covering ϕ_μ is now represented by a rational fraction $\Phi_\mu(X) = W$ such that

$$\Phi_\mu(X) = \frac{K_4(X^2 - r_1 X + r_2)(X - X(V_1))^2(X - X(V_2))^3}{X - 1} \tag{7}$$

and

$$\Phi_\mu(X) - \chi = \frac{K_4 X^4 (X^3 - s_1 X^2 + s_2 X - s_3)}{X - 1} \tag{8}$$

and

$$\Phi_\mu - 1 = \frac{K_4(X - X(V_4))^2(X^5 - u_1 X^4 + u_2 X^3 - u_3 X^2 + u_4 X - u_5)}{X - 1} \tag{9}$$

Where K_4, r_1, r_2, s_1, s_2, s_3, u_1,...,u_5 depend on μ.

As μ tends to zero the covering ϕ_μ tends to f while the coordinates X and W tend to x and y. Therefore the rational fraction Φ_μ tends to F. From the comparison of formulae 1 and 2 on the one hand and 7, 8, 9 on the other hand we deduce the following

$$\begin{aligned}
r_1 &= O(\mu) \\
r_2 &= O(\mu) \\
X(V_1) &= O(\mu) \\
X(V_2) &= O(\mu) \\
s_1 &= O(\mu) \\
s_2 &= O(\mu) \\
s_3 &= O(\mu) \\
X(V_4) &= \frac{7}{6} + O(\mu) \\
u_1 &= -\frac{7}{3} + O(\mu) \\
u_2 &= \frac{49}{12} + O(\mu) \\
u_3 &= -\frac{343}{54} + O(\mu) \\
u_4 &= \frac{12005}{1296} + O(\mu) \\
u_5 &= -\frac{16807}{1296} + O(\mu) \\
K_4 &= \frac{6^6}{7^7} + O(\mu)
\end{aligned}$$

Further if we set $X = 0$ in equations 7 and 8 we find

$$\chi = K_4 r_2 X(V_1)^2 X(V_2)^3. \tag{10}$$

We shall denote by ν_μ the valuation associated to μ. We know that $\nu_\mu(\chi) = 7$ and from the above

$$\nu_\mu(r_2) + 2\nu_\mu(X(V_1)) + 3\nu_\mu(X(V_2)) = 7. \tag{11}$$

On the other hand since V_3 and all the points above U_1 coalesce we know that $\nu_\mu(r_2)$, $\nu_\mu(X(V_1))$ and $\nu_\mu(X(V_2))$ are positive integers. We deduce that $\nu_\mu(r_2) = 2$ and $\nu_\mu(X(V_1)) = \nu_\mu(X(V_2)) = 1$. We write

$$X(V_2) = \mu(v_{2,0} + v_{2,1}\mu + v_{2,2}\mu^2 + \ldots)$$

In order to complete the picture we now look at the situation from the point of view of the coordinate Z. We see that $Z(U_3) = \lambda$ tends to $\infty = Z(U_4)$ when μ tends to 0 while $Z(U_1) = 0$ and $Z(U_2) = 1$.

At the end we get a sphere minus three points U_1, U_2 and $U_3 = U_4$ and a fundamental group $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{U_1, U_2, U_3 = U_4\}, b)$ generated by $\Gamma_1, \Gamma_2, \Gamma_{3,4} = \Gamma_3\Gamma_4$ as on figure 8. Indeed, when U_3 and U_4 coalesce, turning around the resulting point is just turning around U_3 then U_4. Topologically, this is equivalent to punching a big grey hole containing U_3 and U_4 or equivalently removing the segment $[U_3, U_4]$. Turning around this big hole is equivalent to turning around U_3 and then around U_4.

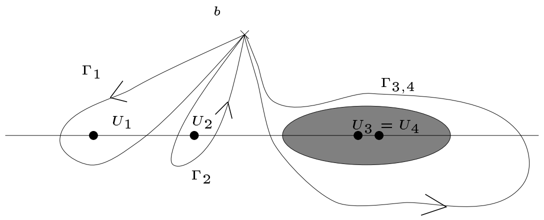

Figure 8: $\pi_1(\mathbb{P}_1 - \{U_1, U_2, U_3 = U_4)\}, b)$

We conclude that when μ tends to zero the covering ϕ_μ tends to a covering of the sphere minus three points with monodromy $(\zeta_1, \zeta_2, \zeta_4\zeta_3)$ in the basis $(\Gamma_1, \Gamma_2, \Gamma_{3,4})$ of $\pi_1(\mathbb{P}_1 - \{U_1, U_2, U_3 = U_4\}, b)$ shown on figure 8. Coming back to section 2 we see that $\zeta_1 = \tau_1$ and $\zeta_2 = \tau_2$ and $\zeta_4\zeta_3 = \tau_3$. Therefore ϕ_μ tends to g when μ tends to zero.

In order to take advantage of this we shall write down a model for ϕ_μ with adapted coordinates on each side. As for the right-hand side we shall of course consider Z-coordinates. We note that when μ tends to zero Z tends to the coordinate z of section 2 since it takes values 0, 1 and ∞ at the three ramification points of the limit covering.

On the left-hand side we must pin down again three points that do not pairwise coalesce, for example $\{V_3, V_2, V_6\}$. We pick a coordinate Y that takes values 0, 1 and ∞ at V_3, V_2 and V_6 respectively. This implies that X/Y is constant equal to $X(V_2)/Y(V_2)$. We set $\kappa = \frac{X(V_2)}{\mu} = v_{2,0} + O(\mu)$ and we have

$$X = \mu\kappa Y \tag{12}$$

Note also that Y tends to the coordinate y of section 2.

The covering ϕ_μ can be now represented by a rational fraction $\Psi_\mu(Y) = Z$ which is related to $\Phi_\mu(X) = W$ by $Z = W\lambda$ and $X = \mu\kappa Y$. We replace X by $\mu\kappa Y$ in equations 7, 8 and divide out by χ and find

$$\Psi_\mu(Y) = \frac{K_3^7 K_4 \kappa^7 (Y^2 - \frac{r_1}{\mu\kappa}Y + \frac{r_2}{\mu^2\kappa^2})(Y - \frac{X(V_1)}{\mu\kappa})^2 (Y-1)^3}{\mu\kappa Y - 1} \tag{13}$$

and

$$\Psi_\mu(Y) - 1 = \frac{K_3^7 K_4 \kappa^7 Y^4 (Y^3 - \frac{s_1}{\mu\kappa}Y^2 + \frac{s_2}{\mu^2\kappa^2}Y - \frac{s_3}{\mu^3\kappa^3})}{\mu\kappa Y - 1} \tag{14}$$

As μ tends to zero Ψ_μ tends to the rational fraction G given in section 2. Comparing the leading coefficients in 13 and 3, and using 6 we find

$$K_3^7 \frac{6^6}{7^7} v_{2,0}^7 = \frac{60}{7f-4}.$$

We now take advantage of the freedom in chosing K_3 assuming K_3 is the unique real root of

$$K_3^7 = \frac{60.7^7}{6^6(7f-4)}$$

so that

$$v_{2,0}^7 = 1.$$

Since our covering is real we deduce that $v_{2,0} = 1$ so

$$X(V_2) = \mu(1 + O(\mu)).$$

Comparing 13, 14 and 5 we find

$$\begin{aligned}
r_1 &= \mu(-\frac{2}{3} - \frac{5}{6}f + O(\mu)) \\
r_2 &= \mu^2(\frac{2}{5} + \frac{19}{30}f + \frac{2}{3}f^2 + O(\mu)) \\
X(V_1) &= \mu(f + O(\mu)) \\
X(V_4) &= \mu(1 + O(\mu))
\end{aligned}$$

$$
\begin{aligned}
s_1 &= \mu(\frac{7}{3} + \frac{7}{6}f + O(\mu)) \\
s_2 &= \mu^2(\frac{7}{5} + \frac{14}{5}f + O(\mu)) \\
s_3 &= \mu^3(\frac{7}{4}f + O(\mu))
\end{aligned}
$$

This time we have first order approximations for all the coefficients arising in our algebraic model. Computing higher order approximations now reduces to linear algebra by Hensel's Lemma. We just plug formal developements into equations 7, 8, 9 and develop in the variable μ. We get a non singular linear system of equations in the next order terms etc.

For example we find

$$
\begin{aligned}
X(V_2) &= \mu - (8/21 + 5f/42)\mu^2 + (101/2205 + 529f/4410 - 5f^2/252)\mu^3 + ... \\
X(V_1) &= f\mu - (5f/21 + 11f^2/42)\mu^2 + (29/4410 + 229f/17640 + 811f^2/4410)\mu^3 + ...
\end{aligned}
$$

5 Looking for algebraic dependencies

In this section we shall derive an algebraic model from the analytic one obtained in section 4.

5.1 General procedure

In section 4 we obtained an analytic model for the covering ϕ_μ with coefficients in the complete field $\mathbb{Q}(f)((\mu))$. We know that this model is actually defined over the algebraic closure of $\mathbb{Q}(\lambda)$ in $\mathbb{Q}(f)((\mu))$. We shall now look for algebraic dependencies between the various coefficients arising in the expression for ϕ_μ. This will give us a model for the Hurwitz space $\mathcal{H}$ defined in section 3.

We pick two functions ($X(V_1)$ and $X(V_2)$ for example) and look for algebraic dependencies between them. Let us call $\mathbb{C}(\mathcal{H})$ the field of functions on the curve $\mathcal{H}$ (our Hurwitz space). This is a genus zero function field over $\mathbb{C}$. To ease notations we shall set $v_1 = X(V_1)$ and $v_2 = X(V_2)$. Assume the degree of the extension $\mathbb{C}(\mathcal{H})/\mathbb{C}(v_1)$ is d_1 and the degree of $\mathbb{C}(\mathcal{H})/\mathbb{C}(v_2)$ is d_2. Then there exists a polynomial $E(X_1, X_2)$ in two variables, with coefficients in $\mathbb{C}$ and degree in X (resp. in Y) equal to d_2 (resp. d_1) such that $E(v_1, v_2) = 0$. If we know the expansions of v_1 and v_2 with enough accuracy (i.e. more than $(d_1+1)(d_2+1)$) finding such a relation is just a matter of linear algebra. We try successive increasing values for d_1 and d_2. For $d_1 = d_2 = 3$ we find

$$
E(v_1, v_2) = -36v_2v_1^3 + 24v_2^3 + 60v_1^3 + 54v_2^2v_1 + 72v_1^2v_2 - 18v_2^3v_1 - 36v_2^2v_1^2 = 0. \quad (15)
$$

We shall see in section 6 that the degrees d_1 and d_2 can be computed *a priori* from the monodromy ζ. We therefore need not compute infinitly many terms in our expansions. With bounded accuracy we can obtain enough dependencies to determine all the coefficients in our equation. This will provide us with a proof that the equation holds since we *a priori* know that such an equation does exist. Considerations in section 6 will enable us to choose functions like v_1 and v_2 that make the degrees d_1 and d_2 minimal.

The curve given by equation 15 is expected to be of genus 0. We therefore look for a parametrization using an algorithm due to Noether, Poincaré and Vessiot—see [18] for a complete algorithmic survey on this question, including problems of rationality. In our case, of course, finding a parameter is particularly trivial since 15 has a unique triple point.

We find that $T = v_2/v_1$ is a parameter and

$$v_1 = \frac{10 + 12T + 9T^2 + 4T^3}{3T(2 + 2T + T^2)} \text{ and } v_2 = Tv_1.$$

Let d_0 be the degree of the field extension $\mathbb{C}(\mathcal{H})/\mathbb{C}(v_1, v_2)$. We may reasonably expect this d_0 to be small (the irreducibility of equation 15 implies $d_0 = 1$). Let d_3 be the degree of $\mathbb{C}(\mathcal{H})/\mathbb{C}(r_1)$. We now look for algebraic dependencies between T and r_1 with degree d_0 in r_1 and d_3 in T. For $d_0 = 1$ and $d_3 = 6$ we find

$$r_1 = -\frac{2(T + 4)(4T^3 + 9T^2 + 12T + 10)(T + 1)^2}{3(T^4 + 6T^3 + 21T^2 + 16T + 6)}.$$

Similarly we find

$$r_2 = \frac{(4T^3 + 9T^2 + 12T + 10)^2}{3(T^2 + 2T + 2)(T^4 + 6T^3 + 21T^2 + 16T + 6)}$$

and this is enough for our purpose since $\mathbb{C}(\mathcal{H}) = \mathbb{C}(v_1, v_2) = \mathbb{C}(T)$.

Indeed we set $\gamma_T(X) = (X^2 - r1X + r2)(X - v_1)^2(X - v_2)^3/(X - 1)$ and factor its derivative with respect to X. This derivative has roots v_1, v_2 with multiplicity 2, 0 with multiplicity 3 and $v_4 = X(V_4)$. We deduce an expression for v_4 :

$$v_4 = \frac{2(T^6 + 6T^5 + 21T^4 + 56T^3 + 51T^2 + 30T + 10)}{3T(T^4 + 6T^3 + 21T^2 + 16T + 6)}.$$

We have $K_4 = 1/\gamma_T(v_4)$ and $\chi = \gamma_T(0)/\gamma_T(v_4)$ and $\Phi_T(X) = \gamma_T(X)/\gamma_T(v_4)$. Note that we now write Φ_T rather than Φ_μ since T is the right algebraic parameter.

The singular value χ is given as a rational fraction in T which is unramified outside $\{0, 1, \infty\}$ i.e. a Belyi function. The associated dessin is the one on figure 5 and the monodromy is the one described in section 3. Note in particular that the factorisation of χ and $\chi - 1$ fits with fact 1. Indeed $\chi = \frac{\chi_0}{\chi_\infty}$ and $\chi - 1 = \frac{\chi_1}{\chi_\infty}$ with

$$\chi_0 = -(T^4 + 6T^3 + 21T^2 + 16T + 6)^5 T^4 (4T^3 + 9T^2 + 12T + 10)^7$$

and

$$\chi_\infty = (T-1)^5 (T^4+8T^3+36T^2+40T+20)^4 (T+1)^2 (2T^2+T+2)^2 (2T^4+8T^3+15T^2+8T+2)^3$$
$$(T^3+3T^2+6T+10)(2T^6+12T^5+51T^4+94T^3+111T^2+60T+20)$$

and

$$\chi_1 = -16(-80-288T-48T^2+2432T^3+7896T^4+13776T^5+15656T^6+12432T^7+6711T^8+2222T^9$$
$$+477T^{10}+60T^{11}+4T^{12})(2+2T+T^2)^3(T^6+6T^5+21T^4+56T^3+51T^2+30T+10)^5.$$

and

$$K_4 = -729 \frac{T^6(T^4 + 6T^3 + 21T^2 + 16T + 6)^6 (T^2 + 2T + 2)^6}{\chi_\infty}.$$

We can now replace T by values in $\mathbb{Q}$ and find coverings defined over $\mathbb{Q}$ in our family. The key point here is that our family is defined over $\mathbb{Q}$. More precisely the Hurwitz space is irreducible and defined over $\mathbb{Q}$ and further has many rational points. To check our computations, we just make sure that formulae 7, 8 and 9 hold.

5.2 Using numerical approximations

In the previous paragraph we computed an algebraic model for our family of coverings from an analytic one using linear algebra computations over the field $\mathbb{Q}(f)$. Indeed we were dealing with series in $\mathbb{Q}(f)((\mu))$ with bounded accuracy. The computations, however, will be greatly accelerated if we work in $\mathbb{C}((\mu))$ instead of $\mathbb{Q}(f)((\mu))$ approximating complex numbers to some fixed accuracy. This accuracy depends now on the height of the equations we expect to find. We have no nice upper bound for this height. We just try. We compute an estimate for f and write down expansions for v_1 and v_2 with approximate coefficients in $\mathbb{C}$. For the linear algebra part (looking for dependencies) we no longer use Gauss algorithm but least squares. This gives a vector $\mathcal{V}$ that minimizes the L_2-norm of $M\mathcal{V} - b$ for a given matrix M and vector b. Of course least squares always give a solution and we need a criterion for this solution to be relevant, depending on the accuracy. We may give a quantified criterion but there is a very simple and efficient qualitative one: something interesting is happening if and only if a small perturbation in the data (i.e. close to zero according to current accuracy) induces a big gap in the minimal norm (i.e. change of magnitude). We thus obtain approximations for the coefficients in the linear equation we are looking for. Since these coefficients are expected to belong to $\mathbb{Q}(f)$ we then look for integer linear relations between 1, θ, θ^2 and any such coefficient c. This is achieved using the famous LLL algorithm [19]. Indeed such a relation corresponds to a small vector in the orthogonal lattice to the vector $(N, \lfloor N\theta \rceil, \lfloor N\theta^2 \rceil, \lfloor Nc \rceil)$ where N is a large integer (close to the inverse of the accuracy). And LLL is designed for finding such small vectors. Another possibility is to perform all the computations using successively all the conjugates of f in $\mathbb{C}$ and then form the symmetric functions of the results. We then obtain approximations of *rational* numbers and may find their exact values thanks to continued fraction algorithm. This is the strategy adopted in [1].

There is an important variant to the method described above. Once we have computed Φ_μ as a rational fraction with coefficients in $\mathbb{C}((\mu))$ with a small accuracy (in the μ-adic topology) we may stop there the computation of expansions and replace μ by a small complex number μ_0. This will give us an approximation according to the ordinary absolute value in $\mathbb{C}$ of the rational fraction Φ_{μ_0} branched at $\{0, 1, \infty, \lambda_0\}$. We then plug this approximation into equations 7, 8 and 9 (where λ is fixed to the value λ_0) and apply an iterative numerical method like Newton's method to get an arbitrarily accurate approximation. We may then replace λ_0 by a very close value λ_1 in equations 7 and 8. The rational fraction Φ_{μ_0} is then a close approximation of a rational fraction Φ_{μ_1} branched at $\{0, 1, \infty, \lambda_1\}$ and the latter one is found by applying Newton's method to equations 7 and 8 with λ fixed to λ_1 and initial value Φ_{μ_0} for Φ_{μ_1}. We can move slowly this way in our Hurwitz space and list vectors of values of the various functions r_1, r_2, v_1,... at many points in it. If Q_1, Q_2, ..., Q_k are these points we represent any such function by an element in the algebra $\mathbb{C}^k$ and we can look for algebraic relations as before. We have replaced Taylor expansion at one point by interpolation at many points. We may of course mix the two approaches. We have to be careful that our points Q_i should be well distributed on the Hurwitz space. Otherwise we are going to loose much accuracy since too points that are close to each other lead to almost the same equations. We use least squares as before with the same criterion for testing the relevance of the result. This approach was used succesfully in [14].

6 Stable curves

In this section we shall give elements for the generalization of the method presented in section 4. We shall also explain how one can compute the degrees of coefficients in some algebraic model for a family of coverings by mere consideration of its monodromy. This will be useful when looking for algebraic depencies since it will tell us what is the degree of the relations we are looking for. In particular we shall be able to select functions that satisfy algebraic relations with smallest possible degree.

The key point in section 4 was that the degeneracy of a sphere minus four points could be seen from two different points of views (namely according to Z or W coordinates). There is a standard way to reconciliate these two points of views. It is connected with the compactification of the moduli space of genus zero r-pointed curves. This compactification is very explicitely described in [12] in terms of trees of projective lines. In this section we shall assume that the reader has some familiarity with this work. We just recall that the authors construct a fine moduli space for r-pointed curves from the consideration of all possible cross-ratios between four marked points on the sphere. One defines the cross-ratio $[U_1, U_2, U_3, U_4]$ as $(z_3 - z_1)(z_4 - z_2)/(z_3 - z_2)/(z_4 - z_1)$ where z_1, z_2, z_3, z_4 are the value of any coordinate z at U_1, U_2, U_3, U_4. Note that $[0, \infty, x, 1] = x$. The relations between these cross-ratios are of two types. Those coming from the action of the symmetric group $\mathcal{S}_4$ on the cross-ratio $[U_1, U_2, U_3, U_4]$ by permutation of the indices plus a relation involving five points

$$[U_2, U_5, U_3, U_4].[U_1, U_2, U_3, U_4] = [U_1, U_5, U_3, U_4].$$

This last relation is connected with the algebraic group law on $\mathbb{P}_1 - \{U_3, U_4\}$. If we consider the projective variant of these equations (introducing numerators and denominators for all cross-ratios) we obtain a projective variety $\bar{M}_{0,r}$ which contains the moduli space $M_{0,r}$ as an open subset.

There also exists a universal curve $\bar{M}_{0,r+1} \to \bar{M}_{0,r}$ corresponding to "forgetting the last point".

The degenerate curves can be described as follows. Assume that we have a projective line minus four points U_1, U_2, U_3, U_4 and let U_1 and U_2 coalesce. The object we get at the end is obtained in the following way : replace U_1 and U_2 by another line crossing the first one and put U_1 and U_2 on it.

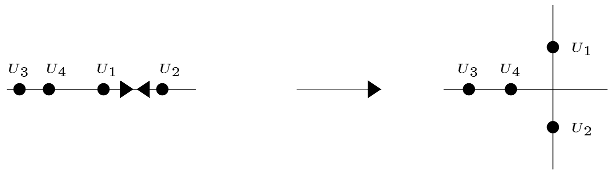

Figure 9: A 4-pointed tree of two projective lines

Now let $A = \mathbb{C}[[\mu]]$ be the local ring of Laurent series in the parameter μ and $Q = \mathbb{C}((\mu))$ its quotient ring. Let $\phi_\mu : \mathcal{C}_\mu \to \mathbb{P}_1 - \{U_1, U_2, U_3, U_4\}$ be a covering defined over Q (i.e. a family of coverings parametrized by the local parameter μ). We assume that the cross-ratio $[U_1, U_4, U_2, U_3]$ is equal to μ^e for some positive integer e. Let W be the coordinate that takes values 0, μ^e, 1, ∞ at U_1, U_2, U_3, U_4 respectively. Let $Z = W/\mu^e$ be the coordinate that takes values 0, 1, μ^{-e}, ∞ at U_1, U_2, U_3, U_4 respectively. The associated field extension to ϕ_μ is

$Q(W) \subset Q(\mathcal{C}_\mu)$. By restricting μ to values in $[0,1]$ we may define the monodromy $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ of ϕ_μ as in section 4. We assume $(\zeta_2\zeta_1)^e = 1$. We can always reduce to this case after base change (i.e. replacing μ by $\mu^{\frac{1}{o}}$ for some integer o). We now consider the ring $\mathcal{R}_0$ generated by W and $T = 1/Z$ over A. This ring $\mathcal{R}_0$ is equal to $\mathbb{C}[[\mu]][W,T]/(WT - \mu^e)$. Let also $\mathcal{R}_1 = A[W]$ and $\mathcal{R}_2 = A[T]$. Then $Spec(\mathcal{R}_1)$ and $Spec(\mathcal{R}_2)$ glue together along $Spec(\mathcal{R}_0)$ to form a fibered surface $\mathcal{S}$ over $Spec(A)$ the generic fiber of which is a smooth curve of genus zero while the special fiber $\mathcal{S}_s$ is made of two genus zero curves $\mathcal{V}$ and $\mathcal{W}$ crossing at the point O with coordinates $W = T = \mu = 0$. The local ring at O is $\mathbb{C}[[\mu, W, T]]/(WT - \mu^e)$. The Zarisky closures of the points U_1, U_2, U_3 and U_4 define horizontal divisors on $\mathcal{S}$. We represent the situation in figure 10.

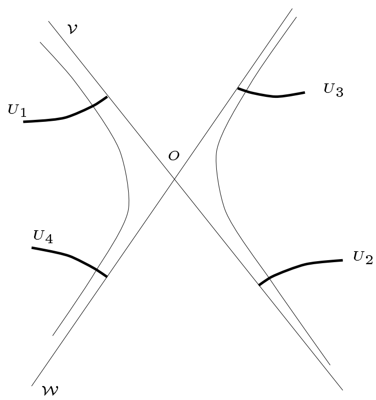

Figure 10: The fibered surface $\mathcal{S}$

We may now take the normal closure of $\mathcal{S}$ in the field extension $Q(W) \subset Q(\mathcal{C}_\mu)$ and obtain a fibered surface $\mathcal{T}$ over $Spec(A)$ and a map $\varphi : \mathcal{T} \to \mathcal{S}$ whose generic fiber is just ϕ_μ. The point is that $\mathcal{T}$ can be described quite sharply in terms of the monodromy ζ. First of all, the special fiber $\mathcal{T}_s$ is a connected curve made of several irreducible components meeting each other transversally and with no other singularity than these crossing points between distinct irreducible components.

The irreducible components of $\mathcal{T}$ fall in two parts. Those that are mapped onto $\mathcal{V}$ by φ which we call $\mathcal{V}_i$ for $1 \le i \le I$ and those that are mapped onto $\mathcal{W}$ which we call $\mathcal{W}_j$ for $1 \le j \le J$. The components $\mathcal{V}_i$ correspond to the orbites $\mathcal{O}_i$ of $(\zeta_1, \zeta_2, \zeta_4\zeta_3)$ and the restriction of φ to $\mathcal{V}_i$ is a covering ramified over three points with monodromy $(\zeta_1, \zeta_2, \zeta_4\zeta_3)|_{\mathcal{O}_i}$ where the $|$ means restriction to $\mathcal{O}_i$. Similarly the components $\mathcal{W}_i$ correspond to orbites $\mathcal{Q}_i$ of $(\zeta_2\zeta_1, \zeta_3, \zeta_4)$. Two components $\mathcal{V}_i$ and $\mathcal{W}_j$ intersect if they share a cycle of $\zeta_2\zeta_1 = (\zeta_4\zeta_3)^{-1}$ and these cycles are in bijection with the crossing points O_k for $1 \le k \le K$ on $\mathcal{T}_s$. These points are the points above O.

If the crossing point O_k of $\mathcal{V}_i$ and $\mathcal{W}_j$ is associated to a cycle of length ℓ_k in $\zeta_2\zeta_1$ then the local ring at O_k is isomorphic to $\mathbb{C}[[\mu, w, t]]/(wt - \mu^{\frac{e}{\ell_k}})$ where t and w

are local parameters at O_k on $\mathcal{V}_i$ and $\mathcal{W}_j$ respectively. The ratio $\theta_k = \frac{e}{\ell_k}$ is called the thickness of the intersection point O_k. Note that θ_k is an integer because we assumed $(\zeta_2\zeta_1)^e = 1$. Note also that θ_k depends on the local parameter μ in the sense that if we replace μ by $\mu^{\frac{1}{o}}$ for some integer o (base change) the thickness θ_k is multiplied by o.

The points mapped to U_1 and U_2 by ϕ_μ correspond to cycles of ζ_1 and ζ_2 and their Zarisky closures in $\mathcal{T}$ cross the $\mathcal{V}_i$'s while the points mapped to U_3 and U_4 by ϕ_μ correspond to cycles of ζ_3 and ζ_4 and their Zarisky closures in $\mathcal{T}$ cross the $\mathcal{W}_j$'s.

In case the generic genus is zero, the special fiber $\mathcal{T}_s$ is a r-pointed tree of projective lines as in [12] and $\mathcal{T}$ is a deformation of it over $Spec(A)$.

We show two examples of such a situation. These examples will be two degeneracies of the covering ϕ_T studied before.

Assume first that ζ is the monodromy c_5 studied in section 4, corresponding to point C on figure 5. Both $(\zeta_1, \zeta_2, \zeta_4\zeta_3)$ and $(\zeta_2\zeta_1, \zeta_3, \zeta_4)$ have a single orbite. Therefore we have a single component $\mathcal{V}_1$ above $\mathcal{V}$ and a single component $\mathcal{W}_1$ above $\mathcal{W}$. Since $\zeta_2\zeta_1$ is a full 7-cycle we have $\ell_1 = 7$. On the other hand, the order e of the braid $t_{1,2}$ acting on ζ is also 7. Thus the thickness of the unique point O_1 above O is just $\theta_1 = 1$.

We draw the corresponding situation on figure 11.

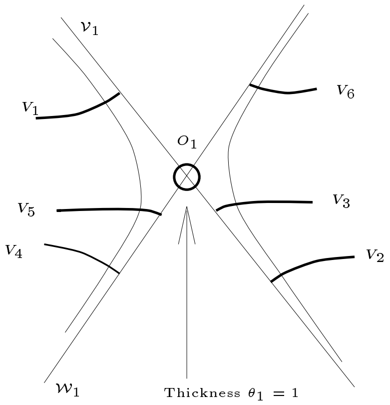

Figure 11: The special fiber at point $C \in \mathcal{H}$

Assume now that ζ is the monodromy d_1 given in definition 2.

$$\begin{aligned}
\zeta_1 &= [1], [2], [3, 4], [5, 6, 7], \\
\zeta_2 &= [1, 2, 3, 4], \\
\zeta_3 &= [1, 5], \\
\zeta_4 &= (\zeta_3\zeta_2\zeta_1)^{-1}.
\end{aligned}$$

This monodromy corresponds to point D in figure 5. This time $(\zeta_1, \zeta_2, \zeta_4\zeta_3)$ has two orbites $\mathcal{O}_1 = \{1, 2, 3, 4\}$ and $\mathcal{O}_2 = \{5, 6, 7\}$ while $(\zeta_2\zeta_1, \zeta_3, \zeta_4)$ has two orbites $\mathcal{Q}_1 = \{1, 2, 3, 5, 6, 7\}$ and $\mathcal{Q}_2 = \{4\}$.

We have three points O_1, O_2 and O_3 above O corresponding to the three cycles $(1,2,3)$, $(5,6,7)$ and (4) of $\zeta_2\zeta_1$.

The order e of braid action is 3 thus the thicknesses are $\theta_1 = 1$, $\theta_2 = 1$, $\theta_3 = 3$. We draw the corresponding picture on figure 12.

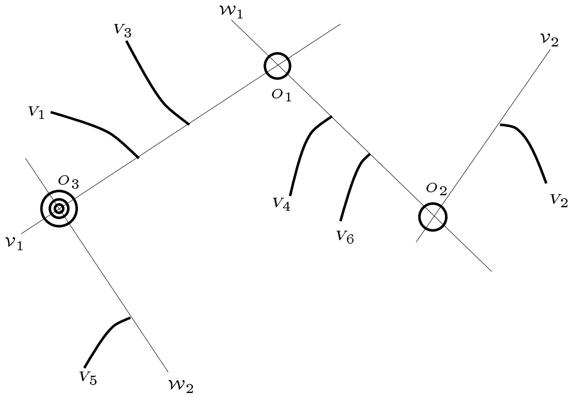

Figure 12: The special fiber at point $D \in \mathcal{H}$

Thicknesses are very useful to compute the order of vanishing (μ-adic valuation) of cross-ratios. For example, in figure 12 the thickness θ_1 of O_1 is nothing but the valuation of the cross-ratio $[V_4, V_1, V_6, V_3]$.

More generally, recall that associated to each component $\mathcal{K}$ of a stable tree of projective lines there is a projection $P_{\mathcal{K}}$ of the full curve onto this component. There is also a unique *median* component $\mathcal{L}_\delta$ associated to any triple of non-singular points $\delta = (\delta_1, \delta_2, \delta_3)$. This component is the unique one on which the three points have pairwise distinct projections.

If we have four points V_1, V_2, V_3, V_4 crossing the special fiber at non-singular distinct points, then there are only two possibilities.

Either the intersections of these points with the special fiber have distinct projections on some component of it. In this case the cross-ratio $\rho = [V_1, V_3, V_2, V_4]$ is a unit in $A = \mathbb{C}[[\mu]]$ and so is $\rho - 1$. This is the case for V_1, V_3, V_4, V_5 on figure 12 since their instersections with the special fiber have distinct projections on $\mathcal{V}_1$.

If the intersections of V_1, V_2, V_3, V_4 don't project on distinct points on any component, then there are two components $\mathcal{K}_1$ and $\mathcal{K}_2$ with associated projections $P_{\mathcal{K}_1}$ and $P_{\mathcal{K}_2}$ such that for example $P_{\mathcal{K}_1}(V_1), P_{\mathcal{K}_1}(V_2), P_{\mathcal{K}_1}(V_3)$ are pairwise distinct while $P_{\mathcal{K}_1}(V_3) = P_{\mathcal{K}_1}(V_4)$ and $P_{\mathcal{K}_2}(V_3), P_{\mathcal{K}_2}(V_4), P_{\mathcal{K}_2}(V_1)$ are pairwise distinct while $P_{\mathcal{K}_2}(V_1) = P_{\mathcal{K}_2}(V_2)$. In this case, the cross-ratio $[V_1, V_3, V_2, V_4]$ has μ-adic valuation equal to the distance between $\mathcal{K}_1$ and $\mathcal{K}_2$ which is defined as the sum of the thicknesses of all intersections between $\mathcal{K}_1$ and $\mathcal{K}_2$.

In particular, we see that the multiplicity of this cross-ratio is bounded by e times the number of intersections. And there are no more intersections than the degree d_ϕ of the covering ϕ_μ.

We deduce that if we have a genus zero covering ϕ of degree d_ϕ of $\mathbb{P}_1$ ramified over four points U_1, U_2, U_3, U_4 and if we pick V_1, V_2, V_3, V_4 above U_1, U_2, U_3, U_4 respectively, then the cross-ratio $\rho = [V_1, V_3, V_2, V_4]$ only vanishes when $\lambda = [U_1, U_3, U_2, U_4]$ vanishes and the multiplicity of ρ is at most d_ϕ times the multiplicity of λ. The exact multiplicity can be obtained from the geometry of the special fiber as we just did.

Let us now call d_Λ the degree of the Hurwitz map from the moduli space $\mathcal{H}$ for a family of coverings to $M_{0,4} = \mathbb{P}_1 - \{0, 1, \infty\}$. We represent below this Hurwitz map and the covering of universal curves as a fibration above it.

$$\begin{tikzcd}
\mathcal{H} \arrow[d, "\Lambda"] & \mathcal{T} \arrow[l, "\Gamma"'] \arrow[d, "\Phi"] \\
M_{0,4} & M_{0,5} \arrow[l]
\end{tikzcd}$$

The degree of the extension $\mathbb{C}(M_{0,4}) = \mathbb{C}(\lambda) \subset \mathbb{C}(\mathcal{H})$ is d_Λ and from the calculation above we deduce that the degree of the extension $\mathbb{C}(\rho) \subset \mathbb{C}(\mathcal{H})$ is bounded above by $d_\Lambda \times d_\phi$.

We thus have a bound for the degree of coefficients appearing in some algebraic model for ϕ.

This bound is indeed very pessimistic since it assumes that we always have many orbites and thick intersections between them. In practice one rather expects degrees like d_Λ/d_ϕ. In our example we have $d_\phi = 7$ and $d_\Lambda = 48$ so that our estimate gives 48×7. If we look at formulae in section 5 we find that all quantities are rational fractions of degree $6 = 48/7$ of the parameter.

In all cases, the extensive combinatorial study of all degenerations of a genus zero covering gives the divisor of zeros and poles of any cross-ratio between branched points. One can then cook a multiplicative combination of these cross-ratios with smallest possible degree. This amounts to finding the shortest vector in the linear space generated by the divisors of all cross-ratios.

Incidentally we obtain a criterion for the Hurwitz space $\mathcal{H}$ to be rational : it is rational if there exists a (combination of) cross-ratio with a single zero. This may be checked easily on the monodromy ζ in some cases.

The method presented here applies to any genus zero covering of the sphere minus r points. When r is bigger than 4 one has to consider maximally degenerate covers. The local ring at the corresponding points in the moduli space $\bar{M}_{0,r}$ is generated by cross-ratios. The ramification at the minimal primes corresponding to these cross-ratios is computed in terms of braid action and the order of vanishing of cross-ratios of branched points along the corresponding divisors can be evaluated as before. This gives a method for computing the *partial* degrees of any coefficient in the algebraic model.

7 Conclusion

We have shown how the computation of a family of coverings can efficiently reduce to the computation of degenerate coverings of the sphere minus three points. The most degenerate will be the better. Reciprocally, if we want to compute a covering of the sphere minus three points, we may find it as a special fiber in a family of higher dimension. The latter family may well be easy to compute provided it admits an(other) simple special fiber. We also have given a method for computing the degree of a coefficient c in some algebraic model (the degree of $\mathbb{C}(\mathcal{H})/\mathbb{C}(c)$) in terms of the monodromy ζ. These degrees depend on the geometry of the degenerate coverings in the family.

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