Counting closed geodesics on flat surfaces

Elise Goujard[1]

September 18, 2014

[1] IRMAR, University of Rennes 1

What is a flat surface ?

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Translation surface

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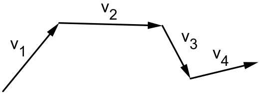

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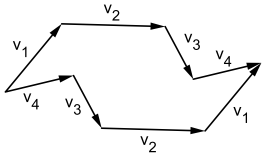

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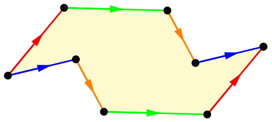

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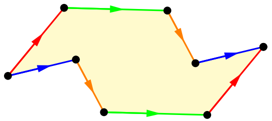

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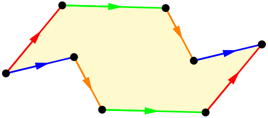

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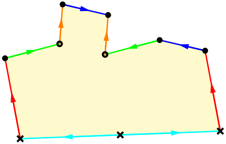

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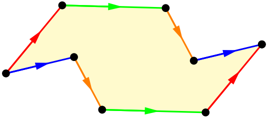

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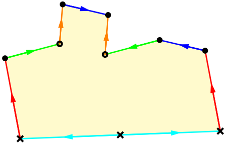

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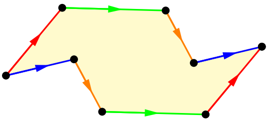

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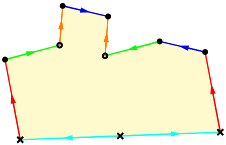

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Both types of surfaces inherits from $\mathbb{C}$ of a flat metric with conical singularities.

	Translation surfaces	Half-translation surfaces
singularity order	d	k
singularity angle	$2\pi(d+1)$	$\pi(k+2)$
moduli space	$\mathcal{H}_g$	$\mathcal{Q}_g$
strata	$\mathcal{H}(d_1, d_2, \ldots, d_n)$	$\mathcal{Q}(k_1, k_2, \ldots, k_n)$
hypersurface	$\mathcal{H}_1(\underline{d})$	$\mathcal{Q}_1(\underline{k})$

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Connected components of $\mathcal{H}_1(\underline{d})$ and $\mathcal{Q}_1(\underline{k})$ carry invariant finite measures (Masur-Veech).

Flat surfaces VS Billiards

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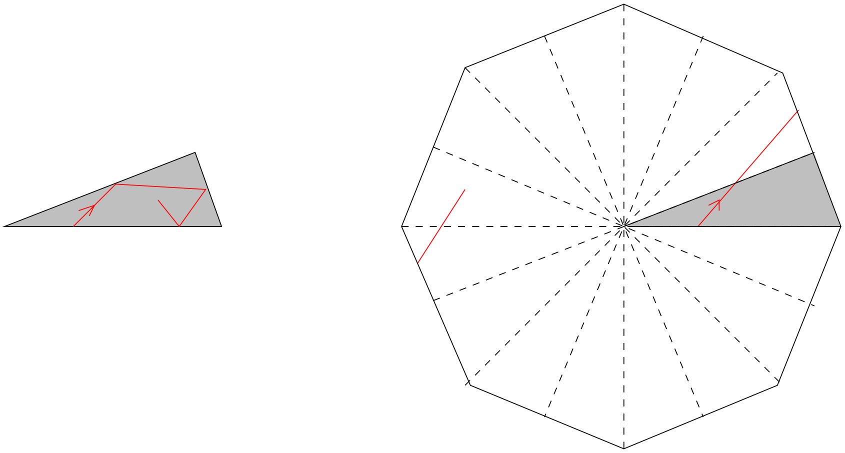

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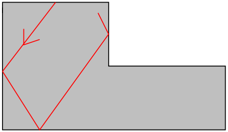
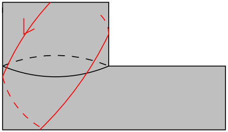
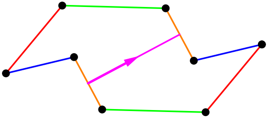
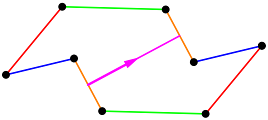
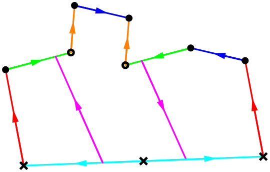
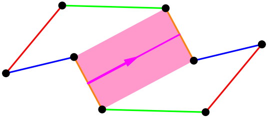


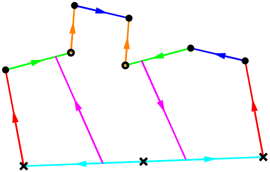




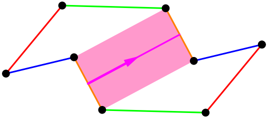


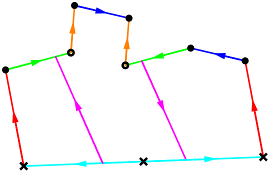






For S a flat surface ($= (X,\omega)$ or (X,q)), we introduce $N(S,L)$ the number of (families of) closed geodesics on S of length $\leq L$.





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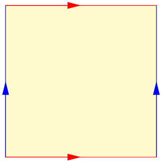
Problem: find the asymptotic of $N(S, L)$ as L goes to ∞.

Toy example: the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$



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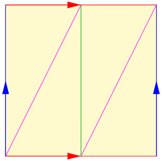

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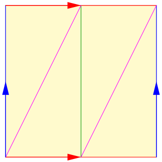

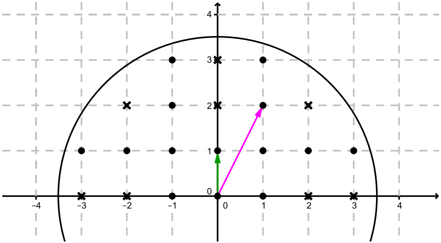

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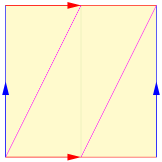

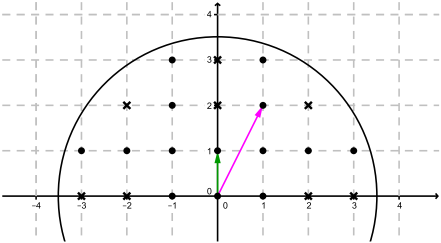

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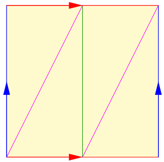

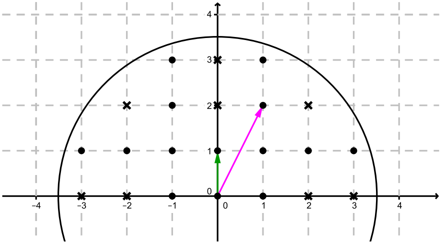

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$$N(\mathbb{T}, L) \sim \frac{1}{\zeta(2)} \pi L^2 \text{ as } L \to \infty$$

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c is called Siegel–Veech constant for K.

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The top g Lyapunov exponents of the Hodge bundle over $\mathcal{H}(d_1, \ldots, d_n)$ along the Teichmüller flow satisfy

$$\lambda_1 + \cdots + \lambda_g = \frac{1}{12} \sum_i \frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3} c_{area}(\mathcal{H}(\underline{d}))$$

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- Works for any invariant sub-orbifold.
- Similar result in the quadratic case.

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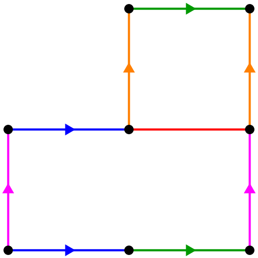
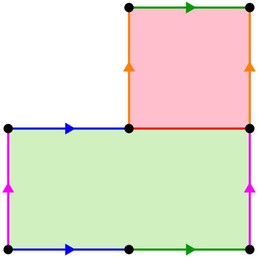
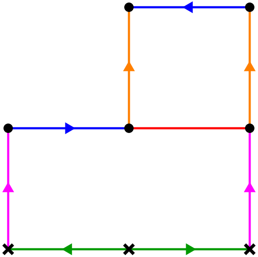
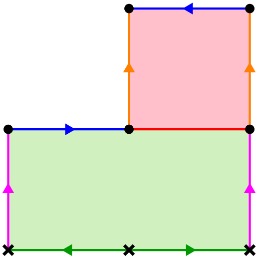
Finally

$$c = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \frac{\mathrm{Vol}\, K^{1,\varepsilon}}{\mathrm{Vol}\, K_1}.$$









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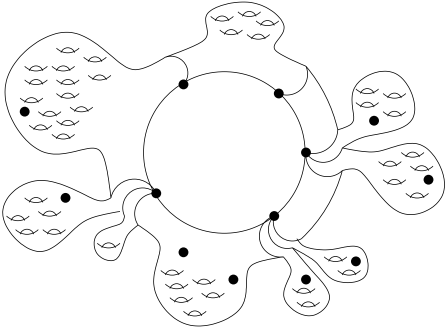
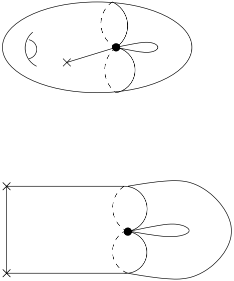
In the Abelian case, it corresponds to maximal collection of homologous saddle connections, and in the quadratic case it corresponds to maximal collections of $\hat{h}$omologous saddle connections.
Configurations are classified by their geometric data.

Examples of configurations with cylinders





× poles, • zeroes

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Assume now that $K = \mathcal{H}(\underline{d})$ or $K = \mathcal{Q}(\underline{k})$ is a connected stratum.
For each geometric type of configuration $\mathcal{C}$, we can define the associate Siegel–Veech constant $c(\mathcal{C})$:

$$c(\mathcal{C}) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \frac{\operatorname{Vol} K^{1,\varepsilon}(\mathcal{C})}{\operatorname{Vol} K_1}$$

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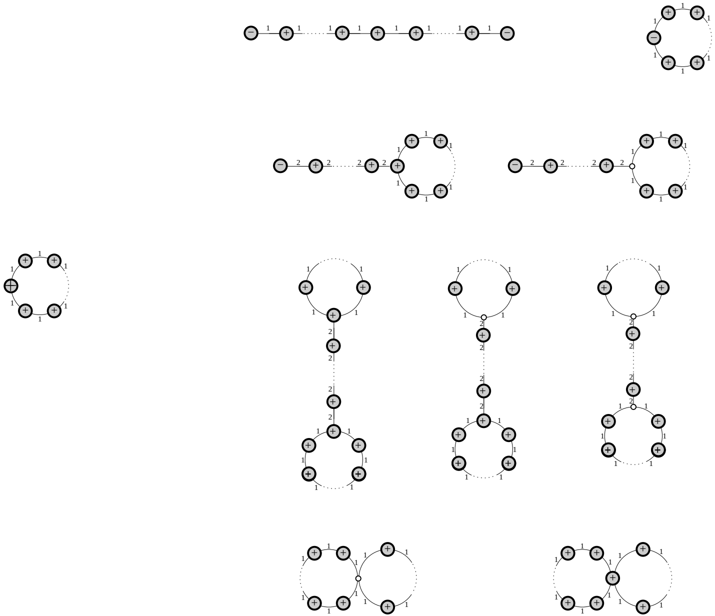
We have $\operatorname{Vol} K^{1,\varepsilon}(\mathcal{C}) = M(\mathcal{C}) \cdot 2\pi\varepsilon^2 \cdot \prod \operatorname{Vol}(\text{boundary strata})$, where $M(\mathcal{C})$ is a combinatorial constant.
These constants were computed by Eskin–Masur–Zorich in the Abelian case and Athreya–Eskin–Zorich in the quadratic case for genus 0.



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Explicit formula for the Siegel–Veech constants $c(\mathcal{C})$, $c_{cyl}(\mathcal{C})$, *and* $c_{area}(\mathcal{C})$, *in the quadratic case, genus* ≥ 1.

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- Computation of Lyapunov exponents are programmed (Zorich, Trevino, ...): numerical values can be easily obtained
- In the Abelian case Eskin's program gives numerical values of $c(\mathcal{C})$ for each configuration.
- Volumes are known explicitly only in the Abelian case (Eskin–Okounkov) and in the quadratic case in genus 0 (Athreya–Eskin–Zorich)

Computing volumes of moduli spaces

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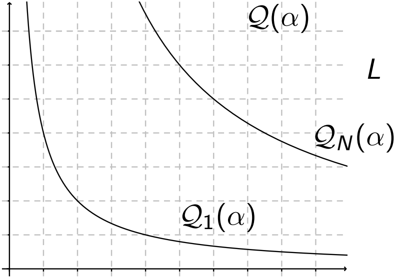

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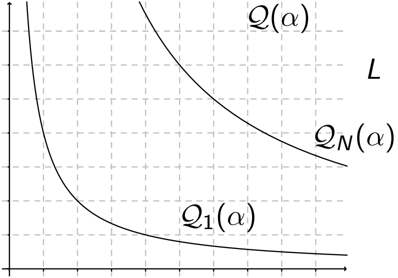

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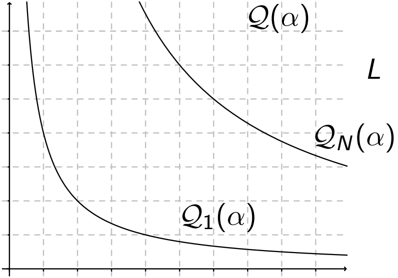

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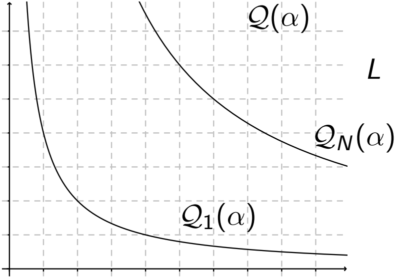

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Integer points in the moduli space (surfaces $S \in L$) correspond to square-tiled surfaces / pillowcases covers.

Conventions (corresponding to [AEZ]):

- labelled zeroes
- $\mathcal{Q}_1(\alpha)$ correspond to surfaces of area $1/2$
- $L = \left(H_1^-(\hat{S}, \hat{\Sigma}; \mathbb{Z})\right)_*$

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In this convention, all saddle connections and loops representing non trivial cycles in the relative homology are "half-integer". Trivial cycles are integer.

Counting integer points of area $\leq N/2$: counting square-tiled surfaces with $2N$ squares of size $1/2 \times 1/2$.

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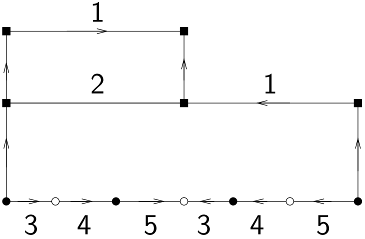

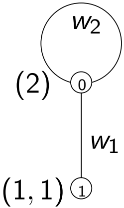

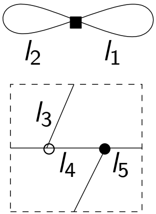

An example

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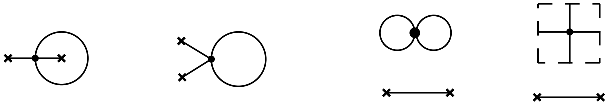

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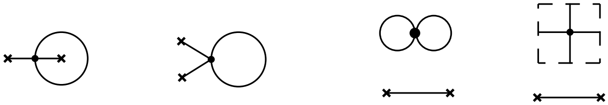

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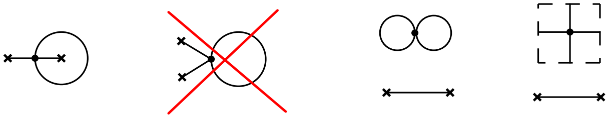

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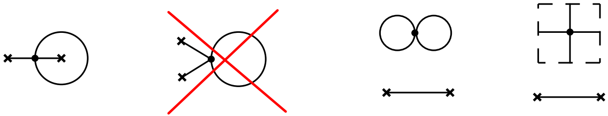
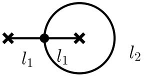
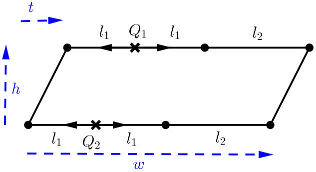
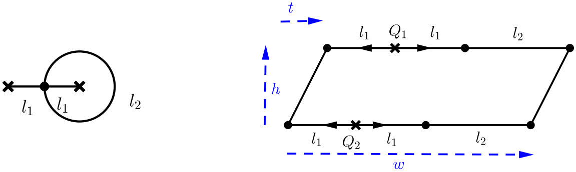


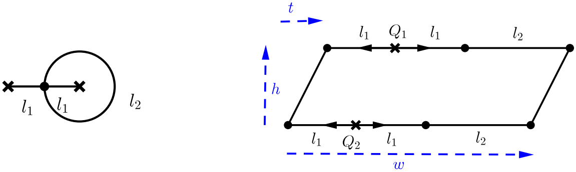
2. Eliminate ribbon graphs with no admissible gluing of cylinders.
3. For each diagram, count the number of square-tiled surfaces of this type with at most $2N$ squares.


l_1
l_1
l_2

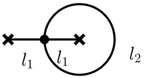

t
l_1
Q_1
l_1
l_2
h
l_1
Q_2
l_1
l_2
w

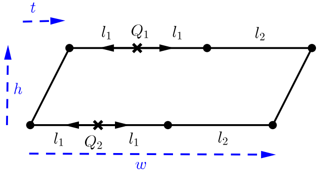


All parameters are "half integer".



All parameters are “half integer”. Equation $w = 2l_1 + l_2$ has $\simeq w$ solutions.

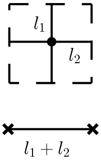


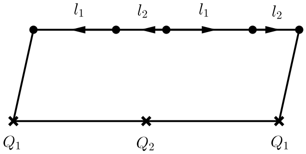


All parameters are "half integer". Equation $w = 2l_1 + l_2$ has $\simeq w$ solutions. The number of square-tiled surfaces with area at most $N/2$ of this type is:

$$\sum_{wh \leq N/2} 2w^2 = \sum_{WH \leq 2N} \frac{W^2}{2} \sim \frac{1}{2}\frac{(2N)^3}{3}\zeta(3) = \frac{4N^3}{3}\zeta(3)$$

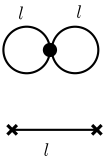
(with $W = 2w$, $H = 2h$, integers).

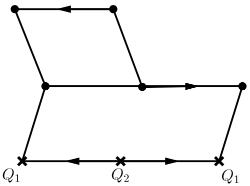




$w = W$ is an integer and all other parameters are "half integer". Equation $W = 2(l_1 + l_2)$ has $\simeq W$ solutions. Here we have a factor $1/4$ responsible for the symmetries of the ribbon graph. The number of square-tiled surfaces with area at most $N/2$ of this type is:

$$\frac{1}{4} \sum_{Wh \le N/2} 2W \cdot W = \frac{1}{2} \sum_{WH \le N} W^2 \sim \frac{N^3}{6} \zeta(3)$$





All parameters are "half integer". Equation $w = l$ has 1 solution. Here we have a factor 1/2 responsible for the symmetries of the ribbon graph.
The number of square-tiled surfaces with area at most $N/2$ of this type is:

$$\frac{1}{2}\sum_{w(2h_1+h_2)\le N/2} 2w(4w) = \sum_{W(2H_1+H_2)\le 2N} W^2 \sim \frac{N^3}{6}(8\zeta(2) - 9\zeta(3))$$

$\dim_{\mathbb{C}} \mathcal{Q}(2,-1^2) = 3$
We obtain

$$\operatorname{Vol}\mathcal{Q}_1(2,-1^2) = \underbrace{8\zeta(3)+\zeta(3)}_{1-cyl\ diag} + (\underbrace{8\zeta(2)-9\zeta(3)}_{2-cyl\ diag}) = 8\zeta(2)$$

Stratum	1*cyl*	2*cyl*	3*cyl*	*Vol*
$\mathcal{Q}(1,-1^5)$	$40\zeta(4)$	$50\zeta(4)$		$90\zeta(4)=\pi^4$
$\mathcal{Q}(1^2,-1^6)$	$140\zeta(6)$	$210\zeta(6)$	$\frac{245}{2}\zeta(6)$	$\frac{945}{2}\zeta(6)=\frac{\pi^6}{2}$
$\mathcal{Q}(2,-1^2)$	$9\zeta(3)$	$8\zeta(2)-9\zeta(3)$		$8\zeta(2)=\frac{4\pi^2}{3}$
$\mathcal{Q}(1^2,-1^2)$	$\frac{50}{3}\zeta(4)$	$\frac{40}{3}\zeta(4)$		$30\zeta(4)=\frac{\pi^4}{3}$
$\mathcal{Q}(3,-1^3)$	$30\zeta(4)$	$20\zeta(4)$		$50\zeta(4)=\frac{5\pi^4}{9}$
$\mathcal{Q}(2,1^2)$	$\frac{11}{2}\zeta(5)$	$-\frac{11}{2}\zeta(5)$ $+3\zeta(2)\zeta(3)$ $+\frac{16}{3}\zeta(4)$	$-3\zeta(2)\zeta(3)$ $+\frac{20}{3}\zeta(4)$	$12\zeta(4)=\frac{2\pi^4}{15}$
$\mathcal{Q}(5,-1)$	$12\zeta(4)$	$\frac{20}{3}\zeta(4)$		$\frac{56}{3}\zeta(4)=\frac{28\pi^4}{135}$

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- For small strata use Eskin–Okounkov's work to obtain explicit volumes

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- For small strata use Eskin–Okounkov's work to obtain explicit volumes
- Compute proportion of 1-cylinder diagrams experimentally (Zorich and Delecroix), and compute (exactly) contribution of 1-cylinder diagrams (Zograf, Zorich)

Thank you for your attention !

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