Modeling very oscillating signals. Application to image processing.

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Abstract

This article is a companion paper of a previous work [6] where we have developed the numerical analysis of a variational model first introduced by L. Rudin, S. Osher and E. Fatemi [22] and revisited by Y. Meyer [18] for removing the noise and capturing textures in an image. The basic idea in this model is to decompose an image f into two components $(u+v)$ and then to search for (u,v) as a minimizer of an energy functional. The first component u belongs to BV and contains geometrical informations while the second one v is sought in a space G which contains signals with large oscillations, i.e. noise and textures. In [18] Y. Meyer carried out his study in the whole $\mathbb{R}^2$ and his approach is rather built on harmonic analysis tools. We place ourselves in the case of a bounded set Ω of $\mathbb{R}^2$ which is the proper setting for image processing and our approach is based upon functional analysis arguments. We define in this context the space G, give some of its properties and then study in this continuous setting the energy functional which allows us to recover the components u and v. We present some numerical experiments to show the relevance of the model for image decomposition and for image denoising.

Key-words: Sobolev spaces, functions of bounded variations, PDEs, oscillating patterns, image decomposition, convex analysis, optimization, calculus of variations.

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1. Introduction

1.1 Rudin-Osher-Fatemi model

Image restoration is an important and challenging inverse problem in image analysis. The problem consists in reconstructing an image u from a degraded data f. The most common model linking u to f is the following one: $f = Ru + \eta$, where R is a linear operator typically modeling blur and η is the noise. Energy minimization has demonstrated to be a powerful approach to tackle this kind of problem (see [4] and references therein for instance). Here we examine a pure denoising situation, i.e. R is the identity operator. The underlying energy is generally composed of two terms: a fidelity term to the data and a regularizing-cost function. One of the most effective method is the total variation minimization as proposed in [22]. This model relies on the assumption that $BV(\Omega)$, the space of functions with bounded variation, is a good space to study images (even if it is known that such an assumption is too restrictive [2]). In [22], the authors decompose an image f into a component u belonging to $BV(\Omega)$ and a component v in $L^2(\Omega)$. In this model v is supposed to be the noise. In such an approach, they minimize:

$$\inf_{(u,v)\in BV(\Omega)\times L^2(\Omega)/f=u+v} \left(\int |Du| + \frac{1}{2\lambda}\|v\|^2_{L^2(\Omega)} \right) \tag{1.1}$$

where $\int |Du|$ stands for the total variation of u. In practice, they compute a numerical solution of the Euler-Lagrange equation associated to (1.1). The mathematical study of (1.1) has been done in [11].

In [18], Y. Meyer shows some limitations of the model proposed in [22]. In particular, if f is the characteristic function of a bounded domain with a C^∞-boundary, then f is not preserved by the Rudin-Osher-Fatemi model (contrary to what should be expected).

1.2 Meyer model

In [18], Y. Meyer suggests a new decomposition. He proposes the following model:

$$\inf_{(u,v)\in BV(\mathbb{R}^2)\times G(\mathbb{R}^2)/f=u+v} \left(\int |Du| + \alpha \|v\|_{G(\mathbb{R}^2)} \right) \tag{1.2}$$

where the Banach space $G(\mathbb{R}^2)$ contains signals with large oscillations, and thus in particular textures and noise. We give here the definition of $G(\mathbb{R}^2)$.

Definition 1.1. $G(\mathbb{R}^2)$ is the Banach space composed of distributions f which can be written

$$f = \partial_1 g_1 + \partial_2 g_2 = \operatorname{div}(g) \tag{1.3}$$

with g_1 and g_2 in $L^\infty(\mathbb{R}^2)$. The space $G(\mathbb{R}^2)$ is endowed with the following norm:

$$\begin{aligned} \|v\|_{G(\mathbb{R}^2)} \quad = \quad & \inf \Big\{ \|g\|_{L^\infty(\mathbb{R}^2)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} |g(x)| \ / v = \operatorname{div}(g), \ g = (g_1, g_2), \\ & g_1 \in L^\infty(\mathbb{R}^2), g_2 \in L^\infty(\mathbb{R}^2), |g(x)| = \sqrt{(|g_1|^2 + |g_2|^2)(x)} \Big\} \end{aligned} \tag{1.4}$$

$BV(\mathbb{R}^2)$ has no simple dual space (see [3]). However, as shown by Y. Meyer [18], $G(\mathbb{R}^2)$ is the dual space of the closure in $BV(\mathbb{R}^2)$ of the Schwartz class. So it is very related to the dual space of $BV(\mathbb{R}^2)$. This is a motivation to decompose a function f on $BV(\mathbb{R}^2) + G(\mathbb{R}^2)$. This is also why the divergence operator naturally appears in the definition of $G(\mathbb{R}^2)$, since the gradient and the divergence operators are dual operators.

A function belonging to G may have large oscillations and nevertheless have a small norm. Thus the norm on G is well-adapted to capture the oscillations of a function in an energy minimization method.

In [18], the author works on the whole $\mathbb{R}^2$. This is not the proper setting for image processing (it skips the problem of boundary conditions). We intend here to give a definition of G for a bounded domain $\Omega \subset \mathbb{R}^2$. Moreover, [18] gives no way to compute numerically a solution of (1.2). To fill this gap, some numerical models have been proposed in the literature [24, 21, 6]. We review below some of them.

1.3 Vese-Osher model

L. Vese and S. Osher were the first authors to numerically tackle Meyer program [24]. They actually solve the problem:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)} \left(\int |Du| + \lambda \|f - u - v\|_2^2 + \mu \|v\|_{G(\Omega)} \right) \tag{1.5}$$

where Ω is a bounded open set. To compute their solution, they replace the term $\|v\|_{G(\Omega)}$ by $\|\sqrt{g_1^2 + g_2^2}\|_p$ (where $v = \operatorname{div}(g_1, g_2)$). Then they formally derive the Euler-Lagrange equations from (1.5). For numerical reasons, the authors use the value $p = 1$ (they claim they made experiments for $p = 1 \dots 10$, and that they did not see any visual difference). They report good numerical results. See also [21] for another related model concerning the case $\lambda = +\infty$ and $p = 2$.

1.4 Aujol-Aubert-Blanc-Féraud-Chambolle model

Inspired from the work by A. Chambolle [10], the authors of [6, 5] propose a relevant approach to solve Meyer problem. They consider the following functional defined on $L^2(\Omega) \times L^2(\Omega)$:

$$F_{\lambda,\mu}(u,v) = \begin{cases} \int_\Omega |Du| + \frac{1}{2\lambda}\|f - u - v\|_{L^2(\Omega)}^2 & \text{if } (u,v) \in BV(\Omega) \times G_\mu(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{1.6}$$

where $G_\mu(\Omega) = \left\{ v \in G(\Omega) \ / \ \|v\|_{G(\Omega)} \leq \mu \right\}$. And the problem to solve is:

$$\inf_{L^2(\Omega)\times L^2(\Omega)} F_{\lambda,\mu}(u,v) \tag{1.7}$$

The authors of [6] present their model in a discrete framework. They carry out a complete mathematical analysis of their discrete model, showing how it approximately solves Meyer problem.

The aim of this paper is to carry out the mathematical analysis of (1.2) and (1.7) in a continuous setting. This will clarify a few facts about Meyer ideas developed in [18]. We first need to give the proper definition of $G(\Omega)$ when Ω is a bounded domain of $\mathbb{R}^2$. This will be done in Section 2, where we fix notations, define $G(\Omega)$, and then give some properties, explaining in particular why this space is so interesting to model signals with strong oscillations. Thanks to these preliminaries, we solve Meyer problem in Section 3, showing the existence of a solution for (1.2). In Section 4, we carry out the mathematical analysis of (1.7) in the continuous framework we have introduced. This proves the relevance of the numerical algorithm developed in [6]. We end this paper by showing some numerical results on images.

2. A space for modeling oscillating patterns in bounded domains

2.1 Preliminaries

Throughout our study, we will use the following classical distributional spaces. $\Omega \subset R^N$ will denote an open bounded set of R^N.

- $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is the set of functions in $C^\infty(\Omega)$ with compact support in Ω. We denote by $\mathcal{D}'(\Omega)$ the dual space of $\mathcal{D}(\Omega)$, i.e. the space of distributions on Ω. $\mathcal{D}(\bar{\Omega})$ is the set of restriction to Ω of functions in $\mathcal{D}(\mathbb{R}^N) = C_c^\infty(\mathbb{R}^N)$.
- $W^{m,p}(\Omega)$ denotes the space of functions in $L^p(\Omega)$ whose distributional derivatives $D^\alpha u$ are in $L^p(\Omega)$, $p \in [1, +\infty)$, $m \geq 1$, $m \in \mathbb{N}$, $|\alpha| \leq m$. $W_0^{m,p}(\Omega)$ denotes the space of functions in $W^{m,p}(\Omega)$ with compact support in Ω. For further details on these spaces, we refer the reader to [1, 14].
- $BV(\Omega)$ is the subspace of functions $u \in L^1(\Omega)$ such that the following quantity is finite:

$$J(u) = \sup \left\{ \int_\Omega u(x) \mathrm{div}\, (\xi(x)) dx / \xi \in C_c^\infty(\Omega, \mathbb{R}^N), \|\xi\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq 1 \right\} \tag{2.1}$$

  $BV(\Omega)$ endowed with the norm $\|u\|_{BV} = \|u\|_{L^1} + J(u)$ is a Banach space. If $u \in BV(\Omega)$, the distributional derivative Du is a bounded Radon measure and (2.1) corresponds to the total variation $|Du|(\Omega)$.
  For $\Omega \subset R^2$, we have [3]: $BV(\Omega) \subset L^2(\Omega)$. For further details on $BV(\Omega)$, we refer the reader to [3].

2.2 Definition and properties of $G(\Omega)$

In all the sequel, we denote by Ω a bounded connected open set of $\mathbb{R}^2$ with a Lipschitz boundary. We adapt Definition 1.1 concerning the space G to the case of Ω. We are going to consider a subspace of the Banach space $W^{-1,\infty}(\Omega) = \left(W_0^{1,1}(\Omega)\right)'$ (the dual space of $W_0^{1,1}(\Omega)$).

Definition 2.1. $G(\Omega)$ is the subspace of $W^{-1,\infty}(\Omega)$ defined by:

$$G(\Omega) = \left\{ v \in L^2(\Omega) \ / \ v = \mathrm{div}\, \xi \ , \ \xi \in L^\infty(\Omega, \mathbb{R}^2) \ , \ \xi.N = 0 \text{ on } \partial\Omega \right\} \tag{2.2}$$

On $G(\Omega)$, the following norm is defined:

$$\|v\|_{G(\Omega)} = \inf \left\{ \|\xi\|_{L^\infty(\Omega,\mathbb{R}^2)} \ / v = \mathrm{div}\, \xi \ , \ \xi.N = 0 \text{ on } \partial\Omega \right\} \tag{2.3}$$

Remark: In Definition 2.1, since $\mathrm{div}\, \xi \in L^2(\Omega)$ and $\xi \in L^\infty(\Omega, \mathbb{R}^2)$, we can define $\xi.N$ on $\partial\Omega$ (in this case, $\xi.N \in H^{-1/2}(\partial\Omega)$, see [23, 17] for further details).

The next lemma was stated in [18]. Using approximations with $C_c^\infty(\Omega)$ functions [3], the proof is straightforward:

Lemma 2.1. *Let $u \in BV(\Omega)$ and $v \in G(\Omega)$. Then: $\int_\Omega uv \leq J(u)\|v\|_{G(\Omega)}$ (where $J(u)$ is defined by (2.1)).*

We have the following simple characterization of $G(\Omega)$:

Proposition 2.1.

$$G(\Omega) = \left\{ v \in L^2(\Omega) \ / \ \int_\Omega v = 0 \right\} \tag{2.4}$$

Proof: Let us denote by $H(\Omega)$ the right-hand side of (2.4). We split the proof into two steps.
Step 1: Let v be in $G(\Omega)$. Then from(2.2) it is immediate that $\int_\Omega v = 0$, i.e. $v \in H(\Omega)$.
Step 2: Let v be in $H(\Omega)$. Then from [7] (Theorem 3') (see also [8]), there exists $\xi \in C^0(\bar{\Omega}, \mathbb{R}^2) \cap W^{1,2}(\Omega, \mathbb{R}^2)$ such that $v = \operatorname{div} \xi$ and $\xi = 0$ on $\partial\Omega$. In particular, we have $\xi \in L^\infty(\Omega, \mathbb{R}^2)$ and $\xi.N = 0$ on $\partial\Omega$. Thus $v \in G(\Omega)$.

■

Remark: Let us stress here how powerful the result in [8, 7] is. It deals with the limit case v in $L^q(\Omega)$, $q = 2$, when the dimension of the space is $N = 2$. The classical method for tackling the equation $\operatorname{div} \xi = v$ with $\xi.N = 0$ on $\partial\Omega$ consists in solving the problem $\Delta u = v$ with $\frac{\partial u}{\partial N} = 0$ on $\partial\Omega$, and in setting $\xi = \nabla u$. If v is in $L^q(\Omega)$ with $q > 2$ this problem admits a unique solution (up to a constant) in $W^{2,q}(\Omega)$. Moreover, thanks to standard Sobolev embeddings (see [14, 15]), $\xi = \nabla u$ belongs to $L^\infty(\Omega, \mathbb{R}^2)$. If $q = 2$, the result is not true and the classical approach does not work. So the result by Bourgain and Brezis is very sharp.

We next introduce a family of convex subsets of $G(\Omega)$. These convex sets will be useful for approximating Meyer problem.

Definition 2.2. Let $G_\mu(\Omega)$ the family of subsets defined by ($\mu > 0$):

$$G_\mu(\Omega) = \left\{ v \in G(\Omega) \ / \ \|v\|_{G(\Omega)} \leq \mu \right\} \tag{2.5}$$

Lemma 2.2. *$G_\mu(\Omega)$ is closed for the $L^2(\Omega)$-strong topology.*

Proof of Lemma 2.2 Let (v_n) be a sequence in $G_\mu(\Omega)$ such that there exists $\hat{v} \in L^2(\Omega)$ with $v_n \to \hat{v}$ in $L^2(\Omega)$-strong. We have $v_n = \operatorname{div} \xi_n$, with ξ_n such that $\|\xi_n\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \mu$ and $\xi_n.N = 0$ on $\partial\Omega$. As $\|\xi_n\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \mu$, there exists $\hat{\xi} \in L^\infty(\Omega, \mathbb{R}^2)$ such that, up to an extraction: $\xi_n \rightharpoonup \hat{\xi}$ in $L^\infty(\Omega, \mathbb{R}^2)$ weak * , and $\|\hat{\xi}\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \mu$.
Moreover if $\phi \in \mathcal{D}(\bar{\Omega})$: $\int_\Omega v_n \phi \, dx = \int_\Omega \operatorname{div} \xi_n \phi \, dx = -\int_\Omega \xi_n \nabla \phi \, dx$. Thus as $n \to +\infty$, we get:

$$\int_\Omega \hat{v}\phi \, dx = -\int_\Omega \hat{\xi} \nabla \phi \, dx = \int_\Omega \operatorname{div} \hat{\xi} \, \phi \, dx - \int_{\partial\Omega} \hat{\xi}.N \phi \tag{2.6}$$

By choosing first a test function in $C_c^\infty(\Omega)$, we deduce from (2.6) that $\hat{v} = \operatorname{div} \hat{\xi}$ in $\mathcal{D}'(\Omega)$, and since $\hat{v} \in L^2(\Omega)$, the equality holds in $L^2(\Omega)$. Then for a general $\phi \in \mathcal{D}(\bar{\Omega})$, it comes $\hat{\xi}.N = 0$ on $\partial\Omega$ (in $H^{-1/2}(\partial\Omega)$).

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The next result is a straightforward consequence of Lemma 2.2.

Corollary 2.1. *The indicator function of $G_\mu(\Omega)$ is lsc (lower-semicontinuous) for the $L^2(\Omega)$-strong topology (and for the $L^2(\Omega)$-weak topology since G_μ is convex).*

Remarks:

1. Let us denote by $K(\Omega)$ the closure in $L^2(\Omega)$ of the set:
$$\left\{ \operatorname{div} \xi \ , \ \xi \in C_c^\infty(\Omega, \mathbb{R}^2) \ , \ \|\xi\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq 1 \right\} \tag{2.7}$$
Using Lemma 2.2 and some results in [23], one can prove that $K(\Omega) = G_1(\Omega)$.
Moreover, one can also show in the same way that $G(\Omega)$ is the closure in $L^2(\Omega)$ of the set:
$$\left\{ \operatorname{div} \xi \ , \ \xi \in C_c^\infty(\Omega, \mathbb{R}^2) \ \right\} \tag{2.8}$$

2. From the proof of Lemma 2.2, one easily deduces that $\|.\|_G$ is lower semi continuous (lsc).

We also have the following result:

Lemma 2.3. *If $v \in G(\Omega)$, then there exists $\xi \in L^\infty(\Omega, \mathbb{R}^2)$ with $v = \operatorname{div} \xi$ and $\xi.N = 0$ on $\partial\Omega$, and such that $\|v\|_G = \|\xi\|_{L^\infty(\Omega,\mathbb{R}^2)}$.*

Proof: Let $v \in G(\Omega)$. Let us consider a sequence $\xi_n \in L^\infty(\Omega, \mathbb{R}^2)$ with $v = \operatorname{div} \xi_n$ and $\xi_n.N = 0$ on $\partial\Omega$, and such that $\|\xi_n\|_{L^\infty(\Omega)} \to \|v\|_G$. There exists $\xi \in L^\infty(\Omega, \mathbb{R}^2)$ such that, up to an extraction, $\xi_n \rightharpoonup \xi$ in $L^\infty(\Omega, \mathbb{R}^2)$ weak *. Then, as in the proof of Lemma 2.2, we can show that $\xi.N = 0$ on $\partial\Omega$ and that $v = \operatorname{div} \xi$.

■

Main property: The following lemma is due to Y. Meyer [18]. But it was stated in the case of $\Omega = \mathbb{R}^2$, and the proof relied upon harmonic analysis tools. Thanks to our definition of $G(\Omega)$, we formulate it in the case when Ω is bounded. Our proof relies upon functional analysis arguments.

Lemma 2.4. *Let Ω be a Lipschitz bounded open set, and let f_n , $n \geq 1$ be a sequence of functions in $L^q(\Omega) \bigcap G(\Omega)$ with the following two properties:*

1. *There exists $q > 2$ and $C > 0$ such that $\|f_n\|_{L^q(\Omega)} \leq C$.*
2. *The sequence f_n converges to 0 in the distributional sense (i.e. in $\mathcal{D}'(\Omega)$).*

Then $\|f_n\|_G$ converges to 0 when n goes to infinity.

This result explains why the norm in $G(\Omega)$ is a good norm to tackle signals with strong oscillations. It will be easier with this norm to capture such signals in a minimization process than with a classical L^2-norm.

Remark: Hypotheses 1. and 2. are equivalent to the simpler one: *there exists $q > 2$ such that $f_n \rightharpoonup 0$ in $L^q(\Omega)$-weak.*

Proof of Lemma 2.4: Let us consider a sequence $f_n \in L^q(\Omega) \bigcap G(\Omega)$ satisfying assumption 1. and let us define the Neumann problem:
$$\begin{cases} \Delta u_n = f_n \ in \ \Omega \\ \frac{\partial u_n}{\partial N} = 0 \ on \ \partial\Omega \end{cases} \tag{2.9}$$
We recall that as $f_n \in G(\Omega)$, we also have $\int_\Omega f_n \, dx = 0$. We know (see [16, 20, 12]) that problem (2.9) admits a solution $u_n \in W^{2,q}(\Omega)$. From [20, 19], we also know that there exists a constant $B > 0$ such that: $\|u_n\|_{W^{2,q}(\Omega)} \leq B\|f_n\|_{L^q(\Omega)}$. And as we assume that $\|f_n\|_{L^q(\Omega)} \leq C$, we get:
$$\|u_n\|_{W^{2,q}(\Omega)} \leq BC \tag{2.10}$$

Since $q > 2$ and Ω bounded, we know (see [1]) that there exists $\theta \in (0, 1)$ such that $W^{2,q}(\Omega)$ is compactly embedded in $C^{1,\theta}(\Omega)$. We denote by $g_n = \nabla u_n$. We have $\|g_n\|_{W^{1,q}(\Omega)} \leq \|u_n\|_{W^{2,q}(\Omega)} \leq BC$. And it is also standard that $W^{1,q}(\Omega)^2$ is compactly embedded in $C^{0,\theta}(\Omega)^2$.

Hence, up to an extraction, we get that there exists u and $g \in C^{0,\theta}$ such that $u_n \to u$ and $g_n \to g$ (for the $C^{0,\theta}$ topology). It is then standard to pass to the limit in (2.9) to deduce that $g_n \to 0$ uniformly (we recall that $g_n = \nabla u_n$). The previous reasonning being true for any subsequence extracted from u_n, we conclude that the whole sequence ∇u_n is such that $\nabla u_n \to 0$ as $n \to +\infty$ in $L^\infty(\Omega, \mathbb{R}^2)$-strong, i.e. $g_n = \nabla u_n \to 0$ in $L^\infty(\Omega, \mathbb{R}^2)$-strong. Since $f_n = \operatorname{div} g_n$, we easily deduce that $\|f_n\|_G \to 0$.

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3. Study of Meyer problem

Thanks to Section 2, we are now in position to carry out the mathematical study of Meyer problem [18].

Let $f \in L^q(\Omega)$ (with $q \geq 2$). We recall that the considered problem is:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)/f=u+v} \left(J(u) + \alpha \|v\|_{G(\Omega)} \right) \tag{3.1}$$

where $J(u)$ is the total variation $|Du|$ defined by (2.1).

Remark: Since f is an image, we know that $f \in L^\infty(\Omega)$. Thus it is not restrictive to suppose $q \geq 2$.

Before considering problem (3.1), we first need to show that we can always decompose a function $f \in L^q(\Omega)$ into two components $(u, v) \in BV(\Omega) \times G(\Omega)$.

Lemma 3.1. *Let $f \in L^q(\Omega)$ (with $q \geq 2$). Then there exists $u \in BV(\Omega)$ and $v \in G(\Omega)$ such that $f = u + v$.*

Proof: Let us choose $u = \frac{1}{|\Omega|}\int_\Omega f$ and $v = f - u = f - \frac{1}{|\Omega|}\int_\Omega f$. We therefore have $u \in BV(\Omega)$ (since Ω is bounded), and $v \in L^2(\Omega)$. Moreover, since $\int_\Omega v = 0$ we deduce from Proposition 2.4 that $v \in G(\Omega)$.

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We now show that problem (3.1) admits at least one solution.

Proposition 3.1. *Let $f \in L^q(\Omega)$ (with $q \geq 2$). Then there exists $\hat{u} \in BV(\Omega)$ and $\hat{v} \in G(\Omega)$ such that $f = \hat{u} + \hat{v}$, and:*

$$J(\hat{u}) + \alpha\|\hat{v}\|_G = \inf_{(u,v)\in BV(\Omega)\times G(\Omega)/f=u+v} \left(J(u) + \alpha\|v\|_G \right) \tag{3.2}$$

Proof: Let us first remark that the functional to minimize in (3.1) is convex with respect to its two variables. Moreover, the infimum in (3.1) is finite (thanks to Lemma 3.1).

Now, let (u_n, v_n) be a minimizing sequence for (3.1) We thus have for some constant C

$$J(u_n) \leq C \text{ and } \|v_n\|_G \leq C \tag{3.3}$$

From Poincaré inequality (see [3]), there exists a constant $B > 0$ such that: $\|u_n - \int_\Omega u_n\|_{L^2(\Omega)} \leq BJ(u_n)$. Thus from (3.3), we get $\|u_n - \int_\Omega u_n\|_{L^2(\Omega)} \leq BC$. But as $u_n + v_n = f$, we have:

$$\int_\Omega u_n + \underbrace{\int_\Omega v_n}_{=0 \text{ since } v_n \in G(\Omega)} = \int_\Omega f \tag{3.4}$$

Hence u_n is bounded in $L^2(\Omega)$. From (3.3), we deduce that u_n is bounded in $BV(\Omega)$. Thus there exists $\hat{u} \in BV(\Omega)$ such that $u_n \rightharpoonup \hat{u}$ in $BV(\Omega)$ weak *. And as $u_n + v_n = f$, we deduce that v_n is also bounded in $L^2(\Omega)$. Therefore, there exists $\hat{v} \in L^2(\Omega)$ such that, up to an extraction, $v_n \rightharpoonup \hat{v}$ in $L^2(\Omega)$ weak.

To conclude, there remains to prove that $(\hat{u}, \hat{v})$ is a minimizer of $J(u) + \alpha\|v\|_{G(\Omega)}$. And this last point comes from the fact that J is lower semi-continuous (lsc) with respect to the BV weak * topology [3], and from the fact that $\|.\|_G$ is lsc with respect to the L^2-weak topology.

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Remark: The uniqueness of a solution for Meyer problem is an open question.

4. An algorithm to solve Meyer problem

In Section 3, we proved the existence of a solution to Meyer problem. We now want to give an algorithm to compute a solution.

4.1 Total variation minimization as a projection

Recently, a projection algorithm to solve Rudin-Osher-Fatemi model [22] has been proposed in [10]. Precisely, the problem studied in [10] is:

$$\inf_{u \in BV(\Omega)} \left(J(u) + \frac{1}{2\lambda} \|f - u\|^2_{L^2(\Omega)} \right) \tag{4.1}$$

In [10], the following result is shown (in fact, the author in [10] considers the discrete case, but as he mentions, the continuous case is similar). It relies on standard convex analysis arguments [13].

Proposition 4.1. *The solution of (4.1) is given by:* $u = f - P_{G_\lambda(\Omega)}(f)$ *, where* $P_{G_\lambda(\Omega)}$ *is the orthogonal projector on* $G_\lambda(\Omega)$*.*

Remark: $P_{G_\lambda(\Omega)}$ is well defined since $G_\lambda(\Omega)$ is a closed convex set of $L^2(\Omega)$.

4.2 Solving Meyer problem as a projection

We now study the algorithm presented in [6] in the continuous setting (whereas in [6], the authors carried out their analysis in a discrete framework).

Let us introduce the following functional defined on $L^2(\Omega) \times L^2(\Omega)$:

$$F_{\lambda,\mu}(u,v) = \begin{cases} J(u) + \frac{1}{2\lambda}\|f - u - v\|^2_{L^2(\Omega)} & \text{if } (u,v) \in BV(\Omega) \times G_\mu(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{4.2}$$

We can rewrite $F_{\lambda,\mu}$ in the following way:

$$F_{\lambda,\mu}(u,v) = J(u) + \frac{1}{2\lambda}\|f - u - v\|^2_{L^2(\Omega)} + \chi_{G_\mu(\Omega)}(v) \tag{4.3}$$

with the convention that $J(u) = +\infty$ if $u \in L^2(\Omega) \backslash BV(\Omega)$, and where $\chi_{G_\mu(\Omega)}$ is the indicator function of $G_\mu(\Omega)$.

The problem we want to solve is:

$$\inf_{L^2(\Omega) \times L^2(\Omega)} F_{\lambda,\mu}(u,v) \tag{4.4}$$

4.3 Existence and uniqueness

We show here that problem (4.4) admits a unique solution.

Proposition 4.2. *Let* $f \in L^2(\Omega)$*. There exists a unique couple* $(\hat{u}, \hat{v})$ *which minimizes* $F_{\lambda,\mu}(u,v)$ *on* $L^2(\Omega) \times L^2(\Omega)$*.*

Proof: We split the proof into two steps.
Step 1 (Uniqueness): The proof is the same as in [6]. We just put it here for the sake of completeness.

To get the uniqueness, we first remark that F is strictly convex on $BV(\Omega) \times G_\mu(\Omega)$, as the sum of a convex function and of a strictly convex function, except in the direction $(u, -u)$. Hence it suffices to check that if $(\hat{u}, \hat{v})$ is a minimizer of F then for $t \neq 0$, $(\hat{u} + t\hat{u}, \hat{v} - t\hat{u})$ is not a minimizer of F (indeed, since F is convex, we can restrict ourselves to small perturbations, and we will assume that $|t| < 1$). The result is obvious if $\hat{v} - t\hat{u} \in L^2(\Omega) \backslash G_\mu(\Omega)$. Let us show that if $\hat{v} - t\hat{u} \in G_\mu(\Omega)$ then the result is still true. Indeed, if $\hat{v} - t\hat{u} \in G_\mu(\Omega)$, we have:

$$F(\hat{u} + t\hat{u}, \hat{v} - t\hat{u}) = F(\hat{u}, \hat{v}) + (|1 + t| - 1)J(\hat{u}) \tag{4.5}$$

By contradiction, let us assume that there exists $\hat{t} < 1$, $\hat{t} \neq 0$ such that $\hat{v} - \hat{t}\hat{u} \in G_\mu(\Omega)$ and

$$F(\hat{u} + \hat{t}\hat{u}, \hat{v} - \hat{t}\hat{u}) \leq F(\hat{u}, \hat{v}) \tag{4.6}$$

As $(\hat{u}, \hat{v})$ minimizes F, (4.6) is an equality. From (4.5), we deduce that $(|1+\hat{t}| - 1)J(\hat{u}) = 0$. And as $\hat{t} \neq \{-2, 0\}$, we get that $J(\hat{u}) = 0$. There exists therefore $\gamma \in \mathbb{R}$ such that $\hat{u} = \gamma$ a.e. (we suppose Ω connected).

1. If $\gamma = 0$, then $\hat{u} = 0$. Thus $(\hat{u} + \hat{t}\hat{u}, \hat{v} - \hat{t}\hat{u}) = (\hat{u}, \hat{v})$.

2. If $\gamma \neq 0$, then $\hat{v} - \hat{t}\hat{u}$ cannot belong to $G_\mu(\Omega)$ since its mean is not 0 (see (2.4)). This contradicts our assumption.

Step 2 (Existence): Let (u_n, v_n) a minimizing sequence for $F_{\lambda,\mu}(u,v)$, i.e.:

$$F_{\lambda,\mu}(u_n, v_n) \to \inf_{L^2(\Omega)\times L^2(\Omega)} F_{\lambda,\mu}(u,v) \tag{4.7}$$

It is clear that $v_n \in G_\mu$ and:

$$J(u_n) \leq C \text{ and } \|f - u_n - v_n\|^2_{L^2(\Omega)} \leq C \tag{4.8}$$

Since $v_n \in G_\mu$, we get that $v_n = \operatorname{div} \xi_n$, with ξ_n such that $\|\xi_n\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \mu$ and $\xi_n.N = 0$ on $\partial\Omega$.

We also have:

$$\int_\Omega u_n v_n = \int_\Omega u_n \operatorname{div} \xi_n \leq \sup_{\xi\in G_\mu(\Omega)} \int_\Omega u_n \operatorname{div} \xi \leq \mu \sup_{\xi\in G_1(\Omega)} \int_\Omega u_n \operatorname{div} \xi \tag{4.9}$$

Hence (using (4.8)):

$$\int_\Omega u_n v_n \leq \mu J(u_n) \leq C \tag{4.10}$$

From (4.8) and (4.10), we deduce that u_n and v_n are bounded in $L^2(\Omega)$. Thanks to (4.8), and since u_n and v_n are bounded in $L^2(\Omega)$, we get that there exists $(\hat{u}, \hat{v})$ such that, up to an extraction: $u_n \rightharpoonup \hat{u}$ in $BV(\Omega)$ weak *, and $v_n \rightharpoonup \hat{v}$ in $L^2(\Omega)$ weak.

From Lemma 2.2, we know that $\chi_{G_\mu(\Omega)}$ is L^2-weakly lsc, where:

$$\chi_{G_\mu(\Omega)}(v) = \begin{cases} 0 \text{ if } v \in G_\mu(\Omega) \\ +\infty \text{ otherwise} \end{cases} \tag{4.11}$$

Since the total variation is lsc for the BV weak * topology, we obtain:

$$\begin{aligned} \inf F_{\lambda,\mu} &= \underline{\lim} \left(J(u_n) + \frac{1}{2\lambda}\|f - u_n - v_n\|^2_{L^2(\Omega)} + \chi_{G_\mu(\Omega)}(v_n) \right) \\ &\geq J(\hat{u}) + \frac{1}{2\lambda}\|f - \hat{u} - \hat{v}\|^2_{L^2(\Omega)} + \chi_{G_\mu(\Omega)}(\hat{v}) \end{aligned}$$

i.e. $(\hat{u}, \hat{v})$ is a minimizer of $F_{\lambda,\mu}(u,v)$ on $L^2(\Omega) \times L^2(\Omega)$.

■

4.4 Characterization of the minimizer $(\hat{u}, \hat{v})$

The following result gives a characterization of the minimizer of $F_{\lambda,\mu}$. We will use it in the next subsection to show the convergence of our algorithm.

Proposition 4.3. *$(\hat{u}, \hat{v})$ is a minimizer of $F_{\lambda,\mu}(u,v)$ if and only if:*

$$\begin{cases} \hat{u} = f - \hat{v} - P_{G_\lambda(\Omega)}(f - \hat{v}) \\ \hat{v} = P_{G_\mu(\Omega)}(f - \hat{u}) \end{cases} \tag{4.12}$$

Proof: We split the proof into two steps:
Step 1 (Necessary condition):Let $(\hat{u}, \hat{v})$ be a minimizer of $F_{\lambda,\mu}(u,v)$. The fact that $\hat{u} = f - \hat{v} - P_{G_\lambda(\Omega)}(f - \hat{v})$ comes from Proposition 4.1. The proof of $\hat{v} = P_{G_\mu(\Omega)}(f - \hat{u})$ is even simpler: since $\hat{v}$ is a minimizer of $F(\hat{u}, .)$, we have for all v in $L^2(\Omega)$: $F_{\lambda,\mu}(\hat{u}, \hat{v}) \leq F_{\lambda,\mu}(\hat{u}, v)$, which implies: $\hat{v}$ minimizes $\frac{1}{2\lambda}\|f - \hat{u} - v\|^2_{L^2(\Omega)} + \chi_{G_\mu(\Omega)}(v)$ for all $v \in L^2(\Omega)$. And this precisely means that $\hat{v} = P_{G_\mu(\Omega)}(f - \hat{u})$.
Step 2 (Sufficient condition): Let $(\hat{u}, \hat{v})$ verifying (4.12). We therefore have:

$$\begin{cases} 0 \in \hat{u} + \hat{v} - f + \lambda\partial J(\hat{u}) \\ 0 \in \hat{u} + \hat{v} - f + \lambda\partial\chi_{G_\mu(\Omega)}(\hat{v}) \end{cases} \tag{4.13}$$

The first line of (4.13) comes from the fact that $\hat{u} = f - \hat{v} - P_{G_\lambda(\Omega)}(f - \hat{v})$ (see [10] and [6]), and the second one from the fact that $\hat{v} = P_{G_\mu(\Omega)}(f - \hat{u})$. We can rewrite (4.13):

$$\begin{cases} \frac{f-\hat{u}-\hat{v}}{\lambda} \in \partial J(\hat{u}) \\ \frac{f-\hat{u}-\hat{v}}{\lambda} \in \partial\chi_{G_\mu(\Omega)}(\hat{v}) \end{cases} \tag{4.14}$$

which means:

$$\left\{ \begin{array}{c} J(u) \geq J(\hat{u}) + \left(u - \hat{u}, \frac{f-\hat{u}-\hat{v}}{\lambda}\right)_{L^2(\Omega)\times L^2(\Omega)} \\ \chi_{G_\mu(\Omega)}(v) \geq \chi_{G_\mu(\Omega)}(\hat{v}) + \left(v - \hat{v}, \frac{f-\hat{u}-\hat{v}}{\lambda}\right)_{L^2(\Omega)\times L^2(\Omega)} \end{array} \right. \tag{4.15}$$

We add the two lines in (4.15), and we obtain:

$$J(u) + \chi_{G_\mu(\Omega)}(v) \geq J(\hat{u}) + \chi_{G_\mu(\Omega)}(\hat{v}) + \left(u - \hat{u}, \frac{f - \hat{u} - \hat{v}}{\lambda}\right) + \left(v - \hat{v}, \frac{f - \hat{u} - \hat{v}}{\lambda}\right) \tag{4.16}$$

We then add $\frac{1}{2\lambda}\|f - u - v\|^2_{L^2(\Omega)}$ to the two sides of (4.16), and we get:

$$\begin{array}{rcl} F_{\lambda,\mu}(u,v) & \geq & F_{\lambda,\mu}(\hat{u},\hat{v}) - \frac{1}{2\lambda}\|f - \hat{u} - \hat{v}\|^2_{L^2(\Omega)} + \frac{1}{2\lambda}\|f - u - v\|^2_{L^2(\Omega)} \\ & & + \left(u - \hat{u}, \frac{f - \hat{u} - \hat{v}}{\lambda}\right) + \left(v - \hat{v}, \frac{f - \hat{u} - \hat{v}}{\lambda}\right) \end{array}$$

To conclude, there remains to show that:

$$\|f - u - v\|^2_{L^2(\Omega)} - \|f - \hat{u} - \hat{v}\|^2_{L^2(\Omega)} + 2\left(u - \hat{u}, f - \hat{u} - \hat{v}\right) + 2\left(v - \hat{v}, f - \hat{u} - \hat{v}\right) \geq 0 \tag{4.17}$$

Let us denote by $L(u,v) = \|f - u - v\|^2_{L^2(\Omega)}$. Since L is convex, we have:

$$L(u,v) \geq L(\hat{u},\hat{v}) + \left(u - \hat{u}, \frac{\partial L}{\partial u}(\hat{u},\hat{v})\right) + \left(v - \hat{v}, \frac{\partial L}{\partial v}(\hat{u},\hat{v})\right) \tag{4.18}$$

And (4.18) is just (4.17) which we wanted to show.

■

The conclusions of Proposition 4.3 naturally lead to the following algorithm for computing the minimizer $(\hat{u},\hat{v})$. This algorithm was developed and applied in [6].

Algorithm:

1. Initialization:

$$u_0 = v_0 = 0 \tag{4.19}$$

2. Iterations:

$$v_{n+1} = P_{G_\mu(\Omega)}(f - u_n) \tag{4.20}$$

$$u_{n+1} = f - v_{n+1} - P_{G_\lambda(\Omega)}(f - v_{n+1}) \tag{4.21}$$

3. Stopping test: we stop if

$$\max(|u_{n+1} - u_n|, |v_{n+1} - v_n|) \leq \epsilon \tag{4.22}$$

Corollary 4.1. *The sequence (u_n, v_n) (defined by (4.19)-(4.22)) is bounded in $BV(\Omega)\times G_\mu(\Omega)$. If in addition v_n is bounded in $L^{2+\delta}(\Omega)$ for some $\delta > 0$, then (u_n, v_n) converges to $(\hat{u},\hat{v}) \in BV(\Omega)\times G_\mu(\Omega)$ (for the $BV(\Omega)-$weak * $\times L^2(\Omega)$-weak topology) the unique minimizer of $F_{\lambda,\mu}(u,v)$.*

Proof: For the reader convenience, we split the proofs into several steps. Our aim is to pass to the limit in:

$$\left\{ \begin{array}{l} v_{n+1} = P_{G_\mu(\Omega)}(f - u_n) \\ u_{n+1} = f - v_{n+1} - P_{G_\lambda(\Omega)}(f - v_{n+1}) \end{array} \right. \tag{4.23}$$

Step 1: By definition of the sequence (u_n, v_n), we have $F_{\lambda,\mu}(u_n, v_n) \geq F_{\lambda,\mu}(u_n, v_{n+1}) \geq F_{\lambda,\mu}(u_{n+1}, v_{n+1})$. Since a decreasing bounded from below sequence in $\mathbb{R}_+$ is convergent, we deduce that there exists $m \geq 0$ such that

$$\lim_{n\to+\infty} F_{\lambda,\mu}(u_n, v_n) = \lim_{n\to+\infty} F_{\lambda,\mu}(u_n, v_{n+1}) = \lim_{n\to+\infty} F_{\lambda,\mu}(u_{n+1}, v_{n+1}) = m \tag{4.24}$$

which in particular implies that

$$\lim_{n\to+\infty}\left(F_{\lambda,\mu}(u_n,v_n)-F_{\lambda,\mu}(u_n,v_{n+1})\right)=0 \tag{4.25}$$

Moreover since the sequence $F_{\lambda,\mu}(u_n,v_n)$ is bounded we get that $J(u_n)$ and $\|f-u_n-v_n\|^2_{L^2(\Omega)}$ are also bounded. We then deduce as in the proof of Proposition 4.2 that v_n (respectively u_n) is bounded in $L^2(\Omega)$ (respectively in $BV(\Omega)$).

Step 2: We now want to show that:

$$\lim_{n\to+\infty}\|v_n-v_{n+1}\|^2_{L^2(\Omega)}=0 \tag{4.26}$$

$$\lim_{n\to+\infty}\|u_n-u_{n+1}\|^2_{L^2(\Omega)}=0 \tag{4.27}$$

and

$$\lim_{n\to+\infty}\left(J(u_n)-J(u_{n+1})\right)=0 \tag{4.28}$$

To prove (4.26), we examine the difference:

$$\begin{aligned}F_{\lambda,\mu}(u_n,v_n)-F_{\lambda,\mu}(u_n,v_{n+1}) &= \frac{1}{2\lambda}\left(\|f-u_n-v_n\|^2_{L^2(\Omega)}-\|f-u_n-v_{n+1}\|^2_{L^2(\Omega)}\right)\\ &= \frac{1}{2\lambda}\left(\|v_n\|^2_{L^2(\Omega)}-\|v_{n+1}\|^2_{L^2(\Omega)}-2(f-u_n,v_n-v_{n+1})\right)\end{aligned} \tag{4.29}$$

where $(.,.)$ denotes the $L^2(\Omega)$ scalar-product. Let us recall that if K is a closed convex subset of $L^2(\Omega)$, then $u=P_K(g)$ if and only if we have (see [9] for instance):

$$(g-u,v-u)\leq 0\ ,\ \forall v\in K \tag{4.30}$$

Here we have $v_{n+1}=P_{G_\mu(\Omega)}(f-u_n)$. Hence, using (4.30), we get

$$(f-u_n-v_{n+1},v-v_{n+1})\leq 0\ \forall v\in G_\mu(\Omega) \tag{4.31}$$

If we choose, $v=v_n$, this implies: $(f-u_n-v_{n+1},v_n-v_{n+1})\leq 0$, i.e. $(f-u_n,v_n-v_{n+1})\leq(v_{n+1},v_n-v_{n+1})$. Using this last inequality in (4.29), we get:

$$\begin{aligned}F_{\lambda,\mu}(u_n,v_n)-F_{\lambda,\mu}(u_n,v_{n+1}) &\geq \frac{1}{2\lambda}\left(\|v_n\|^2_{L^2(\Omega)}-\|v_{n+1}\|^2_{L^2(\Omega)}-2(v_{n+1},v_n-v_{n+1})\right)\\ &\geq \frac{1}{2\lambda}\|v_n-v_{n+1}\|^2_{L^2(\Omega)}\end{aligned} \tag{4.32}$$

And by passing to the limit as $n\to+\infty$, we obtain (4.26). We get (4.27) in the same way, by using that $w_{n+1}=P_{G_\lambda}(f-v_{n+1})$ (with $w_n=f-u_n-w_n$) and (4.26).

We then obtain (4.28) by considering the difference $F_{\lambda,\mu}(u_n,v_{n+1})-F_{\lambda,\mu}(u_{n+1},v_{n+1}$.

Step 3: From Step 1, we know that v_n is bounded in $L^2(\Omega)$. Thus there exists an extraction ϕ and $\hat{v}\in L^2(\Omega)$ such that $v_{\phi(n)}\rightharpoonup\hat{v}$ in $L^2(\Omega)$-weak. Let us now consider the subsequence $v_{\phi(n)+1}$. Since it is bounded in $L^2(\Omega)$, we know that there exists an extraction ψ and $\tilde{v}$ such that $v_{\phi o\psi(n)+1}\rightharpoonup\tilde{v}$ in $L^2(\Omega)$-weak. But from (4.26), we deduce that $\hat{v}=\tilde{v}$. Therefore, we have up to a subsequence that v_n and $v_{n+1}\rightharpoonup\hat{v}$ in $L^2(\Omega)$ weak.

Using (4.27), we can show in the same way that up to a subsequence u_n and $u_{n+1}\rightharpoonup\hat{u}$ in $BV(\Omega)$-weak *

Step 4: We are now in position to pass to the limit in (4.23). Let us first focus on the equation: $v_{n+1}=P_{G_\mu(\Omega)}(f-u_n)$. This equality holds if and only if: $(f-u_n-v_{n+1},v-v_{n+1})\leq 0\ ,\ \forall v\in G_\mu(\Omega)$, i.e.

$$(f,v-v_{n+1})-(u_n,v)+(u_n,v_{n+1})-(v,v_{n+1})+\|v_{n+1}\|^2_{L^2(\Omega)}\leq 0\ ,\ \forall v\in G_\mu(\Omega) \tag{4.33}$$

When $n\to+\infty$, we have $(f,v-v_{n+1})\to(f,v-\hat{v})$, $(u_n,v)\to(\hat{u},v)$ and $(v,v_{n+1})\to(v,\hat{v})$. Moreover, we also have $\underline{\lim}\|v_{n+1}\|^2_{L^2(\Omega)}\geq\|v\|^2_{L^2(\Omega)}$. There remains just to check what happens for (u_n,v_{n+1}). By assumption, we have that v_n is bounded in $L^{2+\delta}(\Omega)$. Therefore, we can assume that up to a subsequence $v_{n+1}\rightharpoonup\hat{v}$ in $L^{2+\delta}(\Omega)$ weak. Let us set $1\leq p<2$ such that $\frac{1}{p}+\frac{1}{2+\delta}=1$. By the compact embedding of $BV(\Omega)$ into $L^p(\Omega)$ [3], we get that $u_n\to\hat{u}$ for the $L^p(\Omega)$-strong topology. Thus we have $(u_n,v_{n+1})\to(\hat{u},\hat{v})$.

Therefore all the terms pass to the limit in (4.33), and we obtain: $(f-\hat{u}-\hat{v},v-\hat{v})\leq 0\ ,\ \forall v\in G_\mu(\Omega)$ which means $\hat{v}=P_{G_\mu(\Omega)}(f-\hat{u})$.

And we can show in the same way that $\hat{u}=f-\hat{v}-P_{G_\lambda(\Omega)}(f-\hat{v})$.

We conclude the proof thanks to Proposition 4.3.

■

4.5 Study of the Limit problem when $\lambda \to 0$

Let us consider the following problem:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)/f=u+v} J(u) + \chi_{G_\mu}(v) \tag{4.34}$$

In the next proposition, we show that problem (4.34) is the limit problem we get when $\lambda \to 0$ in (4.4).

Proposition 4.4. *Let us assume $f \in L^2(\Omega)$, and that there exists $(\tilde{u}, \tilde{v}) \in BV(\Omega) \times G_\mu(\Omega)$ such that $f = \tilde{u} + \tilde{v}$, then:*

1. *There exists $(\hat{u}, \hat{v}) \in BV(\Omega) \times G_\mu(\Omega)$ solution of (4.34).*
2. *Moreover, if $(\hat{u}, \hat{v})$ is unique, and if (u_λ, v_λ) denotes the solution of problem (4.4), then (u_λ, v_λ) converges to $(\hat{u}, \hat{v}) \in BV(\Omega) \times G_\mu(\Omega)$ as λ goes to 0.*

Proof: As we assume the existence of $(\tilde{u}, \tilde{v})$, the infimum in (4.34) is finite. Using Lemma 2.2, the proof of the first point is then straightforward. Let us now focus on the second one. The existence and uniqueness of (u_λ, v_λ) is given by Proposition 4.2. Since (u_λ, v_λ) is the solution of problem (4.4), we have $v_\lambda \in G_\mu$, and: $F_{\lambda,\mu}(u_\lambda, v_\lambda) \leq F_{\lambda,\mu}(\hat{u}, \hat{v})$, which means (since $\hat{u} + \hat{v} = f$)

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) \leq J(\hat{u}) \tag{4.35}$$

And the left-hand side of (4.35) is given by:

$$F_{\lambda,\mu}(u_\lambda, v_\lambda) = J(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|^2_{L^2(\Omega)} + \chi_{G_\mu}(v_\lambda) = J(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|^2_{L^2(\Omega)} \tag{4.36}$$

Hence

$$J(u_\lambda) + \frac{1}{2\lambda}\|f - u_\lambda - v_\lambda\|^2_{L^2(\Omega)} \leq J(\hat{u}) \tag{4.37}$$

which implies

$$\|f - u_\lambda - v_\lambda\|^2 \leq 2\lambda J(\hat{u}) \tag{4.38}$$

We then show as in the proof of Proposition 4.2 that if $\lambda \in [0;1]$, $\|u_\lambda\|_{L^2(\Omega)}$ and thus $\|v_\lambda\|_{L^2(\Omega)}$ are bounded by a constant $C > 0$ which does not depend on λ.

Consider a sequence (λ_n) which goes to 0 as $n \to +\infty$. Then, up to an extraction, there exists $(u_0, v_0) \in BV(\Omega) \times G_\mu(\Omega)$ such that $(u_{\lambda_n}, v_{\lambda_n})$ converges to (u_0, v_0) in $L^2(\Omega)$ weak. By passing to the limit in (4.38), we get: $\|f - u_0 - v_0\|_{L^2(\Omega)} = 0$, i.e. $f = u_0 + v_0$.

To conclude the proof of the proposition, there remains to show that (u_0, v_0) is a solution of problem (4.34). We first notice that as for all $\lambda \in [0,1]$, $\|v_\lambda\|_G \leq \mu$, so we get thanks to Lemma 2.2 that $v_0 \in G_\mu$. And from (4.37), we have for all $\lambda > 0$: $J(u_\lambda) \leq J(\hat{u})$. Hence $u_0 \in BV(\Omega)$ (since $J(u_0) \leq \underline{\lim} J(u_\lambda) \leq J(\hat{u})$).

Let $(u, v) \in BV(\Omega) \times G_\mu(\Omega)$ such that $f = u + v$. We have:

$$\begin{aligned} J(u) + \chi_{G_\mu}\left(\frac{v}{\mu}\right) + \frac{1}{2\lambda}\underbrace{\|f - u - v\|^2}_{=0} &\geq J(u_{\lambda_n}) + \chi_{G_\mu}\left(\frac{v_{\lambda_n}}{\mu}\right) + \frac{1}{2\lambda_n}\|f - u_{\lambda_n} - v_{\lambda_n}\|^2 \\ &\geq J(u_{\lambda_n}) + \chi_{G_\mu}\left(\frac{v_{\lambda_n}}{\mu}\right) \end{aligned}$$

By passing to the limit as $\lambda \to 0$, we obtain: $J(u) + \chi_{G_\mu}\left(\frac{v}{\mu}\right) \geq J(u_0) + \chi_{G_\mu}\left(\frac{v_0}{\mu}\right)$. Hence (u_0, v_0) is a solution of problem (4.34). And as we have assumed that problem (4.34) has a unique solution, we deduce that $(u_0, v_0) = (\hat{u}, \hat{v})$, i.e. (u_0, v_0) is the solution of problem (4.34).

■

4.6 Link between the limit problem and Meyer problem

We now want to show the link between (4.34) and (3.1).

Proposition 4.5. *Let us fix $\alpha > 0$ in problem (3.1). Let $(\hat{u}, \hat{v})$ a solution of problem (3.1). We fix $\mu = \|\hat{v}\|_G$ in (4.34). Then:*

- *$(\hat{u}, \hat{v})$ is also a solution of problem (4.34).*
- *Conversely, any solution $(\tilde{u}, \tilde{v})$ of (4.34) (with $\mu = \|\hat{v}\|_G$) is a solution of (3.1). Moreover, we have $\|\tilde{v}\| = \mu$ and $J(\tilde{u}) = J(\hat{u})$.*

Proof: The proof is exactly the same as in [6].

■

In fact, we can say more about the link between Meyer problem (3.1) and our limit problem (4.34). We denote by

$$H_\alpha(u, v) = J(u) + \alpha\|v\|_G \tag{4.39}$$

Thus we can write Meyer problem as:

$$\inf_{(u,v)\in BV(\Omega)\times G(\Omega)\ /\ f=u+v} H_\alpha(u, v) \tag{4.40}$$

α being fixed, let us denote by

$$Z_\alpha = \left\{v_\alpha,\ v_\alpha \text{ is a solution of the problem } \inf_{v\in G(\Omega)} H_\alpha(f - v, v)\ \right\} \tag{4.41}$$

$$S_\alpha = \left\{\|v_\alpha\|_G,\ v_\alpha \text{ is a solution of the problem } \inf_{v\in G(\Omega)} H_\alpha(f - v, v)\ \right\} \tag{4.42}$$

We know that Z_α and S_α are not empty thanks to Proposition 3.1. We consider the two multi-valued maps $Y : \mathbb{R}_+ \to \mathbb{P}(G(\Omega))$ (resp. $T : \mathbb{R}_+ \to \mathbb{P}(\mathbb{R}_+)$) such that $Y(\alpha) = Z_\alpha$ (resp. $T(\alpha) = S_\alpha$), where $\mathbb{P}(G(\Omega))$ (resp. $\mathbb{P}(\mathbb{R}_+)$) stands for the set of subsets of $G(\Omega)$ (resp. $\mathbb{R}_+$).

We want to show a kind of reciprocal result to Proposition 4.5, i.e. that, for a certain range of μ, there exists α such that $\mu \in T(\alpha)$. The following result holds:

Proposition 4.6.

1. *T is a nonincreasing multi-valued map.*
2. *$Y(0) = \{f - \bar{f}\}$ and $T(0) = \|f - \bar{f}\|_G$ (where $\bar{f}$ stands for the mean value of f over Ω).*
3. *If α goes to $+\infty$, then $T(v_\alpha)$ goes to $\{a\}$, where a is defined by:*

$$a = \inf\left\{\|v\|_G\ /\ f - v \in BV(\Omega),\ v \in G(\Omega)\right\} \tag{4.43}$$

Remark: If $f \in BV(\Omega)$, then $a = 0$.

Proof: We successively show the three points of the proposition.

1. Let $\alpha_2 > \alpha_1 > 0$. Let us pick v_{α_1} in Z_{α_1} and v_{α_2} in Z_{α_2}. Let us denote by $u_{\alpha_1} = f - v_{\alpha_1}$ and $u_{\alpha_2} = f - v_{\alpha_2}$. Then, as v_{α_1} in Z_{α_1}, we have in particular: $J(u_{\alpha_1}) + \alpha_1\|v_{\alpha_1}\|_G \leq J(u_{\alpha_2}) + \alpha_1\|v_{\alpha_2}\|_G$. And as v_{α_2} in Z_{α_2}, we also have: $J(u_{\alpha_2}) + \alpha_2\|v_{\alpha_2}\|_G \leq J(u_{\alpha_1}) + \alpha_2\|v_{\alpha_1}\|_G$. Adding the two last inequalities, we get: $\alpha_1\|v_{\alpha_1}\|_G + \alpha_2\|v_{\alpha_2}\|_G \leq \alpha_1\|v_{\alpha_2}\|_G + \alpha_2\|v_{\alpha_1}\|_G$. And then

$$\underbrace{(\alpha_2 - \alpha_1)}_{>0 \text{ by hypothesis}} \quad (\|v_{\alpha_2}\|_G - \|v_{\alpha_1}\|_G) \leq 0 \tag{4.44}$$

Hence $\|v_{\alpha_2}\|_G \leq \|v_{\alpha_1}\|_G$, which proves the first point of the proposition.

2. Let us now prove the second point of the proposition.

   We have (see (4.39)) $H_0(f-v,v) = J(f-v) \geq 0$ for all $v \in G(\Omega)$. Choosing $v_0 = f - \bar{f}$ (thanks to Proposition 2.1, $v_0 \in G(\Omega)$ since $\bar{v}_0 = 0$), we get $H_0(f-v_0,v_0) = J(f-v_0) = J(\bar{f}) = 0$. Hence $0 = \inf_{v \in G(\Omega)} H_0(f-v,v)$, and $v_0 \in Z_0$. Moreover, $J(u) = 0$ if and only if $u = \bar{u}$. Let v_1 be a solution of $\inf_{v \in G(\Omega)} H_0(f-v,v)$. We thus have $\overline{f-v_1} = f - v_1$. And as $v_1 \in G(\Omega)$, we also have $\bar{v}_1 = 0$. Then $f - v_1 = \overline{f-v_1} = \bar{f} - \bar{v}_1 = \bar{f}$, i.e. $v_1 = v_0$. We conclude that $\{v_0\} = Z_0$. This shows the second point of the proposition.

3. Let us now prove the third point of the proposition.

   Let $\epsilon > 0$. From (4.43), there exists $\hat{v}_\epsilon \in G(\Omega)$ such that $\hat{u}_\epsilon = f - \hat{v}_\epsilon \in BV(\Omega)$ and:

$$a \leq \|\hat{v}_\epsilon\|_G \leq a + \epsilon \tag{4.45}$$

   Let us pick v_α in Z_α, and let us denote by $u_\alpha = f - v_\alpha$. By definition of Z_α, we have for all $(u,v) \in BV(\Omega) \times G(\Omega)$ such that $f = u + v$: $J(u_\alpha) + \alpha\|v_\alpha\|_G \leq J(u) + \alpha\|v\|_G$. By choosing $u = \hat{u}_\epsilon$, and $v = \hat{v}_\epsilon$, we get:

$$J(u_\alpha) + \alpha\|v_\alpha\|_G \leq J(\hat{u}_\epsilon) + \alpha\|\hat{v}_\epsilon\|_G \leq J(\hat{u}_\epsilon) + \alpha a + \alpha\epsilon \tag{4.46}$$

   (4.46) implies that $\|v_\alpha\|_G \leq \frac{J(\hat{u}_\epsilon)}{\alpha} + a + \epsilon$. By passing to the limit $\alpha \to +\infty$, and using (4.45), we get:

$$a \leq \liminf_{\alpha \to +\infty} \|v_\alpha\|_G \leq a + \epsilon \tag{4.47}$$

■

Remark: In fact, we have shown that $T : \mathbb{R}_+ \to [a, \|f - \bar{f}\|_G]$. In particular, T has uniformly bounded values:

1. $T(0) = \{\|f - \bar{f}\|_G\}$
2. If $\alpha \geq 0$, then if $v_\alpha \in Y(\alpha)$, we have

$$\|v_\alpha\|_G \leq \|f - \bar{f}\|_G \tag{4.48}$$

Proposition 4.7. *T is u.s.c. (upper semi continuous) (i.e. T has a closed graph and convex compact values).*

Proof: We split the proof into two steps:
Step1: Let α be in $\mathbb{R}_+$. By definition of S_α, one easily checks that $T(\alpha)$ is convex and closed in $\mathbb{R}$. Moreover we have shown that $T(\alpha)$ is uniformly bounded (see (4.48)). Therefore, $T(\alpha)$ has compact values.
Step 2: Let us now consider a sequence (α_n, v_{α_n}) where $\alpha_n \in \mathbb{R}_+$ and $v_{\alpha_n} \in Z_{\alpha_n}$. Assume that there exists (α_0, v_0) in $\mathbb{R}_+ \times G(\Omega)$ such that $(\alpha_n, v_{\alpha_n}) \to (\alpha_0, v_0)$ as n goes to $+\infty$. As v_{α_n} in Z_{α_n}, we have for all $(u,v) \in X \times G(\Omega)$ such that $f = u + v$: $J(f - v_{\alpha_n}) + \alpha_n\|v_{\alpha_n}\|_G \leq J(u) + \alpha_0\|v\|_G$. By passing to the limit as n goes to $+\infty$ (using the fact that J and $\|.\|_G$ are lsc), we get: $J(f - v_0) + \alpha_0\|v_0\|_G \leq J(u) + \alpha_0\|v\|_G$. Hence v_0 belongs to Z_{α_0}, and therefore $\|v_0\|_G$ is in S_{α_0}. This shows that T has a closed graph.

■

Corollary 4.2. *For all μ in $(a, \|f - \bar{f}\|_G)$, there exists α in $\mathbb{R}_+$ such that there exists (u,v) in $X \times G(\Omega)$ with $\|v\|_G = \mu$ and solving Meyer problem (3.1).*

Proof: This a consequence of Proposition 4.6, Proposition 4.7 and the next theorem (applied to the multi-valued map $T_\mu = T - \mu$) which we state without proof.

Theorem 4.1. *Let us consider a multi-valued map L:*

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{P}(\mathbb{R}) \\ \alpha & \mapsto & [L_{\min}(\alpha), L_{\max}(\alpha)] \end{array}$$

Let us assume that L is such that:

1. *L is u.s.c (upper semi-continuous).*

Figure 1: Barbara image

2. *There exists* $b \in \mathbb{R}$ *(resp.* $c \in \mathbb{R}$*) such that* $L_{min}(b) \leq 0$ *(resp.* $L_{max}(c) \geq 0$*).*

Then there exists $d \in [b, c]$ *such that* $0 \in L(d)$.

■

Remarks:

1. Corollary 4.2 completes the result of Proposition 4.5. It closes the link between Meyer problem (3.1) and our limit problem (4.34).
2. From the results of Subsections 4.5 and 4.6, we deduce that problem (4.4) is a good way to approximate Meyer problem.

4.7 Numerical results

We show here examples of what we obtain with the algorithm (4.19)-(4.22) developed in [6]. Denoising an image, or splitting an image into two components (a first one containing the structures, and a second one containing the oscillatory part) are two difficult inverse problems. They are both well-solved by the algorithm (4.19)-(4.22), as one can sees on the next two examples.

Image decomposition: we consider the Barbara image (see Figure 1). We first show the decomposition we get for this image on Figure 2: it displays both the BV component and the G component of the original image. Depending on the value we set for μ, the textures present in Figure 1 are separated from the BV component of the image. The larger μ is, the more the G-component contains information. The parameter λ controls the L^2 norm of the residual $f - u - v$. The smaller it is set, the smaller the residual is. To tune the parameters, one first set λ to a small value. Then one chooses μ so that the G component contains the desired amount of information.

Denoising: on Figure 3, we add a random Gaussian noise of variance $\sigma = 30.0$ to the Barbara image, and we then perform the algorithm developed in [6]. As expected by our mathematical analysis, the oscillating patterns of the original Barbara image are put into the G-component where their norms are less penalized.

References

[1] R. Adams. *Sobolev Spaces.* Pure and applied Mathematics. Academic Press, Inc, 1975.

[2] L. Alvarez, Y. Gousseau, and J.M. Morel. Scales in natural images and a consequence on their bounded variation norm. In *Scale-Space '99*, volume 1682 of *Lecture Notes in Computer Science*, 1999.

[3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variations and free discontinuity problems.* Oxford mathematical monographs. Oxford University Press, 2000.

[4] G. Aubert and P. Kornprobst. *Mathematical Problems in Image Processing*, volume 147 of *Applied Mathematical Sciences.* Springer-Verlag, 2002.

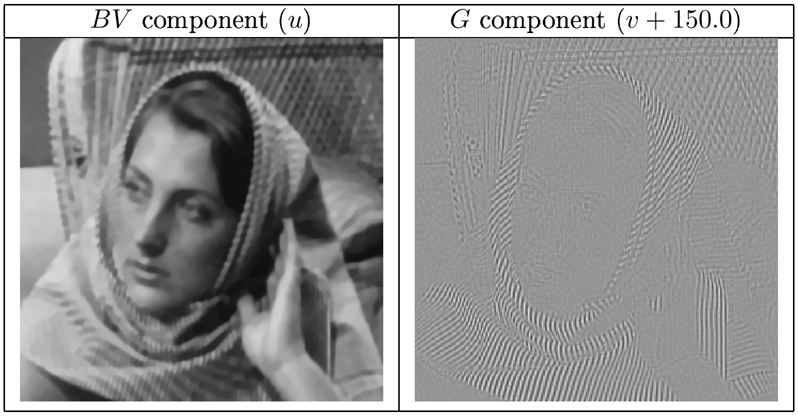

Figure 2: Example of decomposition ($\lambda = 0.1$, $\mu = 10$)

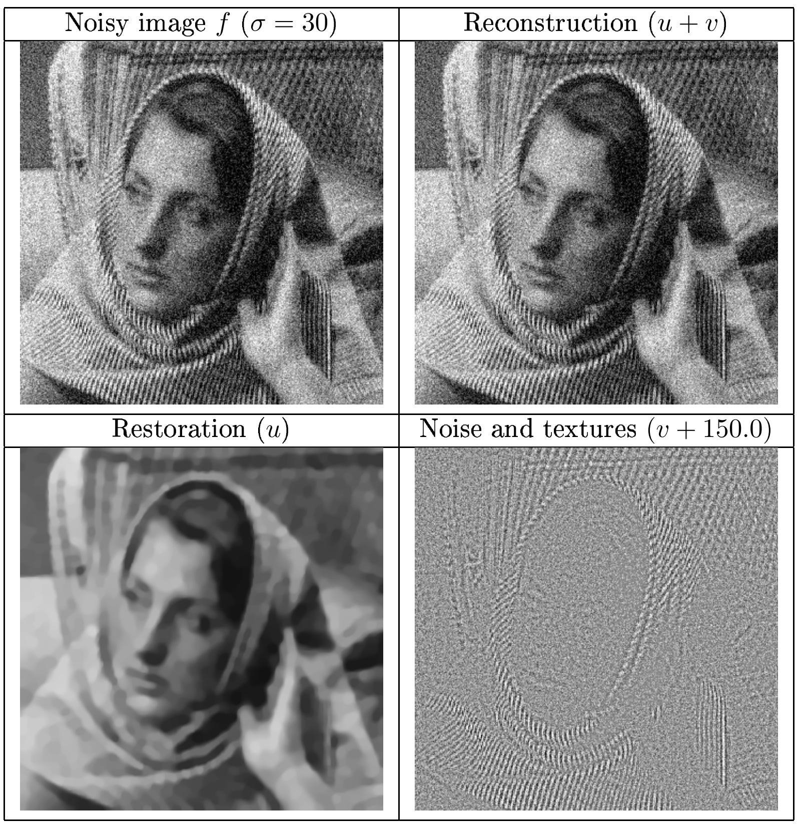

Figure 3: Example of denoising ($\lambda = 2.0$, $\mu = 120$)

[5] J.F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle. Decomposing an image: Application to SAR images. In *Scale-Space '03*, volume 1682 of *Lecture Notes in Computer Science*, 2003.

[6] J.F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle. Decomposing an image: Application to textured images and SAR images, 2003. INRIA Research Report 4704, to appear in JMIV, http://www.inria.fr/rrrt/rr-4704.html.

[7] J. Bourgain and H. Brezis. On the equation div y=f and application to control of phases, October 2002. R02022, Laboratoire J.L. Lions, Université Paris 6, http://www.ann.jussieu.fr/publications/2002.php3.

[8] J. Bourgain and H. Brezis. Sur l' équation *div u=f*. *C.R.Math.Acad.Sci.Paris*, 334(11):973–976, 2002.

[9] H. Brezis. *Analyse fonctionnelle. Théorie et applications.* Mathématiques appliquées pour la maitrise. Masson, 1983.

[10] A. Chambolle. An algorithm for total variation minimization and applications. *JMIV*, 20:89–97, 2004.

[11] A. Chambolle and P.L. Lions. Image recovery via total variation minimization and related problems. *Numerische Mathematik*, 76(3):167–188, 1997.

[12] R. Dautray and J.L. Lions. *Analyse mathématique et calcul numérique pour les sciences et les techniques*, volume 4. Masson, 1988.

[13] I. Ekeland and R. Temam. *Analyse convexe et problèmes variationnels*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Dunod, second edition, 1983.

[14] L.C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 1991.

[15] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 28 of *Princeton Mathematical Series*. Springer-Verlag, 1970.

[16] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and studies in Mathematics*. Pitman, 1985.

[17] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes*, volume 1. Dunod, 1968.

[18] Yves Meyer. Oscillating patterns in image processing and in some nonlinear evolution equations, March 2001. The Fifteenth Dean Jacquelines B. Lewis Memorial Lectures.

[19] A.A. Miljutin. A priori estimates for solutions of second order linear elliptic equations. *Mat. Sbornik*, pages 459–474, 1960. Russian.

[20] C. Miranda. *Partial differential equations of elliptic type*. Springer-Verlag, second edition, 1970.

[21] S.J. Osher, A. Sole, and L.A. Vese. Image decomposition and restoration using total variation minimization and the H^{-1} norm. *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, 1(3):349–370, 2003.

[22] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[23] R. Temam. *Navier Stokes equations*. Elsevier Science Publishers B.V., 1984.

[24] L.A. Vese and S.J. Osher. Modeling textures with total variation minimization and oscillating patterns in image processing. *Journal of Scientific Computing*, 19:553–572, 2003.