STOCHASTIC LAGRANGIAN FLOWS ON SOME COMPACT MANIFOLDS

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Abstract. We investigate ponctual as well as L^2 distances of some stochastic processes with values in the group of homeomorphisms of a compact manifold including processes modelling time evolution of fluids. These processes are associated with operators of the form Laplace-Beltrami plus a first order term. Several constructions are presented, in particular via coupling methods, the corresponding behaviour of the distance depending on the constuction and on the drift properties.

Contents

1. Introduction

The Navier-Stokes system

$$\frac{\partial}{\partial t}u = -(u.\nabla)u + \nu\Delta u - \nabla p$$

is believed to describe the time evolution of the velocity field of fluids with viscosity $\nu \geq 0$. When the fluid is incompressible the condition $\mathrm{div} u = 0$ must be added to the system.

The Lagrangian approach to hydrodynamics studies the position of the underlying particles. In the case of vanishing viscosity V. I. Arnold ([4]) described this position by a geodesic flow on a (infinite dimensional) space of diffeomorphisms. More precisely the relevant space for a fluid on a Riemannian manifold M is the infinite-dimensional group of diffeomorphisms of M (conserving the volume element, in the incompressible case) and its relevant metric is L^2. The study of the corresponding geometry is delicate and gives rise to divergent quantities. Restricting to H^s Sobolev maps with $s > \frac{d}{2} + 1$ introduces sufficient regularity to define a Hilbert manifold; this structure, that considers a H^s topology and simultaneously a L^2 metric, was considered by D. Ebin and J. Marsden ([7]) and named "weak Riemannian structure".

The geometrical approach to Lagrangian hydrodynamics initiated by Arnold allowed to derive the instability of trajectories of the Euler flow (c.f.[3] and [8]) among other properties.

By introducing stochastic Lagrangian flows we can interpret the Navier-Stokes field as a "mean velocity". In this perspective, initiated in works such as [9], we can obtain a generalized notion of geodesic and derive a corresponding variational principle: this has been done in [6] for the torus and in a [1] for a general compact Riemannian manifold.

The stability properties of the Lagrangian stochastic flows were studied in [1], with a special emphasis for the case where the underlying space is the torus.

In this work we consider three different constructions of Lagrangian stochastic processes on compact Riemannian manifolds. We investigate two coupling procedures for some stochastic processes with values in the group of homeomorphisms of a compact manifold. The first one is issued from the so-called mirror coupling of particles in the manifold, and it is proved that it can be defined beyond the hitting time of the diagonal by pairs of particles. The second one is issued from parallel coupling of particles in the manifold. When the manifold is a sphere it is proved that both processes have infinite lifetime. Moreover, when the drift is sufficiently small, parallel coupling in the sphere yields processes in the group of homeomorphism with a L^2 distance converging to 0.

We also consider a stochastic flow approach which has already been described in [1]. Then we study more particularly the case of the d-dimensional sphere.

We notice that this study goes beyond its motivation in hydrodynamics: we can consider stochastic processes with a general drift, not necessarily related to the Navier-Stokes problem.

2. Preliminaries

Let $(M, \mathbf{g})$ be a compact oriented Riemannian manifold without boundary.

Recall that the Itô differential of an M-valued semimartingale Y is defined by

$$dY_t = P(Y)_t \, d\left(\int_0^{\cdot} P(Y)_s^{-1} \circ dY_s\right)_t \tag{2.1}$$

where

$$P(Y)_t : T_{Y_0}M \to T_{Y_t}M \tag{2.2}$$

is the parallel transport along $t \mapsto Y_t$. Alternatively, in local coordinates,

$$dY_t = \left(dY_t^i + \frac{1}{2}\Gamma^i_{jk}(Y_t)dY_t^j \otimes dY_t^k\right)\partial_i \tag{2.3}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection.

If the semimartingale Y_t has an absolutely continuous drift, we denote it by $DY_t\, dt$: for every 1-form $\alpha \in \Gamma(T^*M)$, the finite variation part of

$$\int_0^{\cdot} \langle \alpha(Y_t),\ dY_t\rangle \tag{2.4}$$

is

$$\int_0^{\cdot} \langle \alpha(Y_t),\ DY_t\, dt\rangle\,. \tag{2.5}$$

Let G^0 be the infinite dimensional group of homeomorphisms on M.

Consider a time-dependent C^1 vector field $u(t,x)$ on M.

We consider G^0-valued processes g_t such that for all $x \in M$ the M-valued process $g_t(x)$ has quadratic variation

$$dg_t(x) \otimes dg_t(x) = 2\nu \mathbf{g}^{-1}\left(g_t(x)\right) dt, \tag{2.6}$$

and and absolutely continuous drift satisfying $Dg_t(x) = u(t, g_t(x))$.

An example of such semimartingales is given by an incompressible Brownian flow $g_t \in G_V^0$ with covariance $a \in \Gamma(TM \odot TM)$ and time dependent drift $u(t,\cdot) \in \Gamma(TM)$. We assume that for all $x \in M$, $a(x,x) = 2\nu \mathbf{g}^{-1}(x)$ for some $\nu > 0$. This means that

$$dg_t(x) \otimes dg_t(y) = a\left(g_t)(x), g_t(y)\right) dt, \tag{2.7}$$

$$dg_t(x) \otimes dg_t(x) = 2\nu \mathbf{g}^{-1}\left(g_t(x)\right) dt, \tag{2.8}$$

the drift of $g_t(x)$ is absolutely continuous and satisfies $Dg_t(x) = u(t, g_t(x))$.

The generator of $g_t(x)$ is

$$L_u = \frac{\partial}{\partial t} + \nu\Delta + u.\nabla$$

where Δ is the Laplacian on M.

The distance $\rho(\varphi,\psi)$ between two elements $\varphi, \psi \in G^0$ is defined by

$$\rho^2(\varphi,\psi) = \int_M \rho_M^2(\varphi(x), \psi(x))\, dx \tag{2.9}$$

where ρ_M is the distance in M.

We are interested in estimating the evolution in time of this L^2 distance of two processes starting from two different maps, as well as the evolution in time of the distance between two particles.

3. The distance between two particles: a coupling approach

In this section we consider some coupling processes in G^0 constructed as in [5]. Pointwise construction yields a G^0-valued process.

Consider a stochastic process g_t with values in G^0, such that the pointwise projections are M-valued diffusions associated with L_u. For instance one can take a stochastic flow satisfying (2.7) and (2.8). Let $g_0 = \varphi \in G^0$. For $x \in M$ denote by $d_m g_t(x)$ the martingale part of the Itô differential $dg_t(x)$.

Consider now the so-called "mirror map" on M; this is the map $m_{x,y} : T_x(M) \to T_y(M)$, defined by parallel transporting a vector along the unit speed geodesic $\gamma_{x,y}$ joining x and y (whenever it is unique) and then reflecting into the hyperplane of $T_y(M)$ which is perpendicular to the incoming geodesic. Then we solve the equation

$$d\tilde{g}_t(x) = m_{g_t(x),\tilde{g}_t(x)} d_m g_t(x) + u(t, \tilde{g}_t(x))\, dt \tag{3.1}$$

with $\tilde{g}_0(x) = \psi(x)$, $\psi \in G^0$, and such that $\psi(x)$ does not belong to $C(\varphi(x))$, the cut-locus of $\varphi(x)$. The process $\tilde{g}_t$ is defined up to $t_C(g)$, the first time either $\tilde{g}_t(x)$ hits the cut-locus $C(g_t(x))$, or $(g_t(x), \tilde{g}_t(x))$ hits the diagonal of $M \times M$ for some $x \in M$. The stopping time is in general small, it can even be equal to 0 if for some $x \in M$, $\varphi(x) = \psi(x)$.

Proposition 3.1. *The construction of $\tilde g$ by coupling extends after the hitting time of the diagonal by $(g_t(x), \tilde g_t(x))$ for any $x \in M$. More precisely, replacing $\tilde g_t(x)$ by (r_t, e_t) where r_t is a real-valued process and e_t is a unitary vector in $T_{g_t(x)}M$ satisfying $\tilde g_t(x) = \exp(r_t e_t)$, the equation for (r_t, e_t) smoothly extends on $\{r = 0\}$.*

Proof. For simplicity we let $x_t = g_t(x)$, $y_t = \tilde g_t(x)$. Since we want to extend the construction of the coupled process after the hitting time of the diagonal, we always assume that the distance from x_t to y_t is small. We let $\rho_t(x) = \rho_M(x_t, y_t)$. Let $a \mapsto \gamma_a(x, y)$ the minimal geodesic in time 1 from x to y ($\gamma_0(x, y) = x$, $\gamma_1(x, y) = y$)), and $T_a = T_a(t) = \dot\gamma_a(x_t, y_t)$. For $a \in [0, 1]$ we let $J_a = T\gamma_a$ the tangent map to γ_a. In other words, for $v \in T_xM$ and $w \in T_yM$, $J_a(v, w)$ is the value at time a of the Jacobi field along γ. which takes the values v at time 0 and w at time 1.

For simplicity we write $\gamma_a(t) = \gamma_a(x_t, y_t)$ and $\dot\gamma_a(t) = \dot\gamma_a(x_t, y_t)$. Letting $P_{x_t,\gamma_a(t)}$ be the parallel transport along $\gamma_a(t)$, we have for the Itô covariant differential

$$\begin{aligned}\mathscr{D}\dot\gamma_a(t) &:= P(\gamma_a(\cdot))_t d\left(P(\gamma_a(\cdot))_t^{-1}\dot\gamma_a(t)\right)\\ &= \nabla_{d\gamma_a(t)}\dot\gamma_a + \frac{1}{2}\nabla_{d\gamma_a(t)}\cdot\nabla_{d\gamma_a(t)}\dot\gamma_a(t).\end{aligned}$$

On the other hand the Itô differential $d\gamma_a(t)$ satisfies

$$d\gamma_a(t) = J_a(dx_t, dy_t) + \frac{1}{2}\left(\nabla_{(dx_t,dy_t)}J_a\right)(dx_t, dy_t).$$

So we get

$$\mathscr{D}\dot\gamma_a(t) = \nabla_{J_a(dx_t,dy_t)}\dot\gamma_a + \nabla_{\frac{1}{2}\left(\nabla_{(dx_t,dy_t)}J_a\right)(dx_t,dy_t)}\dot\gamma_a + \frac{1}{2}\nabla_{d\gamma_a(t)}\cdot\nabla_{d\gamma_a(t)}\dot\gamma_a(t). \tag{3.2}$$

Let $e_1(t) \in T_{x_t}M$ be the unit vector satisfying $T_a(t) = \rho_t(x)e_1(t)$. Then we let $e_i(t)$, $i = 2, ..., d$ such that $(e_i(t))_{1\le i\le d}$ is an orthonormal basis, and we define $a \mapsto J_a^i(t)$, $i = 1, ..., d$ the Jacobi field such that $J_0^i(t) = e_i(t)$, $J_1^i(t) = m_{x_t,y_t}e_i(t)$. Moreover we assume that $\nabla_{J_0^i(t)}J_0^i(t) = 0$, $\nabla_{J_1^i(t)}J_1^i(t) = 0$.

With these notations, equation (3.2) rewrites as

$$\begin{aligned}\mathscr{D}T_a &= \nabla_{J_a(dx_t,dy_t)}T_a + \frac{1}{2}\sum_{i=1}^d \nabla_{\nabla_{J_a^i}J_a^i}T_a\,dt + \frac{1}{2}\sum_{i=1}^d \nabla_{J_a^i}\cdot\nabla_{J_a^i}T_a\,dt\\ &= \dot J_a(dx_t, dy_t) + \frac{1}{2}\sum_{i=1}^d \nabla_{J_a^i}\nabla_{J_a^i}T_a\,dt.\end{aligned}$$

Now define the real Brownian motion b_t as

$$db_t = \langle d_m x_t, e_1(t)\rangle \tag{3.3}$$

and let

$$d_m x_t^N = d_m x_t - \langle d_m x_t, e_1(t)\rangle e_1(t) = \sum_{i=2}^d \langle d_m x_t, e_i(t)\rangle e_i(t). \tag{3.4}$$

Note

$$J_a^1 = (1 - 2a)P_{x_t,\gamma_a(t)}e_1(t), \qquad \dot J_a^1 = -2P_{x_t,\gamma_a(t)}e_1(t). \tag{3.5}$$

We have

(3.6)
$$\mathscr{D}T_a = -2db_t P_{x_t,\gamma_a(t)} e_1(t) + \dot J_a(d_m x_t^N, P_{x_t,y_t} d_m x_t^N) + \dot J_a(u(t,x_t), u(t,y_t))\, dt + \frac{1}{2}\sum_{i=2}^d \nabla_{J_a^i}\nabla_{J_a^i} T_a\, dt.$$

From this we easily get the Itô equation for the distance $\rho_t(x)$:

$$\begin{aligned} d\rho_t(x) &= d\left(\left(\int_0^1 \langle T_a(t), T_a(t)\rangle\, da\right)^{1/2}\right)\\ &= \frac{1}{2\rho_t(x)}\left(2\int_0^1 \langle \mathscr{D}T_a(t), T_a(t)\rangle\, da + \int_0^1 \langle \mathscr{D}T_a(t), \mathscr{D}T_a(t)\rangle\, da\right)\\ &- \frac{1}{8\rho_t(x)^3} d\left(\|T_0\|^2\right)\cdot d\left(\|T_0\|^2\right)\\ &= \frac{1}{2\rho_t(x)}\left(-4\rho_t(x)db_t + 2\left\langle P_{y_t,x_t}(u(t,y_t)) - u(t,x_t), T_0\right\rangle dt\right)\\ &+ \frac{1}{2\rho_t(x)}\left(\int_0^1 \sum_{i=1}^d \left\langle \nabla_{J_a^i}\nabla_{J_a^i}T_a, T_a\right\rangle da\, dt + 4\, dt + \sum_{i=2}^d \int_0^1 \|\dot J_a^i\|^2\, da\right)\\ &- \frac{1}{8\rho_t(x)^3} 16\rho_t(x)^2\, dt. \end{aligned}$$

Note

$$\begin{aligned} \int_0^1 \left\langle \nabla_{J_a^i}\nabla_{J_a^i}T_a, T_a\right\rangle da &= \int_0^1 \left\langle \nabla_{J_a^i}\nabla_{T_a}J_a^i, T_a\right\rangle da\\ &= \int_0^1 \left\langle \nabla_{T_a}\nabla_{J_a^i}J_a^i, T_a\right\rangle da - \int_0^1 \left\langle R(T_a, J_a^i)J_a^i, T_a\right\rangle da\\ &= \int_0^1 T_a\left\langle \nabla_{J_a^i}J_a^i, T_a\right\rangle da - \int_0^1 \left\langle R(T_a, J_a^i)J_a^i, T_a\right\rangle da\\ &= \left[\left\langle \nabla_{J_a^i}J_a^i, T_a\right\rangle\right]_0^1 - \int_0^1 \left\langle R(T_a, J_a^i)J_a^i, T_a\right\rangle da\\ &= -\int_0^1 \left\langle R(T_a, J_a^i)J_a^i, T_a\right\rangle da \end{aligned}$$

using the fact that $\nabla_{J_a^i}J_a^i = 0$ for $a = 0, 1$. So finally,

(3.7)
$$\begin{aligned} d\rho_t(x) =& -2db_t + \langle P_{y_t,x_t}(u(t,y_t)) - u(t,x_t), e_1(t)\rangle\, dt\\ &+ \frac{\rho_t(x)}{2}\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1}J_a^i\|^2 - R(e_1, J_a^i)J_a^i, e_1\right) da. \end{aligned}$$

Now writing

(3.8)
$$T_0(t) = \rho_t(x) e_1(t)$$

we get

(3.9)
$$\mathscr{D}T_0 = \rho_t(x)\mathscr{D}e_1(t) + d\rho_t(x) e_1(t) + \frac{1}{2} d\rho_t(x)\mathscr{D}e_1(t)$$

again with $\mathscr{D}e_1(t) = P(x.)_t d\left(P(x.)_t^{-1} e_1(t)\right)$ and this yields

$$\begin{aligned}
\mathscr{D}e_1(t) &= \frac{1}{\rho_t(x)}\mathscr{D}T_0 - \frac{1}{\rho_t(x)}d\rho_t(x)e_1(t) - \frac{1}{2}\frac{1}{\rho_t(x)}d\rho_t(x)\mathscr{D}e_1(t)\\
&= \frac{-2}{\rho_t(x)}db_t e_1(t) + \frac{1}{\rho_t(x)}\dot{J}_0(d_m x_t^N, P_{x_t,y_t} d_m x_t^N)\\
&+ \frac{1}{\rho_t(x)}\dot{J}_0(u(t,x_t), u(t,y_t))\,dt + \frac{1}{2\rho_t(x)}\sum_{i=2}^d \nabla_{J_0^i}\nabla_{J_0^i} T_0\,dt\\
&+ \frac{2}{\rho_t(x)}db_t e_1(t) - \frac{1}{\rho_t(x)}\left\langle P_{y_t,x_t}(u(t,y_t)) - u(t,x_t), e_1(t)\right\rangle e_1(t)\\
&- \frac{1}{2}\left(\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1} J_a^i\|^2 - R(e_1, J_a^i)J_a^i, e_1\right)\,da\right) e_1(t)\\
&- \frac{1}{2}\frac{1}{\rho_t(x)}d\rho_t(x)\mathscr{D}e_1(t)\\
&= \frac{1}{\rho_t(x)}\dot{J}_0(d_m x_t^N, P_{x_t,y_t} d_m x_t^N) + \frac{1}{\rho_t(x)}\dot{J}_0(u^N(t,x_t), u^N(t,y_t))\\
&+ \frac{1}{2\rho_t(x)}\sum_{i=2}^d \nabla_{J_0^i}\nabla_{J_0^i} T_0\,dt\\
&- \frac{1}{2}\left(\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1} J_a^i\|^2 - \langle R(e_1, J_a^i)J_a^i, e_1\rangle\right)\,da\right) e_1(t)
\end{aligned}$$

where we used the fact that $d\rho_t(x)\mathscr{D}e_1(t) = 0$, which can be seen from the martingale part of $\mathscr{D}e_1(t)$ calculated at the third equality.

Now as before

$$\begin{aligned}
\nabla_{J_0^i}\nabla_{J_0^i} T_0 &= \nabla_{T_0}\nabla_{J_0^i} J_0^i - R(T_0, J_0^i)J_0^i\\
&= \rho_t(x)\left(\nabla_{e_1(t)}\nabla_{J_0^i} J_0^i - R(e_1(t), J_0^i)J_0^i\right)\\
&= -\rho_t(x) R(e_1(t), J_0^i)J_0^i
\end{aligned}$$

since $\nabla_{J_0^i} J_0^i \equiv 0$, and $y_t = \exp\left(\rho_t(x)e_1(t)\right)$, so

$$\begin{aligned}
\mathscr{D}e_1(t) =& \frac{1}{\rho_t(x)}\dot{J}_0(d_m x_t^N, P_{x_t,\exp(\rho_t(x)e_1(t))} d_m x_t^N)\\
&+ \frac{1}{\rho_t(x)}\dot{J}_0(u^N(t,x_t), u^N(t, \exp\left(\rho_t(x)e_1(t)\right)))\,dt\\
&- \frac{1}{2}\sum_{i=2}^d R(e_1(t), J_0^i)J_0^i\,dt\\
&- \frac{1}{2}\left(\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1} J_a^i\|^2 - \langle R(e_1, J_a^i)J_a^i, e_1\rangle\right)\,da\right) e_1(t)\,dt.
\end{aligned}$$

Consider the process (r_t, e_t) solution of the system

$$
\begin{aligned}
dr_t =& -2db_t + \left\langle P_{\exp(r_te_t),x_t}(u(t,\exp(r_te_t))) - u(t,x_t), e_t\right\rangle\, dt\\
&+ \frac{r_t}{2}\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_t}J_a^i\|^2 - \langle R(e_t,J_a^i)J_a^i, e_t\rangle\right)\, da\, dt,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\mathscr{D}e_t =& \frac{1}{r_t}\dot{J}_0(d_mx_t^N, P_{x_t,\exp(r_te_t)}d_mx_t^N)\\
&+ \frac{1}{r_t}\dot{J}_0(u^N(t,x_t), u^N(t,\exp(r_te_t)))\, dt\\
&- \frac{1}{2}\sum_{i=2}^d R(e_t,J_0^i)J_0^i\, dt\\
&- \frac{1}{2}\left(\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_t}J_a^i\|^2 - \langle R(e_t,J_a^i)J_a^i, e_t\rangle\right)\, da\right) e_t\, dt,
\end{aligned} \tag{3.11}
$$

with J_a^i, $d_mx_t^N$ defined as before, letting $y_t = \exp(r_te_t)$. Then the coefficients of the system smoothly extend to $r_t = 0$, due to the fact that for $v \in T_{x_t}M$,

$$\frac{1}{r}\dot{J}_0(v, P_{x_t,\exp(r_te_t)}v) = O(1).$$

As a consequence, the pair of coupled processes $(x_t, y_t) = (g_t(x), \tilde{g}_t(x))$ can be defined after the hitting time of the diagonal (note $y_t = \exp(r_te_t)$ with a possibly negative r_t). This is true for all $x \in M$, so this achieves the proof of Proposition 3.1 □

Now consider the problem of the cutlocus.

Proposition 3.2. *When $M = S^d$ the d-dimensional sphere, the construction of the coupled process $\tilde{g}$ from g smoothly extends after the hitting time of the cutlocus by any $(g_t(x), \tilde{g}_t(x))$, $x \in M$. As a consequence, the lifetime of $\tilde{g}$ is the same as the lifetime of g. In addition, $\tilde{g}_t$ is a G^0-valued process.*

Proof. When $M = S^d$ the d-dimensional sphere, just note that the map $m_{x,y}$ is the reflection with respect to the hyperplane in $\mathbb{R}^{d+1}$ containing 0 and orthogonal to $y - x$. It is well defined even if y is in the cutcolus of x, and depends smoothly on (x,y) outside the diagonal. Consequently the mirror coupling of the processes in the sphere can be defined for all times. □

Considering again a general Riemannian manifold M, we let $(\rho_t(x), e_1(x))$ solve (3.10) and (3.11),

$$
\begin{aligned}
d\rho_t(x) =& -2db_t(x) + \left\langle P_{\tilde{g}_t(x),g_t(x)}(u(t,\tilde{g}_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle\, dt\\
&+ \frac{\rho_t(x)}{2}\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1(x)}J_a^i(x)\|^2 - R(e_1(x),J_a^i(x))J_a^i(x), e_1(x)\right)\, da\, dt\\
&- dL_t(x)
\end{aligned} \tag{3.12}
$$

with an additional term $-L_t(x)$ in (3.10), the local time of $\rho_t(x)$ when $(g_t(x), \tilde{g}_t(x))$ visits the cutlocus. See e.g. [5] for a construction around the cutlocus. Denote $[b(x), b(y)]$ the quadratic covariation of $b(x)$ and $b(y)$.

Proposition 3.3. *The Itô differential of the distance ρ_t between g_t and $\tilde g_t$ is given by*

(3.13)
$$\begin{aligned}
d\rho_t = &-\frac{2}{\rho_t}\int_M \rho_t(x)\,db_t(x) - \frac{1}{\rho_t}\int_M \rho_t(x) dL_t(x)\,dx\\
&+\frac{1}{2\rho_t}\int_M \rho_t(x)^2\int_0^1\sum_{i=2}^d \left(\|\nabla_{e_1(x)}J_a^i(x)\|^2 - \langle R(e_1(x),J_a^i(x))J_a^i(x),e_1(x)\rangle\right)\,da\,dx\,dt\\
&+\frac{1}{\rho_t}\int_M \rho_t(x)\left\langle P_{\tilde g_t(x),g_t(x)}(u(t,\tilde g_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle\,dx\,dt\\
&+\frac{2}{\rho_t}\,dt - \frac{2}{\rho_t^3}\int_{M\times M}\rho_t(x)\rho_t(y)d[b(x),b(y)]_t\,dx\,dy.
\end{aligned}$$

There exists an adapted process σ_t bounded in absolute value by 2, a real-valued Brownian motion β_t, a bounded adapted process process c_t and a process v_t satisfying

$$v_t \ge \frac{\rho_t^2 - \left(\int_M \rho_t(x)\,dx\right)^2}{\rho_t^3} \tag{3.14}$$

such that

(3.15)
$$\begin{aligned}
d\rho_t = &\rho_t(\sigma_t\,d\beta_t + c_t\,dt) + v_t\,dt - \frac{1}{\rho_t}\int_M \rho_t(x)dL_t(x)\,dx\\
&+\frac{1}{2\rho_t}\int_M \rho_t(x)^2\int_0^1\sum_{i=2}^d \left(\|\nabla_{e_1(x)}J_a^i(x)\|^2 - R(e_1(x),J_a^i(x))J_a^i(x),e_1(x)\right)\,da\,dx\,dt.
\end{aligned}$$

Proof. From equation (3.12) we easily get

$$\begin{aligned}
d\rho_t^2 = &-4\int_M \rho_t(x)\,db_t(x) - 2\int_M \rho_t(x)dL_t(x)dx\\
&+\int_M \rho_t(x)^2\int_0^1\sum_{i=2}^d \left(\|\nabla_{e_1(x)}J_a^i(x)\|^2 - R(e_1(x),J_a^i(x))J_a^i(x),e_1(x)\right)\,da\,dx\,dt\\
&+2\int_M \rho_t(x)\left\langle P_{\tilde g_t(x),g_t(x)}(u(t,\tilde g_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle\,dx\,dt\\
&+4\,dt.
\end{aligned}$$

Then writing $Y_t = \rho_t^2$ and using

$$d\rho_t = \frac{1}{2\rho_t}dY_t - \frac{1}{8\rho_t^3}d[Y,Y]_t$$

we get equation (3.13).

Now letting $d[b(x),b(y)]_t = h(x,y)_t\,dt$ we have $|h(x,y)_t| \le 1$, and the quadratic variation of ρ_t satisfies

$$d[\rho,\rho]_t = \frac{4}{\rho_t^2}\left(\int_{M\times M}\rho_t(x)\rho_t(y)h(x,y)_t\,dxdy\right)\,dt \le 4\,dt,$$

which yields the existence of σ_t and β_t.

Then clearly

$$\frac{1}{\rho_t}\int_M \rho_t(x)\left\langle P_{\tilde g_t(x),g_t(x)}(u(t,\tilde g_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle\,dx\,dt = \rho_t c_t\,dt$$

with a bounded process c_t.

Next using again

$$\int_{M\times M}\rho_t(x)\rho_t(y)d[b(x),b(y)]_t\,dx\,dy = \left(\int_{M\times M}\rho_t(x)\rho_t(y)h(x,y)_t\,dxdy\right)\,dt$$
$$\le\left(\int_M \rho_t(x)\,dx\right)^2\,dt$$

we get the existence of v_t satisfying (3.14). □

Remark 3.4. Due to the positive drift v_t which is singular at 0 we would expect that if $\varphi \neq \psi$ then there is no coupling of the two processes in G^0 at any time.

Proposition 3.5. *In the case $M = S^d$, $d \geq 2$, the equation for $\rho_t(x)$ writes*

$$\begin{aligned} d\rho_t(x) =& -2db_t(x) + \left\langle P_{\tilde g_t(x),g_t(x)}(u(t,\tilde g_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle\,dt \\ & -(d-1)\tan\frac{\rho_t(x)}{2}\,dt. \end{aligned} \tag{3.16}$$

An equivalent for the drift when $\rho_t(x)$ is close to π (resp. $-\pi$) is $\dfrac{-2(d-1)}{\pi-\rho_t(x)}\,dt$ (resp. $\dfrac{2(d-1)}{\pi+\rho_t(x)}\,dt$). As a consequence almost surely the process $(g_t(x),\tilde g_t(x))$ never hits the cutlocus.

Proof. For $i = 2,\ldots,d$, the boundary conditions for J_a^i yield

$$J_a^i = \left(\cos(\rho_t(x)a) + \tan\frac{\rho_t(x)}{2}\sin(\rho_t(x)a)\right)P_{x_t,\gamma_a(t)}e_i(t).$$

From this we compute

$$
\begin{aligned}
&\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1(x)} J_a^i(x)\|^2 - \langle R(e_1(x), J_a^i(x))J_a^i(x), e_1(x)\rangle\right) da\\
=&(d-1)\int_0^1 \left(\tan\frac{\rho_t(x)}{2}\cos(\rho_t(x)a) - \sin(\rho_t(x)a)\right)^2 da\\
&-(d-1)\int_0^1 \left(\tan\frac{\rho_t(x)}{2}\sin(\rho_t(x)a) + \cos(\rho_t(x)a)\right)^2 da\\
&=(d-1)\int_0^1 \left(\left(\tan^2\frac{\rho_t(x)}{2} - 1\right)\cos(2\rho_t(x)a) - 2\tan\frac{\rho_t(x)}{2}\sin(2\rho_t(x)a)\right) da\\
&=(d-1)\left(\left(\tan^2\frac{\rho_t(x)}{2} - 1\right)\frac{\sin(2\rho_t(x))}{2\rho_t(x)} + \tan\frac{\rho_t(x)}{2}\left(\frac{\cos(2\rho_t(x)) - 1}{\rho_t(x)}\right)\right)\\
&=-\frac{d-1}{\rho_t(x)}\left(\frac{\cos(\rho_t(x))}{\cos^2\frac{\rho_t(x)}{2}}\sin(\rho_t(x))\cos(\rho_t(x)) + 2\tan\frac{\rho_t(x)}{2}\sin^2(\rho_t(x))\right)\\
&=-2\frac{d-1}{\rho_t(x)}\left(\frac{\cos\frac{\rho_t(x)}{2}\sin\frac{\rho_t(x)}{2}}{\cos^2\frac{\rho_t(x)}{2}}\left(\cos^2(\rho_t(x)) + \sin^2(\rho_t(x))\right)\right)\\
&=-2\frac{d-1}{\rho_t(x)}\tan\frac{\rho_t(x)}{2}.
\end{aligned}
$$

and this yields the expression for $d\rho_t(x)$.

From this it is easy to get the equivalents around π and $-\pi$. Then we can compare $\frac{1}{2}\rho_t(x)$ to a Bessel process of dimension δ satisfying $\frac{\delta-1}{2} = 2(d-1)$, namely $\delta = 4d-3$, to conclude that for $d \geq 2$ the cutlocus is not reached.

□

Corollary 3.6. *When $M = S^d$, $d \geq 2$, the Itô differential of the distance ρ_t between g_t and $\tilde{g}_t$ is given by*

$$
\begin{aligned}
&d\rho_t\\
&= -\frac{2}{\rho_t}\int_M \rho_t(x)\, db_t(x)\\
&- \frac{d-1}{\rho_t}\int_M \rho_t(x)\tan\frac{\rho_t(x)}{2}\, dx\, dt\\
&+ \frac{1}{\rho_t}\int_M \rho_t(x)\left\langle P_{\tilde{g}_t(x), g_t(x)}(u(t, \tilde{g}_t(x))) - u(t, g_t(x)), e_1(t)(x)\right\rangle dx\, dt\\
&+ \frac{2}{\rho_t}\, dt - \frac{2}{\rho_t^3}\int_{M\times M} \rho_t(x)\rho_t(y) d[b(x), b(y)]_t\, dx\, dy.
\end{aligned} \tag{3.17}
$$

We investigate parallel coupling $\hat{g}_t$ from g_t on a general compact Riemannian manifold M. It consists in replacing $m_{x,y}$ by $P_{x,y}$, the parallel transport from T_xM to T_yM along the minimal geodesic (whenever it is unique). Then equation (3.1) becomes

$$
d\hat{g}_t(x) = P_{g_t(x), \hat{g}_t(x)} d_m g_t(x) + u(t, \hat{g}_t(x))\, dt \tag{3.18}
$$

with $\hat{g}_0(x) = \psi(x)$, $\psi \in G^0$. An advantage of this construction is that it is well-defined around the diagonal since $P_{x,y}$ depends smoothly on x and y. Moreover the technique for construction around the cutlocus is the same as for mirror coupling, see e.g. [11] for details. Contrarily to mirror coupling there is no simplification in the case of the sphere, parallel transport does not have a smooth behaviour around the cutlocus, but we will see that the cutlocus is not reached. We briefly give the formulas for the distance $\rho_t(x)$ from $\gamma_t(x)$ to $\hat{g}_t(x)$ and the L^2 distance from g_t to $\hat{g}_t$.

Proposition 3.7. *The distance $\rho_t(x)$ from $g_t(x)$ to $\hat{g}_t(x)$ satisfies*

$$
\begin{aligned}
d\rho_t(x) &= \left\langle P_{\hat{g}_t(x),g_t(x)}(u(t,\hat{g}_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle \, dt - dL_t(x) \\
&\quad + \frac{\rho_t(x)}{2}\int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1(x)} J_a^i(x)\|^2 - R(e_1(x), J_a^i(x))J_a^i(x), e_1(x)\right) \, da\, dt,
\end{aligned} \tag{3.19}
$$

in particular it never vanishes.

The L^2 distance between g_t and $\hat{g}_t$ solves

$$
\begin{aligned}
d\rho_t &= -\frac{1}{\rho_t}\int_M \rho_t(x) dL_t(x)\, dx \\
&\quad + \frac{1}{2\rho_t}\int_M \rho_t(x)^2 \int_0^1 \sum_{i=2}^d \left(\|\nabla_{e_1(x)} J_a^i(x)\|^2 - R(e_1(x), J_a^i(x))J_a^i(x), e_1(x)\right) \, da\, dx\, dt \\
&\quad + \frac{1}{\rho_t}\int_M \rho_t(x) \left\langle P_{\hat{g}_t(x),g_t(x)}(u(t,\hat{g}_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle \, dx\, dt.
\end{aligned} \tag{3.20}
$$

In the case of the sphere the situation is even simpler.

Proposition 3.8. *When $M = S^d$ with $d \geq 2$, assuming $\varphi(x) \neq \psi(x)$ a.e., we have*

$$
\begin{aligned}
d\rho_t(x) &= \left\langle P_{\hat{g}_t(x),g_t(x)}(u(t,\hat{g}_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle \, dt \\
&\quad - (d-1)\tan\frac{\rho_t(x)}{2}\, dt.
\end{aligned} \tag{3.21}
$$

The second line is nonpositive, and almost surely the process $(g_t(x), \hat{g}_t(x))$ never hits the cutlocus.

Furthemore,

$$
\begin{aligned}
d\rho_t &= \frac{1}{\rho_t}\int_M \rho_t(x) \left\langle P_{\hat{g}_t(x),g_t(x)}(u(t,\hat{g}_t(x))) - u(t,g_t(x)), e_1(t)(x)\right\rangle \, dx\, dt \\
&\quad - \frac{d-1}{\rho_t}\int_M \rho_t(x)\tan\frac{\rho_t(x)}{2}\, dx\, dt.
\end{aligned} \tag{3.22}
$$

Again the second line is nonpositive. Moreover we have the bound

$$
d\rho_t \leq \rho_t\left(\|\nabla u(t,\cdot)\|_\infty - \frac{d-1}{2}\right) dt. \tag{3.23}
$$

Remark 3.9. It is clear from equation (3.23) that if there exists t_0 such that for all $t \geq t_0$, $\|\nabla u(t,\cdot)\|_\infty \leq \frac{d-1}{2} - \varepsilon$ for some $\varepsilon > 0$, then the L^2 distance between g_t and $\hat{g}_t$ converges to 0 exponentially fast as t goes to infinity.

To finish this section we investigate the rotation of $\hat{g}_t(x)$ around $g_t(x)$, represented by the behaviour of the unit vector $e_1(t)$ such that $\hat{g}_t(x) = \exp(\rho_t(x)e_1(t)$ when $M = S^d$.

Proposition 3.10. *The covariant Itô differential of* $e_1(t)$ *satisfies*

(3.24)
$$\mathscr{D}e_1(t) = \tan\frac{\rho_t(x)}{2} d_m x_t^N + \text{cotan}\rho_t(x)\left(P_{\hat g_t(x),g_t(x)}u^N(t,\hat g_t(x)) - u^N(t,g_t(x))\right) + \frac{d-1}{2}\left(1 - \frac{2}{\rho_t(x)}\tan\frac{\rho_t(x)}{2}\right)e_1(t).$$

If $\|\nabla u(t,\cdot)\|_\infty$ *converges to* 0 *as* t *goes to infinity then* $e_1(t)$ *converges in law to a parallel transport along* $g_t(x)$.

Proof. Equation (3.24) is a direct consequence of (3.11) (which is the same for mirror and parallel coupling). If $\|\nabla u(t,\cdot)\|_\infty$ converges to 0 as t goes to infinity then all coefficients of equation (3.24) converge to 0. Applying Corollary 11.1.5 in [10] for Feller continuity of solutions of stochastic differential equations yields the convergence in law to a parallel motion. □

4. The distance between two particles: a stochastic flow approach

Let $B_t = (B_t^\ell)_{\ell\ge 0}$ be a family of independent real Brownian motions, $\sigma = (\sigma_\ell)_{\ell\ge 0}$, with, for all $\ell \ge 0$, σ_ℓ a divergence free vector field on M. We furthermore assume that

$$\sigma(x)\sigma^*(y) = a(x,y). \tag{4.1}$$

In particular

$$\sigma(x)\sigma^*(x) = 2\nu\mathbf{g}^{-1}(x). \tag{4.2}$$

We let $\varphi,\psi \in G_V^0$. In this section we assume that

$$dg_t(x) = \sigma(g_t(x))\,dB_t + u(t,g_t(x))\,dt, \quad g_0 = \varphi \tag{4.3}$$

and

$$d\tilde g_t(x) = \sigma(\tilde g_t(x))\,dB_t + u(t,\tilde g_t(x))\,dt, \quad \tilde g_0 = \psi. \tag{4.4}$$

For simplicity we let $x_t = g_t(x)$, $y_t = \tilde g_t(x)$ and

$$\rho_t(x) = \rho_M(x_t,y_t).$$

For $x,y \in M$ such that y does not belong to the cutlocus of x, we let $a \mapsto \gamma_a(x,y)$ be the minimal geodesic in time 1 from x to y ($\gamma_0(x,y) = x$, $\gamma_1(x,y) = y$)). For $a \in [0,1]$ we let $J_a = T\gamma_a$ the tangent map to γ_a. In other words, for $v \in T_xM$ and $w \in T_yM$, $J_a(v,w)$ is the value at time a of the Jacobi field along $\gamma_\cdot$ which takes the values v at time 0 and w at time 1.

For this construction we have proved in [1] the following formula for the distance of two Lagrangian processes,

Proposition 4.1. *The Itô differential of the distance ρ_t between g_t and $\tilde{g}_t$ is given by*

$$
\begin{aligned}
d\rho_t = & \frac{1}{\rho_t}\sum_{\ell\geq 0}\left(\int_M \rho_t(x)\left(P_{\tilde{g}_t(x),g_t(x)}(\sigma_\ell^T(\tilde{g}_t(x))) - \sigma_\ell^T(g_t(x)\right) dx\right) dB_t^\ell \\
& + \frac{1}{\rho_t}\int_M \rho_t(x)\left(P_{\tilde{g}_t(x),g_t(x)}(u^T(\tilde{g}_t(x)))) - u^T(g_t(x))\right) dx\, dt - \frac{1}{\rho_t}\int_M \rho_t(x) dL_t(x)\, dx \\
& + \frac{1}{2\rho_t}\left(\int_M \sum_{\ell\geq 0}\left(\int_0^1\left(\|\dot{J}_a^{\ell,N}\|^2 - \left\langle R(T_a(t,x), J_a^{\ell,N}(t,x))J_a^{\ell,N}(t,x), T_a(t,x)\right\rangle\right) da\right) dx\right) dt \\
& + \frac{1}{2\rho_t}\sum_{\ell\geq 0}\left(1-\cos^2\left(\dot{J}_0^{\ell,T}(t,\cdot), T_0(t,\cdot)\right)\right)\int_M \left\|\dot{J}_0^{\ell,T}(t,x)\right\|^2 dx\, dt
\end{aligned}
$$

where

$$
\cos\left(\dot{J}_0^{\ell,T}(t,\cdot), T_0(t,\cdot)\right) = \frac{\int_M \left\langle \dot{J}_0^{\ell,T}(t,x), T_0(t,x)\right\rangle dx}{\rho_t\left(\int_M \left\|\dot{J}_0^{\ell,T}(t,x)\right\|^2 dx\right)^{1/2}}.
$$

We consider the case of the two dimensional sphere. Let S^2 be the sphere of radius one defined in spherical coordinates as

$$
S = \{(\theta,\phi) : \theta \in [0,\pi[, \phi \in S^1\}
$$

where S^1 is the one dimensional sphere parametrized by the angle. Then the metric tensor is given by

$$
\mathbf{g}_{1,1} = 1,\ \mathbf{g}_{1,2} = 0,\ \mathbf{g}_{2,2} = sin^2\theta
$$

Consider the volume measure $dm = \sin\theta\ d\theta\ d\phi$ and the Laplacian:

$$
\begin{aligned}
\Delta f &= \frac{1}{\sqrt{|det\mathbf{g}|}}\partial_i(\sqrt{|det\mathbf{g}|}\mathbf{g}^{i,j}\partial_j f) \\
&= \partial_\theta^2 f + \frac{\cos\theta}{\sin\theta}\partial_\theta f + \frac{1}{\sin^2\theta}\partial_\phi^2 f
\end{aligned}
$$

The Lagrangian flows associated to the Brownian motion on the sphere are defined by the following stochastic differential equations,

$$
\begin{aligned}
dg_t^1(x) &= dw_t^1 + \frac{\cos g_t^1(x)}{\sin g_t^1(x)}dt + u^1(t, g_t(x))dt \\
dg_t^2(x) &= \frac{1}{\sin g_t^1(x)}dw_t^2 + u^2(t, g_t(x))dt
\end{aligned}
$$

with (w^1, w^2) a two-dimensional Brownian motion, $x \in S^2$. Here $g_t^2(x)$ is considered as an $\mathbb{R}$-valued process.

We study the ponctual distance,

$$
\rho_t^2 = [\rho_t^1]^2 + [\rho_t^2]^2
$$

with $[\rho_t^1] = |g_t^1(x) - g_t^1(y)|$, $[\rho_t^2] = \rho_{S^1}(g_t^2(x), g_t^2(y))$, $x, y \in S^2$. Define $x_t = g_t(x)$ and $y_t = g_t(y)$.

Since points are polar, almost surely for all t, x_t^1 and y_t^1 never visit $\{0,\pi\}$. Moreover it is clear that if $x_0^1 \neq y_0^1$ then $x_t^1 - y_t^1$ never vanishes. We can assume that $x_0^1 > y_0^1$. So we have for all t $[\rho_t^1] = x_t^1 - y_t^1$ and

$$\frac{d[\rho_t^1]}{dt} = \frac{\cos x_t^1}{\sin x_t^1} - \frac{\cos y_t^1}{\sin y_t^1} + u^1(t,x_t) - u^1(t,y_t)$$
$$\leq -[\rho_t^1] + 2\|u(t,\cdot)\|_\infty.$$

Suppose that for all t,

$$\|u(t,.)\|_\infty \leq ce^{-at}.$$

Then

$$\frac{d[\rho_t^1]}{dt} \leq -[\rho_t^1] + 2ce^{-at}.$$

We therefore have the following decay behaviour

$$[\rho_t^1] \leq [\rho_0^1]e^{-t} + \frac{2c}{1-a}\left(e^{-at} - e^{-t}\right).$$

Concerning the second component, we have

$$\begin{aligned} d[\rho_t^2] &= \frac{(x_t^2 - y_t^2)}{[\rho_t^2]}(\frac{1}{\sin x_t^1} - \frac{1}{\sin y_t^1})dw_t^2 + \frac{1}{2[\rho_t^2]}(\frac{1}{\sin x_t^1} - \frac{1}{\sin y_t^1})^2 dt \\ &+ \frac{1}{[\rho_t^2]}(x_t^2 - y_t^2)(u^2(t,x_t) - u^2(t,y_t)dt + dL_t \\ &- \frac{1}{2[\rho_t^2]^3}(x_t^2 - y_t^2)^2(\frac{1}{\sin x_t^1} - \frac{1}{\sin y_t^1})^2 dt \\ &= \frac{(x_t^2 - y_t^2)}{[\rho_t^2]}(\frac{1}{\sin x_t^1} - \frac{1}{\sin y_t^1})dw_t^2 + \frac{1}{[\rho_t^2]}(x_t^2 - y_t^2)(u^2(t,x_t) - u^2(t,y_t)dt + dL_t \end{aligned}$$

where L_t is the local time of the distance in S^1 at 0 and π. We have

$$\frac{1}{[\rho_t^2]}(x_t^2 - y_t^2)(u^2(t,x_t) - u^2(t,y_t)) \leq 2\|u(t,\cdot)\|_\infty \leq 2ce^{-at},$$

however it is not clear what is the behaviour of the second component ρ_t^2.

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