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On the separation cut-off phenomenon for Brownian motions on high dimensional spheres

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This paper proves that the separation convergence toward the uniform distribution abruptly occurs at times around $\ln(n)/n$ for the (time-accelerated by 2) Brownian motion on the sphere with a high dimension n. The arguments are based on a new and elementary perturbative approach for estimating hitting times in a small noise context. The quantitative estimates thus obtained are applied to the strong stationary times constructed in (Arnaudon, Coulibaly-Pasquier and Miclo (2020)) to deduce the wanted cut-off phenomenon.

Keywords: Hitting times; separation discrepancy; small noise one-dimensional diffusions; spherical Brownian motions; strong stationary times

1. Introduction

Consider the Brownian motion $X := (X(t))_{t\geq 0}$ on the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ of dimension $n + 1 \geq 2$, time-accelerated by a factor 2, so the generator of X is the Beltrami-Laplacian $\triangle$ on $\mathbb{S}^{n+1}$. Starting from a point, the time marginal laws of X spread over $\mathbb{S}^{n+1}$ and approach the uniform distribution in large times. A traditional question is to estimate corresponding speeds of convergence, or mixing times, especially for large n. The answer depends on the way the difference between the time marginal and the uniform distribution is measured. Saloff-Coste (1994) proved that for the total variation, the mixing time is equivalent to $\ln(n)/(2n)$ and furthermore a cut-off phenomenon occurs (see also Méliot (2014) for extensions). In contrast to the more traditional setting of discrete-time Markov chains, there is no difficulty with very small mixing times in the framework of continuous time, since time can be scaled by any positive factor. If we were working with finite state spaces, general arguments based on reversibility, see (1.5) in Hermon, Lacoin and Peres (2016), associated with the cut-off in total variation, would show that for the separation discrepancy the mixing time asymptotically belongs to the interval $[\ln(n)/(2n), \ln(n)/n]$. This observation would imply at once the upper bound on τ_n in Theorem 2 below. The convergence of X to the uniform distribution can be brought back to a one-dimensional question, by considering its radial part (with respect to the starting point), since its “angular part” is at once at equilibrium by symmetry. One-dimensional diffusions are quite close to birth and death processes, so we can expect from the results of Diaconis and Saloff-Coste (2006) and Ding, Lubetzky and Peres (2010) that a cut-off phenomenon equally occurs in the separation sense. Our goal here is to check that this is indeed the case and that this abrupt convergence occurs at times round $\ln(n)/n$. Our proof is based on two ingredients: (1) the resort to the strong stationary times for X presented in Arnaudon, Coulibaly-Pasquier and Miclo (2020) and (2) quantitative estimates on the hitting times for one-dimensional diffusion processes, obtained via elementary calculus (and a very restricted dose of stochastic calculus).

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This alternative point of view on cut-off differs from the traditional approach based on spectral analysis and could be extended to other situations where less spectral information is available. A more general purpose of this paper is to advertise the resort to strong stationary times for diffusion processes, subject which has been much less investigated than its finite state space counterpart, maybe because such finite times may not exist in full generality.

Without loss of generality, we can assume that X starts from $x_0 := (1,0,0,\ldots,0) \in \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. It was seen in Coulibaly-Pasquier and Miclo (2021) that X can be algebraically intertwined with a Markov process $D := (D(t))_{t\geq 0}$ taking values in the closed balls of $\mathbb{S}^{n+1}$ centered at x_0, starting at $\{x_0\}$ and absorbed in finite time in the whole set $\mathbb{S}^{n+1}$. Denote

$$\tau_n := \inf\{t \geq 0 : D(t) = \mathbb{S}^{n+1}\}$$

the absorption time. More precisely, the *algebraic intertwining relation* means that the commutation relation $\mathcal{L}\Lambda = \Lambda\triangle$ holds (when applied to functions belonging to the domain of $\triangle$, see Coulibaly-Pasquier and Miclo (2021)), where $\mathcal{L}$ is the generator of the absorbed process D and Λ is the Markov kernel from the state space $\mathcal{D}$ of D (i.e. the set of all closed balls of $\mathbb{S}^{n+1}$ centered at x_0) to $\mathbb{S}^{n+1}$ which associates to any $B \in \mathcal{D}$, $\Lambda(B,\cdot)$ the normalized uniform law over B.

In Arnaudon, Coulibaly-Pasquier and Miclo (2020), this algebraic intertwining relation was extended into a *probabilistic intertwining relation* which is a coupling of X and D so that

- at any time $t \geq 0$, the conditional law of $X(t)$ knowing the trajectory $D([0,t]) := (D(s))_{s\in[0,t]}$ is given by $\Lambda(D(t),\cdot)$,
- D is progressively measurable with respect to X, in the sense that for any $t \geq 0$, $D([0,t])$ depends on X only through $X([0,t])$.

In general, such couplings are not unique, and in Arnaudon, Coulibaly-Pasquier and Miclo (2020) several couplings of X and D were constructed (two of them are recalled in Corollary 2 below). Due to these couplings and to general arguments from Diaconis and Fill (1990), τ_n is a strong stationary time for X, meaning that τ_n and $X(\tau_n)$ are independent and $X(\tau_n)$ is uniformly distributed over $\mathbb{S}^{n+1}$. As a consequence, see Diaconis and Fill (1990), we have

$$\forall\, t \geq 0, \qquad \mathfrak{s}(\mathcal{L}(X(t)), \mathcal{U}_{n+1}) \leq \mathbb{P}[\tau_n \geq t],$$

where the left hand side is the separation discrepancy between the law of $X(t)$ and the uniform distribution $\mathcal{U}_{n+1}$ over $\mathbb{S}^{n+1}$. Recall that the separation discrepancy between two probability measures μ and ν defined on the same measurable space is given by

$$\mathfrak{s}(\mu,\nu) = \operatorname*{ess\,sup}_{\nu} 1 - \frac{d\mu}{d\nu},$$

where $d\mu/d\nu$ is the Radon-Nikodym density of μ with respect to ν.

Remark 1. Note that for any $t \in [0,\tau_n)$, the "opposite pole" $(-1,0,0,\ldots,0)$ does not belong to the support of $\Lambda(D(t),\cdot)$. It follows from an extension of Remark 2.39 of Diaconis and Fill (1990) that τ_n is even a sharp strong stationary time for X, meaning that

$$\forall\, t \geq 0, \qquad \mathfrak{s}(\mathcal{L}(X(t)), \mathcal{U}_{n+1}) = \mathbb{P}[\tau_n \geq t].$$

Thus the understanding of the convergence in separation of X toward $\mathcal{U}_{n+1}$ amounts to understanding the distribution of τ_n. From the bibliographical survey given above, it can be expected that τ_n is of order $\ln(n)/n$.

In confirmation of the above observation, a first purpose of this note is to prove the following result.

Theorem 1. *For all n large, we have* $\mathbb{E}[\tau_n] \sim \ln(n)/n$.

Let us go further by showing a cut-off phenomenon, namely that in the scale $\ln(n)/n$, the random variable τ_n is in fact close to its mean $\mathbb{E}[\tau_n]$. This property can be written under several forms, see e.g. the review of Diaconis (1996) or the book Levin, Peres and Wilmer (2009) (both in the context of finite Markov chains). We consider the following simple formulation:

Theorem 2. *For any* $r > 0$, *we have*

$$\lim_{n\to\infty} \mathbb{P}\left[\tau_n > (1+r)\frac{\ln(n)}{n}\right] = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}\left[\tau_n < (1-r)\frac{\ln(n)}{n}\right] = 0.$$

As an immediate consequence of Remark 1 and Theorem 2, we get

Corollary 1. *For any* $r > 0$, *we have*

$$\lim_{n\to\infty} \mathfrak{s}\left(\mathcal{L}\left(X\left((1+r)\frac{\ln(n)}{n}\right)\right), \mathcal{U}_{n+1}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{s}\left(\mathcal{L}\left(X\left((1-r)\frac{\ln(n)}{n}\right)\right), \mathcal{U}_{n+1}\right) = 1.$$

For any $t \geq 0$, denote $R(t)$ the Riemannian radius of $D(t)$ in $\mathbb{S}^{n+1}$, so that $R(0) = 0$ and

$$\tau_n = \inf\{t \geq 0 \,:\, R(t) = \pi\}. \tag{1}$$

It was seen in Coulibaly-Pasquier and Miclo (2021) that $R := (R(t))_{t\geq 0}$ solves the stochastic differential equation

$$\forall\, t \in (0, \tau_n), \qquad dR(t) = \sqrt{2}dB(t) + b_n(R(t))dt, \tag{2}$$

where $(B(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ and the mapping b_n is given by

$$\forall\, r \in (0, \pi), \qquad b_n(r) := 2\frac{\sin^n(r)}{\int_0^r \sin^n(u)\,du} - n\frac{\cos(r)}{\sin(r)}. \tag{3}$$

It is not difficult to check, see e.g. (37) of the Supplementary Material (Arnaudon, Coulibaly-Pasquier and Miclo (2024)), which is an equivalent as $x \to 0_+$, that as r goes to 0_+, $b_n(r) \sim (n+2)/r$, and this is sufficient to ensure that 0 is an entrance boundary for R, so that starting from 0, it will never return to 0 at positive times.

In the following corollary we present two intertwinings, which were constructed in Arnaudon, Coulibaly-Pasquier and Miclo (2020) Theorems 3.5 and 4.1.

Corollary 2. *Let* $(X_t)_{t\geq 0}$ *be a Brownian motion in* $\mathbb{S}^{n+1}$ *started at* x_0. *For* $x \in \mathbb{S}^{n+1}\setminus\{x_0, -x_0\}$, *denote by* $N(x)$ *the unit vector at* x *normal to the circle with radius* $\rho(x_0, x)$ *where* ρ *is the distance in the sphere, pointing towards* x_0: $N(x) = -\nabla\rho(x_0, \cdot)(x)$.

(i) *Full coupling. Let* $D_1(t)$ *be the ball in* $\mathbb{S}^{n+1}$ *centered at* x_0 *with radius* $R_1(t)$ *solution started at* 0 *to the Itô equation*

$$dR_1(t) = -\sqrt{2}\langle N(X_t), dX_t\rangle + n\left[2\cot(\rho(x_0, X_t)) - \cot(R_1(t))\right]dt. \tag{4}$$

This evolution equation is considered up to the hitting time $\tau_n^{(1)}$ *of* π *by* $R_1(t)$.

(ii) *Full decoupling, reflection of D on X. Let $D_2(t)$ be the ball in $\mathbb{S}^{n+1}$ centered at x_0 with radius $R_2(t)$ solution started at* 0 *to the Itô equation*

$$dR_2(t) = -\sqrt{2}dW_t + 2dL_t^{R_2}(\rho(x_0,\cdot))(X) - n\cot(R_2(t))\,dt, \tag{5}$$

where $(W_t)_{t\geq 0}$ is a real-valued Brownian motion independent of $(X_t)_{t\geq 0}$ and $L_t^{R_2}[\rho(x_0,X)]$ is the local time at 0 *of the process $R_2 - \rho(x_0,X)$. These considerations are valid up to the hitting time $\tau_n^{(2)}$ of π by $R_2(t)$.*

Let $D(t)$ be the ball in $\mathbb{S}^{n+1}$ centered at x_0 with radius $R(t)$, defined in (2)*, and let τ_n be the stopping time defined in* (1)*. Then we have:*

(iii) *for $i = 1,2$ $X_{\tau_n^{(i)}}$ is uniformly distributed in $\mathbb{S}^{n+1}$,*

(iv) *the pairs $(\tau_n^{(1)},(D_1(t))_{t\in[0,\tau_n^{(1)}]})$, $(\tau_n^{(2)},(D_2(t))_{t\in[0,\tau_n^{(2)}]})$ and $(\tau_n,(D(t))_{t\in[0,\tau_n]})$ have the same law. In particular $\tau_n^{(1)}$ and $\tau_n^{(2)}$ satisfy Theorems* 1 *and* 2*.*

The terms *full coupling* and *full decoupling* come from the facts that in (i), D_1 and X are always in interaction, due to the term $\langle N(X_t), dX_t\rangle$ in (4), while in (ii), this interaction is restricted to the times where X encounters the boundary of D_2, due to the term $dL_t^{R_2}(\rho(x_0,\cdot))(X)$ in (5). Heuristically, in (i) the motions of the boundary and the "radial part" (with respect to the skeleton) of X tend to be synchronous, while in (ii) they are independent as long as X is not on the boundary of D_2. We refer to Arnaudon, Coulibaly-Pasquier and Miclo (2020) for more details.

Heuristically speaking, the mapping b_n is mainly of order n (see Lemma 1, except that close to $\pi/2$, the order is rather $\sqrt{n}$, see Proposition 2 and Lemma 2), thus renormalizing time by a factor $1/n$, we end up with a small noise diffusion, so large deviation estimates could lead to the desired result. Indeed, in the next section we will show that $\ln(n)/n$ is an equivalent of the time needed to go from 0 to π for the dynamical system obtained by removing the Brownian motion in (2). But instead of subsequently resorting to the large deviation theory, which cannot be directly applied here due to the existence of two scales $1/n$ and $1/\sqrt{n}$ mentioned above, we present in Section 3 an alternative direct perturbative argument to estimate hitting times, leading to curious optimization problems over *surrogates* of the drift. The latter are approximatively solved in Section 4, leading to the proofs of Theorems 1 and 2. The additional material section justifies the resort to surrogates, by showing that the cut-off phenomenon cannot be deduced by only working with the initial drift.

2. Corresponding dynamic systems

In the spirit of the small noise approach alluded to above, we give here a heuristic justification of the $\ln(n)/n$ term by forgetting the Brownian motion in (2). Nevertheless the following computations are not disconnected from our main goal, as they will be re-used later on.

The dynamical system associated to (2) is defined by

$$\begin{cases} x_0 = 0 \\ \dot{x}_t = b_n(x_t), \end{cases} \tag{6}$$

up to the deterministic time T_n when it hits π (Proposition 2 below will imply in particular that $(x_t)_{t\in[0,T_n]}$ is increasing and that T_n is finite). The goal of this section is to show the following behavior for this hitting time:

Theorem 3. *For large n we have* $T_n \sim \ln(n)/n$.

This bound can serve as an "explanation" for the quantity $\ln(n)/n$ as Theorem 1 will be obtained via perturbative arguments around this result. The proof of Theorem 3 consists of the two matching lower and upper bounds separately presented in the next subsections. In both cases, b_n will be replaced by more manageable drifts.

2.1. The upper bound

Our goal here is to show one "half" of Theorem 3. This inequality will be used for a similar bound on the mixing time, which is interesting in a sampling context, since it serves as a guarantee for convergence.

Proposition 1. *We have* $\limsup_{n\to\infty} nT_n/\ln(n) \le 1$.

In order to prove Proposition 1, we replace b_n by a simpler drift $\widetilde{b}_n \le b_n$, whose corresponding hitting time $\widetilde{T}_n$ of π will furnish a time satisfying $\widetilde{T}_n \ge T_n$. Here is the first step in this direction:

Lemma 1. *For any* $x \in (0,\pi)$*, we have* $b_n(x) \ge n|\cot(x)|$.

Proof. First consider the case where $x \in [\pi/2,\pi)$. Since $\sin^n(x) \ge 0$ and $\int_0^x \sin^n(u)\,du \ge 0$, we get

$$b_n(x) \ge -n\frac{\cos(x)}{\sin(x)} \ = \ n|\cot(x)|.$$

Next consider the case where $x \in (0,\pi/2]$. Define for such fixed x,

$$\forall\, 0 \le v \le x, \qquad f(v) := \sin(x-v) - \sin(x) + \cos(x)v.$$

We compute

$$f'(v) = -\cos(x-v) + \cos(x) \ \le \ 0,$$

and since $f(0) = 0$, we deduce that $\sin(x-v) \le \sin(x) - \cos(x)v$, for any $0 \le v \le x$. It follows that

$$\begin{aligned}\int_0^x \left(\frac{\sin(u)}{\sin(x)}\right)^n du &= \int_0^x \left(\frac{\sin(x-v)}{\sin(x)}\right)^n dv \ \le \ \int_0^x (1-\cot(x)v)^n\, dv \\ &\le \int_0^x \exp(-n\cot(x)v)\,dv \ = \ \frac{1}{n\cot(x)}[1-\exp(-n\cot(x)x)].\end{aligned}$$

Coming back to b_n, we get

$$\begin{aligned}b_n(x) &\ge 2n\cot(x)\frac{1}{1-\exp(-n\cot(x)x)} - n\cot(x) \ = \ n\cot(x)\left(\frac{2}{1-\exp(-n\cot(x)x)} - 1\right) \\ &= n\cot(x)\frac{1+\exp(-n\cot(x)x)}{1-\exp(-n\cot(x)x)} \ \ge \ n\cot(x) \ = \ n|\cot(x)|.\end{aligned}$$

□

The previous bound has the drawback to vanish at $x = \pi/2$, which is problematic for the hitting time of π. So we need another lower bound for b_n:

Proposition 2. *There exists a constant $\widetilde{c} > 0$ such that for all n large enough,*

$$\forall\, x \in (0,\pi), \qquad b_n(x) \geq \widetilde{c}\sqrt{n}.$$

Fix some $A > 0$ and note that for $x \in (0,\pi)$ outside $[\pi/2 - A/\sqrt{n}, \pi/2 + A/\sqrt{n}]$, we have

$$|\cot(x)| \geq |\cos(x)| \;\geq\; \cos\left(\frac{\pi}{2} - \frac{A}{\sqrt{n}}\right) \;\sim\; \frac{A}{\sqrt{n}}. \tag{7}$$

It follows from Lemma 1 that to prove Proposition 2, it sufficient to investigate the behavior of $b_n(x)$ on $[\pi/2 - A/\sqrt{n}, \pi/2 + A/\sqrt{n}]$. We begin with the point $\pi/2$:

Lemma 2. *For large n, we have $b_n(\pi/2) \sim 2\sqrt{2n/\pi}$.*

Proof. By definition, for any $n \in \mathbb{N}$ we have $b_n(\pi/2) = 2/\iota_n$, with

$$\iota_n := \int_0^{\pi/2} \sin^n(u)\, du.$$

By integration by part, it appears that this quantity satisfies,

$$\forall\, n \geq 2, \qquad \iota_n = \frac{n-1}{n}\iota_{n-2},$$

from which we get that for n large, $\iota_n \sim \sqrt{\pi/(2n)}$, and we deduce the wanted equivalent. □

For the other points $x \in [\pi/2 - A/\sqrt{n}, \pi/2 + A/\sqrt{n}]$ (with $n > 4A^2/\pi^2$), we are to systematically consider the change of variables

$$a := \sqrt{n}\left(x - \frac{\pi}{2}\right) \;\in\; [-A, A]. \tag{8}$$

We need the following ingredients.

Lemma 3. *With the parametrization (8), we get for large n, uniformly over $a \in [-A, A]$,*

$$\cos(x) \;\sim\; -\frac{a}{\sqrt{n}}, \quad \sin^n(x) \;\sim\; e^{-a^2/2} \text{ and } I_n(x) \;\sim\; \frac{h(a)}{\sqrt{n}},$$

where

$$\forall\, x \in [0,\pi],\; I_n(x) := \int_0^x \sin^n(u)\, du \quad \text{and} \quad \forall\, a \in \mathbb{R},\; h(a) := \int_{-\infty}^a e^{-u^2/2}\, du.$$

Proof. Writing $x = \frac{\pi}{2} + \frac{a}{\sqrt{n}}$ as in (8), the first equivalent is obtained via an immediate expansion around $\pi/2$. For the second equivalent, note that, keeping using the same change of variables in the sequel,

$$\sin^n(x) = \left(\sqrt{1 - \cos^2(x)}\right)^n \;=\; \exp\left(\frac{n}{2}\ln\left(1 - \cos^2\left(\frac{\pi}{2} + \frac{a}{\sqrt{n}}\right)\right)\right)$$

$$\sim \exp\left(-\frac{n}{2}\cos^2\left(\frac{\pi}{2} + \frac{a}{\sqrt{n}}\right)\right) \;\sim\; e^{-a^2/2}.$$

For the last equivalent, write

$$I_n(x) = \int_0^{\pi/2} \sin^n(y)\, dy + \int_{\pi/2}^{x} \sin^n(y)\, dy.$$

From the previous computation, especially its uniformity, we deduce

$$\int_{\pi/2}^{x} \sin^n(y)\, dy \sim \int_0^a e^{-v^2/2} \frac{dv}{\sqrt{n}}.$$

From the proof of Lemma 2 we have for large n,

$$\int_0^{\pi/2} \sin^n(y)\, dy \sim \sqrt{\frac{\pi}{2n}} = \frac{1}{\sqrt{n}} \int_{-\infty}^{0} e^{-v^2/2}\, dv,$$

and thus finally the wanted equivalent. □

We can now come to the

Proof of Proposition 2. Recalling the definition of b_n given in (3) and the change of variables (8), we deduce from Lemma 3 that uniformly for $a \in [-A, A]$,

$$b_n(x) \sim \sqrt{n}\beta(a), \tag{9}$$

with

$$\forall\, a \in \mathbb{R}, \qquad \beta(a) := 2\frac{e^{-a^2/2}}{h(a)} + a. \tag{10}$$

This mapping will be precisely investigated in Section 4, but for the moment just note that by continuity we can choose $A > 0$ sufficiently small so that

$$\forall\, a \in [-A, A], \qquad \beta(a) \geq \frac{\beta(0)}{2} = \sqrt{\frac{2}{\pi}}.$$

Proposition 2 then follows from this bound and (7), for any given $\widetilde{c} \in (0, \sqrt{2/\pi} \wedge A)$. □

The previous lower bounds on b_n lead us to introduce a new function $\widetilde{b}_n$ on $(0, \pi)$ via

$$\forall\, x \in (0, \pi), \qquad \widetilde{b}_n(x) := \begin{cases} \widetilde{c}\sqrt{n} & \text{if } x \in [\pi/2 - A/\sqrt{n}, \pi/2 + A/\sqrt{n}] \\ n|\cot(x)| & \text{otherwise.} \end{cases}$$

Our interest in $\widetilde{b}_n$ is its simplicity and the fact that $b_n \geq \widetilde{b}_n$. Indeed, it is the primary ingredient in the

Proof of Proposition 1. Replacing (6) by

$$\begin{cases} \widetilde{x}_0 = 0 \\ \dot{\widetilde{x}}_t = \widetilde{b}_n(\widetilde{x}_t), \end{cases} \tag{11}$$

defined up to the time $\widetilde{T}_n$ it hits π, we get $T_n \leq \widetilde{T}_n$, for any $\in \mathbb{N}$. Proposition 1 is an immediate consequence of this bound and of the next lemma. □

Lemma 4. *For n large, we have $\widetilde{T}_n \sim \ln(n)/n$.*

Proof. We decompose $\widetilde{T}_n$ into $\widetilde{T}_n^{(1)} + \widetilde{T}_n^{(2)} + \widetilde{T}_n^{(3)}$ where

$$\widetilde{T}_n^{(1)} := \inf\left\{t \geq 0 : \widetilde{x}_t = \frac{\pi}{2} - \frac{A}{\sqrt{n}}\right\}$$

$$\widetilde{T}_n^{(2)} := \inf\left\{t \geq 0 : \widetilde{x}_{\widetilde{T}_n^{(1)}+t} = \frac{\pi}{2} + \frac{A}{\sqrt{n}}\right\}$$

$$\widetilde{T}_n^{(3)} := \inf\{t \geq 0 : \widetilde{x}_{\widetilde{T}_n^{(1)}+\widetilde{T}_n^{(2)}+t} = \pi\},$$

for any fixed $A > 0$. Let us analyse each of these times separately

- For $t \in [0, \widetilde{T}_n^{(1)})$, we rewrite the second equation of (11) as

$$\frac{\sin(\widetilde{x}_t)}{\cos(\widetilde{x}_t)}\dot{\widetilde{x}}_t = n, \quad \text{i.e.} \quad -\frac{d}{dt}\ln(\cos(\widetilde{x}_t)) = n.$$

Integrating between 0 and $\widetilde{T}_n^{(1)}$ we get

$$n\widetilde{T}_n^{(1)} = \ln(\cos(0)) - \ln\left(\cos\left(\frac{\pi}{2} - \frac{A}{\sqrt{n}}\right)\right) = -\ln\left(\cos\left(\frac{\pi}{2} - \frac{A}{\sqrt{n}}\right)\right).$$

For large n, we have $\cos\left(\pi/2 - A/\sqrt{n}\right) \sim A/\sqrt{n}$, and it follows that

$$-\ln\left(\cos\left(\frac{\pi}{2} - \frac{A}{\sqrt{n}}\right)\right) \sim \frac{\ln(n)}{2},$$

and as a consequence $\widetilde{T}_n^{(1)} \sim \ln(n)/(2n)$.

- For $t \in (\widetilde{T}_n^{(1)}, \widetilde{T}_n^{(1)} + \widetilde{T}_n^{(2)})$, (11) writes $\dot{\widetilde{x}}_t = \widetilde{c}\sqrt{n}$, and we get

$$\widetilde{T}_n^{(2)} = \frac{\frac{\pi}{2} + \frac{A}{\sqrt{n}} - (\frac{\pi}{2} - \frac{A}{\sqrt{n}})}{\widetilde{c}\sqrt{n}} = \frac{2\frac{A}{\sqrt{n}}}{\widetilde{c}\sqrt{n}} = \frac{2A}{\widetilde{c}n}.$$

- For $t \in (\widetilde{T}_n^{(2)} + \widetilde{T}_n^{(2)}, \widetilde{T}_n^{(1)} + \widetilde{T}_n^{(2)} + \widetilde{T}_n^{(3)})$, by symmetry of $\widetilde{b}$ through $[0, \pi/2 - A\sqrt{n}] \ni x \mapsto \pi - x \in [\pi/2 + A/\sqrt{n}, \pi]$, we have

$$\widetilde{T}_n^{(3)} = \widetilde{T}_n^{(1)} \sim \frac{\ln(n)}{2n}.$$

Putting together these estimates, we deduce the desired result. $\square$

Remark 2. In a similar spirit, if we were interested in total variation instead of separation, we would rather be considering the time for the dynamical system $(\widetilde{x}_t)_{t\geq 0}$ to go from 0 to $\pi/2$, since from the concentration of measure phenomenon, “most the mass of the sphere is around the (generalized) equator”. From the previous computation, it appears that this time is of order $\ln(n)/(2n)$, this gives a heuristic explanation for the fact that the mixing times in total variation and in separation differ by a factor 2.

2.2. The lower bound

Our goal here is to show the second "half" of Theorem 3:

Proposition 3. *We have* $\liminf_{n\to\infty} nT_n/\ln(n) \geq 1$.

As in the previous section, we are to replace b_n by a simpler drift $b_n \leq \widehat{b}_n$, whose corresponding hitting time $\widehat{T}_n$ of π will furnish a time satisfying $\widehat{T}_n \leq T_n$. We start by noting that from (9), we deduce at once

Lemma 5. *For any* $A > 0$*, we can find a constant* $\widehat{c}_A > 0$ *such that for all* n *large enough,*

$$\forall\, a \in [-A, A], \qquad b_n\left(\frac{\pi}{2} + \frac{a}{\sqrt{n}}\right) \leq \widehat{c}_A \sqrt{n}.$$

Fix $A > 0$. Here is an analogue of Lemma 1.

Lemma 6. *There exists a quantity* $\epsilon(A) > 0$ *such that for all* n *sufficiently large, depending on* A*,*

$$\forall\, x \in (0, \pi) \setminus (\pi/2 - A/\sqrt{n}, \pi/2 + A/\sqrt{n}), \qquad b_n(x) \leq (1 + \epsilon(A))\, n |\cot(x)|.$$

Furthermore, we have

$$\lim_{A\to+\infty} \epsilon(A) = 0. \tag{12}$$

Proof. Two cases are treated separately:

• For fixed $x \in [\pi/2 + A/\sqrt{n}, \pi)$, we have on one hand,

$$\sin^n(x) \leq \sin^n\left(\pi/2 + \frac{A}{\sqrt{n}}\right) \sim e^{-A^2/2},$$

for n large, and on the other hand

$$I_n(x) \geq I_n(\pi/2) \sim \sqrt{\frac{\pi}{2n}},$$

(where I_n was defined in Lemma 3). It follows that for n sufficiently large, we have

$$b_n(x) \leq 3\sqrt{\frac{2}{\pi}} e^{-A^2/2} \sqrt{n} + n|\cot(x)|.$$

Furthermore we have for large n,

$$|\cot(x)| \geq \left|\cot\left(\pi/2 + \frac{A}{\sqrt{n}}\right)\right| \sim \frac{A}{\sqrt{n}}.$$

It follows that for n large enough,

$$3\sqrt{\frac{2}{\pi}} e^{-A^2/2} \sqrt{n} \leq 4\sqrt{\frac{2}{\pi}} \frac{e^{-A^2/2}}{A} n|\cot(x)|,$$

implying $b_n(x) \leq (1+\epsilon_+(A))\, n|\cot(x)|$, with

$$\epsilon_+(A) := 4\sqrt{\frac{2}{\pi}}\frac{e^{-A^2/2}}{A}.$$

- For fixed $x \in (0, \pi/2 - A/\sqrt{n}]$, we have

$$I_n(x) \geq \int_0^x \cos(u)\sin^n(u)\,du \;=\; \frac{\sin^{n+1}(x)}{n+1},$$

so that

$$b_n(x) \leq \frac{2(n+1)}{\sin(x)} - n\cot(x).$$

Introduce $x_A \in (0, \pi/4)$ so that $1 \leq (1+1/A)\cos(x_A)$. For any $x \in (0, x_A]$, we have $\cos(x) \geq \cos(x_A)$ and thus

$$b_n(x) \leq \left(\frac{2(n+1)}{n}\left(1+\frac{1}{A}\right) - 1\right) n\cot(x) \;\leq\; \left(1+\frac{3}{A}\right) n\cot(x),$$

for n large enough. Denote $\eta_n := 1/\sqrt{n}$ and assume that n is sufficiently large so that $\eta_n \leq x_A$. For $x \in [x_A, \pi/2 - A/\sqrt{n}]$, we have

$$\begin{aligned}
I_n(x) &\geq \int_{x-\eta_n}^x \sin^n(u)\,du \;\geq\; \frac{1}{\cos(x-\eta_n)}\int_{x-\eta_n}^x \cos(u)\sin^n(u)\,du \\
&= \frac{1}{\cos(x-\eta_n)}\left[\frac{\sin^{n+1}(u)}{n+1}\right]_{x-\eta_n}^x \;=\; \frac{1}{\cos(x-\eta_n)}\left[\frac{\sin^{n+1}(x)}{n+1} - \frac{\sin^{n+1}(x-\eta_n)}{n+1}\right] \\
&= \frac{\cos(x)}{\cos(x-\eta_n)}\left[1 - \left(\frac{\sin(x-\eta_n)}{\sin(x)}\right)^{n+1}\right]\frac{\sin^{n+1}(x)}{(n+1)\cos(x)}.
\end{aligned}$$

Note that

$$\min\left\{\frac{\cos(x)}{\cos(x-\eta_n)} \;:\; x \in (x_A, \pi/2 - A/\sqrt{n})\right\} \;=\; \frac{\cos(\pi/2 - A/\sqrt{n})}{\cos(\pi/2 - A/\sqrt{n} - \eta_n)},$$

and the right hand side converges toward $A/(A+1)$ for large n. We also have

$$\max\left\{\left(\frac{\sin(x-\eta_n)}{\sin(x)}\right)^n \;:\; x \in (x_A, \pi/2 - A/\sqrt{n})\right\} \;=\; \left(\frac{\sin(\pi/2 - A/\sqrt{n} - \eta_n)}{\sin(\pi/2 - A/\sqrt{n})}\right)^n,$$

and the right hand side converges toward $e^{-(A+1)^2/2}e^{A^2/2} = e^{-(A+1/2)}$ for large n. It follows that for n sufficiently large,

$$I_n(x) \geq \frac{A}{A+2}(1-e^{-A})\frac{\sin^{n+1}(x)}{(n+1)\cos(x)},$$

and we deduce that for $x \in [x_A, \pi/2 - A/\sqrt{n})]$,

$$b_n(x) \leq \left(2\frac{A+2}{A(1-e^{-A})} - 1\right) n\cot(x) \;=\; (1+\epsilon_-(A))n\cot(x),$$

with

$$\epsilon_{-}(A) := 2\frac{2 + Ae^{-A}}{A(1 - e^{-A})}.$$

The wanted bound follows with $\epsilon(A) := \epsilon_{-}(A) \vee \epsilon_{+}(A)$, satisfying (12). $\square$

The two previous upper bounds on b_n lead us to introduce a new function $\widehat{b}_n$ on $(0,\pi)$ via

$$\forall\, x \in (0,\pi), \qquad \widehat{b}_n(x) := \begin{cases} \widehat{c}_A\sqrt{n} & \text{if } x \in [\pi/2 - A/\sqrt{n}, \pi/2 + A/\sqrt{n}] \\ (1+\epsilon(A))n|\cot(x)| & \text{otherwise,} \end{cases}$$

satisfying $b_n \leq \widehat{b}_n$. Replacing (6) by

$$\begin{cases} \widehat{x}_0 = 0 \\ \dot{\widehat{x}}_t = \widehat{b}_n(\widehat{x}_t), \end{cases} \tag{13}$$

defined up to the time $\widehat{T}_n$ it hits π, we get $T_n \geq \widehat{T}_n$ for any $n \in \mathbb{N}$. By decomposing $\widehat{T}_n$ into $\widehat{T}_n^{(1)} + \widehat{T}_n^{(2)} + \widehat{T}_n^{(3)}$, where $\widehat{T}_n^{(1)}$, $\widehat{T}_n^{(2)}$ and $\widehat{T}_n^{(3)}$ are defined as in the proof of Lemma 4, with the dynamical system $(\widetilde{x}_t)_{t\geq 0}$ replaced by $(\widehat{x}_t)_{t\geq 0}$, we get via similar estimates,

$$\lim_{n\to\infty} \frac{n}{\ln(n)}\widehat{T}_n = 1 + \epsilon(A).$$

We deduce that for any $A > 0$,

$$\liminf_{n\to\infty} \frac{n}{\ln(n)} T_n \geq 1 + \epsilon(A),$$

and letting A go to $+\infty$, we deduce

$$\liminf_{n\to\infty} \frac{n}{\ln(n)} T_n \geq 1.$$

In conjunction with Proposition 1, this bound ends the proof of Theorem 3.

3. Perturbative arguments for absorption

We present here general and very simple perturbative arguments for the expectation and the concentration of a hitting time.

Consider a diffusion $X := (X(t))_{t\geq 0}$ on $[0,\pi]$ whose evolution is given on $(0,\pi)$ by the stochastic differential equation

$$dX(t) = \sqrt{2}dB(t) + \frac{1}{\varphi'(X(t))}dt, \tag{14}$$

where $\varphi : [0,\pi] \to \mathbb{R}_+$ is twice continuously differentiable and increasing on $[0,\pi]$ and such that 0 is an entrance boundary (ensured by $\liminf_{x\to 0_+} x/\varphi'(x) \geq 1$), and where $(B(t))_{t\geq 0}$ is a standard Brownian motion. We start with $X_0 = 0$ and the above diffusion is defined up to the hitting time τ of π. By the above assumptions τ is a.s. finite and our first objective here is to give a simple upper bound of $\mathbb{E}[\tau]$ in terms of φ.

Lemma 7. *Assume that* $\min_{[0,\pi]} \varphi'' > -1$. *Then we have*

$$\mathbb{E}[\tau] \leq \frac{\varphi(\pi) - \varphi(0)}{1 + \min_{[0,\pi]} \varphi''}.$$

Proof. Due to entrance boundary assumption, the process X stays in $(0,\pi]$ at any positive time and by Itô's formula, we have for $t \in (0,\tau)$

$$\begin{aligned} d\varphi(X(t)) &= \varphi'(X(t))dX(t) + \frac{\varphi''(X(t))}{2} d\langle X\rangle_t \\ &= \sqrt{2}\varphi'(X(t))dB(t) + dt + \varphi''(X(t))dt. \end{aligned}$$

Thus integrating between 0 and τ, we get

$$\varphi(X_\tau) - \varphi(0) = \int_0^\tau \varphi'(X(t))\, dB(t) + \int_0^\tau 1 + \varphi''(X(t))\, dt. \tag{15}$$

Taking the expectation, we deduce

$$\varphi(\pi) - \varphi(0) = \mathbb{E}\left[\int_0^\tau 1 + \varphi''(X(t))\, dt\right] \geq \left(1 + \min_{[0,\pi]} \varphi''\right) \mathbb{E}[\tau],$$

which implies the desired bound. □

The above arguments equally lead to a reverse bound:

Lemma 8. *Assume that* $\max_{[0,\pi]} \varphi'' > -1$. *Then we have*

$$\mathbb{E}[\tau] \geq \frac{\varphi(\pi) - \varphi(0)}{1 + \max_{[0,\pi]} \varphi''}.$$

These two results will be the unique insertion into the field of stochastic calculus needed to deduce Theorem 1. They will be reinforced by Lemmas 9 and 10 below to get Theorem 2. We would like to apply them with $\varphi' = 1/b_n$, but as we will see in the Supplementary Material (Arnaudon, Coulibaly-Pasquier and Miclo (2024)), this is not a good idea. It is better to first slightly improve the bounds of Lemmas 7 and 8. Consider

$$\Psi_+(\varphi) := \left\{\psi \in C^2([0,\pi],\mathbb{R}_+) : \psi' \geq \varphi', \min_{[0,\pi]} \psi'' > -1 \text{ and } \limsup_{x\to 0_+} \psi'(x)/x \leq 1\right\}.$$

For any $\psi \in \Psi_+(\varphi)$, which should be seen as a *surrogate* of φ, consider the diffusion starting with $Y(0) = 0$ and satisfying

$$dY(t) = \sqrt{2}dB(t) + \frac{1}{\psi'(Y(t))}dt, \tag{16}$$

up to the hitting time σ of π. The definition of Ψ_+ ensures that 0 is an entrance boundary and that $\tau \leq \sigma$. Applying Lemma 7 to the diffusion $(Y(t))_{t\in[0,\sigma]}$, we get

$$\mathbb{E}[\sigma] \leq \frac{\psi(\pi) - \psi(0)}{1 + \min_{[0,\pi]} \psi''},$$

and we deduce the upper bound

$$\mathbb{E}[\tau] \le \mathbb{E}[\sigma] \le \inf_{\psi\in\Psi_+(\varphi)} \frac{\psi(\pi)-\psi(0)}{1+\min_{[0,\pi]}\psi''}. \tag{17}$$

To evaluate the right hand side seems an interesting optimisation problem. We will not investigate it here in general, but we will see that for our particular problem it leads to the right equivalent (while only considering $\psi = \varphi \in \Psi_+$ does not). Similarly, introduce

$$\Psi_-(\varphi) := \left\{\psi \in C^2([0,\pi],\mathbb{R}_+) \,:\, \psi' \le \varphi',\, \max_{[0,\pi]}\psi'' > -1 \text{ and } \limsup_{x\to 0_+}\psi'(x)/x \le 1\right\}.$$

Then we have

$$\mathbb{E}[\tau] \ge \sup_{\psi\in\Psi_-(\varphi)} \frac{\psi(\pi)-\psi(0)}{1+\max_{[0,\pi]}\psi''}. \tag{18}$$

Both (17) and (18) will enable us to get the equivalent given in Theorem 1 for the expectation of the strong stationary time τ_n, by exhibiting two appropriate families of surrogates $(\psi_{+,n})_{n\in\mathbb{Z}_+}$ and $(\psi_{-,n})_{n\in\mathbb{Z}_+}$, respectively from Ψ_+ and Ψ_- and with $\min_{[0,\pi]}\psi''_{+,n}$ and $\max_{[0,\pi]}\psi''_{-,n}$ going to zero as n goes to infinity.

By going a little further, it is possible to deduce the cut-off phenomenon of Theorem 2: instead of using that the expectation of a martingale is zero, as in Lemmas 7 and 8, we can evaluate its variance via its bracket. It leads to the following result for the hitting time τ of π by the diffusion (14) starting from 0.

Lemma 9. *Assume that* $\varphi(0) = 0$ *and* $\min_{[0,\pi]}\varphi'' > -1/3$. *Then we have for any* $r > 0$,

$$\mathbb{P}\left[\tau > \frac{\varphi(\pi)}{1+\min_{[0,\pi]}\varphi''}(1+r)\right] \le \frac{1}{r^2\varphi^2(\pi)(1+3\min_{[0,\pi]}\varphi'')}\int_0^\pi (\varphi'(u))^3\,du.$$

Proof. From (15), we deduce

$$(1+\min_{[0,\pi]}\varphi'')\tau \le \varphi(\pi) + Z, \tag{19}$$

where $Z := -\int_0^\tau \varphi'(X(t))\,dB(t)$, so that

$$\begin{aligned}\mathbb{P}\left[\tau > \frac{\varphi(\pi)}{1+\min_{[0,\pi]}\varphi''}(1+r)\right] &= \mathbb{P}\left[\left(1+\min_{[0,\pi]}\varphi''\right)\tau > \varphi(\pi)(1+r)\right] \le \mathbb{P}[Z > \varphi(\pi)r]\\ &\le \frac{1}{(\varphi(\pi)r)^2}\mathbb{E}[Z^2] = \frac{1}{(\varphi(\pi)r)^2}\mathbb{E}\left[\int_0^\tau (\varphi'(X(s)))^2\,ds\right].\end{aligned}$$

Let us evaluate the last expectation as we have done for $\mathbb{E}[\tau]$. Denote γ the function on $[0,\pi]$ satisfying $\gamma(0) = 0$ and

$$\forall\, x \in [0,\pi], \qquad \gamma'(x) := (\varphi'(x))^3,$$

so that, taking into account that $\gamma'' = 3(\varphi')^2\varphi''$,

$$(\varphi')^2 = \gamma'' + \gamma'/\varphi' - 3(\varphi')^2\varphi'' \le \gamma'' + \gamma'/\varphi' - 3\left(\min_{[0,\pi]}\varphi''\right)(\varphi')^2.$$

It follows that

$$\left(1+3\left(\min_{[0,\pi]}\varphi''\right)\right)\mathbb{E}\left[\int_0^\tau (\varphi'(X(s)))^2\,ds\right] \le \mathbb{E}\left[\int_0^\tau [\gamma''+\gamma'/\varphi'](X(s))\,ds\right]$$
$$= \mathbb{E}\left[\gamma(X_\tau)-\gamma(X_0)-\int_0^\tau \gamma'(X(s))\,dB(s)\right] \;=\; \gamma(\pi).$$

The wanted result follows. □

Replacing (19) by $(1+\max_{[0,\pi]}\varphi'')\tau \ge \varphi(\pi)+Z$, the same arguments show:

Lemma 10. *Assume that* $\varphi(0)=0$ *and* $\min_{[0,\pi]}\varphi'' > -1/3$. *Then we have for any* $r>0$,

$$\mathbb{P}\left[\tau < \frac{\varphi(\pi)}{1+\max_{[0,\pi]}\varphi''}(1-r)\right] \le \frac{1}{r^2\varphi^2(\pi)(1+3\min_{[0,\pi]}\varphi'')}\int_0^\pi (\varphi'(u))^3\,du.$$

The comparison with diffusions of the form (16) leads to the following extensions of the two previous lemmas: for any $\psi\in\Psi_+(\varphi)$, such that $\min_{[0,\pi]}\psi'' > -\frac{1}{3}$,

$$\mathbb{P}\left[\tau > \frac{\psi(\pi)-\psi(0)}{1+\min_{[0,\pi]}\psi''}(1+r)\right] \le \frac{1}{r^2(\psi(\pi)-\psi(0))^2(1+3\min_{[0,\pi]}\psi'')}\int_0^\pi (\psi'(u))^3\,du, \tag{20}$$

and for any $\psi\in\Psi_-(\varphi)$, such that $\min_{[0,\pi]}\psi'' > -\frac{1}{3}$,

$$\mathbb{P}\left[\tau < \frac{\psi(\pi)-\psi(0)}{1+\max_{[0,\pi]}\psi''}(1-r)\right] \le \frac{1}{r^2(\psi(\pi)-\psi(0))^2(1+3\min_{[0,\pi]}\psi'')}\int_0^\pi (\psi'(u))^3\,du. \tag{21}$$

4. Construction of appropriate surrogates

We come back to the diffusion defined in (2). We would like to apply the bounds of the previous section with $\varphi_n' = 1/b_n$, for given $n\in\mathbb{N}$. It leads us to construct appropriate surrogates $\psi_n\in\Psi_+(\varphi_n)$ and $\psi_{n,-}\in\Psi_-(\varphi_n)$, whose corresponding bounds will imply Theorems 1 and 2.

As suggested by the computations of Section 2, it is important to understand the behavior of b_n at the scale $1/\sqrt{n}$: fix any $A>0$ and consider the change of variable $x=\pi/2+a/\sqrt{n}$ for $a\in[-A,A]$. Here is a first result about the mapping β defined in (10):

Lemma 11. *There exists a unique* $a_0\in\mathbb{R}$ *such that* $\beta'(a_0)=0$. *Furthermore, we have* $a_0>0$.

Proof. We compute

$$\forall\, a\in\mathbb{R},\qquad \beta'(a) = -2\frac{ae^{-a^2/2}}{h(a)} - 2\frac{e^{-a^2}}{h^2(a)} + 1.$$

Denote $X := e^{-a^2/2}/h(a)$, so that $\beta'(a)=0$ is equivalent to the equality $2aX+2X^2-1=0$. Furthermore we compute

$$\forall\, a\in\mathbb{R},\qquad \beta''(a) = -2X[1-a^2-3aX-2X^2].$$

It follows that if $a \in \mathbb{R}$ is such that $\beta'(a) = 0$, then

$$\beta''(a) = 2aX(a + X). \tag{22}$$

We examine separately three cases:

- If $a > 0$, then $\beta''(a) > 0$, namely the critical point a is a local minimum.
- If $a = 0$, we verify directly that

$$\beta'(0) = -2\frac{1}{h^2(0)} + 1 \ = \ -\frac{4}{\pi} + 1 \ < \ 0.$$

- If $a < 0$, let us show that $a + X > 0$. Indeed, for $u < a < 0$, we have $1/u > 1/a$ and thus

$$h(a) = \int_{-\infty}^{a} \frac{u}{u} e^{-u^2/2}\, du \ < \ \frac{1}{a}\int_{-\infty}^{a} u e^{-u^2/2}\, du \ = \ -\frac{1}{a}e^{-a^2/2}, \tag{23}$$

implying $a + X > 0$. We deduce from (22) that $\beta''(a) < 0$, i.e. the critical point a is a local maximum. Since two different local minima (respectively maxima) are necessarily separated by a local maximum (resp. minimum), we deduce there is at most one point a in $(0, +\infty)$ (resp. $(-\infty, 0)$) satisfying $\beta'(a) = 0$. Note that as a goes to $+\infty$ we have $\beta(a) \sim a$ and that as a goes to $-\infty$, $\beta(a) \sim -a$. The latter relation comes from the fact that (23) is well known to be an equivalent for $h(a)$ as $a \to -\infty$ (this is proven by an integration by parts). It follows that coming from $-\infty$ and going to $+\infty$, β cannot have first a local maximum. Since β must have at least one local minimum, it appears finally that β has a unique critical point a_0, which is a local minimum. We also infer that $a_0 > 0$. □

As a consequence of $\beta''(a_0) > 0$, seen in the above proof, we get:

Lemma 12. *There exists $\varepsilon_0 > 0$ sufficiently small so that the following quantities are finite for any $\varepsilon \in (0, \varepsilon_0)$:*

$$a_+(\varepsilon) := \min\{a > a_0 \ : \ \beta'(a)/\beta^2(a) = \varepsilon\}$$
$$a_-(\varepsilon) := \max\{a < a_0 \ : \ \beta'(a)/\beta^2(a) = -\varepsilon\}.$$

In the sequel $\varepsilon_0 > 0$ is fixed as in the above lemma. Define

$$\phi(a) := \frac{1}{|a|} \tag{24}$$

and for $\varepsilon \in (0, \varepsilon_0)$, consider

$$m_+(\varepsilon) := \sup\left\{m > a_+(\varepsilon) \ : \ \frac{1}{\beta(a_+(\varepsilon))} - \varepsilon(m - a_+(\varepsilon)) = \phi(m)\right\}$$
$$m_-(\varepsilon) := \inf\left\{m < a_-(\varepsilon) \ : \ \frac{1}{\beta(a_-(\varepsilon))} + \varepsilon(m - a_-(\varepsilon)) = \phi(m)\right\}.$$

For ε_0 chosen sufficiently small, $m_+(\varepsilon)$ and $m_-(\varepsilon)$ are respectively a maximum and a minimum as illustrated by Figure 1. More rigorously concerning $m_+(\varepsilon)$, note that $1/\beta$ is always below ϕ and the mapping $\mathbb{R} \ni m \mapsto \frac{1}{\beta(a_+(\varepsilon))} - \varepsilon(m - a_+(\varepsilon))$ is the graph of the tangent line to $1/\beta$ at $a_+(\varepsilon)$. For $\varepsilon > 0$ sufficiently small, this tangent is almost horizontal and hits the graph of ϕ restricted to $(0, +\infty)$. Since

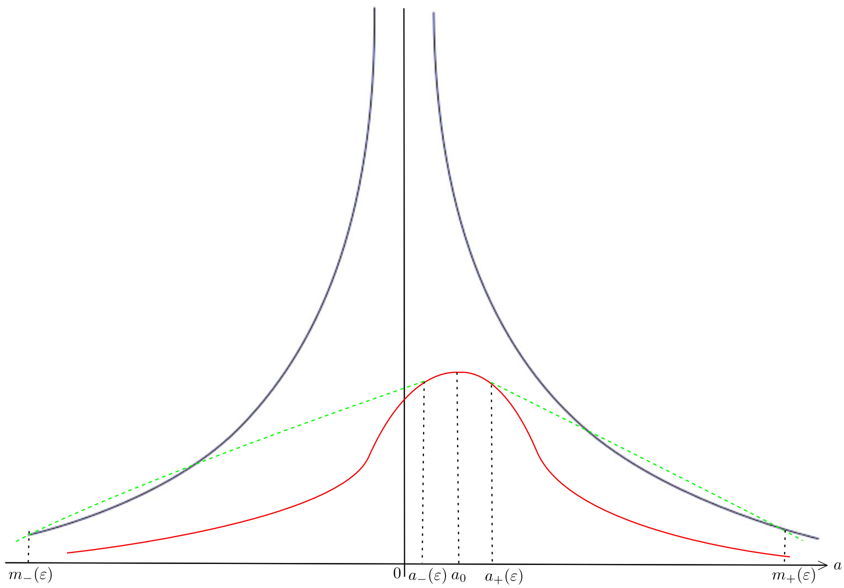

Figure 1. The mappings ϕ and $1/\beta$ are respectively in blue and red. The half-tangents with slope $-\varepsilon$ and ε are in green.

the value of the tangent goes to $-\infty$ for large values of the abscissa a, the tangent hits ϕ at a last value of a, which is defined to be $m_+(\varepsilon)$. A similar reasoning is valid for $m_-(\varepsilon)$.

The following observation will be important:

Lemma 13. *We have*

$$\lim_{\varepsilon\to 0_+} m_+(\varepsilon) = +\infty \quad \textit{and} \quad \lim_{\varepsilon\to 0_+} m_-(\varepsilon) = -\infty.$$

Proof. Fix any $M > 2\beta(a_0)$. Taking into account that $\lim_{\varepsilon\to 0_+} a_+(\varepsilon) = a_0$, for $\varepsilon > 0$ sufficiently small, we have

$$\frac{1}{\beta(a_+(\varepsilon))} - \varepsilon(M - a_+(\varepsilon)) > \frac{1}{2\beta(a_0)} > \phi(M).$$

It follows there exists $m \in (M, 1/(a_+(\varepsilon)\varepsilon) + a_+(\varepsilon))$ such that

$$\frac{1}{\beta(a_+(\varepsilon))} - \varepsilon(m - a_+(\varepsilon)) = \phi(m),$$

and we deduce $\liminf_{\varepsilon\to 0_+} m_+(\varepsilon) \geq M$, and finally the first desired divergence.

The second one is obtained in the same way. $\square$

For any $\varepsilon \in (0,\varepsilon_0)$, consider the function θ_ε defined on $\mathbb{R}$ by

$$\forall\, a \in \mathbb{R}, \qquad \theta_\varepsilon(a) := \begin{cases} \phi(a) & \text{if } a < m_-(\varepsilon) \text{ or } a > m_+(\varepsilon) \\ \dfrac{1}{\beta(a_-(\varepsilon))} + \varepsilon(a - a_-(\varepsilon)) & \text{if } a \in [m_-(\varepsilon), a_-(\varepsilon)] \\ \dfrac{1}{\beta(a_+(\varepsilon))} - \varepsilon(a - a_+(\varepsilon)) & \text{if } a \in [a_+(\varepsilon), m_+(\varepsilon)] \\ 1/\beta(a) & \text{if } a \in [a_-(\varepsilon), a_+(\varepsilon)]. \end{cases}$$

Next results will show θ_ε stays between $1/\beta$ and ϕ and has a small derivative, as it can be guessed from Figure 1. Up to a regularization, it will serve as the derivative of a convenient surrogate.

Lemma 14. *We have $|\theta'_\varepsilon(a)| \le \varepsilon$ for any $a \in \mathbb{R} \setminus \{m_-(\varepsilon), m_+(\varepsilon)\}$. In particular, we get*

$$\lim_{\varepsilon \to 0_+} \sup_{\mathbb{R}\setminus\{m_-(\varepsilon),m_+(\varepsilon)\}} |\theta'_\varepsilon| = 0.$$

Proof. By construction, θ_ε is differentiable on $\mathbb{R}$, except maybe at $m_-(\varepsilon)$ and $m_+(\varepsilon)$, where the left and right derivates may differ. By definition of $a_-(\varepsilon)$ and $a_+(\varepsilon)$ in Lemma 12, we have

$$\forall\, a \in [a_-(\varepsilon), a_+(\varepsilon)], \qquad |\theta'_\varepsilon(a)| \;=\; |(1/\beta)'(a)| \le \varepsilon.$$

Furthermore, note that

$$\forall\, a \in (m_-(\varepsilon), a_-(\varepsilon)] \sqcup [a_+(\varepsilon), m_+(\varepsilon)), \qquad |\theta'(a)| = \varepsilon.$$

Finally, we have

$$\forall\, a > m_+(\varepsilon), \qquad |\theta'_\varepsilon(a)| = |\phi'(a)| \;=\; \frac{1}{a^2},$$

so that

$$\forall\, a > m_+(\varepsilon), \qquad |\theta'_\varepsilon(a)| \le \frac{1}{m_+^2(\varepsilon)},$$

and similarly

$$\forall\, a < m_-(\varepsilon), \qquad |\theta'_\varepsilon(a)| \le \frac{1}{m_-^2(\varepsilon)}.$$

We deduce $|\theta'_\varepsilon(a)| \le \max(1/m_-^2(\varepsilon), 1/m_+^2(\varepsilon), \varepsilon)$. To conclude the desired bound, note that at $m_+(\varepsilon)$, we have $-\varepsilon \le \phi'(m_+(\varepsilon)) \le 0$, since after $m_+(\varepsilon)$, ϕ is above the line of slope $-\varepsilon$ passing through $\phi(m_+(\varepsilon))$. Thus we get $1/m_+^2(\varepsilon) \le \varepsilon$. Similarly we have $1/m_-^2(\varepsilon) \le \varepsilon$ and the claimed result follows. □

Let us check that for $\varepsilon > 0$ small enough, θ_ε remains above $1/\beta$.

Lemma 15. *There exists $\varepsilon_1 \in (0,\varepsilon_0)$ such that for any $\varepsilon \in (0,\varepsilon_1)$, we have $\theta_\varepsilon \ge 1/\beta$.*

Proof. To simplify the notation, let us write $q := 1/\beta$ and let us work on $[a_0, +\infty)$, similar arguments are valid on $(-\infty, a_0]$. For $\varepsilon \in (0,\varepsilon_0)$, define

$$c_+(\varepsilon) := \min\left\{ m > a_+(\varepsilon) \,:\, \frac{1}{\beta(a_+(\varepsilon))} - \varepsilon(m - a_+(\varepsilon)) = \phi(m) \right\}.$$

On $[a_0,+\infty)$, it is clear from the definition of θ_ε that $\theta_\varepsilon \geq q$, except maybe on $[a_+(\varepsilon),c_+(\varepsilon)]$ (note that on $(c_+(\varepsilon),m_+(\varepsilon))$, $\theta_\varepsilon \geq \phi \geq q$). We have already seen that $\lim_{\varepsilon\to 0_+} a_+(\varepsilon) = a_0$, and we have

$$\lim_{\varepsilon\to 0_+} c_+(\varepsilon) = c_+(0), \tag{25}$$

where $c_+(0) = 1/q(a_0)$ is the unique positive solution a of $\phi(a) = q(a_0)$. We compute that

$$\forall\, a \in \mathbb{R}, \qquad q'(a) = \frac{1}{2} - (1 + \frac{a^2}{2})q^2(a), \tag{26}$$

from which, we get

$$\forall\, a \in \mathbb{R}, \qquad q''(a) = -aq^2(a) - 2(1 + \frac{a^2}{2})q(a)q'(a). \tag{27}$$

Thus we can find $\varepsilon_2 > 0$ such that

$$\forall\, a \in [a_0, a_0 + \varepsilon_2], \qquad q''(a) \leq \frac{q''(a_0)}{2} = -a_0\frac{q^2(a_0)}{2} < 0.$$

Let $\varepsilon_3 > 0$ be such that for $\varepsilon \in (0,\varepsilon_3)$, we have $a_+(\varepsilon) \in (a_0, a_0 + \varepsilon_2/2)$. By the strict concavity of q on $[a_0, a_0 + \varepsilon_2]$, the affinity of θ_ε on $[a + (\varepsilon), m_+(\varepsilon)]$ and the fact that $\theta'_\varepsilon(a_+(\varepsilon)) = q'(a_+(\varepsilon))$, we deduce that for $\varepsilon \in (0,\varepsilon_3)$,

$$\forall\, a \in [a_+(\varepsilon), m_+(\varepsilon) \wedge (a_0 + \varepsilon_2)], \qquad \theta_\varepsilon(a) \geq q(a).$$

Furthermore, up to reducing $\varepsilon_3 > 0$, we can assume that $m_+(\varepsilon) > a_0 + \varepsilon_2$. It remains to consider the situation on the segment $[a_0 + \varepsilon_2, c_+(\varepsilon)]$. Taking into account (25) and the fact that the slope of θ_ε tends to zero as $\varepsilon \to 0_+$, to show that $\theta_\varepsilon \geq q$ on $[a_0 + \varepsilon_2, c_+(\varepsilon)]$ (for $\varepsilon \in (0,\varepsilon_1)$ for some $\varepsilon_1 \in (0,\varepsilon_3)$), it is sufficient to show that $q' < 0$ on $(a_0,+\infty)$. By contradiction, assume there exists $a_1 > a_0$ such that $q'(a_1) = 0$. From (27), we deduce that

$$q''(a_1) = -a_1 q^2(a_1) < 0.$$

From the fact that $q'(a_0) = 0$ and $q''(a_0) = -a_0 q^2(a_0) < 0$, there must exist $a_2 \in (a_0, a_1)$ with $q'(a_2) = 0$ and $q''(a_2) \geq 0$. This is in contradiction with the fact that $q''(a_2) = -a_2 q^2(a_2) < 0$. □

Consider the fonction f_n given by

$$\forall\, x \in [0,\pi] \setminus \{\pi/2\}, \qquad f_n(x) := \frac{|\tan(x)|}{n}.$$

We have for large n and for any given $a \neq 0$, $f_n(x) \sim \phi(a)/\sqrt{n}$, with ϕ defined in (24). Fix $\varepsilon \in (0,\varepsilon_1)$, where ε_1 is as in Lemma 15, and take $A > 0$ large enough, so that $-A < m_-(\varepsilon)$ and $A > m_+(\varepsilon)$. For $n \geq A^2$, define the mapping ξ_n on $[0,\pi]$ satisfying $\xi_n(0) = 0$ and

$$\forall\, x \in (0,\pi), \qquad \xi'_n(x) := \begin{cases} \frac{1}{\sqrt{n}}\theta_\varepsilon(a) & \text{if } a \in [-A,A] \\ f_n(x) & \text{otherwise,} \end{cases} \tag{28}$$

(recall that $a = \sqrt{n}(x - \pi/2)$). The function ξ_n may not be strictly differentiable at $\pi/2 - A/\sqrt{n}$ and $\pi/2 + A/\sqrt{n}$ (the above formulas giving the right derivative at $-A$ and the left derivative at A), nor twice differentiable at $\pi/2 - m_-(\varepsilon)/\sqrt{n}$ and $\pi/2 + m_+(\varepsilon)/\sqrt{n}$. But outside these four points, ξ_n is twice differentiable. Convoluting ξ_n with an approximation of the Dirac mass at 0 and taking into account Lemma 14, we construct an increasing function ψ_n twice differentiable on $(0,\pi)$ such that for n large enough,

$$b_n \geq (1-\varepsilon)\frac{1}{\psi_n'} \tag{29}$$

$$\sup_{(0,\pi)} |\psi_n''| \leq \varepsilon(1+\varepsilon). \tag{30}$$

Furthermore, the computations of Lemma 4 show that for large n, $\xi_n(\pi) \sim \ln(n)/n$, thus for n large enough,

$$\psi_n(\pi) - \psi_n(0) \leq (1+\varepsilon)\frac{\ln(n)}{n}. \tag{31}$$

Taking into account that for $\varepsilon > 0$ small enough, we have for n large enough, $\psi_{n,+} := \psi_n/(1-\varepsilon) \in \Psi_+(\varphi_n)$, we deduce from (17)

$$\limsup_{n\to\infty} \frac{n}{\ln(n)}\mathbb{E}[\tau_n] \leq \frac{1+\varepsilon}{1-\varepsilon-\varepsilon(1+\varepsilon)}$$

(where τ_n is the strong stationary time defined in (1)) and letting ε go to zero, we conclude to the bound

$$\limsup_{n\to\infty} \frac{n}{\ln(n)}\mathbb{E}[\tau_n] \leq 1. \tag{32}$$

To get a reverse bound, it is sufficient to apply (18) with appropriate surrogates $\psi_{n,-} \in \Psi_-(\varphi_n)$. Inspired by the computations of Section 2.2, we first find $A_0 > 0$ in Lemma 6 such for any $A > A_0$ the quantity $\epsilon(A) > 0$ is well-defined there. Given $\varepsilon > 0$, the above arguments are still valid, except that (29) and (31) can respectively be replaced by

$$b_n \leq (1+\varepsilon)\frac{1+\epsilon(A)}{\psi_n'}$$

$$\psi_n(\pi) - \psi_n(0) \geq (1-\varepsilon)\frac{\ln(n)}{n}, \tag{33}$$

for any $A > A_0'$, for some $A_0' \geq A_0$ and for any $n \geq n_0$ (depending on A_0'). It follows in particular that for n large enough, $\psi_{n,-} := \psi_n/[(1+\epsilon(A))(1+\varepsilon)] \in \Psi_-(\varphi_n)$ and we deduce from (18),

$$\liminf_{n\to\infty} \frac{n}{\ln(n)}\mathbb{E}[\tau_n] \geq \frac{1-\varepsilon}{(1+\varepsilon)(1+\epsilon(A)) - \varepsilon(1+\varepsilon)}.$$

Letting ε go to zero and A to to $+\infty$, we deduce $\liminf_{n\to\infty} n\mathbb{E}[\tau_n]/\ln(n) \geq 1$. In conjunction with (32), this ends the proof of Theorem 1.

To end this section, let us show Theorem 2. We begin by its first convergence, where $r > 0$ is fixed from now on. For $\varepsilon > 0$ sufficiently small, consider again the mapping $\psi_{n,+} \in \Psi_+(\varphi_n)$ defined above. According to (20), we have for any $r > 0$,

$$\mathbb{P}\left[\tau_n > \frac{\psi_{n,+}(\pi) - \psi_{n,+}(0)}{1 + \min_{[0,\pi]} \psi_{n,+}''}(1 + r/2)\right]$$

$$\leq \frac{4}{r^2(\psi_{n,+}(\pi)-\psi_{n,+}(0))^2(1+3\min_{[0,\pi]}\psi''_{n,+})}\int_0^\pi (\psi'_{n,+}(u))^3\,du.$$

Up to choosing $\varepsilon > 0$ even smaller, (30) and (31) ensure that for all n sufficiently large, we have

$$\frac{\psi_{n,+}(\pi)-\psi_{n,+}(0)}{1+\min_{[0,\pi]}\psi''_{n,+}}(1+r/2) < (1+r)\frac{\ln(n)}{n},$$

implying

$$\mathbb{P}\left[\tau_n > (1+r)\frac{\ln(n)}{n}\right] \leq \frac{4}{r^2(\psi_{n,+}(\pi)-\psi_{n,+}(0))^2(1+3\min_{[0,\pi]}\psi''_{n,+})}\int_0^\pi (\psi'_{n,+}(u))^3\,du.$$

Thus the first convergence of Theorem 2 is a consequence of (33) and

Lemma 16. *We have*

$$\lim_{n\to\infty}\frac{n^2}{\ln^2(n)}\int_0^\pi (\psi'_{n,+}(u))^3\,du = 0.$$

Proof. The above convergence is equivalent to

$$\lim_{n\to\infty}\frac{n^2}{\ln^2(n)}\int_0^\pi (\psi'_n(u))^3\,du = 0. \tag{34}$$

Since differentiation and convolution commute and convolution is a contraction in $\mathbb{L}^p$, for $p \geq 1$ (recall that $\psi'_n > 0$), (34) is itself implied by

$$\lim_{n\to\infty}\frac{n^2}{\ln^2(n)}\int_0^\pi (\xi'_n(u))^3\,du = 0. \tag{35}$$

Coming back to Definition (28), we write

$$\begin{aligned}\int_0^\pi (\xi'_n(u))^3\,du &= \int_{(0,\pi)\setminus[\pi/2-A/\sqrt{n},\pi/2+A/\sqrt{n}]}(\xi'_n(u))^3\,du + \int_{[\pi/2-A/\sqrt{n},\pi/2+A/\sqrt{n}]}(\xi'_n(u))^3\,du\\ &= \frac{1}{n^3}\int_{(0,\pi)\setminus[\pi/2-A/\sqrt{n},\pi/2+A/\sqrt{n}]}|\tan(u)|^3\,du + \frac{1}{n^2}\int_{-A}^{A}\theta_\varepsilon^3(a)\,da.\end{aligned}$$

Note that the first term of the right hand side is equal to

$$\begin{aligned}\frac{2}{n^3}\int_0^{\pi/2-A/\sqrt{n}}\tan^3(u)\,du &= \frac{2}{n^3}\int_{A/\sqrt{n}}^{\pi/2}\cot^3(u)\,du \;\leq\; \frac{2}{n^3}\int_{A/\sqrt{n}}^{\pi/2}\frac{1}{u^3}\,du\\ &= \frac{1}{n^3}\left[-\frac{1}{u^2}\right]_{A/\sqrt{n}}^{\pi/2} \;\leq\; \frac{1}{n^3}\frac{n}{A^2} \;=\; \frac{1}{(An)^2},\end{aligned}$$

and thus

$$\frac{n^2}{\ln^2(n)}\frac{1}{n^3}\int_{(0,\pi)\setminus[\pi/2-A/\sqrt{n},\pi/2+A/\sqrt{n}]}|\tan(u)|^3\,du \leq \frac{1}{(A\ln(n))^2}$$

converging toward 0 for large n. Similarly we have

$$\frac{n^2}{\ln^2(n)}\frac{1}{n^2}\int_{-A}^{A}\theta_\varepsilon^3(a)\,da = \frac{1}{\ln^2(n)}\int_{-A}^{A}\theta_\varepsilon^3(a)\,da$$

converging toward 0 for large n and ending the proof of (35). □

The proof of the second convergence of Theorem 2 follows a similar pattern, via (21) applied to $\psi_{n,-} \in \Psi_-(\varphi_n)$. Indeed, $r \in (0,1)$ being fixed, we first find $A > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small so that for all large enough n,

$$\frac{\psi_{n,-}(\pi) - \psi_{n,-}(0)}{1 + \min_{[0,\pi]}\psi''_{n,-}}(1 - r/2) > (1 - r)\frac{\ln(n)}{n},$$

and we get

$$\mathbb{P}\left[\tau_n < (1-r)\frac{\ln(n)}{n}\right] \le \frac{4}{r^2(\psi_{n,-}(\pi) - \psi_{n,-}(0))^2(1 + 3\min_{[0,\pi]}\psi''_{n,-})}\int_0^\pi (\psi'_{n,-}(u))^3\,du.$$

This bound implies the second convergence of Theorem 2 via the analogue of Lemma 16, where $\psi_{n,+}$ is replaced by $\psi_{n,-}$, and which is proven in exactly the same way. We also deduce the following consequences from the proof of Lemma 16:

Corollary 3. *For any $x \in \mathbb{S}^{n+1}$, let $X^x := (X_t^x)_{t\ge 0}$ be the Brownian motion on the sphere $\mathbb{S}^{n+1}$ (time-accelerated by a factor 2), starting with $X_0^x = x$. There exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $r > 0$ and for all $n \ge n_0$,*

$$\left\|\mathcal{L}\left(X^x_{(1+r)\frac{\ln(n)}{n}}\right) - \mu_{\mathbb{S}^{n+1}}\right\|_{\mathrm{tv}} \le \frac{C}{r^2\ln^2(n)}$$

$$\forall\, y \in \mathbb{S}^{n+1}, \qquad P^{(n+1)}_{(1+r)\frac{\ln(n)}{n}}(x,y) \ge \left(1 - \frac{C}{r^2\ln^2(n)}\right)\frac{1}{\mathrm{vol}(\mathbb{S}^{n+1})},$$

where $\|\cdot\|_{\mathrm{tv}}$ stands for the total variation norm, $\mathcal{L}(X_t^x)$ is the law of X_t^x, $\mu_{\mathbb{S}^{n+1}}$ is the uniform measure in $\mathbb{S}^{n+1}$, and $P_t^{(n+1)}(\cdot,\cdot)$ is the heat kernel density at time $t > 0$ associated to the Laplacian on $\mathbb{S}^{n+1}$.

Proof. From the computations in the proof of Lemma 16, there exist a constant C depending on the quantity $\max\{\int_{-A}^{A}\theta_\varepsilon^3(a)\,da, \frac{1}{A^2}\}$, and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\mathbb{P}\left[\tau_n > (1+r)\frac{\ln(n)}{n}\right] \le \frac{C}{r^2\ln^2(n)}.$$

The first conclusion follows, since

$$\left\|\mathcal{L}\left(X^x_{(1+r)\frac{\ln(n)}{n}}\right) - \mu_{\mathbb{S}^{n+1}}\right\|_{\mathrm{tv}} \le \mathfrak{s}\left(\mathcal{L}\left(X^x_{(1+r)\frac{\ln(n)}{n}}\right), \mu_{\mathbb{S}^{n+1}}\right) \le \mathbb{P}\left[\tau_n > (1+r)\frac{\ln(n)}{n}\right].$$

The second conclusion follows by definition of the separation discrepancy, since for all $y \in \mathbb{S}^{n+1}$ and $t > 0$,

$$1 - P_t^{(n+1)}(x,y)\mathrm{vol}(\mathbb{S}^{n+1}) \le \mathfrak{s}(\mathcal{L}(X_t^x), \mu_{\mathbb{S}^{n+1}}).$$

□

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Supplementary Material

Supplement to "On the separation cut-off phenomenon for Brownian motions on high dimensional spheres" (DOI: 10.3150/23-BEJ1622SUPP; .pdf). The necessity of the notion of surrogate introduced above is discussed there.

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