POINCARÉ INEQULALITY WITH EXPLICIT CONSTANT IN DIMENSION $d \geq 1$

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GLOBAL SENSITIVITY ANALYSIS AND POINCARÉ INEQUALITIES

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Contents

1. Introduction

Let μ be a probability measure on $\mathbb{R}^d$ or on a domain Ω of $\mathbb{R}^d$. We say that the Poincaré inequality holds for μ if there exists a finte constant $C > 0$ such that for all "smooth function" f,

$$\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 d\mu. \tag{1.1}$$

Recall that

$$\begin{aligned}\mathrm{Var}_\mu(f) &= \int \left(f - \int f d\mu\right)^2 d\mu \\ &= \int f \left(f - \int f d\mu\right) d\mu \\ &= \int f^2 d\mu - \left(\int f d\mu\right)^2.\end{aligned}$$

Actually we want the Poincaré inequality to hold for all $f \in H_1(\mu)$ with

$$H_1(\mu) = \{f \in L^2(\mu), |\nabla f| \in L^2(\mu)\}.$$

The smallest constant C for which (1.1) holds will be denoted by $C_P(\mu)$ or C_P if the context is clear.

The goal of these notes is to review some results on the Poincaré inequality. In view of their possible application in sensitivity analysis, we shall concentrate here in methods that provide explicit constants in the Poincaré inequality. We do not have the pretention to be exhaustivethe term .

In dimension $d \ge 2$, finding an explicit and good Poincaré constant is not an easy task. We will see why the log-concavity or better the uniform log-concavity of the measure helps a lot and in the compact case that the convexity of the boundary is also important.

The plan of these notes are the following. In Section 2, we make the link between the Poincaré inequality, the spectral gap of the associated diffusion operator and the exponential convergence (decay variance) of the associated semi-group.

In Section 3, we recall the Bakry-Emery argument in the uniformly convex case.

In Section 4, we present and discuss the tensorisation property of the Poincaré ineqaultiy and two perturbation arguments: the classical argument of Holley-Stroock and the stability by transport Lipschitz. This permits to deduce new Poincaré inequalities from existing ones.

In Section 5, we present two variance inequalities close to the Bakry-Emery argument. The key point here is indeed the intertwining between the gradient and the semi-group. The first result is the standard Brascamp-Lieb variance inequality for which provide two proofs: one with the study of the semi-group on gradients and another by duality (where we carefully take into account the boundary). The second result is the Veysseire's inequality where we relax the infimum of the uniform convexity of the potential into its harmonic mean.

In Section 6, we provide explicit bounds obtained in the log-concave case by localization argument. The first one are due to Payne and Weinberger in the compact case and were extended by Kannan, Lovász and Simonovits and Bobkov on $\mathbb{R}^d$. We also discuss the famous KLS conjecture and its very last development.

In Section 7, we explain how symmetries can improve a lot the Poincaré constant. We discuus the unconditional and the radial cases.

The final Section presents a generalisation of the Brascamp-Lieb inequalities which might be useful in the non log-concave case. .

2. The diffusion operator associated to the measure

2.1. **Link with a diffusion operator.** Here we consider a smooth probability measure μ on $\mathbb{R}^d$ or on a domain Ω of $\mathbb{R}^d$; that is μ admits a density with respect to the Lebesgue measure

$$d\mu = \frac{e^{-V}}{Z} dx$$

with V a smooth function.

We shall consider the following diffusion operator

$$L = \Delta - \nabla V \cdot \nabla. \tag{2.1}$$

In the case of the standard Gaussian measure, $d\gamma(x) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}|x|^2} dx$, the potential V is given by $V(x) = \frac{1}{2}|x|^2$ and the diffusion operator (called the Ornstein-Uhlenbeck operator) is given by

$$L^{\mathrm{OU}} = \Delta - x \cdot \nabla.$$

In case of $\mathbb{R}^d$, $-L$ is symmetric and ≥ 0 with respect to the measure μ on the space of smooth and compact supported functions $\mathcal{C}_c^\infty(\mathbb{R}^d)$:

$$\int f(-Lg) d\mu = \int (-Lf) g d\mu = \int \nabla f \cdot \nabla g d\mu, \quad f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d). \tag{2.2}$$

Proof. By a standard integration by parts,

$$\begin{aligned} \int \Delta f g \, d\mu &= \int \Delta f (g e^{-V}) \, dx \\ &= \int -\nabla f \cdot \nabla (g e^{-V}) \, dx \\ &= \int -\nabla f \cdot \nabla g \, d\mu + \int \nabla f \cdot \nabla V \, g \, d\mu. \end{aligned}$$

Thus

$$\int Lf\, g\, d\mu = -\int \nabla f \cdot \nabla g\, d\mu$$

□

Note that this implies the following *invariance property of L*:

$$\int Lf\, d\mu = 0, \quad \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

In the case of a domain of $\mathbb{R}^d$ with a (smooth) boundary, one has to add the Neumann condition on the function. $-L$ is symmetric and ≥ 0 with respect to the measure μ on

$$\{f \in \mathcal{C}_c^\infty(\bar{\Omega}), \nabla f \cdot \eta = 0 \text{ on } \partial\Omega\}. \tag{2.3}$$

where η is the outer normal vector of Ω

Indeed for general $f, g \in \mathcal{C}_c^\infty(\bar{\Omega})$,

$$\int_\Omega g(-Lf)\, d\mu = \int_\Omega \nabla g \cdot \nabla f\, d\mu - \int_{\partial\Omega} g\, (\nabla f \cdot \eta)\, d\mu|_{\partial\Omega}.$$

In both case, it is possible to extend $-L$ to a non-negative self-adjoint operator on a domain of the Hilbert space $L^2(\mu)$.

Actually , it is easier to start with the quadratic form:

$$\mathcal{E}(f,g) := \int \nabla f \cdot \nabla g d\mu, \quad f, g \in H^1(\mu); \tag{2.4}$$

and then to consider the self-adjoint operator associated to it.

In the case of $\mathbb{R}^d$, the domain of $(-L)$ is

$$\mathcal{D}(-L) = \{f \in L^2(\mu), Lf \in L^2(\mu)\}$$

where in the last line Lf is defined in the distributional sense.

In the sequel, we still denote by L the self-adjoint extension.

2.2. **The spectrum and the semi-group of the diffusion operator.** $-L$ is a self-adjoint operator on a domain of $L^2(\mu)$. One can thus define its spectrum. Since, $-L$ is non-negative, its spectrum $\sigma(-L)$ is included in $[0, \infty)$.

First, note that 0 is always an eigenvalue of $-L$. Indeed since μ is probability measure, the constants are in the domain of $-L$ and are eigenfunctions associated to the eigenvalue 0. Actually, by integration by parts, still valid on the domain $\mathcal{D}(-L)$,

$$\int (-Lf) f\, d\mu = \int |\nabla f|^2 d\mu,$$

one has the equivalence:

$$f \in \mathcal{D}(-L) \text{ and } -Lf = 0 \Leftrightarrow f = cte.$$

Now by the spectral Theorem, there exists some projectors E_λ in $L^2(\mu)$ for $\lambda \in \sigma(-L)$ such that for

$$f = \int_0^\infty dE_\lambda(f) \left(= \int_{\sigma(-L)} dE_\lambda(f) \right), \quad f \in L^2(\mu)$$

and

$$-Lf = \int_0^\infty \lambda dE_\lambda(f), \quad f \in \mathcal{D}(-L)$$

This decomposition also gives:

$$\|f\|_\mu^2 = \int f^2 d\mu = \int_0^\infty dE_\lambda(f,f), \quad f \in L^2(\mu),$$

$$\mathcal{E}(f,f) = \int |\nabla f|^2 d\mu = \int f\,(-Lf)\,d\mu = \int_0^\infty \lambda dE_\lambda(f,f), \quad f \in H^1(\mu),$$

$$\|(-Lf)\|_\mu^2 = \int (-Lf)^2 d\mu = \int_0^\infty \lambda^2 dE_\lambda(f,f), \quad f \in \mathcal{D}(-L).$$

In a lot of situation, the spectrum is discrete with an orthonormal basis of eigenfunctions $(\phi_k)_{k\geq 0}$. For example, in the case of the Ornstein-Uhlenbeck case, the spectrum $\sigma(-L^{\mathrm{OU}}) = \mathbb{N}$ and the eigenfunctions are given by the Hermite polynomials. In this case, the above formulae are a generalisation of

$$f = \sum_{k\geq 0} < f, \phi_k > \phi_k, \quad f \in L^2(\mu),$$

and

$$-Lf = \sum_{k\geq 0} \lambda_k < f, \phi_k > \phi_k, \quad f \in \mathcal{D}(-L)$$

and for example

$$\|f\|_\mu^2 = \sum_{k\geq 0} < f, \phi_k >^2, \quad f \in L^2(\mu).$$

It is now possible to define the semi-group associated to L by the functional calculus:

$$P_t f := e^{tL} f = \int_0^\infty e^{-t\lambda} dE_\lambda(f), \quad t \geq 0.$$

This is a semi-group since for $s, t \geq 0$,

$$P_{t+s} f = P_t(P_s f).$$

Note that $P_0 f = f$ and that in our setting, the semi-group is *ergodic* meaning that for each f in L^2,

$$P_t f \xrightarrow[t\to\infty]{L^2(\mu)} P_\infty f = \int f d\mu.$$

Moreover the semi-group satisfies the diffusion equation

$$\begin{aligned}\partial_t P_t f &= L P_t f, \quad f \in L^2(\mu)\\ &= P_t L f, \qquad \text{if moreover } f \in \mathcal{D}(-L).\end{aligned}$$

Note also that since by the invariance of L, $\int Lf d\mu = 0$, then

$$\int P_t f d\mu = \int f d\mu, \quad f \in L^2.$$

Remark 2.1. *It is also possible to define the semi-group using a probabilistic argument. One has*

$$P_t(f)(x) = \mathbb{E}[f(X_t^x)]$$

where X_t^x is the Markov process with generator L starting in x. It can be also seen as the solution starting in x of the stochastic differential equation

$$dX_t^x = -\nabla V(X_t^x) + \sqrt{2} dB_t$$

where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion. Under mild assumption (for example if $\mathrm{Hess}(V)$ is bounded below), the two semi-groups coincide.

2.3. **The Poincaré inequality, the spectral gap and the convergence of the semi-group to equilibrium.**

Theorem 2.2. *Below, let us connect C_P and λ_1 by $\lambda_1 = \frac{1}{C_P}$. The following assertion are equivalent.*

(1) *The Poincaré inequality* (1.1) *holds with constant C_P.*
(2) *The spectrum of $-L$ lies in $\{0\} \cup [\lambda_1, \infty)$.*
(3) *For all $f \in L^2(\mu)$: $\mathrm{Var}_\mu(P_t f) \leq e^{-2\lambda_1 t} \mathrm{Var}_\mu(f)$.*

Note that even, if the Poincaré inequality holds, λ_1 is not necessarily a true eigenvalue.

Note that the point (3) expresses an exponential convergence of the semi-group in $L^2(\mu)$ to equilibrium since

$$\mathrm{Var}_\mu(P_t f) = \| \left(P_t f - \int P_t f d\mu\right) \|_\mu^2 = \| \left(P_t f - \int f d\mu\right) \|_\mu^2$$

and since $P_\infty f = \int f d\mu$.

Proof. We first prove the equivalence between (2) and (3). Assume (2), then

$$P_t f - \int f d\mu = \int_{\lambda \geq \lambda_1} e^{-\lambda t} dE_\lambda(f).$$

and thus

$$\| \left(P_t f - \int f d\mu\right) \|_\mu^2 \leq e^{-2\lambda_1 t} \int_{\lambda \geq \lambda_1} dE_\lambda(f) = e^{-2\lambda_1 t} \mathrm{Var}_\mu(f).$$

Reciprocally, for all $f \in L^2(\mu)$ taking $g = \int_{(0,\lambda_1)} dE_\lambda(f)$, one has

$$\mathrm{Var}_\mu(P_t g) = \int_{(0,\lambda_1)} e^{-2\lambda t} dE_\lambda(f,f) > e^{-2\lambda_1 t} \int_{(0,\lambda_1)} dE_\lambda(f,f) = e^{-2\lambda_1 t} \mathrm{Var}_\mu(g).$$

Assuming (3) implies that $g = 0$. Since this holds for all $f \in L^2(\mu)$ one has $\sigma(-L) \cap (0, \lambda_1) = \emptyset$ and (2) follows.

We turn to the proof of the equivalence between (1) and (3).
First one has

$$\mathrm{Var}_\mu(P_t f) = \int \left(P_t f - \int f d\mu\right)^2 d\mu = \int \left(P_t f - P_\infty f\right)^2 d\mu$$

and thus

$$\frac{d}{dt} \mathrm{Var}_\mu(P_t f) = \int 2 \left(P_t f - P_\infty f\right) \left(-L P_t f\right) d\mu = -\int |\nabla P_t f|^2 d\mu. \tag{2.5}$$

Therefore if (1) holds; that is if the Poincaré inequality holds, then

$$\frac{d}{dt} \mathrm{Var}_\mu(P_t f) = -\int |\nabla P_t f|^2 \, d\mu \leq -\frac{2}{C_P} \mathrm{Var}_\mu(P_t f).$$

With $\phi(t) = \mathrm{Var}_\mu(P_t f)$, this writes $\phi'(t) \leq -\frac{2}{C_P} \; \phi(t)$ and by Gronwall Lemma, this implies

$$\phi(t) \leq e^{-\frac{2}{C_P} t} \phi(0);$$

that is (3).

Reciprocally, assume (3) holds. Since (3) is an equality in 0, taking derivatives in $t = 0$ for $f \in H^1(\mu)$ gives

$$-2 \int |\nabla f|^2 d\mu = \phi'(0) \leq -2\lambda_1 \phi(0) = -2\lambda_1 \mathrm{Var}_\mu(f).$$

or equivalently

$$\lambda_1 \mathrm{Var}_\mu(f) \leq \int |\nabla f|^2 d\mu.$$

□

Remark 2.3. *Actually, since here the semi-group P_t is symmetric, it only suffices to prove (see* [BGL14] *Theorem 4.2.5) that, for all $f \in L^2(\mu)$ there exists a constant $c(f) > 0$ such that for all $t \geq 0$,*

$$\mathrm{Var}_\mu(P_t f) \leq c(f) e^{-2\lambda_1 t}.$$

In the above proof, (2.5) is reminiscent of the following important representation of the variance. This representation is based on the interpolation by the diffusion semi-group. Formally, one has:

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= \int \left(f - \int f d\mu\right)^2 d\mu \\
&= \int (P_0 f - P_\infty f)^2 \, d\mu \\
&= \int \int_0^\infty -\frac{d}{ds}(P_s f)^2 \, ds \, d\mu \\
&= \int_0^\infty \int 2 P_s f(-L P_s f) \, d\mu \, ds \\
&= 2 \int_0^\infty \int |\nabla P_s f|^2 \, d\mu \, ds
\end{aligned} \tag{2.6}$$

and for $t \geq 0$,

$$\begin{aligned}
\mathrm{Var}_\mu(P_t f) &= \int (P_t f - P_\infty f)^2 \, d\mu \\
&= 2 \int_t^\infty \int |\nabla P_s f|^2 \, d\mu \, ds.
\end{aligned}$$

It is also possible to produce a representation of the variance with a one-side interpolation of the semi-group. Formally, one has

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= \int f \left(f - \int f d\mu\right) d\mu \\
&= \int f \, (P_0 f - P_\infty f) \, d\mu \\
&= \int f \int_0^\infty -\frac{d}{dt} P_t f \, dt \, d\mu \\
&= \int_0^\infty \int f(-L P_t f) \, d\mu \, dt \\
&= \int_0^\infty \int \nabla f \cdot \nabla P_t f \, d\mu \, dt.
\end{aligned} \tag{2.7}$$

2.4. **Convexity assumption of the boundary.**

This interpretation in term of the convergence of equilibrium of the Poincaré inequality illustrates the following fact:

If the boundary is not convex, the convergence to equilibrium can be very slow and the Poincaré constant will be very big!

Moreover, it is clear that if the domain is not connected that the Poincaré inequality can not hold (take a function constant on each connected component of the domain).

In consequence, contrary to the dimension 1, there is no monotonicity of the Poincaré constant with respect the inclusion of domains.

Moreover, we will see in Remark 6.4 that even for convex domain (with the Lebesgue measure), there is also no monotonicity of the Poincaré constant with respect the inclusion.

We only have:

Proposition 2.4. *If $\cup_n \Omega_n = \Omega$ is an increasing sequence of domain, (and if $d\mu_n = \frac{\mathbf{1}_{\Omega_n}}{\mu(\Omega_n)} d\mu$) then:*

$$C_P(\Omega) \leq \liminf_n C_P(\Omega_n).$$

Proof. Indeed, for $f \in H_1(\Omega)$, applying the Poincare inequality for $f_n = f\mathbf{1}_{\Omega_n}$ in $L^2(\Omega_n)$

$$\frac{1}{\mu(\Omega_n)} \int_\Omega f^2 \mathbf{1}_{\Omega_n} d\mu - \frac{1}{\mu(\Omega_n)^2} \left(\int_\Omega f \mathbf{1}_{\Omega_n} d\mu \right)^2 \leq C_P(\Omega_n) \int |\nabla f|^2 \frac{\mathbf{1}_{\Omega_n}}{\mu(\Omega_n)} d\mu,$$

and the result follows by the dominated convergence theorem. $\square$

Let us present however the following gluing property of the Poincaré constant see [BGL14] Proposition 4.6.4.

Before we state the result, we that if μ is a finite positive measure on Ω, the normalized measure $\frac{1}{\mu(\Omega)}\mu$ satisfies the Poincaré inequality with constant C if and only if

$$\int f^2 d\mu - \frac{1}{\mu(\Omega)} \left(\int f d\mu \right)^2 \leq C \int |\nabla f|^2 d\mu, \quad f \in H^1(\mu).$$

Proposition 2.5. *Let μ be a finite on E and let $K, L \subset E$ with $\mu(K \cap L) > 0$. Then*

$$C_P(K \cup L) \leq \frac{\mu(K \cup L)}{\mu(K \cap L)} (C_P(K) + C_P(L)).$$

Proof. Let f with $\int_{K\cup L} f d\mu = 0$, then

$$\begin{aligned}\int_{K\cup L} f^2 d\mu &\leq \int_K f^2 d\mu + \int_L f^2 d\mu \\ &\leq C_P(K)\int_K |\nabla f|^2 d\mu + C_P(L)\int_L |\nabla f|^2 d\mu \\ &\quad + \frac{1}{\mu(K)}\left(\int_K f d\mu\right)^2 + \frac{1}{\mu(L)}\left(\int_L f d\mu\right)^2 \\ &\leq (C_P(K)+C_P(L))\int_{K\cup L} |\nabla f|^2 d\mu \\ &\quad + \frac{1}{\mu(K)}\left(\int_K f d\mu\right)^2 + \frac{1}{\mu(L)}\left(\int_L f d\mu\right)^2\end{aligned}$$

An the result follows from the technical result that for such a function f with $\int_{K\cup L} f d\mu = 0$,

$$\frac{1}{\mu(K)}\left(\int_K f d\mu\right)^2 + \frac{1}{\mu(L)}\left(\int_L f d\mu\right)^2 \leq \left(1 - \frac{\mu(K\cup L)}{\mu(K\cap L)}\right)\int_{K\cup L} f^2 d\mu.$$

□

3. Bakry-Emery criterion and Γ_2 calculus

In this section we only consider the case of $\mathbb{R}^d$. For all this section, we refer for the monograph [BGL14]. The case of domain of $\mathbb{R}^d$ is really more technical. For example in the stochastic representation of the semi-group P_t one has to consider the local time of the process at the boundary. We deal therefore for the case of $\mathbb{R}^d$. Note however that for convex domain in $\mathbb{R}^d$, it is sometimes possible to approach them by a convex measure on $\mathbb{R}^d$.

As it will be clear below, in this section we are interested in measure $d\mu = e^{-V}dx$ on $\mathbb{R}^d$ with a potential V uniformly convex.

Given a diffusion operator L, we first define the "carré du champ" operator Γ:

$$\Gamma(f,g) = \frac{1}{2}\left(L(fg) - fLg - gLf\right). \tag{3.1}$$

This operator depends only on the second order part of L and in our setting $L = \Delta - \nabla V \cdot \nabla$, one has

$$\Gamma(f,g) = \nabla f \cdot \nabla g$$

We now define the "carré du champ itéré" operator Γ_2 by:

$$\Gamma_2(f,g) = \frac{1}{2}\left(L(\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\right). \tag{3.2}$$

In our setting on $\mathbb{R}^d$, one has:

$$\Gamma_2(f,f) = |\nabla\nabla f|^2 + \nabla f \cdot \mathrm{Hess}V \nabla f.$$

Here $|\nabla\nabla f|^2 = \sum_{i,j=1}^d (\partial_{i,j} f)^2$. This formula is a particular case of the Bochner-Weizenbock formula. For a diffusion operator on a general Riemannian manifold, there is also a potential term with the Ricci curvature.

For $\rho \in \mathbb{R}$, one says that the Bakry-Emery curvature criterion $CD(\rho,\infty)$ is satisfied if for all function $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f). \tag{3.3}$$

This criterion can also be reinforced with a dimension term and one says that the Bakry-Emery curvature-dimension criterion $CD(\rho,n)$ is satisfied if for all function $f \in \mathcal{C}_c^\infty$,

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{n}(Lf)^2. \tag{3.4}$$

Here we shall concentrate on the $CD(\rho,\infty)$ criterion.

Lemma 3.1. *Let $L = \Delta - \nabla V \cdot \nabla$ on $\mathbb{R}^d$. The operator L satisfies the $CD(\rho,\infty)$ criterion if and only if for all $x \in \mathbb{R}^d$*

$$\mathrm{Hess} V(x) \geq \rho I$$

in the quadratic form sense.

The typical example is thus the Gaussian for which $\mathrm{Hess}V = I$ and which satisfies thus the $CD(1,\infty)$ criterion.

The operator Γ and Γ_2 appear in the derivatives of

$$\psi(s) = P_s\Big((P_{t-s}f)^2\Big), \quad 0 \leq s \leq t.$$

Formally, one has with $g = P_{t-s}f$

$$\begin{aligned}\psi'(s) &= P_s\Big(L(g^2) - 2gLg\Big)\\ &= 2P_s\Big(\Gamma(g,g)\Big)\end{aligned}$$

and

$$\begin{aligned}\psi''(s) &= 2P_s\Big(L\Gamma(g,g) - 2\Gamma(g,Lg)\Big)\\ &= 4P_s\Big(\Gamma_2(g,g)\Big).\end{aligned}$$

Theorem 3.2. *If the $CD(\rho,\infty)$ is satisfied, then*

$$|\nabla P_t f|^2 \leq e^{-2\rho t} P_t |\nabla f|^2$$

Proof. With the above calculation, under $CD(\rho,\infty)$ one has:

$$\psi''(s) \geq 2\rho\psi'(s)$$

and by Gronwall lemma,

$$\psi'(s) \geq e^{2\rho s}\psi'(0).$$

In particular, since $\psi'(0) = \Gamma(P_t f)$ and $\psi'(t) = P_t(\Gamma(f))$, the result follows. □

As a consequence, we obtain the following result.

Theorem 3.3. *If the $CD(\rho, \infty)$ criterion is satisfied with $\rho > 0$, then the Poincaré inequality holds with constant $C_P \le \frac{1}{\rho}$.*

$$\mathrm{Var}_\mu(f) \le \frac{1}{\rho} \int |\nabla f|^2 d\mu.$$

Proof. Combining the above result with the representation of the variance (2.6), one has

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= 2 \int_0^\infty \int |\nabla P_t f|^2 \, d\mu \, dt \\
&\le 2 \int_0^\infty \int e^{-2\rho t} P_t |\nabla f|^2 \, d\mu \, dt \\
&= 2 \int_0^\infty e^{-2\rho t} \int P_t |\nabla f|^2 \, d\mu \, dt \\
&= 2 \int_0^\infty e^{-2\rho t} dt \ \int |\nabla f|^2 \, d\mu \\
&= \frac{1}{\rho} \int |\nabla f|^2 d\mu.
\end{aligned}$$

Note that we used the invariance of the semi-group. □

Remark 3.4. *Actually, if the $CD(\rho, \infty)$ is satisfied, then the stronger sub-commutation holds*

$$|\nabla P_t f| \le e^{-\rho t} P_t |\nabla f|$$

and stronger functional inequalities as the log-Sobolev holds.

Remark 3.5. *Note that for the Orstein-Uhlenbeck semi-group, an explicit formula known as the Mehler formula holds and one actually has*

$$\nabla P_t f = e^{-t} P_t(\nabla f).$$

Remark 3.6. *The Bakry-Emery argument does not require the symmetry of the semi-group $(P_t)_{t \ge 0}$. It can thus be advantageous to consider non-reversible semi-group associated to the (invariant) measure μ (see* [AC00]*).*

4. The tensorisation and two perturbation arguments

In this section, we present some classical methods that permit, knowing some Poincaré inequalities to infer new ones.

4.1. **The tensorisation of the Poincaré inequality.**

We start by the fundamental fact that the Poincaré measure tensorises; that is that the Poincaré constant of a product measure does not deteriorate.

Theorem 4.1. *Let $\mu = \mu_1 \otimes \cdots \otimes \mu_d$ be a product measure. Then*

$$C_P(\mu) \le \max\Big(C_p(\mu_1), \cdots, C_P(\mu_d)\Big).$$

Proof. First we treat the case of $f : \mathbb{R}^2 \to \mathbb{R}$, one has

$$\iint f(x,y)^2 dxdy - \Big(\iint f(x,y)dxdy\Big)^2 = \int_x \Big(\int_y f(x,y)^2 dx\Big)\, dy - \int_x \Big(\int_y f(x,y)dx\Big)^2 dy$$
$$+ \int_x \Big(\int_y f(x,y)dx\Big)^2 dy - \Big(\int_x \Big(\int_y f(x,y)dx\Big)\, dy\Big)^2.$$

More generally, for a function $f : \mathbb{R}^d \to \mathbb{R}$, set for $1 \leq k \leq d$

$$f_k(x_1, \dots x_k) = \iint f(x_1, \dots x_d) d\mu_{k+1}(x_{k+1}) \dots d\mu_d(x_d)$$

and with $f_d = f$ and $f_0 = \int f d\mu$; then one has

$$\begin{aligned}
&\mathrm{Var}_\mu(f)\\
&= \iint f^2(x_1, \dots x_d) d\mu_1(x_1) \dots d\mu_d(x_d) - \Big(\iint f(x_1, \dots x_d) d\mu_1(x_1) \dots d\mu_d(x_d)\Big)^2\\
&= \sum_{k=1}^d \iint f_k^2(x_1, \dots x_d) d\mu_1(x_1) \dots d\mu_d(x_d) - \iint f_{k-1}^2(x_1, \dots x_d) d\mu_1(x_1) \dots d\mu_d(x_d)
\end{aligned}$$

This gives the usual tensorisation of the variance as

$$\mathrm{Var}_\mu(f) = \sum_{k=1}^d \iint \mathrm{Var}_{\mu_k}(f_k) d\mu_1(x_1) \dots d\mu_{k-1}(x_{k-1}) \tag{4.1}$$

Now using the Poincaré inequality for the one-dimensional measure μ_k gives, for each $(x_1, \dots x_{k-1})$

$$\mathrm{Var}_{\mu_k}(f_k) \leq C_P \int_{x_k} (\partial_k(f_k))^2 d\mu_k(x_k)$$

and by the Cauchy-Schwarz inequality:

$$\begin{aligned}
(\partial_k(f_k))^2 &= \Big(\iint \partial_k f(x_1, \dots x_d) d\mu_{k+1}(x_{k+1}) \dots d\mu_d(x_d)\Big)^2\\
&\leq \iint (\partial_k f)^2(x_1, \dots x_d) d\mu_{k+1}(x_{k+1}) \dots d\mu_d(x_d)
\end{aligned}$$

and the result follows with (4.1). □

4.2. **The Holley-Stroock perturbation argument.** In this section we shall present the Holley-Stroock argument which deals with bounded perturbation of measures.

We will use it below to obtain some estimates for the Poincaré constant of measure which are only uniformly log-concave at infinity.

Theorem 4.2. *Let $d\mu = \frac{1}{Z} e^{-V} dx$ and $d\mu' = \frac{1}{Z'} e^{-V'} dx$ and assume that $\|V - V'\|_\infty \leq C$, then*

$$C_P(\mu') \leq e^{4C} C_P(\mu).$$

Proof. First clearly $Z' = \int e^{-V'}dx \leq e^C \int e^{-V}dx = e^C Z$ and similarly $Z \leq e^C Z'$. The variance admits also the variational formula

$$\mathrm{Var}_\mu(f) = \inf_{a\in\mathbb{R}} \int (f-a)^2 d\mu.$$

and thus for $a = \int f d\mu$,

$$\begin{aligned}
\mathrm{Var}_{\mu'}(f) &\leq \int (f-a)^2 \frac{e^{-V'}}{Z'} dx \\
&\leq e^{2C} \int (f-a)^2 \frac{e^{-V}}{Z} dx = e^{2C}\,\mathrm{Var}_\mu(f) \\
&\leq e^{2C} C_P(\mu) \int |\nabla f|^2 d\mu \\
&\leq e^{4C} C_P(\mu) \int |\nabla f|^2 d\mu'
\end{aligned}$$

and the result follows. □

Remark 4.3. *In high dimension, this argument can produce very poor constant. For example, let us consider a product measure $d\mu_1 = \frac{1}{Z_1} e^{-V_1} dx$ on $\mathbb{R}^d$ with $V_1(x) = \sum_{i=1}^d U_1(x_i)$ where U_1 is a function $U_1 : \mathbb{R} \to \mathbb{R}$. Let us perturb it in $d\mu_2(x) = \frac{1}{Z_2} e^{-V_2} dx$ with $V_2(x) = \sum_{i=1}^d U_2(x_i)$. One has*

$$\|V_1 - V_2\|_\infty = d\|U_1 - U_2\|_\infty$$

and the Holley-Stroock argument gives

$$C_P(\mu_2) \leq e^{4d\|U_1-U_2\|_\infty} C_P(\mu_1).$$

However, by tensorisation, both measure are product and $Cp(\mu_1)$ and $C_P(\mu_2)$ do not depend on the dimension d.

We present an application of this Holley-Stroock perturbation argument to the class of measures which are only uniformly log-concave only at infinity:

Theorem 4.4. [Led01] *Let $d\mu = \frac{1}{Z} e^{-V} dx$ with $V = U + W$ with $\mathrm{Hess} U \geq \rho > 0$ and $\|W\|_\infty \leq C$. then*

$$C_P(\mu) \leq \frac{e^{4C}}{\rho}.$$

4.3. **Stability with respect to Lipschitz transport.** Here we present a stability result of the Poincaré inequality with respect to the transport of measures.

Theorem 4.5. *Let μ be a probability measure on $\mathbb{R}^d$ and let T be a L-Lipschitz function from $\mathbb{R}^d$ to possibly $\mathbb{R}^{d'}$. Let $\tilde{\mu} = T_\sharp \mu$ be the image of μ under the application T then*

$$C_P(\tilde{\mu}) \leq L^2 C_P(\mu).$$

Proof. For simplicity we assume that T is smooth. Let $f : \mathbb{R}^{d'} \to \mathbb{R}$. Since

$$\int f(y) d\tilde{\mu}(x) = \int foT(x) d\mu(x),$$

one has

$$\begin{aligned}
\mathrm{Var}_{\tilde{\mu}}(f) &= \mathrm{Var}_{\mu}(foT) \\
&\leq C_P(\mu) \int |\nabla(foT)(x)|^2 d\mu(x) \\
&\leq L^2 C_P(\mu) \int |(\nabla f)oT(x)|^2 d\mu(x) \\
&= L^2 C_P(\mu) \int |\nabla f(y)|^2 d\tilde{\mu}(y).
\end{aligned}$$

The last inequality follows from $\partial_i(foT)(x) = \sum_j (\partial_j f)oT(x)\, \partial_i T_j(x)$ and thus by Cauchy-Schwarz,

$$|\nabla(foT)(x)^2| \leq |(\nabla f)oT|^2 \sum_j |\nabla T_j(x)|^2$$

which, since $|T(y) - T(x)|^2 = \sum_j |T_j(y) - T_j(x)|^2 \leq L^2 d(x,y)^2$, gives

$$\sum_j |\nabla T_j(x)|^2 \leq L^2.$$

□

Note that in this direct transport approach, one does not have to care much about the boundary since one consider functions $f \in H^1(\mu)$. Instead the boundary condition appears directly in the dual formulation.

With this approach, it is possible to recover the (Bakry-Emery) Theorem 3.3 in the uniform convex case. Indeed if $d\mu = e^{-V}dx$ on $\mathbb{R}^d$ with $\mathrm{Hess}V \geq 1$, by a famous result of Caffarelli [**?**], there exists a map T transporting the standard Gaussian measure on μ and which is a *contraction*. Moreover E. Milman [Mil18] remarks that it permits to compare not only the first eigenvalue of L and L^{OU} but all their eigenvalues!

As simple first application, one can deduce that the Poincaré constant of the Gaussian measure $\mathcal{N}(m,\Gamma)^2$ is given

$$C_p(\mathcal{N}(m,\Gamma)) \leq \|\Gamma\|_{\mathrm{op}}$$

In fact, one has equality.

Similarly, this method gives an upper bound of the Poincaré constant of the uniform measure on an ellipsoid knowing the one of the ball.

This approach has also been used in a lot of situations. For example, it has been used for the uniform measure on the simplex or on the l^p balls in $\mathbb{R}^d$ (see [AGB15] and the references therein). From this, one infers weak form of the

Poincaré inequality (where the energy term is replaced by $\|\nabla f\|_\infty^2$). In the log-concave situation, this weak Poincaré inequality is known to be equivalent up to a universal constant to the usual Poincaré inequality, due to an important result of E. Milman [Mil09].

5. Intertwining and more variance inequality

In this section, we present two results that are valid in the log-concave case and that are based on the intertwining between the gradient and the diffusion operator.

5.1. **The Brascamp-Lieb variance inequality.**

The first one is the Brascamp-Lieb variance inequality which is very similar to the Poincaré inequality but with a different energy term.

Theorem 5.1. [BL76]*Let $\mu = e^{-V}dx$ be a probability measure. Assume that V is strictly convex, i.e. $HessV > 0$ and that if it exists, that the boundary is convex. Then*

$$\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^d} (\nabla f) \cdot (\mathrm{Hess}\, V)^{-1} (\nabla f)\, d\mu. \tag{5.1}$$

The original proof proceeds by induction. We shall present below two (related) more global proofs.

Remark 5.2. *Of course, if* $\mathrm{Hess}V \geq \rho > 0$*, one recovers the same Poincaré inequality as under the $CD(\rho,\infty)$ criterion. It is also optimal for the Gaussian. This result can also be interpreted as a weighted Poincaré inequality.*

5.2. **A first approach using the semi-group on gradients.**

We recall that L is the diffusion operator $L = \Delta - \nabla V \cdot \nabla$ on $\mathbb{R}^d$.

The Weizenböck formula expresses the commutation between L and the gradient. It writes:

$$\nabla L f = (\mathcal{L} - \mathrm{Hess}V)(\nabla f), \tag{5.2}$$

with $\mathcal{L}$ the diffusion operator acting on the gradients as follows:

$$\mathcal{L}(\nabla f) = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix} (\nabla f).$$

First, note that $-\mathcal{L}$ is symmetric and ≥ 0 with respect to the gradient (or even vector fields) in $L^2(\mu)$ (equipped with the standard scalar product). The operator $-\mathcal{L} + \mathrm{Hess}V$ is thus called a Schrödinger operator: it writes as a diffusion operator $-\mathcal{L}$ plus a (matrix) potential $\mathrm{Hess}V$. Note that the Bochner formula for the Γ_2 operator is a similar result but for the square norm of the gradient and involves only functions and not gradients.

Under some condition (at least if $\mathrm{Hess}V$ is bounded below on $\mathbb{R}^d$), the operator can be extended as a self adjoint operator, generates a (Feynman-Kac type) semi-group $e^{t(\mathcal{L}-\mathrm{Hess}V)}$ and one has the commutation:

$$\nabla P_t f = e^{t(\mathcal{L} - \mathrm{Hess} V)}(\nabla f). \tag{5.3}$$

Proof. Now the proof of the Brascamp-Lieb inequality proceeds as follows. By the representation of the variance (2.7), one has

$$\begin{aligned} \mathrm{Var}_\mu(f) &= \int_0^\infty \int_{\mathbb{R}^d} \nabla f \cdot \nabla P_t f \, d\mu \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} (\nabla f) \cdot e^{t(\mathcal{L} - \mathrm{Hess} V)}(\nabla f) \, d\mu \, dt \\ &= \int_{\mathbb{R}^d} (\nabla f) \cdot (-\mathcal{L} + \mathrm{Hess}\, V)^{-1} (\nabla f) \, d\mu \end{aligned} \tag{5.4}$$

since by the functional calculus (or the spectral theorem), one has

$$(-\mathcal{L} + \mathrm{Hess}\, V)^{-1} = \int_0^\infty e^{t(\mathcal{L} - \mathrm{Hess} V)} dt.$$

Now, since $-\mathcal{L}$ is symmetric and ≥ 0, one has in the sense of self-adjoint operators:

$$(-\mathcal{L} + \mathrm{Hess}\, V)^{-1} \leq (\mathrm{Hess}\, V)^{-1}.$$

and the Brascamp-Lieb inequality follows. □

Remark 5.3. *The Brascamp-Lieb inequality admits extremal functions given by $f(x) = \langle c, \nabla V \rangle$ where c is a constant vector on $\mathbb{R}^d$. This is not always the case for the Poincaré inequality.*

5.3. **A duality approach.** Here we present a slightly different approach which works by duality and is sometimes called the L^2 Hörmander method.

In this section, we directly deal with the domain case. Of course, all this applies directly and in a easier way to the case of $\mathbb{R}^d$.

We insist on the fact that we consider the Neumann boundary condition for L. We first start with a lemma that deals with the existence of the solution to the Poisson equation $-Lg = f$.

Lemma 5.4. *Let $f \in L^2(\mu)$ such that $\int_\Omega f d\mu = 0$. Then there exists $g \in \mathcal{D}(L)$ such that*

$$-Lg = f.$$

In particular, g satisfies the Neumann boundary condition: $\nabla g \cdot \eta = 0$ on $\partial\Omega$ with η the outer normal on $\partial\Omega$.

We recall the following integration by parts:

Lemma 5.5. *Let f, g be smooth (with no condition on boundary). Then*

$$\int_\Omega g(-Lf) d\mu = \int_\Omega \nabla f \cdot \nabla g d\mu - \int_{\partial\Omega} \nabla f \cdot \eta \, g d\mu.$$

We can now state the duality result.

Lemma 5.6. *Assume that for all g with $\nabla g \cdot \eta = 0$, one has*

$$\int_\Omega (-Lg)^2 d\mu \geq \int \nabla g \cdot K \nabla g d\mu$$

where for each x, $K(x)$ is symmetric positive matrix. Then for every $f \in H^1(\mu)$, one has:

$$\mathrm{Var}_\mu(f) \leq \int \nabla f \cdot K^{-1} \nabla f d\mu.$$

Proof. First one has $\mathrm{Var}_\mu(f) = \int \tilde{f}^2 d\mu$ with $\tilde{f} = f - \int f d\mu$. Moreover since $\int \tilde{f} d\mu = 0$, there exists $g \in \mathcal{D}(L)$ such that

$$-Lg = \tilde{f}.$$

Now, the trick is to decompose the variance as follows:

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= 2 \int_\Omega \tilde{f}^2 d\mu - \int_\Omega \tilde{f}^2 d\mu \\
&= 2 \int_\Omega \tilde{f}(-Lg) d\mu - \int_\Omega (-Lg)^2 d\mu \\
&= 2 \int_\Omega \nabla f \cdot \nabla g d\mu - \int_\Omega (-Lg)^2 d\mu \\
&\leq 2 \int_\Omega K^{-\frac{1}{2}} \nabla f \cdot K^{\frac{1}{2}} \nabla g d\mu - \int_\Omega \nabla g \cdot K \nabla g d\mu \\
&= \int_\Omega \nabla f \cdot K^{-1} \nabla f d\mu - \int_\Omega |K^{\frac{1}{2}} \nabla g - K^{-\frac{1}{2}} \nabla f|^2 \\
&\leq \int_\Omega \nabla f \cdot K^{-1} \nabla f d\mu
\end{aligned}$$

□

Actually, we have the following caracterization of the Poincaré inequality. Since $\forall g \in \mathcal{D}(L)$,

$$\int \Gamma_2(g) d\mu = \int (-Lg)^2 d\mu.$$

The result below is also called the integrated Γ_2 criterion.

Theorem 5.7. *The Poincaré inequality holds with constant C_P if and only if*

$$\int (-Lg)^2 d\mu \geq \frac{1}{C_P} \int |\nabla g|^2 d\mu, \forall g \in \mathcal{D}(L).$$

Proof. The reciprocal has be done just above. For the direct sense, it is sufficient to check it for g which is orthogonal to the constants. Recall that the Poincaré inequality is equivalent to the fact that the spectrum of L is included in $\{0\} \cup [\lambda_1, \infty)$ with $\lambda_1 = \frac{1}{C_P}$. In this case, by the spectral theorem and the Poincaré inequality, one has:

$$\begin{aligned}\int (-Lg)^2 d\mu &= \int_{(0,\infty)} \lambda^2 dE_\lambda(g)\\ &= \int_{[\lambda_1,\infty)} \lambda^2 dE_\lambda(g)\\ &\geq \lambda_1 \int_{[\lambda_1,\infty)} \lambda dE_\lambda(g)\\ &= \lambda_1 \int g(-Lg)d\mu\\ &= \lambda_1 \int |\nabla g|^2 d\mu.\end{aligned}$$

□

The next lemma gives the computation of $\int (-Lg)^2 d\mu$. It is sometimes called the Reilly formula (see e.g. [KM16]).

Lemma 5.8. *Let $g \in \mathcal{C}_c^\infty(\bar{\Omega})$ satisfying the Neumann boundary condition $\nabla g \cdot \eta = 0$ on $\partial\Omega$, then*

$$\begin{aligned}\int_\Omega (-Lg)^2 d\mu = \int_\Omega |\nabla\nabla g|^2 d\mu + \int_\Omega \nabla g \cdot (\mathrm{Hess} V)\, \nabla g d\mu\\ + \int_{\partial\Omega} \nabla g \left[(\partial_j \eta_i)_{i,j}\right] \nabla g d\mu.\end{aligned}$$

Let us explain a bit the present of the curvature term $((Jac\,\eta)_{i,j} = (\partial_j\eta_i)_{i,j}$ on the boundary. It corresponds to the second fundamental form of the boundary. Let us compute this term in a more concrete way, in a standard situation where the boundary $\partial\Omega$ is determined by a smooth algebraic equation $F = 0$.

First, in this case the outer normal η is given by

$$\eta = \frac{\nabla F}{|\nabla F|}$$

and thus

$$\begin{aligned}(Jac\,\eta)_{i,j} &= \partial_j \eta_i\\ &= \frac{\partial^2_{i,j} F}{|\nabla F|} - \frac{\partial_i F\, \partial_j |\nabla F|}{|\nabla F|^2}\end{aligned}$$

Now since on the boundary $\partial\Omega$, the Neumann condition $\nabla g \cdot \eta = 0$ holds, one has:

$$\begin{aligned}\int_{\partial\Omega} \nabla g \cdot Jac\,\eta\, \nabla g\, d\mu &= \int_{\partial\Omega} \nabla g \cdot \frac{\nabla^2 F}{|\nabla F|} \nabla g\, d\mu - \int_{\partial\Omega} \nabla g \cdot \nabla F\, \frac{\nabla^2 F \nabla F}{|\nabla F|^3} \cdot \nabla g\, d\mu\\ &= \int_{\partial\Omega} \nabla g \cdot \frac{\nabla^2 F}{|\nabla F|} \nabla g\, d\mu.\end{aligned}$$

In particular, the boundary term in Lemma 5.8 is non-negative if the function F is convex. And actually, a (smooth) domain in $\mathbb{R}^d$ is convex if and only if it has a convex boundary.

Now, we can give another proof of the Brascamp-Lieb variance inequality. Note that this proof is valid with a domain on $\mathbb{R}^d$ with convex boundary.

Proof of Theorem 5.1. Let $d\mu = e^{-V}dx$ be a strictly log-concave measure on $\mathbb{R}^d$ or on a convex domain of $\mathbb{R}^d$. Here strictly log-concave means $\mathrm{Hess}V(x) > 0$ for all x.

By the Reilly formula and the convexity of the boundary, one has:

$$\begin{aligned}\int_\Omega (-Lg)^2 d\mu &\geq \int_\Omega |\nabla\nabla g|^2 d\mu + \int_\Omega \nabla g \cdot (\mathrm{Hess}V)\, \nabla g d\mu \\ &\geq \int_\Omega \nabla g \cdot (\mathrm{Hess}V)\, \nabla g d\mu.\end{aligned}$$

and the result follows from the duality Lemma 5.6. □

Finally, we prove the Reilly formula.

Proof of Lemma 5.8. Using the Neumann boundary condition and the intertwining

$$\nabla(-Lg) = (-\mathcal{L} + \mathrm{Hess}V)\nabla g,$$

two successive integration by parts give:

$$\begin{aligned}\int_\Omega (-Lg)^2 &= \int_\Omega \nabla g \cdot \nabla(-Lg) d\mu - \int_{\partial\Omega} \nabla g \cdot \eta\, (-Lg) d\mu \\ &= \int_\Omega \nabla g \cdot \nabla(-Lg) d\mu \\ &= \int_\Omega \nabla g \cdot (-\mathcal{L} + \mathrm{Hess}V)\nabla g d\mu \\ &= \int_\Omega |\nabla\nabla g|^2 d\mu - \sum_i \int_{\partial\Omega} \partial_i g\, (\nabla(\partial_i g)\cdot \eta) d\mu|_{\partial\Omega} \\ &\quad + \int_\Omega \nabla g \cdot (\mathrm{Hess}V)\, \nabla g\, d\mu.\end{aligned}$$

In the case where g satisfies the Neumann boundary condition, one has

$$\nabla g \cdot \eta = \sum_k \partial_k g\, \eta_k = 0$$

and taking the derivative in i, one infers that:

$$\nabla(\partial_i g \cdot \eta) + \sum_k \partial_k g\, \partial_i \eta_k = 0$$

and the results follows. □

This duality approach has been used in a quite large number of works. Let us cite only some of them. In [Led01], Ledoux uses it to study some spin systems (that is some perturbation of product measures). It has also been used to study the improvements of the Poincaré constant under symmetries [BCE13, Kla09] or some second order Poincaré inequality [CEFM04].

5.4. **Veysseire's result.** The second result of this section is due to Veysseire and constitutes an improvement of the Bakry-Emery case. Here we do not need a uniform positive lower bound of the smallest eigenvalue of Hess$V$ but we only have to control its harmonic mean.

Theorem 5.9. [Vey10] *Let $d\mu = e^{-V}dx$ be a probability measure with* Hess$V > 0$ *then with $\rho(x)$ the smallest eigenvalue of* Hess$V$*,*

$$C_P(\mu) \leq \int \frac{1}{\rho(x)} d\mu(x).$$

Proof. The main idea has been encountered before: it is to perform the commutation between L and the gradient.

Actually, with the previous notation, one can show:

$$\lambda_1 = \inf \sigma(-L) \setminus \{0\} \geq \inf \sigma(-\mathcal{L} + \mathrm{Hess}V)$$

It is now possible to dominate the semi-group on the gradients by the one on the functions and one has

$$\inf \sigma(-\mathcal{L} + \mathrm{Hess}V) \geq \inf \sigma(-L + \rho).$$

Now the goal is to bound from below the bottom of the spectrum of the Schrödinger operator $-L + \rho$ on the functions. It is given by the variational formula

$$\sigma(-L + \rho) = \inf \left\{ \int |\nabla f|^2 d\mu + \int \rho f^2 d\mu, \ \int f^2 d\mu = 1. \right\}$$

Now by the Poincaré inequality, for f with $\int f^2 d\mu = 1$, one has

$$\int |\nabla f|^2 d\mu + \int \rho f^2 d\mu \geq \lambda_1 \int f^2 d\mu - \lambda_1 \left(\int f d\mu \right)^2 + \int \rho f^2 d\mu.$$

This gives

$$0 \geq \inf \left\{ \int \rho f^2 d\mu - \lambda_1 \left(\int f d\mu \right)^2, \ \int f^2 d\mu = 1 \right\}$$

and since by Cauchy-Schwarz

$$\left(\int f d\mu \right)^2 \leq \int \rho f^2 d\mu \int \frac{1}{\rho} d\mu,$$

one infers

$$0 \geq \inf \left\{ \int \rho f^2 d\mu \left(1 - \lambda_1 \int \frac{1}{\rho} d\mu \right), \ \int f^2 d\mu = 1 \right\}$$

At least in the uniformly convex case, this gives

$$\lambda_1 \geq \frac{1}{\int \frac{1}{\rho} d\mu}.$$

The general case follows by approximation. $\square$

Remark 5.10. *The study between the spectrum of $-L$ on functions and of $-\mathcal{L} + \mathrm{Hess}V$ on gradients is an old and long subject (see e.g.* [Hel02]*). Recently, a careful study has be done in* [BO]*. Here we prove that the Poincaré inequality holds under the condition that infimum of the mean of the potential ρ on some sub-domains is positive. In particular, the result may be applied even if ρ vanishes on some region.*

6. Localization techniques in the log-concave case

First, it is known that a log-concave probability measure satisfies the Poincaré inequality. (see e.g. Theorem 6.6 below). In this section, we present the results obtain in the log-concave by localization arguments. We also present the famous KLS conjecture which asserts, in one of its equivalent formulation, that the Poincaré constant of a measure log-concave depends, up to an universal constant, only on its covariance matrix (and therefore not to the dimension). We describe below some recent and impressive progress.

Remark 6.1. *Note that even on $\mathbb{R}$, the probability measures with potential $V(x) = |x|^\alpha/\alpha, \alpha > 0$, called the Subbotin or exponential power distribution, satisfies the Poincaré inequality only for $\alpha \geq 1$. Note also that for $\alpha = 1$, the Poincaré inequality holds but there is no true eigenvalue associated to the spectral gap.*

It is known that a Poincaré inequality implies some exponential integrability of the distance (see e.g. Proposition 4.4.2 in [BGL14]*). Thus heavy-tailed distributions can not satisfy the usual Poincaré inequality. However, in this situation, it is still possible to obtain some weighted Poincaré inequalities (see* [BL09] *for the cases of Cauchy distribution).*

Of course, it is possible that some non-log concave function at infinity satisfies the Poincaré inequality (see [HN03] *for some examples and also* [BJ21] *for a non-convex perturbation of a product measure).*

6.1. **The Payne-Weinberger result in the compact case.**

The first result concerning Poincaré inequality with explicit constant is due to Payne and Weinberger [PW60]. It was stated for the uniform measure on compact convex sets but directly applies for a log-concave measure on a compact convex set.

Theorem 6.2. [PW60] *Let K be a convex bounded domain of $\mathbb{R}^d$ and μ be a probability measure on K. Then*

$$C_P(\mu) \leq \frac{Diam(K)^2}{\pi^2}.$$

The method of the proof is a *localization technique*. The idea is to perform successive *half-space bissections*. First one can consider only function f such that $\int_K f d\mu = 0$. The idea is now to find a hyperplane H which contains the barycenter of K and thus decompose $K = K_- \cup K+$ with $\mu(K_+) = \mu(K_-)$ and such that

$$\int_{K_+} f d\mu|_{K_+} = \int_{K_-} f d\mu|_{K_-} = 0.$$

Now, if for each $i = +, -$, the desired Poincaré inequality holds for K_i; that is

$$\int_{K_i} f^2 d\mu|_{K_i} \leq \frac{Diam(K_i)^2}{\pi^2},$$

then the desired Poincaré inequality holds for K.

The idea is to iterate this half-space bissection. At the end, each K_i is actually closed to a one-dimensional segment.

Remark 6.3. *The result of Payne and Weinberger is actually optimal, since there is equality for the uniform measure on the one-dimensional segment* $[-R, R]$. *Indeed, in this case, the first eigenfunction is given by* $\sin(\frac{\pi}{2R}x)$ *with eigenvalue* $\frac{\pi^2}{4R^2}$.

Remark 6.4. *An elementary but striking point when one consider high dimension is the following: the diameter of the unit cube in dimension d is* $\sqrt{d}$. *As a consequence, if* μ *is a log-concave measure supported in the unit cube on* $\mathbb{R}^d$ *then*

$$C_P(\mu) \leq \frac{d}{\pi^2}$$

whereas by tensorisation, the Poincaré constant of the uniform measure on the Lebesgue cube is $\frac{\pi^2}{4}$. *The result is optimal if one consider (a measure closed to) the uniform measure on the diagonal of the cube. With this, we see that there is no monotonicity of the Poincaré constant with respect of the inclusion even for convex sets.*

Remark 6.5. *In the previous remark, we discuss the case of the cube. The same phenomena also appears for the ball. The Payne-Weinberger estimate for a a log-concave measure supported on a ball is actually optimal. One can approach the case of the uniform measure on a segment. But, it is known that the Poincaré constant of the uniform measure on the ball of radius R is of order* $\frac{R^2}{n}$ *(see Section 7.2).*

6.2. **Extension of KLS and Bobkov.** The result of Payne-Weinberger has been improved by Kannan, Lovász and Simonovits [KLS95] (see also Bobkov [Bob99]) to the case of general log-concave measure on $\mathbb{R}^d$. Here, we cite the result by KLS:

Theorem 6.6. [KLS95] *Let* μ *be a log-concave measure supported on on* $\mathbb{R}^d$, *Then*

$$C_P(\mu) \leq \frac{4}{\ln^2 2} \Big\| |x - x_0| \Big\|_1.$$

with x_0 *the center of gravity of* μ*; that is the point which minimises* $\int |x - x_0| d\mu(x)$.

In some sense, the quantity $\left\| |x-x_0| \right\|_1$ is an "average diameter" of the measure μ. This quantity can increase with the dimension.

The proof are done through isoperimetrics inequality and the Cheeger's theorem. The result of Cheeger compares the isoperimetric constant and the first eigenvalue λ_1 in the log-concave case. The proof by KLS is based on the four functions theorem: in order that some inequality between four integral holds for any log-concave measure on $\mathbb{R}^d$, it is enough to check that the integral inequality is valid for one-dimensional log-concave segments ("needles"). This explains the term: *localization*.

The proof of Bobkov is done with the Brunn-Minkowski inequalities and produces a worse and not optimal constant: $C_P(\mu) \leq 432\, Var(\mu)$.

6.3. **The KLS conjecture.** Here, we have to mention the famous KLS conjecture stated in [KLS95] and its last development, even if they can not apply directly here because of the presence of an unknown universal constant.

Conjecture 6.7. *There exists a universal constant $C > 0$ such that for every log-concave measure μ in isotropic position (in any dimension), one has*

$$C_P(\mu) \leq C. \tag{6.1}$$

A probability measure is in *isotropic position* if it is centered and if its covariance matrix is the identity.

In their recent work, Klartag and Lehec [KL] prove the remarkable result that the above conjecture is true up to a polylog. Their method is a continuation of the study of the so-called *stochastic localization* due to Eldan [Eld13] and pursued by Y. Chen [Che21].

Theorem 6.8. [KL] *There exists a universal constant $C > 0$ such that for any $d \geq 1$ and any log-concave measure μ in isotropic position in $\mathbb{R}^d$, one has*

$$C_P(\mu) \leq C(\ln n)^{10}. \tag{6.2}$$

Note that in the general case case where the covariance matrix Γ is not the identity, using the stability of the Poincaré constant with respect to Lipschitz transport, this gives that for any log-concave measure μ on $\mathbb{R}^d$:

$$C_P(\mu) \leq C\|\Gamma\|_{op}(\ln n)^{10}$$

with $\|\Gamma\|_{op}$ the biggest eigenvalue of the non-negative symmetric matrix Γ.

7. Symmetries and improvement of the Poincaré constant.

A seen before, the tensorisation property is an important feature of the the Poincaré inequality. We present here some link with some symmetries and their improvement on the Poincaré constants.

7.1. **Unconditional log-concave case.** A convex set K or more generally a measure $d\mu = e^{-V}dx$ is said *unconditional* if it is symmetric with respect to all hyperplanes of coordinates; equivalently that is if for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and all choice of signs $(\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, one has

$$V(\varepsilon_1 x_1, \cdots, \varepsilon_d x_d) = V(x_1, \cdots, x_d).$$

The following result due [Kla09] extends the Payne-Weinberger bound to the class of unconditional convex set K.

Theorem 7.1. [Kla09] *Let μ be an unconditional convex measure on an unconditional set K and assume*

$$K \subset [-R, R]^d.$$

Then

$$C_P(\mu) \leq \frac{4R^2}{\pi^2}.$$

The proof is based on the fact that for a log-concave measure any eigenfunction g associated to the first non-trivial eigenvalue (if it exists) has a "favorite" direction in the sense

Lemma 7.2. *Let μ be a log-concave measure and assume g is an eigenfunction associated to the first non-trivial eigenvalue λ_1: then*

$$\int \nabla g d\mu \neq 0.$$

Proof. The proof is done by contradiction using to the L^2 Hörmander and two times the Poincaré inequality: one time for the term $\int |\nabla\nabla g|^2 d\mu$ and one for the term $\int |\nabla g|^2 d\mu$. Indeed, one has

$$\begin{aligned}
\lambda_1^2 \int g^2 d\mu &= \int |\nabla\nabla g|^2 d\mu + \int \nabla g \cdot (\mathrm{Hess} V)\, \nabla g d\mu \\
&\geq \int |\nabla\nabla g|^2 d\mu \\
&= \sum_{i=1}^d \int |\nabla(\partial_i g)|^2 d\mu \\
&\geq \sum_{i=1}^d \lambda_1 \int (\partial_i g)^2 d\mu \text{ since } \int \partial_i g d\mu = 0 \\
&= \lambda_1 \int |\nabla g|^2 d\mu \\
&\geq \lambda_1^2 \int g^2 d\mu \text{ since } \int g d\mu = 0.
\end{aligned}$$

A little more work is necessary to obtain a strict inequality and the contradiction. □

As a consequence, using the unconditional symmetries, there must exists a first non-trivial eigenvalue g which is *odd with respect to one coordinate* , say i_0.

The proof is then easily finished in the compact case by a Poincaré inequality in dimension 1. Indeed, since for any $(x_1, \dots, x_{i_0-1}, x_{i_0+1}, \dots x_d) \in \mathbb{R}^{d-1}$,

$$\int_{x_{i_0} \in \mathbb{R}} g(x)\ d\mu(x_{i_0} | (x_1, \dots, x_{i_0-1}, x_{i_0+1}, \dots x_d) = 0.$$

Then by Fubini theorem and using the Poincaré inequality in dimension 1, one has (note that $\int g d\mu = 0$ and $\int_K |\nabla g|^2 d\mu(x) = \lambda_1 \int_K g^2 d\mu$):

$$\int_K g^2(x) d\mu \leq \frac{4R^2}{\pi^2} \int_K |\partial_{i_0} g|^2 d\mu(x) \leq \frac{4R^2}{\pi^2} \int_K |\nabla g|^2 d\mu(x).$$

The general case where the true eigenfunction do not necessarily exist has been considered in [BK20]. Moreover, the authors obtain (see precise statement in Theorem 17) that a perturbation of a *symmetric* probability *product* measure by an *unconditional* (and *coordinatewise non-increasing*) perturbation does not deteriorate the Poincaré constant.

Note that a *weighted Poincaré inequality* for the class of *unconditional functions* has been obtain by Klartag (see e.g [Kla13], see also [KM16]).

7.2. **Radial case.** In this section, we consider the case of radial measure on $\mathbb{R}^n$. We have $d\mu = e^{-U(|x|)} dx$ where $U : [0, +\infty) \to \mathbb{R}$ is a smooth function and

$$Lf(x) = \Delta f(x) - U'(|x|) \frac{x}{|x|} \cdot \nabla f(x), \quad x \in \mathbb{R}^d,$$

The following result is due to [Bob03]. The constant was improved [BJM16].

Theorem 7.3. [Bob03, BJM16]
Let μ be a log-concave radial probability measure on $\mathbb{R}^d$ with $d \geq 2$. Then, one has

$$\frac{\int_{\mathbb{R}^d} |x|^2 \mu(dx)}{d} \leq C_P(\mu) \leq \frac{\int_{\mathbb{R}^d} |x|^2 \mu(dx)}{d-1}.$$

The result is completed in dimension 1 by Bobkov [Bob99]:

$$C_P(\mu) \leq 12\,\mathrm{Var}(\mu)$$

Actually, this result applies also to radial measure supported in a ball, outside of a ball or even in an annulus.

Obtaining a lower bound for the Poincaré constant is easy. One has just to plug a (good) function in Poincaré inequality. Note also that the variational caracterisation

$$C_P(\mu) = \sup_{f, f \perp 1} \frac{\mathrm{Var}_\mu(f)}{\int |\nabla f|^2 d\mu}$$

is equivalent to the classical variational caracterisation of the first non-trivial eigenvalue of $-L$:

$$\lambda_1 = \inf_{f, f \perp 1} \frac{\int |\nabla f|^2 d\mu}{\mathrm{Var}_\mu(f)}.$$

The lower bound the Poincaré constant is here just obtained by considering linear functions, for example taking $f(x) = x_1$. Indeed, it satisfies $\int |\nabla f|^2 d\mu = 1$ and by symmetries $\int f d\mu = 0$ and $\int f^2 d\mu = \frac{1}{d} \int |x|^2 d\mu$.

The case of the Subbotin or exponential power measures ($V_\alpha(x) = |x|^\alpha/\alpha$ for $\alpha > 0$) is explicitly given in [BJM16]. As said before, these probability measures satisfy the usual Poincaré inequality if and only if $\alpha \geq 1$. The case $\alpha = 2$ corresponds to the Gaussian. For $\alpha \geq 1$, the measures are log-concave but the behaviour for $\alpha \in [1, 2)$ and $\alpha \in (2, \infty)$ are very different. For $\alpha > 2$, the measures are uniformly log-concave at infinity but have a lack of convexity near the origin. Whereas for $\alpha \in [1, 2)$ this is the contrary, they are not uniformly log-concave at infinity.

In a simple form, this method produces the bounds:

$$\frac{d-1}{d+1} \times d^{1-2/\alpha} \leq \lambda_1(-L_\alpha) \leq \frac{d+2}{d} \times d^{1-2/\alpha}. \tag{7.1}$$

With this method, we also recover that the the Poincaré constant for the uniform measure on the ball of radius R is of order $\frac{R^2}{d}$. The explicit value is given by the first zero of the derivatives of some Bessel functions [Wei56].

This approach works also beyond the log-concave case. Weighted Poincaré inequalities for generalized Cauchy distribution (and also fro the standard Gaussian measure) are given in [BJM16].

We now discuss the proof of the upper bound in Theorem 7.3. The idea is to see that that the radial measure μ is close to a product measure.. More precisely, the measure μ is the image measure of the product measure $\nu \otimes \sigma_{d-1}$ by the mapping $(r,\theta) \in (0,+\infty) \times S^{d-1} \to r\theta$. Here ν is the one-dimensional probability measure on $\mathbb{R}_+$ with density proportional to

$$r^{d-1}e^{-U(r)} = e^{-(U(r)-(d-1)\ln r)}$$

Lemma 7.4. *With these notation, one has:*

$$C_P(\mu) \leq \max\left(C_P(\nu), C_P(\sigma_{d-1}) \int_{\mathbb{R}^d} \|x\|^2 \mu(dx)\right). \tag{7.2}$$

Now the Poincaré constant of the uniform measure on the sphere

$$C_P(\sigma_{d-1}) = \frac{1}{d-1}.$$

The last idea is to take advantage of the term r^{d-1} in the radial measure ν:

$$(U(r) - (d-1)\ln r)'' = U''(r) + \frac{d-1}{r^2}.$$

By the Veysseire's result, this gives

$$C_P(\nu) \leq \frac{\int_0^{+\infty} r^2 \nu(dr)}{d-1} = C_P(\sigma_{d-1}) \int_{\mathbb{R}^d} \|x\|^2 \mu(dx). \tag{7.3}$$

Proof of Lemma 7.2. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and denote g on $(0,+\infty)\times S^{d-1}$ by $g(r,\theta) := f(r\theta)$. First using the Poincaré inequality for the radial operator $-L_\nu$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} f^2 d\mu &= \int_{S^{d-1}} \left(\int_0^{+\infty} g(r,\theta)^2 \nu(dr)\right) \sigma_{n-1}(d\theta) \\
&\leq \int_{S^{d-1}} \left(\int_0^{+\infty} g(r,\theta)\nu(dr)\right)^2 \sigma_{d-1}(d\theta) \\
&\quad + C_P(\nu) \int_{S^{d-1}} \int_0^{+\infty} \left|\partial_r g(r,\theta)\right|^2 \nu(dr)\sigma_{d-1}(d\theta) \qquad (7.4)\\
&= \int_{S^{d-1}} \left(\int_0^{+\infty} g(r,\theta)\nu(dr)\right)^2 \sigma_{d-1}(d\theta) \\
&\quad + C_P(\nu) \int_{S^{d-1}} \int_0^{+\infty} \left|\theta \cdot \nabla f(r\theta)\right|^2 \nu(dr)\sigma_{d-1}(d\theta).
\end{aligned}$$

Now if we set

$$h(\theta) := \int_0^{+\infty} g(r,\theta)\nu(dr),$$

using the Poincaré inequality for the uniform measure on the sphere S^{d-1}

$$\int_{S^{d-1}} h(\theta)^2 \sigma_{d-1}(d\theta) \;\leq\; \left(\int_{\mathbb{R}^d} f d\mu\right)^2 + C_P(\sigma_{d-1}) \int_{S^{d-1}} |\nabla_{S^{d-1}} h(\theta)|^2 \sigma_{n-1}(d\theta)$$

where $\nabla_{S^{d-1}} h(\theta)$ denotes the spherical gradient at θ which can also be written as $\Pi_{\theta^\perp}(\nabla h)(\theta)$.

Now by Cauchy-Schwarz, one has:

$$\begin{aligned} |\nabla_{S^{d-1}} h(\theta)|^2 &= \left|\int_0^{+\infty} r \Pi_{\theta^\perp}(\nabla f)(r\theta)\nu(dr)\right|^2 \\ &\leq \int_0^{+\infty} r^2 \nu(dr) \int_0^{+\infty} |\Pi_{\theta^\perp}(\nabla f)(r\theta)|^2 \,\nu(dr) \\ &= \int_{\mathbb{R}^d} |x|^2 \mu(dx) \int_0^{+\infty} |\Pi_{\theta^\perp}(\nabla f)(r\theta)|^2 \,\nu(dr). \end{aligned} \tag{7.5}$$

Since by Pythagorus, $|\nabla f(r\theta)|^2 = |\theta \cdot \nabla f(r\theta)|^2 + |\Pi_{\theta^\perp}(\nabla f)(r\theta)|^2$, summing (7.4) and (7.5) ends the proof of the lemma. □

8. Generalized Brascamp-Lieb inequalities.

In this final section, we present quickly some opening and a method that can be used in the non-uniformly, non strictly or even non convex situation. The method presented is a generalization of the Brascamp-Lieb inequality on $\mathbb{R}^d$.

Theorem 8.1. [ABJ18] *Let $d\mu = e^{-V} dx$ be a probability measure on $\mathbb{R}^d$. Assume that there exists a smooth diagonal inversible matrix $B(x)$ such that*

$$\mathrm{Hess} V(x) - (LB)(x) B^{-1}(x) > 0 \textit{ for all } x \in \mathbb{R}^d.$$

Then for all function f,

$$\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^d} (\nabla f) \cdot \Big(\mathrm{Hess} V - (LB) B^{-1}\Big)^{-1} (\nabla f)\, d\mu. \tag{8.1}$$

The diagonal assumption can be relaxed but the assumption is not still well understood.

Explicit Poincaré constants for the non-convex potential $V(x,y) = x^4 + y^4 - \beta xy$ (for small β) together with weighted Poincaré for the Subbotin distributions are obtained in [ABJ18].

When $B(x) = b(x) Id$, the method is close to the Lyapunouv method (see e.g. [BCG08] and [BGL14] Theorem 4.6.2) but permits to obtain more explicit and reasonable Poincaré constants.

Moreover, in dimension 1, if b is the derivative of an increasing function h, one has

$$V'' - \frac{Lb}{b} = \frac{(-Lh)'}{h'}.$$

In particular, since (if it exists) the first non trivial eigenfunction g_1 is strictly increasing, taking $b = g_1'$ the new potential is constant to the spectral gap; that is, one has:

$$V'' - \frac{Lb}{b} = \frac{(-Lg_1)'}{g_1'} = \lambda_1.$$

This recovers the *Chen's formula* (see [CW97] and also [DW11, BJ14]) for the first non trivial eigenvalue of $-L$:

$$\lambda_1 = \sup_{h,h'>0} \inf_{x\in\mathbb{R}} \frac{(-Lh(x))'}{h'(x)}. \tag{8.2}$$

Remark 8.2. *Another idea to go beyond the uniformly convex Bakry-Emery case is to change the metric and to hope that with this new metric, the new diffusion operator satisfies some Bakry-Emery curvature criterion. This idea is implemented for example in* [KM16].

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