# Image processing with variational approaches and Partial Differential Equations Practice 3: Optical flow

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## 1. Introduction

The objective of this practice is to implement algorithms dedicated to the estimation of optical flow between successive frames of a video $I(x, y, t)$ defined on a constant $2D$ pixel domain $(x, y) \in \Omega$ for different times $t = 1 \cdots N$. We fill focus on variational methods derived from the seminal work of Horn and Schunck [2]. Assuming that the pixels intensities do not change over time, the optical flow problem between two successive images $I(x, y, t)$ and $I(x, y, t+1)$ can be stated as the estimation of a $2D$ motion field $(u, v)$ so that:

$$I(x + u(x, y), y + v(x, y), t + 1) = I(x, y, t), \tag{1.1}$$

where $u(x, y)$ and $v(x, y)$ stand for the horizontal and vertical components of the motion field and are defined over $\Omega$.

This is illustrated in the following figure:

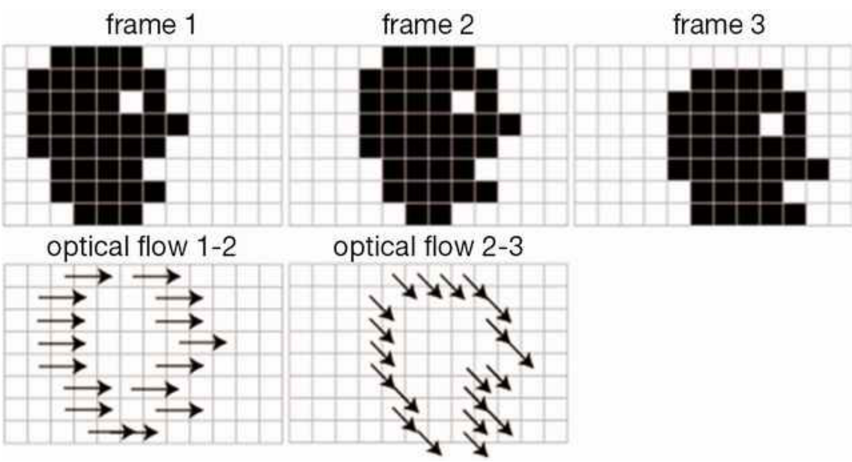



Illustration [3] of motion field $w = (u, v)$ between successive frames.

Relation (1.1) is non linear in $(u, v)$. Assuming that the motion amplitude is small, a linearization can be performed to obtain:

$$I(x+u, y+v, t+1) \approx I(x, y, t) + \partial_x I(x, y, t)u(x, y) + \partial_y I(x, y, t)v(x, y) + \partial_t I(x, y, t), \quad (1.2)$$

Considering the partial temporal derivative of the image $\partial_t I(x, t) = I(x, y, t+1) - I(x, y, t)$ and using relation (1.1), we obtain:

$$\partial_t I + \partial_x I u + \partial_y I v = \partial_t I + \nabla I \cdot w \approx 0, \quad (1.3)$$

where $w = (u, v)$ is the motion field to estimate. This constraint, that can also be derived from $\frac{dI}{dt} = 0$, is known as the optical flow constraint equation. Hence, the data term to minimize for the optical flow estimation can be written as:

$$||\partial_t I + \nabla I \cdot w||^2.$$

By minimizing this squared $L_2$ distance, we are looking for a motion field $w$ that register the image $I(x, y, t+1)$ with respect to the image $I(x, y, t)$.

This problem is nevertheless ill-posed, since two unknowns $u$ and $v$ of total size $2|\Omega|$ have to be estimated from only $|\Omega|$ constraints. As a consequence, Horn and Schunck proposed to add some regularity constraints on the motion components in order to close the problem. Assuming that the underlying motion field is smooth, they consider the Tikhonov regularization:

$$||\nabla w||^2 = ||\nabla u||^2 + ||\nabla v||^2.$$

The final convex functional to minimize thus reads:

$$J(u, v) = \lambda ||\partial_t I + \partial_x I u + \partial_y I v||^2 + \left(||\nabla u||^2 + ||\nabla v||^2\right), \quad (1.4)$$

where $\lambda > 0$ is the regularization parameter.

# 2. Horn and Schunck model for color images

We now consider the optical flow estimation between color images $I$ in the $RGB$ space, we will refer to the index $c = 1, 2, 3$ for each color channel. The problem thus becomes:

$$J(u, v) = \lambda \sum_{c=1}^{3} ||\partial_t I^c + \partial_x I^c u + \partial_y I^c v||^2 + ||\nabla u||^2 + ||\nabla v||^2, \quad (2.1)$$

In order to minimize the convex and differentiable functional (2.1), we rely on the simple gradient descent algorithm, even if more advanced optimization tools could be considered. Initializing $u_0 = v_0 = 0$ and deriving functional (2.1), one obtains, for a time step $\tau > 0$:

$$\begin{cases} u_{k+1} = u_k + \tau(\Delta u_k - \lambda \sum_{c=1}^{3} (\partial_t I^c + \partial_x I^c u_k + \partial_y I^c v_k)\partial_x I^c) \\ v_{k+1} = v_k + \tau(\Delta v_k - \lambda \sum_{c=1}^{3} (\partial_t I^c + \partial_x I^c u_k + \partial_y I^c v_k)\partial_y I^c). \end{cases} \quad (2.2)$$

## Implementation

- Load the two images $I_1$ and $I_2$ from frame10.png and frame11.png
- Precompute the gradients $\partial_x I$ and $\partial_y I$ on the image $I_1$ and the temporal derivative as $\partial_t I = I_2 - I_1$. Be careful that these derivatives are computed on each color channel $c$.
- Implement algorithm (2.2) and test it for $\lambda = 1/400$ and $\tau = 0.02$ with a large number of iterations. A test $||u_{k+1} - u_k||/||u_k|| + ||v_{k+1} - v_k||/||v_k|| < \epsilon = 10^{-3}$ or $10^{-4}$ can be considered to check the convergence and stop the iterations.
- Display/Validation of result:
  - The function flowToColor.m (that needs computeColor.m) takes $u$ and $v$ as arguments and returns an image (to display) representing the optical flow with standard color convention.
  - The vector field can be displayed on the image with the function drawMotion.m that takes as inputs the vector field $(u, v)$ and the image $I_1$.
  - Write a function *Registration.m* that takes as inputs the vector field $(u, v)$ and the second image $I_2$ and realize the registration: $\tilde{I}_2(x, y) = I_2(x + u, y + v)$ using bilinear interpolation. The obtained image should be closed to $I_1$.

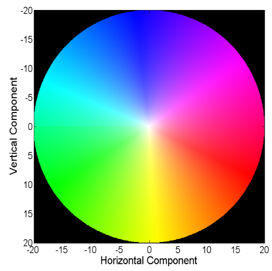
You should obtain the following results:

$I_1$

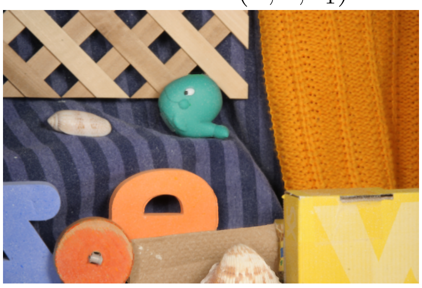
$I_2$

drawMotion$(u, v, I_1)$

flowToColor$(u, v)$




Color convention



$\tilde{I}_2$ =Registration$(u, v, I_2)$

$|\tilde{I}_2 - I_1|$

where most of the registration errors $|\tilde{I}_2 - I_1|$ appears in the occluded/desoccluded areas and for the regions with higher motions.

# 3 Multi-resolution algorithm

The previous algorithm assumes that the motion amplitude is low. This is not a good model for estimating large displacements. In order to circumvent this limitation, multi-resolution (also known as coarse-to-fine) approaches are commonly used.

The idea is to create a pyramid of down-sampled images $I_1$ and $I_2$. We will denote by $I^\ell$ an image at level $\ell$, such that each spatial dimension of the domain of $I^{\ell+1}$ is half the one of $I^\ell$ and $I^0 = I$, i.e. $|\Omega^\ell| = |\Omega|/4^\ell$ and $\Omega^0 = \Omega$.

Hence, at a low resolution (level $l = 5$ for instance), the number of column and lines of $I^\ell$ are divided by a factor $2^\ell$ with respect to the dimension of $I^0$. As a consequence, the assumption of small motion amplitude is checked at this scale and the previous algorithm can be used to estimate the motion $(u^\ell, v^\ell)$ between images $I_1^\ell$ and $I_2^\ell$.

The multi-resolution framework then consists in estimating the motion $(u^{\ell+1}, v^{\ell+1})$ at a pyramid level $\ell + 1$ and using it at the next level, as illustrated in the following figure.

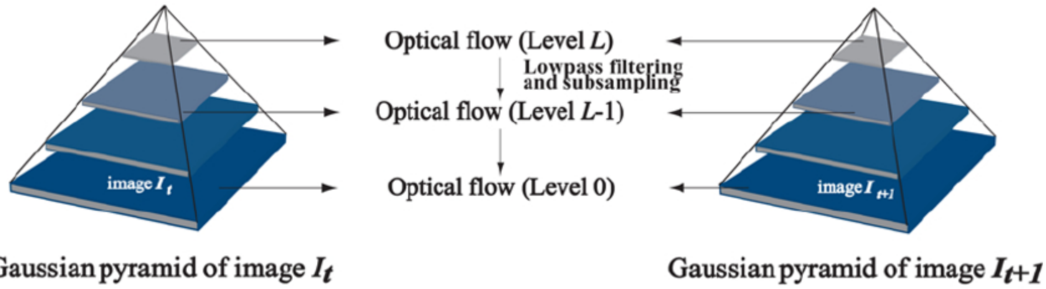



Illustration [5] of the multi-resolution approach.

Hence, we can realize the linearization of the optical flow constraint at the level $\ell$ around the previously estimated motion field $(u^{\ell+1}, v^{\ell+1})$:

$$\begin{aligned}
&I^\ell(x + u^{\ell+1} + u^\ell, y + v^{\ell+1} + v^\ell, t + 1)\\
\approx &I^\ell(x + u^{\ell+1}, y + v^{\ell+1}, t) + \partial_x I^\ell(x + u^{\ell+1}, y + v^{\ell+1}, t)u^\ell\\
&+ \partial_y I^\ell(x + u^{\ell+1}, y + v^{\ell+1}, t)v^\ell + \partial_t I^\ell(x + u^{\ell+1}, y + u^{\ell+1}, t),
\end{aligned} \tag{3.3}$$

Denoting as $\tilde{I}_2^\ell$ the image $I_2^\ell$ registered by the motion field $(u^{\ell+1}, v^{\ell+1})$, the linearization can be discretized to obtain the following optical flow constraint:

$$\partial_t I^\ell + \partial_x I^\ell u^\ell + \partial_y I^\ell v^\ell = 0,$$

where

$$\partial_t I^\ell = \tilde{I}_2^\ell - I_1^\ell, \qquad \partial_x I^\ell = \partial_x \tilde{I}_2^\ell, \qquad \partial_y I^\ell = \partial_y \tilde{I}_2^\ell, \tag{3.4}$$

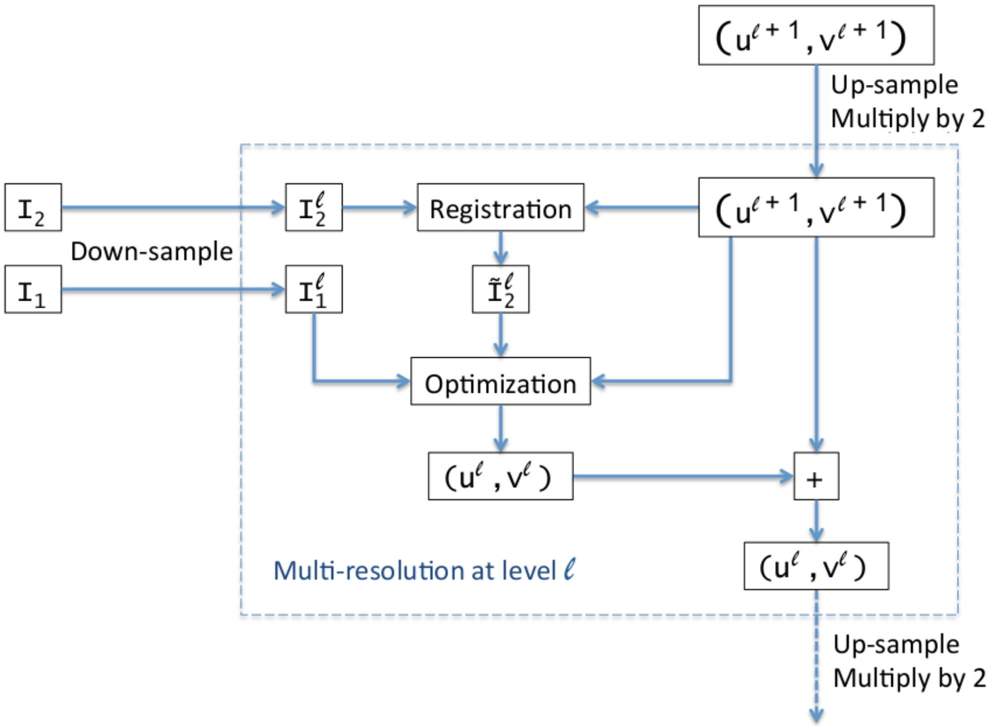
The final convex functional to minimize over $(u^\ell, v^\ell)$ at pyramid level $\ell$ reads:

$$\lambda||\partial_t I^\ell + \partial_x I^\ell u^\ell + \partial_y I^\ell v^\ell||^2 + ||\nabla(u^\ell + u^{\ell+1})||^2 + ||\nabla(v^\ell + v^{\ell+1})||^2.$$

Notice that the regularization is done over the whole current motion field $(u^\ell + u^{\ell+1}, v^\ell + v^{\ell+1})$. It leads to the following iterative gradient descent for each level $\ell$:

$$\begin{cases} u_{k+1}^\ell = u_k^\ell + \tau(\Delta(u_k^\ell + u^{\ell+1}) - \lambda \sum_{c=1}^3 (\partial_t I^{\ell,c} + \partial_x I^{\ell,c} u_k^\ell + \partial_y I^{\ell,c} v_k^\ell)\partial_x I^{\ell,c}) \\ v_{k+1}^\ell = v_k^\ell + \tau(\Delta(v_k^\ell + v^{\ell+1}) - \lambda \sum_{c=1}^3 (\partial_t I^{\ell,c} + \partial_x I^{\ell,c} u_k^\ell + \partial_y I^{\ell,c} v_k^\ell)\partial_y I^{\ell,c}). \end{cases} \tag{3.5}$$

Gathering all elements, the algorithm at level $\ell$ is described in the following diagram. In the next sections, we will only change the "Optimization" box of this framework.




Implement the multi-resolution algorithm and test it for $L = 6$ with the same parameters than before.

- Inputs: $I_1$, $I_2$ and a number of pyramid level $L$
- Initialize $u^L = v^L = 0$ on the domain $\Omega^L$
- For $\ell = L - 1 \cdots 0$
  - Up-sample $u^{\ell+1}$ and $v^{\ell+1}$ to fit the domain of $\Omega^\ell = |\Omega|/4^\ell$ (that is twice bigger in each dimension than the one of $\Omega^{\ell+1}$). Set $u^{\ell+1} = 2u^{\ell+1}$ and $v^{\ell+1} = 2v^{\ell+1}$ (the motion amplitude is twice bigger on the higher resolution domain $\Omega^\ell$.
  - Down-sample $I_1^\ell$ and $I_2^\ell$ to fit the domain $|\Omega^\ell| = |\Omega|/4^\ell$ from $I_1$ and $I_2$
  - Register $I_2^\ell$ with $(u^{\ell+1}, v^{\ell+1})$ to obtain $\tilde{I}_2^\ell$ using your function *Registration.m*
  - Initialize $u_0^\ell = v_0^\ell = 0$. Take $\tau = 0.005$.
  - Realize the gradient descent algorithm (3.5) with (3.4) to obtain $(u^\ell, v^\ell)$.
  - Set $(u^\ell, v^\ell) = (u^\ell, v^\ell) + (u^{\ell+1}, v^{\ell+1})$
- Output: $(u^0, v^0)$

The obtained optical flow is now more accurate as illustrated below.


flowToColor$(u, v)$


$|\tilde{I}_2 - I_1|$

# 4. $L_1$ data term

In the regions of occlusion/desocclusions, the constraint $I(x+u, y+v, t+1) = I(x, y, t)$ can not be verified so that the norm of the previous data term $||\partial_t I + \partial_x I u + \partial_y I v||^2$ may "explode". In order to be more robust to such outliers, we now consider the following $L_1$ data term:

$$||\partial_t I + \partial_x I u + \partial_y I v||,$$

which is still convex in $(u, v)$ but non differentiable, so that gradient descent can not be applied. Hence, we will consider the Forward-Backward algorithm that is dedicated to the minimization of the sum of two convex functionals $F(x) + G(x)$, one of them, say $G$, being differentiable. This algorithm reads:

$$x_{k+1} = Prox_{\tau F}(x_k - \tau \nabla G(x_k)), \tag{4.1}$$

where the proximity operator is defined as:

$$Prox_{\tau F}(\tilde{x}) = \arg\min_x \frac{||x - \tilde{x}||^2}{2\tau} + F(x).$$

While the step $\tilde{x} = x_k - \tau \nabla G(x_k)$ is an explicit (forward) gradient descent over the function $G$, the step $Prox_{\tau F}(\tilde{x})$ corresponds to an implicit (backward) descent and is commonly used when the function $F$ is not differentiable.

With our new data term, we can now define the whole optical flow functional to minimize with $F(u, v) = \lambda ||\partial_t I + \partial_x I u + \partial_y I v||$ and $G(u, v) = ||\nabla u||^2 + ||\nabla v||^2$. Notice that for technical reasons, **we will consider from now grayscale images**. As before, the gradient of $G$ is:

$$\nabla_u G(u, v) = -\Delta u, \qquad \nabla_v G(u, v) = -\Delta v. \tag{4.2}$$

Denoting as $\rho(\tilde{u}, \tilde{v}) = \partial_t I + \partial_x I \tilde{u} + \partial_y I \tilde{v}$, the proximity operator of $F$ can be expressed as [6]:

$$Prox_{\tau F}(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}) + \begin{cases} \tau \lambda \nabla I & if\ \rho(\tilde{u}, \tilde{v}) \leq -\tau\lambda ||\nabla I||^2 \\ -\tau \lambda \nabla I & if\ \rho(\tilde{u}, \tilde{v}) \geq \tau\lambda ||\nabla I||^2 \\ -\rho(\tilde{u}, \tilde{v}) \nabla I / ||\nabla I||^2 & otherwise, \end{cases} \tag{4.3}$$

where $\nabla I = (\partial_x I, \partial_y I)$ is a vector field of the same dimension than $(u, v)$ as we deal with grayscale images. Homework: show relation (4.3). The proximity operator of the color image data term $\sum_c ||\partial_t I^c + \partial I^c_x u + \partial_y I^c v||$ is more complex to compute, see [4] if you are ambitious.

Implement the multi-resolution algorithm for the $L_1$ data term on grayscale images. Do not forget to convert the color images into grayscale ones, using for instance MATLAB's function *rgb2gray*. With respect to the squared $L_2$ model, the gradient descent algorithm (3.5), should now be replaced by the Forward-Backward algorithm (4.1) applied to the new problem. This gives, at pyramid level $\ell$ and iteration $k+1$:

$$\begin{aligned} (\tilde{u}, \tilde{v}) &= (u^\ell_k, v^\ell_k) - \tau \nabla G(u^\ell_k + u^{\ell+1}, v^\ell_k + v^{\ell+1}) \\ (u^\ell_{k+1}, v^\ell_{k+1}) &= Prox_{\tau F}(\tilde{u}, \tilde{v}), \end{aligned} \tag{4.4}$$

using the above relations (4.2) and (4.3). One can take $\tau = 1/8$ and $\lambda = 1/300$.

This new model is more robust to outliers and it also decreases the registration error.

flowToColor$(u, v)$



$|\tilde{I}_2 - I_1|$

# 5. Total variation regularization

All the previous models rely on a Tikhonov regularization of the motion field. This leads to smooth motion transitions. In order to promote piece wise constant motion regions and thus sharper transitions, we now consider the Total Variation regularization of the motion field $(u, v)$, as proposed in [6].

The corresponding functional to minimize, still defined for grayscale images, reads:

$$\lambda||\partial_t I + \partial_x I u + \partial_y I v|| + ||\nabla u|| + ||\nabla v||.$$

Relying on the dual formulation of the total variation (see Practice 2) and introducing the dual variables $z^u = (z^u_x, z^u_y)$ and $z^v = (z^v_x, z^v_y)$, minimizing the previous functional is equivalent to solve the primal-dual saddle point problem:

$$\min_{u,v} \max_{z^u,z^v} \lambda||\partial_t I + \partial_x I u + \partial_y I v|| + \langle \nabla u, z^u\rangle + \langle \nabla v, z^v\rangle - \iota_B(z^u) - \iota_B(z^v),$$

where $\iota_B$ is the characteristic function of the $\ell_2$ ball of radius 1, i.e. $\iota_B(z) = 0$ if $||z|| \leq 1$ and $+\infty$ otherwise. To solve this problem, we rely on the primal-dual algorithm of [1] that reads, at level $\ell$ and iteration $k + 1$:

$$\begin{aligned} z^u_{k+1} &= Proj_B\left(z^u_k + \sigma\nabla(u^\ell_k + u^{\ell+1})\right) \\ z^v_{k+1} &= Proj_B\left(z^v_k + \sigma\nabla(v^\ell_k + v^{\ell+1})\right) \\ (u^\ell_{k+1}, v^\ell_{k+1}) &= Prox_{\tau F}(u^\ell_k + \tau div(z^u_{k+1}), v^\ell_k + \tau div(z^v_{k+1})), \end{aligned} \tag{5.1}$$

where $Prox_{\tau F}$ is still defined in (4.3) and the projection onto the ball $B$ is:

$$Proj_B(z) = \begin{cases} z & if\ ||z|| \leq 1 \\ z/||z|| & otherwise. \end{cases}$$

Implement the multi-resolution algorithm for the $L_1$ data term and Totaal Variation regularization on grayscale images. This corresponds to change the optimization scheme as (5.1). The time step value can be here taken as $\sigma = 1/\left(2 + \max\left(\max(||\nabla u^{\ell+1}||), \max(||\nabla v^{\ell+1}||)\right)\right)$ and $\tau = 1/4$. With parameter $\lambda = 1/2$, the following result, containing sharper motion transitions, should be obtained.

flowToColor$(u, v)$

$|\tilde{I}_2 - I_1|$

Notice that the flowToColor function normalizes the vectors with respect to the highest motion vector. This explains the variations of color intensities in the produced images.

**If you have time.** You can test your algorithms and compare your results with ground truth motion fields using data available here.

# References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