Exponential growth for the wave equation with compact time-periodic positive potential

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Abstract

We prove the existence of smooth positive potentials $V(t,x)$, periodic in time and with compact support in x for which the Cauchy problem for the wave equation $u_{tt} - \Delta_x u + V(t,x)u = 0$ has solutions with exponentially growing global and local energy. Moreover, we show that there are resonances, $z \in \mathbb{C}$, $|z| > 1$, associated to $V(t,x)$. © 2000 Wiley Periodicals, Inc.

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1 Introduction

Consider the Cauchy problem for the wave equation with potential,

$$
\begin{cases} \partial_t^2 u - \Delta_x u + V(t,x)u = 0, & (t,x) \in \mathbb{R}^{1+3}, \\ u\big|_{t=s} = f_1(x), \quad u_t\big|_{t=s} = f_2(x). \end{cases} \tag{1.1}
$$

The potential, $V \in C^\infty(\mathbb{R}^{1+3}; \mathbb{R})$, is *time periodic and compactly supported*, that is

(1.2) $\exists \rho, T > 0,\ \forall (t,x) \in \mathbb{R}^{1+3}, \quad V(t+T,x) = V(t,x), \quad \operatorname{supp} V \subset \big\{|x| \le \rho\big\}.$

Denote by D the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$
\|\varphi\|_D := \Big(\int_{\mathbb{R}^3} |\nabla \varphi(x)|^2 dx\Big)^{1/2}.
$$

For initial data $f = (f_1, f_2)$ in the Hilbert space $H := D \times L^2(\mathbb{R}^3)$ there is a unique solution $u \in C(\mathbb{R}; D) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^3))$ of (1.1). Define the propagator

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$U(t,s)f := (u,u_t)(t,x)$ and denote by $U_0(t,s)$ the propagator in the case $V=0$. We denote by $\big(\cdot\,,\cdot\big)_H$ the scalar product in H. When $V=0$ the coefficients are independent of time so the propagator $U_0(t,s)$ depends only on $t-s$.

For
$$e \;:=\; \frac{1}{2}\big(|u_t|^2+|\nabla_x u|^2\big)$$
one has the differential energy law
$$\partial_t e \;-\; \operatorname{div}_x\big(\mathrm{Re}(\overline{u}_t\,\nabla_x u)\big) \;=\; 2\,\mathrm{Re}\,\big(\overline{u}_t\,(u_{tt}-\Delta u)\big)\,.$$
Integrating over $\mathbb{R}^3_x$ when $V=0$ shows that U_0 *is unitary*.

Since V is compactly supported there is a constant α depending only on V so that,

$$\|Vu\|_{H^1(\mathbb{R}^3)} \;\le\; \alpha\,\|\nabla_x u\|_{L^2(\mathbb{R}^3)}. \tag{1.3}$$

Estimating
$$\Big|\int \overline{u}_t\;Vu\;dx\Big| \;\le\; \|u_t\|_{L^2(\mathbb{R}^3)}\,\|Vu\|_{L^2(\mathbb{R}^3)} \;\le\; \alpha\,\|u_t\|_{L^2(\mathbb{R}^3)}\;\|\nabla_x u\|_{L^2(\mathbb{R}^3)},$$
shows that for solutions of (1.1),
$$\partial_t\int e(t,x)\;dx \le 2\,\alpha\int e(t,x)\;dx,$$
whence

$$\|U(t,s)f\|_H \;\le\; e^{\alpha|t-s|}\,\|f\|_H. \tag{1.4}$$

The perturbation $U(t,s)f-U_0(t,s)f$ is equal to $(w(t),w_t(t))$, where w is the unique solution of the Cauchy problem
$$w_{tt}-\Delta w \;=\; -Vu, \qquad w|_{t=s}=w_t|_{t=s}=0\,.$$
Estimate (1.3) shows that $Vu\in C(\mathbb{R}\,;H^1(\mathbb{R}^3))$ and Vu is supported in $|x|\le\rho$. It follows that
$$w\in C(\mathbb{R}\,;H^2(\mathbb{R}^3))\,\cap\,C^1(\mathbb{R}\,;H^1(\mathbb{R}^3))\,, \qquad \operatorname{supp} w(t)\;\subset\;\big\{|x|\le\rho+|t-s|\big\}\,,$$
$$\exists C,\;\;\forall\,|t-s|\le T, \qquad \big\|\big(w(t)\,,\,w_t(t)\big)\big\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \;\le\; C\,\big\|\big(u(s)\,,\,u_t(s)\big)\big\|_H\,.$$
In particular $U(t,s)$ *is a compact perturbation of the unitary operator* $U_0(t,s)$. Therefore, the spectrum of the Floquet or monodromy operator $U(T,0)$ consists of essential spectrum equal to the unit circle and at most a countable set of eigenvalues of finite algebraic multiplicity. In the present context, it is known that there are at most a finite number of eigenvalues with modulus greater than 1 [12].

These results correspond to the intuitive idea that typical waves radiate to infinity escaping the region where the potential can act. For large time such solutions behave like solutions of the free wave equation. The pure point spectrum off the unit circle, yield exceptional solutions which do not radiate to infinity.

If there is a point $\underline{z} \in \operatorname{spec} U(T,0)$ with $|\underline{z}| > 1$, then choosing f as an eigenfunction one has

$$U(nT,0)f \;=\; \underline{z}^n f\,, \qquad n \in \mathbb{Z}\,.$$

Taking $n \to \infty$ shows that (1.4) cannot hold with $\alpha = 0$, and there are solutions that grow exponentially in time.

It is easy to construct examples of growing solutions when the potential V is permitted to take negative values ([10], [7]). For example, if $C_0^\infty(\mathbb{R}^3) \ni V = V(x) \le 0$ and is not identically equal to zero, then for $g > 0$ sufficiently large, the Schrödinger operator $-\Delta + gV$ has a strictly negative eigenvalue

$$\big(-\Delta + gV\big)\psi = -\lambda^2\psi\,, \qquad \psi \in \mathscr{S}(\mathbb{R}^3)\,, \qquad \lambda > 0\,.$$

Then $u(t,x) := e^{\lambda t}\psi(x)$ is an exponentially growing solution. The propagator $U(T,0)$ for the potential $gV(x)$ satisfies

$$U(T,0)\,(\psi,\lambda\psi) \;=\; e^{\lambda T}(\psi,\lambda\psi)\,, \qquad (\psi,\lambda\psi) \in H\,.$$

Consider the perturbed potential $gV(x) + \varepsilon W(t,x)$ with W smooth, T-periodic, and compactly supported. The new evolution operator $U^\varepsilon(T,0)$ satisfies $\|U^\varepsilon(T,0) - U(T,0)\| \le C\varepsilon$. So for ε small it has pure point spectrum near $e^{\lambda T}$.

When $V \ge 0$ the situation is radically different. For the time independent case, $V = V(x)$, the energy

$$\int \Big[e(t,x) + V(x)\,\frac{|u(t,x)|^2}{2} \Big]\, dx$$

is conserved and there is no growth.

When Cooper and Strauss extended the Lax-Phillips theory to time periodic scatterers [8], [9] more than thirty years ago, they conjectured that there exist periodic potentials $V \ge 0$ with compact support so that $\operatorname{spec} U(T,0) \cap \{|z| > 1\} \neq \emptyset$. Numerical computations [15], [17], [18] supported this conjecture. The conjecture was further supported in Cooper [6] where certain perturbations were proved to lead to slow decay. Cooper conjectured that larger perturbations could lead to growth. Our main result proves the conjecture of Cooper and Strauss.

Theorem 1.1. *There exists a nonnegative, smooth, T-periodic, compactly supported potential $V(t,x)$ for which the Floquet operator $U(T,0)$ has an eigenvalue of modulus greater than one.*

There are other constructions of growing solutions preceding ours, each failing for the case of smooth, compact, positive periodic potentials. If one eschews compact support then one can choose $V = V(t)$ independent of x. Fourier transform in x transforms the problem to

$$\hat{u}_{tt}(t,\xi) + |\xi|^2\hat{u}(t,\xi) + V(t)\,\hat{u}(t,\xi) \;=\; 0\,.$$

One can then choose $V \ge 0$ smooth and periodic so that there are ξ such that the Hill's operator $d^2/dt^2 + (V(t) + |\xi|^2)$ has exponentially growing solutions. Such growth when the frozen operators have strictly positive conserved energies is called *parametric resonance*. Choosing data whose Fourier transform is supported in intervals of ξ where such resonance occurs, yields growing solutions.

For the wave equation with potential, the bicharacteristics are exactly the same as those of the free wave equation. In particular they lie over rays in (t,x) which travel along straight lines at speed one. These rays escape the region where the potential acts and the problem is *non trapping*. There are examples of exponentially growing solutions with compactly supported periodic perturbations which trap rays.

In [5], [14] the Dirichlet problem for periodically moving obstacles having trapped rays is examined. In particular, in [14] it was proved that if we have at least one bicharacteristic with suitable amplifying properties, then there exist solutions with exponentially growing local energy. The existence of a *trapped bicharacteristic* is exploited in [4] for the Cauchy problem for the equation

$$\partial_t^2 u \;-\; \mathrm{div}_x \big(a(t,x)\,\mathrm{grad}\, u\big) \;=\; 0\,,$$

where $a(t,x)$ is periodic in time, $0 < c \le a(t,x) \le C$ and $a(t,x) = 1$ for $|x| \ge \rho$. The growth associated to these trapped rays is connected with the presence of continuous spectrum of the monodromy operator $U(T,0)$ outside the unit disk. In that regard, recall the conjecture (see [13]) that for trapping periodic perturbations, the cut-off resolvent $\chi(x)(U(T,0)-zI)^{-1}\chi(x)$ with $\chi \in C_0^\infty(\mathbb{R}^3)$ equal to 1 for $|x| \le \rho + T$ does not have a meromorphic continuation from $\{z \in \mathbb{C} : |z| \ge A \gg 1\}$ to $\{|z| > 1\}$.

Our strategy is to confine and pump. The pump is based on the equation with potential $q(t)$ in the ball $\{|x| \le L\}$ and Dirichlet boundary conditions on the boundary, $\{|x| = L\}$. Expanding in eigenfunctions of the Dirichlet Laplacian, yields Hill's equations

$$a_n''(t) \;+\; q(t)\, a_n(t) \;+\; \lambda_n a_n(t) = 0$$

for the Fourier coefficients $a_n(t)$. One chooses $q(t)$ so that one of these Hill's equations has exponentially growing solutions. Choose a sequence of cutoff functions $\chi^\delta(x) \in C_0^\infty(|x| < L)$, $0 \le \chi \le 1$, with $\chi = 1$ except in layer of width δ of the boundary. Replacing the potential $q(t)$ by $q(t)\chi^\delta(x)$ changes the evolution operator little in norm, so the operator $K^\delta(T)$ taking Cauchy data at $t = 0$ to data at $t = T$ has an eigenvalue z_1 of modulus greater than 1. That is the *pump*.

Next replace the Dirichlet condition by a potential $b^\varepsilon(x)$ where $b^\varepsilon(x) \in C_0^\infty(\mathbb{R}^3)$ is a barrier supported in $\{L \le |x| \le L+1\}$, and equal to $1/\varepsilon$ for $\{L+\varepsilon \le |x| \le L+1-\varepsilon\}$. Let $V^\varepsilon(t,x) := b^\varepsilon(x) + q(t)\chi^\delta(x)$ with $\varepsilon > 0$. When ε is sufficiently small, initial waves supported in $|x| \le L$ are nearly confined to that ball. There is leakage through the barrier. We prove that the growth from the pump beats the loss by showing that there is an eigenvalue of modulus greater than one.

Our construction is inspired by the analysis of Beale [2] for the Helmholtz resonator. For z_1 and φ the eigenvalue and eigenfunction of $K^\delta(T)$, the main difficulty is to prove that the resolvents $(K^\delta(T)-zI)^{-1}\varphi$ and $(U^\varepsilon(T,0)-zI)^{-1}\varphi$ for z belonging to a small circle γ centered at z_1 are close. Since $K^\delta(T)$ has an eigenvalue z_1, $|z_1|>1$, the Cauchy contour integral of its resolvent over γ is non zero. It follows that the integral of the resolvent of $U^\varepsilon(0,T)$ over γ is non zero, proving the existence of an eigenvalue z_ε, $|z_\varepsilon|>1$ of $U^\varepsilon(0,T)$. For the problem of Beale with time independent coefficients, it sufficed to compare the resolvents of the generators. For that, the analysis leaned heavily on elliptic regularity. The heart of our proof is the convergence of

$$\Big(\varphi\,,\,(U^\varepsilon(T,0))-zI)^{-1}\varphi\Big)_H \;\to\; \Big(\varphi\,,\,(K^\delta(T)-zI)^{-1}\varphi\Big)_H,$$

whose proof is more delicate.

For the problem (1.1) there is a notion of *resonance* and a Corollary of Theorem 1.1 proves the existence of resonances. The existence of such resonances for positive, compact, periodic potentials is conjectured in the same articles cited above. Recall three equivalent definitions of resonances (see [12] [3], for details concerning resonances). The first definition concerns smooth solutions $u\in C^\infty(\mathbb{R};\mathscr{D}'(\mathbb{R}^3))$ of (1.1) which though not necessarily small for $|x|\to\infty$ they are *outgoing*. *Outgoing solutions* are defined as follows. For $\Phi\in C_0^\infty(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3)$ denote $U_0(t,0)\Phi=\big(w_\Phi(t),\partial_t w_\Phi(t)\big)$. A solution u is *outgoing* when for all Φ,

$$\lim_{t\to-\infty}\Big(\big\langle u(t)\,,\,w_\Phi(t)\big\rangle\;+\;\big\langle u_t(t)\,,\,\partial_t w_\Phi(t)\big\rangle\Big)\;=\;0,$$

where $\langle\,,\,\rangle$ means the action in the sense of distributions. A point $z=e^{i\sigma T}$ is a *resonance* if there exists an outgoing solution $u(t,x)$ of (1.1) with data $0\neq f=(f_1,f_2)\in H$ such that $e^{-i\sigma t}u(t,x)$ is periodic in t with period T. In this case $\sigma\in\mathbb{C}$ is called a *scattering frequency*.

The second definition concerns cut-off resolvents. Suppose that $\chi\in C_0^\infty(\mathbb{R}^3)$ is identically equal to one on a neighborhood of $\{x:\operatorname{dist}(x,\operatorname{supp}V)\leq T\}$. Then the cut-off resolvent $\chi(U(T,0)-z)^{-1}\chi$ has a meromorphic extension from $|z|\gg1$ to $z\neq0$ and the poles are the resonances.

Finally, the resonances $z\neq0$ are the eigenvalues of the reduced Floquet operator,

$$Z^b(T,0)\;:=\;P_+^b\,U(T,0)\,P_-^b,$$

where $P_\pm^b$ are the orthogonal projections on the orthogonal complements of Lax-Phillips [10] spaces,

$$D_\pm^b\;:=\;\{f\in H:\,U_0(t)f=0,\quad\text{when}\quad |x|\leq b\pm t,\,\pm t\geq0\},\qquad b>\rho.$$

The spectrum of $Z^b(T,0)$ is independent on $b>\rho$ and in $\mathbb{C}\setminus\{0\}$ it consists of a discrete set of eigenvalues with finite multiplicity which can only accumulate at 0.

For T-periodic, compactly supported, smooth, V, it is known that the following are equivalent (Theorem 5.5.3 in [12]):

i. One has local energy decay in the sense that for any $\chi \in C_0^\infty(\mathbb{R}^3)$ the operator $\chi U(t,0)\chi$ tends strongly to zero as $t \to \infty$.
ii. One has exponential decay of local energy in the sense that for any $\chi \in C_0^\infty(\mathbb{R}^3)$ there is a constant $C = C(\chi) > 0$ so that for $t \geq 0$,

$$\left\|\chi U(t,0)\chi\right\|_{\mathscr{L}(H)} \leq C e^{-Ct}.$$

iii. The operator $Z^b(T,0)$ has no eigenvalues z with $|z| \geq 1$.

The dichotomy given by the presence or absence of eigenvalues of $Z(T,0)$ of modulus greater or equal than one, determines whether the local energy decays or not. There are some sufficient conditions guaranteeing the absence of such point spectrum which do not pretend to be sharp [12]. Theorem 5.4.1 in [12] proves that for $|\lambda| > 1$, the operator P_+^a, $a > \rho$, is an isomorphism from the generalized eigenspace $\mathscr{G}_\lambda$ of $U(T,0)$ corresponding to λ to the generalized eigenspace $\mathscr{F}_\lambda$ of $Z(T,0)$ corresponding to λ. Thus $Z(T,0)$ has no eigenvalue λ, $|\lambda| > 1$, if and only if $U(T,0)$ has no such eigenvalue and we obtain the following.

Corollary 1.2. *There exists a nonnegative, smooth, T-periodic, compactly supported potential $V(t,x)$ for which the problem (1.1) has a resonance with modulus greater than one.*

Remark 1.3. **1.** Our analysis of the confine and pump mechanism works for general domains and dimensions. For ease of reading, we present the case of a ball in $\mathbb{R}^3$.
2. In Section 5 we give a proof independent of [12] that there are resonances near the eigenvalues of the pump.

The paper is organized as follows. Section 2 is devoted to studying the pump. Section 3 proves the weak convergence on space time of the solutions of the equation with potentials V^ε to those of the uncoupled Dirichlet problems for suitably restricted weakly convergent Cauchy data. In Section 4 we establish fixed time weak convergence of the resolvent $(U^\varepsilon(T,0) - zI)^{-1}\varphi$ to the corresponding resolvent of the decoupled Dirichlet problems when φ vanishes for $L \leq |x| \leq L+1$ and z does not meet the spectra of $U^\varepsilon(T,0)$. The key step is a bound on the resolvents independent of ε. The main results are derived from this in Section 5.

2 The pump

Define $B_L := \{x \in \mathbb{R}^3 : |x| \leq L\}$. The starting point of the construction is the mixed initial boundary value problem in $\mathbb{R}_t \times B_L$,

$$\begin{cases} u_{tt} - \Delta_x u + q(t)u \;=\; 0, & (t,x) \in \mathbb{R} \times B_L, \\ u(t,x)\big|_{|x|=L} \;=\; 0. \end{cases} \tag{2.1}$$

Choose a T-periodic smooth potential $q(t) \geq 0$ such that the equation

$$a''(t) + q(t)a(t) + \lambda a(t) = 0 \tag{2.2}$$

has an interval of instability $I =]\alpha,\beta[\subset \mathbb{R}^+$ (see for instance [11]). Choose $L > 0$ and $k \in \mathbb{N}$ so that $\lambda = k^2\pi^2/L^2 \in I$. Then, there is a μ_1 with $|\mu_1| > 1$ and a solution $a(t)$ of (2.2) so that

$$\big(a(T), a'(T)\big) \;=\; \mu_1\,\big(a(0), a'(0)\big).$$

Fix k and L. Then,

$$u(t,x) \;:=\; \begin{cases} |x|^{-1}\,a(t)\,\sin(k\pi|x|/L) & \text{if} \quad x \neq 0, \\[2ex] k\,\pi\,a(t)/L & \text{if} \quad x = 0, \end{cases}$$

is a solution of (2.1) which is exponentially growing as $t \to +\infty$.

Denote by $K(T)$ the operator taking Cauchy data $(u_0,u_1) \in H_0^1(B_L) \times L^2(B_L)$ at time $t = 0$ to Cauchy data $(u(T),u_t(T))$ at time T. Denote by $K_0(T)$ the analogous operator for the problem with $q = 0$. Then $K_0(T)$ is unitary in the norm

$$\|(u(t),u_t(t))\|^2 \;:=\; \frac{1}{2}\int_{B_L} \Big(|\nabla_x u(t,x)|^2 + |u_t(t,x)|^2\Big)\,dx\,.$$

The operator $K(T)$ is a compact perturbation of $K_0(T)$ and, by construction, $K(T)$ has an isolated eigenvalue μ_1 with $|\mu_1| > 1$.

For each $0 < \delta < L/10$, choose $\chi^\delta(x) \in C_0^\infty(B_L)$ with

$$0 \le \chi^\delta(x) \le 1, \qquad \chi^\delta\big|_{B_{L-\delta}} \;=\; 1\,.$$

Denote by $K^\delta(T) \in \mathrm{Hom}(H_0^1(B_L) \times L^2(B_L))$ the evolution operator associated to the differential operator $\partial_t^2 - \Delta_x + q(t)\,\chi^\delta(x)$ with Dirichlet boundary conditions on $|x| = L$. The energy method shows that as $\delta \to 0$ we have

$$\text{(2.3)} \qquad \big\|K^\delta(T) \;-\; K(T)\big\|_{\mathrm{Hom}(H_0^1(B_L)\times L^2(B_L))} \quad\to\quad 0\,.$$

Choose $r < |\mu_1| - 1$, so that μ_1 is the only eigenvalue of $K(T)$ belonging to the disk D_r of radius r and center μ_1. The norm convergence (2.3) implies that one can choose $0 < \delta \ll 1$ so that $K^\delta(T)$ has at least one eigenvalue z_1 inside the disk D_r and no eigenvalue on the boundary ∂D_r. The eigenvalue z_1 may have multiplicity greater than one.

In the following we will assume that $0 < \delta \ll 1$ is fixed so that $K^\delta(T)$ has an eigenvalue $z_1 \in D_r$. Consider the wave equation

$$\text{(2.4)} \qquad u_{tt} - \Delta_x u + V^\varepsilon(t,x)\,u \;=\; 0\,, \qquad V^\varepsilon(t,x) \;:=\; b^\varepsilon(x) + q(t)\chi^\delta(x)$$

with barrier potential

$$0 \le b^\varepsilon(x) \in C_0^\infty(\{L < |x| < L+1\}), \qquad b^\varepsilon(x) = 1/\varepsilon \quad \text{on} \quad \big\{L+\varepsilon < |x| < L+1-\varepsilon\big\}\,.$$

For Cauchy data

$$(u(0,x), u_t(0,x)) = w = (w_1(x), w_2(x))$$

which are supported in B_L, and for $\varepsilon > 0$ sufficiently small, the solution at time T is mostly confined to $\{|x| \le L\}$ and is well approximated by the solution given as the extension by zero of $K^\delta(T)w$. We prove that for $0<\varepsilon\ll 1$ and $t=T$ the evolution operator $U^\varepsilon(T,0)$ for (2.4) has an eigenvalue inside D_r.

3 Weak convergence

The first step is to study the weak limits as $\varepsilon\to 0$ of the differential equations on $\mathbb{R}^{1+3}$ with potentials V^ε. Introduce the energy

$$E^\varepsilon(u,t) \;:=\; \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\nabla_{t,x}u(t,x)|^2+b^\varepsilon(x)|u(t,x)|^2\Big)\,dx\,.$$

When there is little risk of confusion the dependence on u will be suppressed. For $t>0$, the energy identity implies

$$E^\varepsilon(t)\;-\;E^\varepsilon(0)\;=\;-2\,\mathrm{Re}\int_0^t\int_{\mathbb{R}^3} q\,\chi^\delta\,u\bar{u}_t\;dt\,dx\,.$$

For $(w_1,w_2)\in D\times L^2(\mathbb{R}^3)=H$ we use two norms

$$\|(w_1,w_2)\|_\varepsilon^2=\int_{\mathbb{R}^3}\Big(|w_2|^2\;+\;|\nabla_x w_1|^2\;+\;b^\varepsilon|w_1|^2\Big)\,dx\,,$$

and

$$\|(w_1,w_2)\|_H^2\;=\;\int_{\mathbb{R}^3}\Big(|w_2|^2\;+\;|\nabla_x w_1|^2\Big)\,dx.$$

Denote by D_0 the elements of D which vanish for $L\le|x|\le L+1$. On $D_0\times L^2(\mathbb{R}^3)$ the norms $\|\cdot\|_H$, and $\|\cdot\|_\varepsilon$ are equal. A function $u\in C(\mathbb{R}\,;D_0)$ vanishes on $|x|=L$ and $|x|=L+1$. It is in this way that Dirichlet conditions at these boundaries are expressed in the following Proposition.

Proposition 3.1. *Consider a sequence $\varepsilon_n\to 0$ and weakly convergent Cauchy data in H,*

$$w^n\;=\;(w_1^n,w_2^n)\rightharpoonup w\;\in H\,.$$

Suppose in addition that

$$\exists C>0,\;\;\forall n,\quad \|w^n\|_{\varepsilon_n}\;\le\;C\,. \tag{3.1}$$

Denote by u^n be the solution of the equation (2.4) with initial data w^n. Then for any $\underline{T}>0$ there exists u so that,

$$u^n\;\rightharpoonup\;u\quad\textit{weak star in}\quad L^\infty([0,\underline{T}]\,;D)\,,$$

and

$$\partial_t u^n\;\rightharpoonup\;\partial_t u\quad\textit{weak star in}\quad L^\infty([0,\underline{T}]\,;L^2(\mathbb{R}^3))\,.$$

Moreover, $u(t)\in\;C(\mathbb{R}\,;D_0)\;\cap\;C^1(\mathbb{R}\,;L^2(\mathbb{R}^3))$ and $u(t)$ for $|x|\le L$ is the unique solution of the Dirichlet problem in $\mathbb{R}\times B_L$

$$u_{tt}-\Delta_x u+q(t)\chi^\delta(x)u\;=\;0\,,\qquad (u(0),u_t(0))=w\big|_{B_L},\quad u\big|_{|x|=L}=0, \tag{3.2}$$

while $u(t)$ for $|x|\geq L+1$ is the unique solution of the Dirichlet problem in $\mathbb{R}\times\{x: |x|\geq L+1\}$

$$u_{tt}-\Delta_x u \;=\; 0\,, \qquad (u(0),u_t(0))=w\big|_{\{|x|\geq L+1\}}, \quad u|_{|x|=L+1}=0. \tag{3.3}$$

Remark 3.2. The values of the initial time derivative $\partial_t w^n\big|_{L\leq|x|\leq L+1}$ on $\{L\leq|x|\leq L+1\}$ do not influence the limit.

Proof. Fix $\underline{T}>0$. The energy estimate shows that u^n is bounded in $L^\infty([0,\underline{T}]\,;D)$ and $\partial_t u^n$ is bounded in $L^\infty([0,\underline{T}]\,;L^2(\mathbb{R}^3))$. It suffices to show that any weak star limit point vanishes for $L\leq|x|\leq L+1$ and satisfies (3.2) and (3.3).

Passing to a subsequence, we can suppose that u^n converges weak star to v, $\partial_t u^n$ converges weak star to $\partial_t v$ in $L^\infty([0,\underline{T}]\,;D)$ and $L^\infty([0,\underline{T}]\,;L^2(\mathbb{R}^3))$, respectively. The energy estimate implies that $v=0$ for $L\leq|x|\leq L+1$. That is, v takes values in D_0. Therefore $v_t=0$ for $L\leq|x|\leq L+1$.

To identify the limit in $\{|x|\geq L+1\}$ where the lower order terms vanish, suppose that $\varphi\in C_0^\infty(\mathbb{R}_t\times\mathbb{R}^3_x)$ vanishes when $|x|=L+1$ and $t\geq\underline{T}$. An integration by parts shows that

$$\int_0^{\underline{T}}\int_{|x|\geq L+1} u^n\left(\partial_t^2-\Delta_x\right)\varphi\,dt\,dx \;=\; \int_0^{\underline{T}}\int_{|x|=L+1} u^n(t,x)\,\varphi_r(t,x)dtdS_x+\int_{|x|\geq L+1}\Big(w_1^n\,\varphi(0,x)\;-\;w_2^n\,\partial_t\varphi(0,x)\Big)\,dx,$$

where φ_r denotes the radial derivative of φ.

Since u^n is weakly convergent in $L^2([0,\underline{T}]\,;H^1_{\rm loc}\{|x|\leq L+1\})$, its trace on $|x|=L+1$ converges weakly in $L^2([0,\underline{T}]\,;L^2(\{|x|=L+1\}))$ to the trace of v. On the other hand, since the limit v vanishes for $L\leq|x|\leq L+1$ the trace of v vanishes, so passing to the limit yields

$$\int_0^{\underline{T}}\int_{|x|\geq L+1} v\left(\partial_t^2-\Delta_x\right)\varphi\;dt\,dx \;=\; \int_{|x|\geq L+1}\Big(w_1\,\varphi(0,x)\;-\;w_2\,\partial_t\varphi(0,x)\Big)\,dx\,.$$

This is the weak form of the equation (3.2) with Dirichlet boundary condition on $|x|=L+1$. Uniqueness for that problem shows that v is equal to the advertised u on that set.

To identify the limit v in $|x|\leq L$, suppose that $\psi\in C^\infty(\mathbb{R}_t\times B_L)$ vanishes when $|x|=L$ and for $t\geq T$. Integration by parts shows that

$$\int_0^{\underline{T}}\int_{B_L} u^n\left(\partial_t^2-\Delta_x+q(t)\chi^\delta(x)\right)\psi\,dt\,dx=\int_{B_L}\Big(w_1^n(x)\,\psi(0,x)-w_2^n(x)\,\psi_t(0,x)\Big)dx + \int_0^{\underline{T}}\int_{|x|=L} u^n(t,x)\,\psi_r(t,x)\,dt\,dS_x+\int_0^{\underline{T}}\int_{B_L} u^n q(t)\chi^\delta(x)\,\psi\,dt\,dx.$$

Passing to the limit in the first line is easy. On the other hand, we have already shown that the traces of u^n on $\{x : |x| = L\}$ tend to zero weakly in $L^2([0,\underline{T}]; L^2(\{|x| = L\}))$. Thus passing to the limit, we find

$$\int_0^{\underline{T}} \int_{B_L} v \left(\partial_t^2 - \Delta_x + q(t)\chi^\delta(x)\right) \psi \, dt\, dx \; =$$
$$\int_{B_L} \left(w_1(x)\psi_x(0,x) - w_2(x)\, \psi_t(0,x)\right) dx \; + \; \int_0^{\underline{T}} \int_{B_L} v q(t)\chi^\delta(x)\psi \, dt\, dx.$$

This is the weak form of the initial boundary value problem (3.3) with Dirichlet boundary conditions on $\{x : |x| = L\}$. This uniquely determines the restriction of v to B_L as the solution of (3.3). □

Proposition 3.1 proves weak convergence in regions of space time. The next example shows that the sequence of solutions $u^n(t)$ need not converge weakly for t fixed.

Example. *If $w_1^n = 0$ and $w_2^n = 1$ on a neighborhood of the sphere, $\{|x| = L + 1/2\}$ in the center of the barrier, then finite speed of propagation implies that on a space time neighborhood of $\{t = 0\} \times \{|x| = L + 1/2\}$, $u^n(t,x) = y^n(t)$ where y^n is the solution of the ordinary differential equation,*

$$\frac{d^2 y^n}{dt^2} + \frac{y^n}{\varepsilon_n} = 0, \qquad y^n(0) = 0, \qquad \frac{dy^n}{dt}(0) = 1\,.$$

Then, $y^n(t) = \sqrt{\varepsilon_n}\sin(t/\sqrt{\varepsilon_n})$, $dy^n/dt = \cos(t/\sqrt{\varepsilon_n})$. Thus for typical $\varepsilon_n \to 0$, $dy^n(t)/dt$ does not converge weakly for t fixed.

The next corollary shows that one does have weak convergence in the complement of the barrier.

Corollary 3.3. *Suppose that ε_n, w^n, u and u^n are as in Proposition 3.1. Then, for every fixed $t > 0$, the sequence $\left(u^n(t)\,,\, \partial_t u^n(t)\right)$ converges weakly to $\left(u, \partial_t u\right)$ outside the barrier, that is for every $\Phi = (\Phi_1, \Phi_2) \in H$ with $\operatorname{supp}\Phi_i \cap \{L < |x| < L+1\} = \emptyset$, $i = 1,2$,*

$$\lim_{n\to\infty} \left(\left(u^n(t)\,,\, \partial_t u^n(t)\right)\,,\, \Phi\right)_H \quad \longrightarrow \quad \left(\left(u(t)\,,\, \partial_t u(t)\right), \Phi\right)_H.$$

Proof. Since the u^n, u_t^n are bounded in H it suffices to prove the assertion for $\Phi_i \in C_0^\infty\left(\mathbb{R}^3 \setminus \{L \le |x| \le L+1\}\right)$ since such data are dense in the desired Φ.

Estimate (3.1) implies that w_1 vanishes for $L \le |x| \le L+1$.

An integration by parts in x yields,

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^3} \left(\partial_t^2 u^n - \Delta_x u^n + V^{\varepsilon_n} u^n\right) \Phi(x)\; dx \\ &= \int_0^T (\partial_t^2 u^n, \Phi)_{L^2}\; dt + \int_0^T (\nabla_x u^n, \nabla_x \Phi)_{L^2}\; dt + \int_0^T (q(t)\chi^\delta u^n, \Phi)_{L^2}\; dt. \end{aligned}$$

The boundary terms on $|x|=L$ and $|x|=L+1$ vanish since $\operatorname{supp}\Phi\cap\{x: L<|x|<L+1\}=\emptyset$.

Proposition 3.1 implies weak convergence of u^n in $L^2([0,T]\,;D)$ and weak convergence of u^n_t in $L^2([0,T]\,;L^2(\mathbb{R}^3))$ to u and u_t, respectively. The fundamental theorem of calculus implies

$$\int_0^T \partial_t(u^n_t,\Phi)_{L^2}\,dt = (u^n_t(T),\Phi)_{L^2} - (w^n_2,\Phi)_{L^2}.$$

Passing to the limit $n\to\infty$, yields

$$\begin{aligned}\lim_{n\to\infty}(u^n_t(T),\Phi)_{L^2} &= (w_2,\Phi)_{L^2}\\ &\quad -\int_0^T(\nabla_x v,\nabla_x\Phi)_{L^2}\,dt-\int_0^T\int_{|x|\le L} q(t)\chi^\delta v\Phi\,dxdt.\end{aligned}\tag{3.4}$$

Since Φ is supported outside of the barrier, a similar integration by parts yields

$$(u_t(T),\Phi)_{L^2} = (w_2,\Phi)_{L^2} - \int_0^T(\nabla_x u,\nabla_x\Phi)_{L^2}dt - \int_0^T\int_{|x|\le L} q(t)\chi^\delta u\Phi\,dxdt.\tag{3.5}$$

Combining (3.4) and (3.5) shows that that $u^n_t(T)$ converges weakly to $u_t(T)$ in $L^2\big(\mathbb{R}^3\setminus\{L\le|x|\le L+1\}\big)$.

Next

$$\begin{aligned}(\nabla_x u^n(T),\nabla_x\Phi)_{L^2} - (\nabla_x w^n_1,\nabla\Phi)_{L^2} &= -(u^n(T),\Delta_x\Phi)_{L^2} + (w^n_1,\Delta_x\Phi)_{L^2}\\ &= -\int_0^T\partial_t(u^n,\Delta_x\Phi)_{L^2}dt = -\int_0^T(u^n_t,\Delta_x\Phi)_{L^2}dt.\end{aligned}\tag{3.6}$$

The same computation with u^n replaced by u yields

$$(\nabla_x u(T),\nabla_x\Phi)_{L^2} \;=\; (\nabla_x w_1,\nabla\Phi)_{L^2} - \int_0^T(u_t,\Delta_x\Phi)_{L^2}dt.\tag{3.7}$$

In (3.6) use the weak convergence of u^n_t in $L^2\big([0,T]\,;L^2(\mathbb{R}^3)\big)$ and that of w^n_1 in D to find

$$\lim_{n\to\infty}\big(\nabla_x u^n(T),\nabla_x\Phi\big)_{L^2} = \big(\nabla_x w_1,\nabla_x\Phi\big)_{L^2} - \int_0^T\big(u_t,\Delta_x\Phi\big)_{L^2}dt\,.$$

Comparing with (3.7) completes the proof of weak convergence. $\square$

4 Weak resolvent convergence

Denote by $S^\varepsilon(t)=U^\varepsilon(t,0)$ the map from Cauchy data at time zero to Cauchy data at time t for the solutions of the wave equation with periodic potential V^ε. Next denote by $S^\varepsilon_0(t)$ the map from Cauchy data at time zero to Cauchy data at time t for the wave equation with time independent potential $b^\varepsilon(x)$. Finally, given $w=(w_1,w_2)\in H$ such that both components vanish on $\{x: L\le|x|\le L+1\}$, introduce the operator

$$D(t)w \;:=\; \big(u(t)\,,u_t(t)\big)\,,$$

where u is as in Proposition 3.1. More precisely, $D(t)w = 0$ for $L \leq |x| \leq L+1$ and if $D(t)w = (u(t), \partial_t u(t))$, then u is the solution of the Dirichlet problems in $|x| \leq L$ and $|x| \geq L+1$, respectively, for $\partial_t^2 - \Delta_x + q(t)\chi^\delta(x)$ and $\partial_t^2 - \Delta_x$. Notice that for w with the properties above if $z \notin \operatorname{spec} D(t)$, we have $(D(t) - zI)^{-1} w = (g_1, g_2)$, where g_1 and g_2 vanish for $L \leq |x| \leq L+1$.

The operator $S_0^\varepsilon(t)$ is **not** defined in terms of $U_0(t)$. Since $b^\varepsilon(x)$ is independent on t, the operator $S_0^\varepsilon(t)$ is unitary in the ε-dependent norm $\|\cdot\|_\varepsilon$. More precisely, for a bounded operator $A \in \mathscr{L}(H)$ we use the operator norm

$$\|A\|_\varepsilon = \sup_{f \neq 0} \frac{\|Af\|_\varepsilon}{\|f\|_\varepsilon},$$

$\|.\|_\varepsilon$ being the norm in H related to $b^\varepsilon(x)$ defined in Section 3.

Therefore, for $|z| > 1$ we have

$$\left\|(S_0^\varepsilon(t) - zI)^{-1}\right\|_\varepsilon \leq \frac{1}{|z| - 1}.$$

The time dependent lower order term, $q(t)\chi^\delta(x)$, is a bounded perturbation. Therefore,

$$\forall \underline{T}, \ \exists C, \ \forall t \in [0, \underline{T}], \ \varepsilon \in]0,1], \ \varphi \in H, \qquad \|S^\varepsilon(t)\varphi\|_\varepsilon \leq C\|\varphi\|_\varepsilon,$$

with $C > 0$ depending on $\underline{T} > 0$ but not on ε.

Lemma 4.1. *For each $\varepsilon > 0$, $\underline{T} > 0$, the operator $S^\varepsilon(\underline{T}) - S_0^\varepsilon(\underline{T}) \in \mathscr{L}(H)$ is compact. In addition, if $w^n \rightharpoonup 0$ weakly in H and satisfies (3.1), then for $n \to \infty$,*

$$\sup_{0 \leq t \leq \underline{T}} \ \sup_{0 < \varepsilon \leq 1} \ \|(S^\varepsilon(t) - S_0^\varepsilon(t))w^n\|_\varepsilon \quad \to \quad 0.$$

Proof. Fix $0 < \varepsilon \leq 1$ and consider the solutions u_n and v_n of the Cauchy problems

$$(\partial_t^2 - \Delta_x)u_n + V^\varepsilon u_n = 0, \qquad (u_n, \partial_t u_n)(0,x) = w^n,$$
$$(\partial_t^2 - \Delta_x)v_n + b^\varepsilon v_n = 0, \qquad (v_n, \partial_t v_n)(0,x) = w^n.$$

Then $(u_n - v_n)$ satisfies

$$\left(\partial_t^2 - \Delta + b^\varepsilon\right)(u_n - v_n) = -q(t)\chi^\delta(x)u_n, \qquad (u_n - v_n)(0) = \partial_t(u_n - v_n)(0) = 0.$$

The energy identity implies that for $0 \leq \tau \leq \underline{T}$,

$$\|u_n(\tau,x) - v_n(\tau,x)\|_\varepsilon^2 = -2\operatorname{Re}\int_0^\tau \int_{\mathbb{R}^3} u_n q \chi^\delta \, \overline{\partial_t(u_n - v_n)}\, dt dx. \tag{4.1}$$

Proposition 3.1 implies that

$$u_n \to 0 \quad \text{weak star in} \quad L^\infty([0,\underline{T}]\,; D),$$

and

$$\partial_t(u_n - v_n) \to 0 \quad \text{weak star in} \quad L^\infty([0,\underline{T}]\,; L^2(\mathbb{R}^3)).$$

Since $q\chi^\delta$ has compact spatial support it follows that

$$q\chi^\delta\, u_n \;\to\; 0 \quad \text{strongly in} \quad L^2([0,\underline{T}]\,;L^2(\mathbb{R}^3)).$$

On the other hand,

$$\partial_t(u_n - v_n) \;\to\; 0 \quad \text{weakly in} \quad L^2([0,\underline{T}]\,;L^2(\mathbb{R}^3))\,,$$

and we deduce

$$\int_{\mathbb{R}^3} u_n\, q\chi^\delta\, \overline{\partial_t(u_n - v_n)}\, dx \;\to\; 0 \quad \text{in} \quad L^1([0,\underline{T}]\,;\mathbb{C})\,.$$

This together with (4.1) completes the proof. □

Proposition 4.2. *Let $K \subset \{z \in \mathbb{C} :\ |z| > 1\}$ be a compact set disjoint from*

$$\operatorname{spec} D(T) \cup \Big(\cup_{0<\varepsilon\le\varepsilon_0} \operatorname{spec} S^\varepsilon(T) \Big).$$

Then there exist $\varepsilon_0 > 0$ and $C_0 > 0$ so that

$$\forall\, \varepsilon \in]0,\varepsilon_0],\ z \in K, \qquad \|(S^\varepsilon(T) - zI)^{-1}\|_\varepsilon \;\le\; C_0\,. \tag{4.2}$$

Proof. In $\{z \in \mathbb{C} :\ |z| \gg 1\}$, the operators $S^\varepsilon(T) - zI$ are compact perturbations of the invertible operators $S_0^\varepsilon(T) - zI$ both depending analytically on z. The analytic Fredholm theory implies that for each ε, $(S^\varepsilon(T) - zI)^{-1}$ has a meromorphic continuation in $\{z \in \mathbb{C} : |z| > 1\}$.

If the proposition were false, there would exist $\varepsilon_n \to 0$, $z_n \in K$, and w_n with

$$\|w_n\|_{\varepsilon_n} = 1\,, \qquad \|(S^{\varepsilon_n}(T) - z_n I) w_n\|_{\varepsilon_n} \to\ 0\,. \tag{4.3}$$

Both parts of (4.3) give strong control in the barrier. Passing to a subsequence, we may suppose that w_n converges weakly to a limit w in H and $z_n \to z \in K$. As in Proposition 3.1, we deduce that $w = 0$ for $L \le |x| \le L+1$ and the definition of $D(T)$ given in the beginning of this section implies $(D(T) - zI)w = 0$ for $L \le |x| \le L+1$.

We claim that $(D(T) - zI)w = 0$ for every x. Corollary 3.3 implies that $S^{\varepsilon_n}(T)w_n$ converges weakly to $D(T)w$ outside of the barrier and the second expression in (4.3) yields

$$\|S^{\varepsilon_n}(T)w_n - z_n w_n\|_{\varepsilon_n} \to 0.$$

Passing to the limit $n \to \infty$, shows that $D(T)w - zw = 0$ for $x \in \mathbb{R}^3 \setminus \{L \le |x| \le L+1\}$, and the claim is established.

Consequently, $D(T)w = zw$ for all $x \in \mathbb{R}^3$. Since z is not in the spectrum of $D(T)$, it follows that $w = 0$. Next

$$\begin{aligned}
\|(S^{\varepsilon_n}(T) - zI)w_n\|_{\varepsilon_n} &= \|(S_0^{\varepsilon_n}(T) - zI)w_n + (S^{\varepsilon_n}(T) - S_0^{\varepsilon_n}(T))w_n\|_{\varepsilon_n} \\
&\ge \|(S_0^{\varepsilon_n}(T) - zI)w_n\|_{\varepsilon_n} + \|(S^{\varepsilon_n}(T) - S_0^{\varepsilon_n}(T))w_n\|_{\varepsilon_n}\,.
\end{aligned}$$

Lemma 4.1 implies that the second summand tends to zero so

$$\|(S^{\varepsilon_n}(T) - zI)w_n\|_{\varepsilon_n} \ge\ (|z|-1)\|w_n\|_{\varepsilon_n} + o(1) \;=\; (|z|-1) + o(1)\,.$$

This contradicts (4.3) and therefore proves the proposition. □

Remark 4.3. **1.** It follows that for all φ and $0 < \varepsilon \le \varepsilon_0$,

$$\|(S^\varepsilon(T) - zI)^{-1}\varphi\|_H \;\le\; \|(S^\varepsilon(T) - zI)^{-1}\varphi\|_\varepsilon \;\le\; C_0\,\|\varphi\|_\varepsilon \;\le\; \frac{C}{\sqrt{\varepsilon}}\,\|\varphi\|_H.$$

Therefore

$$\|(S^\varepsilon(T) - zI)^{-1}\|_{\mathscr{L}(H)} \le \frac{C}{\sqrt{\varepsilon}}\,. \tag{4.4}$$

On the other hand, for φ vanishing in $L \le |x| \le L+1$ we get

$$\|(S^\varepsilon(T) - zI)^{-1}\varphi\|_H \le C_0\|\varphi\|_H.$$

2. The compactness of the lower order term is related to the fact that it does not affect the leading order effects in the propagation of singularities. On the negative side, this eliminates one possible strategy for amplification and it prevents us from using microlocal techniques. It is a key ingredient in the proofs of this section.

Proposition 4.4. *Suppose that both components of $\varphi \in H$ vanish for $L \le |x| \le L+1$ and $z \in \{z \in \mathbb{C} : |z| > 1\}$ satisfies*

$$z \;\notin\; \operatorname{spec} D(T) \;\cup\; \Big(\cup_{0<\varepsilon\le\varepsilon_0} \operatorname{spec} S^\varepsilon(T)\Big)\,.$$

Then

$$\lim_{\varepsilon\to 0}\,(S^\varepsilon(T) - zI)^{-1}\varphi \;=\; (D(T) - zI)^{-1}\varphi \qquad \textit{weakly in}\, H\,.$$

Proof. Let

$$\psi^\varepsilon \;:=\; (S^\varepsilon(T) - zI)^{-1}\varphi\,. \tag{4.5}$$

Since $\|\varphi\|_\varepsilon$ is bounded independent of ε, the resolvent estimate (4.2) implies that uniformly in ε we have

$$\|\psi^\varepsilon\|_\varepsilon \;=\; \|(S^\varepsilon(T) - zI)^{-1}\varphi\|_\varepsilon \;\le\; C\,. \tag{4.6}$$

Since $\|\psi^\varepsilon\|_H \le \|\psi^\varepsilon\|_\varepsilon$, every subsequence has a subsequence which is weakly convergent. Denote by ψ the weak limit in H of such a weakly convergent subsequence. Clearly, ψ vanishes for $L \le |x| \le L+1$. It suffices to show that $\psi = (D(T) - zI)^{-1}\varphi$.

Applying $S^\varepsilon(T) - zI$ to (4.5), we find

$$(S^\varepsilon(T) - zI)\psi^\varepsilon \;=\; \varphi\,.$$

Since ψ vanishes for $L \le |x| \le L+1$, an application of Corollary 3.3 shows that $S^\varepsilon(T)\psi^\varepsilon$ converges weekly to $D(T)\psi$ in $D_0 \times L^2(\mathbb{R}^3 \setminus \{L \le |x| \le L+1\})$. Thus passing to the limit $\varepsilon \to 0$, we deduce

$$(D(T) - zI)\psi \;=\; \varphi\,.$$

Therefore $\psi = (D(T) - zI)^{-1}\varphi$ and the proof is complete. □

5 Main theorem

Our construction in Section 2 shows that for $r > 0$ small enough the operator $K^{\delta}(T)$ with fixed $\delta > 0$ has exactly one eigenvalue z_1 inside the disk $D_r = \{z \in \mathbb{C} : |z - z_1| \leq r\}$ with eigenfunction φ. The next result is more precise than Theorem 1.1 and Corollary 1.2.

Theorem 5.1. *Suppose that $\gamma = \{z \in \mathbb{C} : |z - z_1| = r,\ |z| > 1\}$ is a circle disjoint from the spectrum of $K^{\delta}(T)$ so that $K^{\delta}(T)$ has exactly one eigenvalue z_1 in D_r. Then there exists an ε_0 so that for $0 < \varepsilon \leq \varepsilon_0$ the operator $S^{\varepsilon}(T)$ has an eigenvalue in D_r. For the same values of ε the problem (1.1) with potential $V^{\varepsilon}(t,x)$ has a resonance in D_r.*

Proof. If $K^{\delta}(T)\varphi = z_1\varphi$, $\varphi \neq 0$, it is clear that we can extend φ as 0 outside B_L and $D(T)\varphi = z_1\varphi$. First we prove that $S^{\varepsilon}(T)$ has an eigenvalue. If $S^{\varepsilon}(T)$ has an eigenvalue on γ, the assertion holds, so we suppose that $S^{\varepsilon}(T)$ has no eigenvalues on γ.

For $|z| \gg 1$ we write

$$(D(T) - zI)^{-1}\varphi = f_1(z) + f_2(z),$$

where $f_1(z) \in H$ vanishes outside B_L, while $f_2(z) \in H$ vanishes for $|x| \leq L + 1$. Applying $D(T) - zI$, we deduce $(D(T) - zI)f_2(z) = 0$ and since the propagator of the Dirichlet problem in $\{x \in \mathbb{R}^3 : |x| \geq L + 1\}$ has no eigenvalues z, $|z| > 1$, we obtain $f_2(z) = 0$. Consequently,

$$(D(T) - zI)^{-1}\varphi = (K^{\delta}(T) - zI)^{-1}\varphi = f_1(z).$$

For $z \in \gamma$ by analytic continuation we deduce

$$(D(T) - zI)^{-1}\varphi = (K^{\delta}(T) - zI)^{-1}\varphi,$$

hence the operator $D(T)$ has no eigenvalues $z \in \gamma$. Next

$$\frac{1}{2\pi i}\oint_{\gamma}(D(T) - zI)^{-1}\varphi\, dz = \frac{1}{2\pi i}\oint_{\gamma}(K^{\delta}(T) - zI)^{-1}\varphi dz = \varphi\,.$$

Since both components of φ vanish on $\{L \leq |x| \leq L + 1\}$, we have weak resolvent convergence given by Corollary 3.3. Moreover,

$$\oint_{\gamma}\big(\varphi\,,\,(D(T) - zI)^{-1}\varphi\big)_H\, dz \neq 0\,.$$

The estimate (4.2) implies that there is a constant C so that

$$\sup_{0<\varepsilon\leq\varepsilon_0}\ \sup_{z\in\gamma}\ \Big|\big(\varphi\,,\,(S^{\varepsilon}(T) - zI)^{-1}\varphi\big)_H\Big| \leq C\,.$$

Together with the weak convergence from Proposition 4.4, the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0} \oint_\gamma \big(\varphi, (S^\varepsilon(T)-zI)^{-1}\varphi\big)_H\, dz = \oint_\gamma \big(\varphi, (D(T)-zI)^{-1}\varphi\big)_H\, dz \neq 0.$$

Therefore, there exists an $\varepsilon_1 > 0$ so that for $0 < \varepsilon \le \varepsilon_1$,

$$\oint_\gamma \big(\varphi, (S^\varepsilon(T)-zI)^{-1}\varphi\big)_H\, dz \neq 0. \tag{5.1}$$

Consequently, for these ε, the operator $(S^\varepsilon(T)-zI)^{-1}$ is not analytic in D_r. Hence $S^\varepsilon(T)$ has an eigenvalue inside D_r.

Theorem 5.4.1 of [12] shows that every eigenvalue z_ε of $U^\varepsilon(T,0)$ with $|z_\varepsilon| > 1$ is automatically a resonance for the problem (1.1) with $V^\varepsilon(t,x)$. Moreover, the algebraic multiplicities of the eigenvalue and the resonance are the same.

We give an independent proof of the weaker fact that there is a resonance in D_r based on the second definition of resonances as the poles of the meromorphic continuation of the cut-off resolvent given in Section 1.

Fix ε so that $0 < \varepsilon \le \varepsilon_1$ and (5.1) holds. Let $\Phi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function such that $\Phi(x) = 1$ for $x \in B_L$. Then

$$\oint_\gamma \big(\Phi\varphi, (S^\varepsilon(T)-zI)^{-1}\Phi\varphi\big)_H\, dz = \oint_\gamma \big(\varphi, \Phi(S^\varepsilon(T)-zI)^{-1}\Phi\varphi\big)_H dz \neq 0.$$

Therefore the cut-off resolvent $\Phi(S^\varepsilon(T)-zI)^{-1}\Phi$ is not analytic in D_r which proves the existence of a resonance in D_r. □

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