EIGENVALUES OF THE REFERENCE OPERATOR AND SEMICLASSICAL RESONANCES

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ABSTRACT. We prove that the estimate of the number of the eigenvalues in intervals $[\lambda-\delta, \lambda+\delta]$, $0 < \frac{h}{C} \leq \delta \leq C$, of the reference operator $L^{\#}(h)$ related to a self-adjoint operator $L(h)$ is equivalent to the estimate of the integral over $[\lambda-\delta, \lambda+\delta]$ of the sum of harmonic measures associated to the resonances of $L(h)$ lying in a complex neighborhood Ω of $\lambda > 0$ and the number of the positive eigenvalues of $L(h)$ in $[\lambda-\delta, \lambda+\delta]$. We apply this result to obtain a Breit-Wigner approximation of the derivative of the spectral shift function near critical energy levels.

1. INTRODUCTION

This paper is devoted to the analysis of the connection between the distribution of the semi-classical resonances $z_j(h)$ of a Schrödinger type operator $L = L(h)$, $0 < h \leq h_0$, and the behavior of the counting function

$$N(L^{\#}(h), [\alpha, \beta]) = \#\{\mu \in \mathbb{R} : \mu \in \mathrm{sp}_{pp} L^{\#}(h),\ \alpha \leq \mu \leq \beta\}$$

of the so called reference operator $L^{\#}(h)$ related to $L(h)$ (see Section 2). Under the general "black box" assumptions (2.1)-(2.7) we may define the semi-classical resonances $w \in \overline{\mathbb{C}}_-$ by complex scaling [26], [23]. Let $\mathrm{Res}\, L$ be the set of resonances of L. Then for every relatively compact open domain $\Omega \subset\subset \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ the estimate

$$N(L^{\#}(h), [-\lambda, \lambda]) = \mathcal{O}\Big(\Big(\frac{\lambda}{h^2}\Big)^{n^{\#}/2}\Big),\ n^{\#} \geq n,\ \lambda \geq 1$$

implies the bound

$$\#\{w \in \mathrm{Res}\, L \cap \Omega\} \leq C(\Omega) h^{-n^{\#}},\ 0 < h \leq h_0 \tag{1.1}$$

(see [26], [23] and, [27] for the classical case). Given an interval $[E_0, E_1]$, $0 < E_0 < E_1$, such that every $\lambda \in [E_0, E_1]$ is a non-critical energy level for the principal symbol of L, a more precise result holds and recently J. F. Bony [1] proved (see also [3] for similar results concerning critical energy levels) that the condition

$$N(L^{\#}(h), [\lambda-\delta, \lambda+\delta]) \leq C\delta h^{-n^{\#}} \tag{1.2}$$

for all $\lambda \in [E_0, E_1]$, $0 < \frac{h}{C_1} \leq \delta \leq C_1$, implies

$$\#\{w \in \mathbb{C} :\ w \in \mathrm{Res}\, L,\ |w-\lambda| \leq \delta\} \leq C\delta h^{-n^{\#}} \tag{1.3}$$

for $\lambda \in [E_0, E_1]$ and $0 < \frac{h}{B} \leq \delta \leq B$ (see also [19] for the case of compact perturbations). Moreover, under the assumption (1.2) we can obtain a Weyl type asymptotics and a Breit-Wigner approximation of the spectral shift function $\xi(\lambda, h)$ (see [8] for more details). Finally, there exists a close relation between the behavior of $\xi(\lambda, h)$ and that of $N(\lambda) = N(L^{\#},]-\infty, \lambda])$. This relation has been studied by S. Nakamura [16] in the case of short range perturbations of the Schrödinger operator $L = -h^2\Delta + V(x)$ and by the authors [7], [8] in the setup of "black box" long range scattering.

It is natural to expect that some information on the distribution of the resonances in a complex neighborhood Ω of $[E_0, E_1]$ will imply (1.2) and via the results in [8] the asymptotics of $\xi(\lambda, h)$. To our best knowledge it seems that there are no such results in the literature. The purpose of this paper is to show that (1.2) is equivalent to a similar condition involving the sum of the *harmonic measures*

$$\omega_{\mathbb{C}_-}(w, J) = \int_J \frac{|\operatorname{Im} w|}{\pi |t-w|^2} dt,\ \ J \subset \mathbb{R} = \partial\mathbb{C}_-$$

related to the resonances $w, \operatorname{Im} w < 0$, lying in a *complex* neighborhood of $[E_0, E_1]$. We refer to [15], [13], [18], [19], [4] for the results concerning the Breit-Wigner approximations and the harmonic measures $\omega_{\mathbb{C}_-}(w, .)$. More precisely, the condition (1.2) is equivalent to the same condition for the function

$$M_\Omega(\lambda) = \sum_{\substack{w \in \operatorname{Res} L,\ w \in \Omega, \\ \operatorname{Im} w \neq 0}} \omega_{\mathbb{C}_-}(w,]-\infty, \lambda]) + \#\{\mu \in]-\infty, \lambda] \cap \Omega :\ \mu \in \operatorname{sp}_{pp} L\}$$

(Theorem 1) which may be considered as an analogue of the counting function of eigenvalues. Notice that the positive eigenvalues $\mu \in \operatorname{sp}_{pp} L$ coincide with the resonances $w \in \mathbb{R}^+$ and the function $M_\Omega(\lambda)$ is completely determined by the resonances in Ω. In particular, from Theorem 1 we obtain a new proof of the implication (1.1) $\Rightarrow$ (1.2) established by J. F. Bony [1] (Corollary 2).

We can define the spectral shift function $\xi(\lambda, h)$ for $L(h)$ and $\tilde{L}(h)$, where $\tilde{L}(h)$ is an intermediate operator defined in Proposition 1. On the other hand, for short range perturbations the spectral shift function $\xi(\lambda, h) = \xi(L, \tilde{L})$ can be defined for the pair of operators $L, L_0 = -h^2\Delta$. The importance of Theorem 1 is that we have the equivalence of three conditions $i) - iii)$ and exploiting $i)$ and $iii)$ we obtain after minor modifications of the arguments of Section 6 in [8] a Breit-Wigner approximation of the derivative of the spectral shift function $\xi(\lambda, h)$ (Theorem 2).

In the case when we have no "black box" and $L(h)$ is a h-pseudodifferential self-adjoint operator in $L^2(\mathbb{R}^n)$ with principal symbol $l_0(x, \xi)$ we should stress that the assumptions (2.1)-(2.9) do not concern the eventual *critical points* of $l_0(x, \xi)$ lying in $\{(x, \xi) \in \mathbb{R}^n :\ |x| \geq R_0 > 0\}$. Thus we can cover the case of critical energy levels choosing an appropriate *weight factor* $r(h)$ with $\inf_{h \in]0, h_0]} r(h) > 0$. For non-degenerate critical points the results of J. F. Bony [3] imply the assumption $i)$ of Theorem 1 with suitable $r(h)$ and combining this with Theorem 2 we obtain some applications for non-degenerate critical points (see Section 4). There are only few results concerning a Breit-Wigner approximation of $\frac{\partial \xi}{\partial \lambda}(\lambda, h)$ near critical energy levels (see [13], [11], [12]). In this direction Theorem 2 and Corollary 3 present some general results. In Section 5 we compare our results in the one dimensional case with those obtained recently by Fujiié and Ramond [11], [12]. In the paper we denote by C positive constants, independent on h, which may change from line to line.

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2. Assumptions and results

We start by the abstract "black box" scattering assumptions introduced in [26], [23] and [25]. The operator $L(h) = L,\ 0 < h \le h_0$, is defined in a domain $\mathcal{D} \subset \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)),\ B(0,R_0) = \{x \in \mathbb{R}^n : |x| \le R_0\},\ R_0 > 0,\ n \ge 1.$$

Below $h > 0$ is a small parameter. We suppose that $\mathcal{D}$ satisfies

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{2.1}$$

uniformly with respect to h in the sense of [23]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0,R_0))$ with the norm $\| < hD >^2 u\|_{L^2}$, $< hD >^2 = 1 + (hD)^2$, and equip $\mathcal{D}$ with the norm $\|(L+i)u\|_{\mathcal{H}}$. Then we require that $1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{D} \longrightarrow H^2(\mathbb{R}^n \setminus B(0,R_0))$ is uniformly bounded with respect to h and this map has a uniformly bounded right inverse.

Assume that

$$1\!\!1_{B(0,R_0)}(L+i)^{-1} \text{is compact} \tag{2.2}$$

and

$$(Lu)|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}} = Q\Big(u|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}}\Big), \tag{2.3}$$

where Q is a formally self-adjoint differential operator

$$Qu = \sum_{|\nu| \le 2} a_\nu(x;h)(hD_x)^\nu u \tag{2.4}$$

with $a_\nu(x;h) = a_\nu(x)$ independent of h for $|\nu| = 2$ and $a_\nu \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to h.

We assume also the following properties:

There exists $C > 0$ such that

$$l_0(x,\xi) = \sum_{|\nu|=2} a_\nu(x)\xi^\nu \ge C|\xi|^2,\ \forall \xi \in \mathbb{R}^n, \tag{2.5}$$

$$\sum_{|\nu| \le 2} a_\nu(x;h)\xi^\nu \longrightarrow |\xi|^2,\ \ |x| \longrightarrow \infty \tag{2.6}$$

uniformly with respect to h.

There exist $\theta_0 \in]0, \frac{\pi}{2}[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_\nu(x;h)$ of Q can be extended holomorphically in x to

$$\Gamma = \{r\omega :\ \omega \in \mathbb{C}^n,\ \mathrm{dist}\,(\omega, S^{n-1}) < \epsilon,\ r \in \mathbb{C}, r \in e^{i[0,\theta_0]}]R_1, +\infty[\} \tag{2.7}$$

and (2.5), (2.6) extend to Γ.

Let $R > R_0$, $T_{\tilde{R}} = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$, $\tilde{R} > 2R$. Set

$$\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(T_{\tilde{R}} \setminus B(0,R_0))$$

and consider a differential operator

$$Q^\# = \sum_{|\nu| \le 2} a_\nu^\#(x;h)(hD)^\nu$$

on $T_{\tilde{R}}$ with $a_\nu^\#(x;h) = a_\nu(x;h)$ for $|x| \leq R$ satisfying (2.3), (2.4), (2.5) with $\mathbb{R}^n$ replaced by $T_{\tilde{R}}$. Consider a self-adjoint operator $L^\# : \mathcal{H}^\# \longrightarrow \mathcal{H}^\#$ defined by

$$L^\# u = L\varphi u + Q^\#(1-\varphi)u, \; u \in \mathcal{D}^\#,$$

with domain

$$\mathcal{D}^\# = \{u \in \mathcal{H}^\# : \; \varphi u \in \mathcal{D}, \; (1-\varphi)u \in H^2\},$$

where $\varphi \in C_0^\infty(B(0,R);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$. Denote by $N(L^\#, [-\lambda,\lambda])$ the number of eigenvalues of $L^\#$ in the interval $[-\lambda,\lambda]$. Then we assume that

$$N(L^\#, [-\lambda,\lambda]) = \mathcal{O}\Big(\Big(\frac{\lambda}{h^2}\Big)^{n^\#/2}\Big), \; n^\# \geq n, \; \lambda \geq 1. \tag{2.8}$$

Finally, we suppose that with some constant $C \geq 0$ independent on h we have

$$\text{sp } L(h) \subset [-C, \infty[, \tag{2.9}$$

where sp (L) denotes the spectrum of L.

Following [23], [25], we define the resonances $w \in \overline{\mathbb{C}}_-$ by the complex scaling method as the eigenvalues of the complex scaling operator L_θ. Denote by $\operatorname{Res} L(h)$, the set of resonances. We will say that $\lambda \in \mathbb{R}$ is a *non-critical energy level* for Q if for all $(x,\xi) \in \Sigma_\lambda = \{(x,\xi) \in \mathbb{R}^{2n} : l(x,\xi) = \lambda\}$ we have $\nabla_{x,\xi} l(x,\xi) \neq 0$, $l(x,\xi)$ being the principal symbol of Q. Since $L(h)$ tends to $-h^2\Delta$, for $\lambda > 0$ fixed, the set of the critical points of the Hamiltonian $l(x,\xi)$ in Σ_λ is compact. Then taking R_0 sufficiently large, we can suppose that λ is non critical for Q and we can construct $Q^\#$ so that λ is non critical for $Q^\#$, too.

We fix $E_1 > E_0 > 0$ and introduce an intermediate operator $\tilde{L}(h)$ having no resonances in a complex neighborhood of $[E_0, E_1]$ and each $\lambda \in [E_0, E_1]$ is a non critical energy level for $\tilde{L}$ (see Proposition 1). Moreover, the estimate (3.1) makes possible to introduce the spectral shift function $\xi(\lambda, h)$ for the pair $(L(h), \tilde{L}(h))$ (see Section 3) and, as in [8], we define

$$\xi(\lambda, h) = \lim_{\epsilon \to 0, \, \epsilon > 0} \xi(\lambda + \epsilon, h).$$

Our main result is the following.

Theorem 1. *Assume that L satisfies the assumptions* (2.1) $-$ (2.9) *and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for Q and $Q^\#$. Then for any real valued function $r(h)$, $h \in]0, h_0]$ such that $\inf_{h \in]0,h_0]} r(h) > 0$, the following assertions are equivalent:*

i) There exist positive constants $B_1, C_1, \; \epsilon_1, \; h_1$ such that for any $\lambda \in [E_0 - \epsilon_1, E_1 + \epsilon_1]$, $h \in]0, h_1]$ and $h/B_1 \leq \delta \leq B_1$ we have

$$\#\{\mu \in \mathbb{R} : \mu \in sp(L^\#(h)) \cap [\lambda - \delta, \lambda + \delta]\} \leq C_1 \delta r(h) h^{-n^\#}.$$

ii) For every complex relatively compact neighborhood $\Omega \subset \{z \in \mathbb{C} : \; \operatorname{Re} z > 0\}$ of $[E_0, E_1]$, independent on h, there exist positive constants $B_2, C_2, \; \epsilon_2, \; h_2$, depending on Ω, such that for any $\lambda \in [E_0 - \epsilon_2, E_1 + \epsilon_2]$, $h \in]0, h_2]$ and $h/B_2 \leq \delta \leq B_2$ we have

$$\sum_{\substack{w \in \text{Res } L(h) \cap \Omega, \\ \text{Im } w \neq 0}} \omega_{\mathbb{C}_-}(w, [\lambda - \delta, \lambda + \delta]) + \#\{\mu \in \mathbb{R} : \mu \in sp_{pp}(L(h)) \cap [\lambda - \delta, \lambda + \delta]\} \leq C_2 \delta r(h) h^{-n^\#}.$$

iii) There exist positive constants $B_3, C_3, \epsilon_3, h_3$ *such that for any* $\lambda \in [E_0 - \epsilon_3, E_1 + \epsilon_3]$, $h \in]0, h_3]$ *and* $\frac{h}{B_3} \leq \delta \leq B_3$ *we have*

$$|\xi(\lambda + \delta, h) - \xi(\lambda - \delta, h)| \leq C_3 \delta r(h) h^{-n^{\#}}.$$

Remarks. 1. In the assertion *ii*) it is sufficient to establish the bound for one complex neighborhood Ω of $[E_0, E_1]$ with constants depending on Ω. Then for every other complex neighborhood $\Omega_1 \supset \Omega$ the sum of the harmonic measures related to the resonances lying in $\Omega_1 \setminus \Omega$ is easily estimated by $\mathcal{O}(h^{-n^{\#}})$ by using the bound of the function counting the resonances. On the other hand, it is clear that if every $\lambda \in [E_0, E_1]$ is a non-critical energy level for Q the same is true for a small neighborhood of $[E_0, E_1]$.

2. The assumption *iii*) does not depend on the choice of the operator $\tilde{L}$. This follows from the equivalence of *ii*) and *iii*), as well as from the observation that if we have two operators $\tilde{L_i}$, $i = 1, 2$, with the properties of Proposition 1, then $\xi(L, \tilde{L_1}) - \xi(L, \tilde{L_2}) = \xi(\tilde{L_2}, \tilde{L_1})$ and for $\xi(\tilde{L_2}, \tilde{L_1})$ we obtain easily *iii*) since the operators $\tilde{L_i}$, $i = 1, 2$, have non-trapping energy levels in $[E_0, E_1]$. In the case of short range perturbations we can take $\tilde{L}(h) = L_0 = -h^2\Delta$ and the estimate (3.1) (see Section 3) holds for the coefficients of L and L_0. Thus we can define the spectral shift function related to L and L_0.

3. If L is h-pseudodifferential operator in $L^2(\mathbb{R}^n)$, then the assumptions (2.2)-(2.9) don't exclude the existence of critical points (x, ξ) of the principal symbol of L lying in $B(0, R_0)$. Thus Theorem 1 covers the case of critical energy levels and we will present some applications in Sections 4 and 5.

The assertion *ii*) is independent of the choice of a reference operator $L^{\#}(h)$ so we obtain the following.

Corollary 1. *Let* $L_1^{\#}$, $L_2^{\#}$ *be two reference operators for* L *satisfying the conditions* (2.1) – (2.9) *and suppose that each* $\lambda \in [E_0, E_1]$ *is a non-critical energy level for* Q, $Q_1^{\#}$ *and* $Q_2^{\#}$. *Then* $L_1^{\#}$ *satisfies i) if and only if* $L_2^{\#}$ *satisfies i).*

From the implication $i) \Rightarrow ii)$ we deduce an upper bound for the counting function of resonances in small domains. In fact, as in the proof of Lemma 6.1 in [19], for $0 < y < \delta$ and $|x - \lambda| < \delta$ we have

$$\int_{\lambda-2\delta}^{\lambda+2\delta} \frac{y}{(x-\mu)^2 + y^2} d\mu \geq \int_{-\delta/y}^{\delta/y} \frac{1}{1+r^2} dr \geq \pi/2.$$

Thus we deduce

$$\sum_{\substack{w \in \mathrm{Res}\, L(h) \cap \Omega, \\ \mathrm{Im}\, w \neq 0}} \omega_{\mathbb{C}_-}(w, [\lambda - 2\delta, \lambda + 2\delta]) + \#\{\mu \in \mathbb{R} : \mu \in \mathrm{sp}_{pp}(L(h)) \cap [\lambda - \delta, \lambda + \delta]\}$$

$$\geq \frac{1}{2} \#\{z \in \mathrm{Res}\, L(h),\ \mathrm{Im}\, z \neq 0,\ |z - \lambda| \leq \delta\} + \#\{z \in \mathrm{Res}\, L(h) \cap [\lambda - \delta, \lambda + \delta]\}$$

and we obtain the following.

Corollary 2. *The assumption i) of Theorem* 1 *implies the existence of positive constants* C, B, b, h_0 *such that for any* $\lambda \in [E_0 - b, E_1 + b]$, $h \in]0, h_0]$ *and* $h/B \leq \delta \leq B$ *we have*

$$\#\{z \in \mathbb{C} : z \in \mathrm{Res}\, L(h),\ |z - \lambda| \leq \delta\} \leq C \delta r(h) h^{-n^{\#}}.$$

In the non-critical case we can take $r(h) = 1$ and this corollary gives a new proof of a recent result of J. F. Bony [1] (see also [19] for the case of compact perturbations). In the critical case the statement of Corollary 2 implies the results of J. F. Bony [3] for differential operators L and dimension $n \geq 2$ (see Section 4). The results in [3] in the case $n = 1$ for h-pseudodifferential operators L are more precise since the upper bounds $r(h)$ is replaced by $r(\delta)$.

We may obtain a Breit-Wigner approximation for the derivative of the spectral shift function $\xi(\lambda, h)$ defined before Theorem 1. In fact, by using the assertions *i)* and *iii)* of Theorem 1 and repeating with minor modifications the arguments of Section 6 in [8], we obtain the following generalization of Corollary 1 in [8].

Theorem 2. *Assume that L satisfies the assumptions* (2.1) – (2.9) *and suppose that* $[E_0, E_1]$ *is a non-critical energy level for Q and $Q^{\#}$. Let $r(h)$, $h \in]0, h_0]$, be a real valued function such that* $\inf_{h\in]0,h_0]} r(h) > 0$. *Then if one of the assumptions i) – iii) of Theorem 1 holds, then for each $E \in]E_0, E_1[$ there exist constants $C_2 > C_1 > 0$, $h_0' > 0$ so that for $|\lambda - E| \leq C_1 h$, $h \in]0, h_0']$, we have*

$$\frac{\partial \xi}{\partial \lambda}(\lambda, h) = -\frac{1}{\pi} \sum_{\substack{|E-w|\leq C_2 h,\\ w\in \mathrm{Res}\, L(h)}} \frac{\operatorname{Im} w}{|\lambda - w|^2} + \sum_{\substack{|E-w|\leq C_1 h,\\ w\in \mathrm{sp}_{pp}\, L(h)}} \delta(\lambda - w) + \mathcal{O}\Big(r(h) h^{-n^{\#}}\Big). \tag{2.10}$$

3. Proof of Theorem 1

The proof of Theorem 1 is based on a representation formula for the spectral shift function (see Theorem 1 in [8]). Given a Hamiltonian $l(x, \xi)$, denote by

$$\exp(tH_l)(x_0, \xi_0) = (x(t, x_0, \xi_0),\ \xi(t, x_0, \xi_0))$$

the trajectory of the Hamilton flow $\exp(tH_l)$ passing through $(x_0, \xi_0) \in \Sigma_\lambda$. Recall that $\lambda \in J$ is a *non-trapping energy level* for $l(x, \xi)$ if for every $R > 0$ there exists $T(R) > 0$ such that for $(x_0, \xi_0) \in \Sigma_\lambda$, $|x_0| < R$, the x-component of the trajectory of $\exp(tH_l)$ passing through (x_0, ξ_0) satisfies

$$|x(t, x_0, \xi_0)| > R,\ \ \forall |t| > T(R).$$

We introduce an intermediate operator exploiting the following result of J. F. Bony (see also [24]).

Proposition 1 ([2]). *Let L satisfy the assumptions of Section* 2 *and let* $0 < E_0 < E_1$. *Then there exists a differential operator*

$$\tilde{L}(h) = \sum_{|\nu|\leq 2} \tilde{a}_\nu(x; h)(hD_x)^\nu,$$

satisfying the assumptions (2.4) – (2.7) *and the following properties:*

(*a*) *There exists $\overline{n} > n$ such that we have*

$$\Big| a_\nu(x; h) - \tilde{a}_\nu(x; h) \Big| \leq \mathcal{O}(1) \langle x \rangle^{-\overline{n}},\ \ |\nu| \leq 2 \tag{3.1}$$

for $x \in \Gamma$ introduced in (2.7), *uniformly with respect to h,* *(b) The operator $\tilde{L}$ has no resonances in a complex neighborhood Ω_0 of $[E_0, E_1]$ and Ω_0 is independent on h,*

(c) There exists an open interval $I_0 \subset]0, +\infty[$ containing $[E_0, E_1]$, such that each $\mu \in I_0$ is non-trapping energy level for $\tilde{L}$.

The property (a) guarantees that for every $f \in C_0^\infty(\mathbb{R})$ the operator $f(L) - f(\tilde{L})$ is "trace class near infinity". More precisely, if we denote $L_2 = L$ and $L_1 = \tilde{L}$, given $f \in C_0^\infty(\mathbb{R})$, independent on h, and $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $\overline{B(0, R_0)}$ we can define $\mathrm{tr_{bb}}[f(L_j)]_{j=1}^2$, as in [23], [25], by the equality

$$\mathrm{tr_{bb}}\Big(f(L_2) - f(L_1)\Big) = [\mathrm{tr}(\chi f(L_j)\chi + \chi f(L_j)(1-\chi) + (1-\chi)f(L_j)\chi)]_{j=1}^2$$
$$+\mathrm{tr}[(1-\chi)f(L_j)(1-\chi)]_{j=1}^2\,,$$

where we use the notation $[a_j]_{j=1}^2 = a_2 - a_1$. The spectral shift function $\xi(\lambda, h)$ is a distribution in $\mathcal{D}'(\mathbb{R})$ such that

$$< \xi'(\lambda, h), f(\lambda) >_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} = \mathrm{tr_{bb}}\Big(f(L(h)) - f(\tilde{L}(h))\Big), f(\lambda) \in C_0^\infty(\mathbb{R})\,.$$

Applying Theorem 1 of [8] in the domain Ω_0, we deduce that there exists a function $g_+(z, h)$, holomorphic in Ω_0, such that for $\mu \in I_0 = W_0 \cap \mathbb{R}$, $W_0 \subset\subset \Omega_0$ we have

$$\xi'(\mu, h) = \frac{1}{\pi}\operatorname{Im} g_+(\mu, h) + \sum_{\substack{w \in \mathrm{Res}\, L \cap \Omega_0, \\ \operatorname{Im} w \neq 0}} \frac{-\operatorname{Im} w}{\pi|\mu - w|^2} + \sum_{w \in \mathrm{Res}\, L \cap I_0} \delta(\mu - w), \tag{3.2}$$

where $g_+(z, h)$ satisfies the estimate

$$|g_+(z, h)| \leq C(W_0) h^{-n^\#}, \ \ z \in W_0 \tag{3.3}$$

with $C(W_0) > 0$ independent on $h \in]0, h_0]$.

Property (c) shows that $\tilde{L}$ has no critical energy levels $\lambda \in [E_0, E_1]$. In the following, we fix an open interval $I_0 \subset \mathbb{R}^+ \cap \Omega_0$ containing $[E_0, E_1]$ so that each $\lambda \in I_0$ is a non-critical energy level for the operators Q, $\tilde{L}$ and we introduce open intervals $I_2 \subset\subset I_1 \subset\subset I_0$ containing $[E_0, E_1]$. We suppose that $|\lambda - z| \geq \eta_0 > 0$ for $\lambda \in I_1$, $z \notin \Omega_0$.

Consider a function $\theta \in C_0^\infty(] - \epsilon_4, \epsilon_4[)$, $\theta(0) = 1$, $\theta(-t) = \theta(t)$ such that the Fourier transform of θ satisfies $\hat{\theta}(\lambda) \geq 0$ on $\mathbb{R}$. Assume that there exist $\epsilon_0 > 0$, $\delta_0 > 0$ so that $\hat{\theta}(\lambda) \geq \delta_0 > 0$ for $|\, \lambda \,| \leq \epsilon_0$ and introduce the function

$$\Big(\mathcal{F}_h^{-1}\theta\Big)(\lambda) = (2\pi h)^{-1}\int e^{it\lambda/h}\theta(t)dt = (2\pi h)^{-1}\hat{\theta}(-h^{-1}\lambda).$$

The next lemma, established in [8], yields a connection between the derivatives of the functions M_{φ, Ω_0} and $N_\varphi^\#$.

Proposition 2 ([8]). *Let $\varphi \in C_0^\infty(I_1; \mathbb{R}^+)$ and let*

$$N_\varphi^\#(\mu) = \mathrm{tr}\Big(\varphi(L^\#)\mathbf{1}_{]-C^\#, \mu]}(L^\#)\Big),$$

$$G_\varphi(\mu) = \frac{1}{\pi}\int_{]-\infty, \mu]} \operatorname{Im} g_+(\nu, h)\varphi(\nu)d\nu,$$

$$M_{\varphi,\Omega_0}(\mu) = \sum_{\substack{w\in \text{Res } L\cap\Omega_0,\\ \operatorname{Im} w\neq 0}} \int_{]-\infty,\mu]} \frac{-\operatorname{Im} w}{\pi|\nu-w|^2}\varphi(\nu)d\nu + \sum_{w\in\text{Res } L\cap]-\infty,\mu]} \varphi(w). \tag{3.4}$$

Then there exists $\omega_\varphi \in C_0^0(\mathbb{R})$ such that

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi,\Omega_0})(\mu) = \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N_\varphi^{\#})(\mu) - G'_\varphi(\mu) + \omega_\varphi(\mu)h^{-n} + \mathcal{O}(h^{1-n^{\#}}), \tag{3.5}$$

where $\mathcal{O}(h^{1-n^{\#}})$ is uniform with respect to $\mu \in \mathbb{R}$.

For our argument we need a Tauberian theorem involving the factor $r(h)$. A such theorem can be obtained by modifying the proof of the Tauberian theorem in [17], [20]. For the sake of completeness we present a version of the Tauberian theorem related to a real valued function $r(h)$, $h \in]0, h_0]$ such that $\inf_{h\in]0,h_0]} r(h) > 0$.

Theorem 3. *Let $\sigma(\lambda, h)$, $h \in]0, h_0]$, be a set of real valued increasing functions. Assume that there exist $a, b, c \in \mathbb{R}$ and $d \in \mathbb{N}$ independent of h so that*
$\sigma(\lambda, h) = 0$ for $\lambda \leq a$, $\sigma(\lambda, h) = c$ for $\lambda \geq b$,
$\sigma(\lambda, h) = \mathcal{O}(h^{-d})$ uniformly with respect to $\lambda \in \mathbb{R}$ and $h \in]0, h_0]$.

Then the following assertions are equivalents:

i) There exists positive constant C_1 such that for any $\lambda \in \mathbb{R}$, $h \in]0, h_0]$ we have

$$\Big|\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\lambda)\Big| \leq C_1 r(h)h^{-d}.$$

ii) There exists positive constant C_2 such that for any $\lambda \in \mathbb{R}$, $h \in]0, h_0]$ and $\eta \geq 0$ we have

$$\sigma(\lambda+\eta, h) - \sigma(\lambda-\eta, h) \leq C_2(\eta + h)r(h)h^{-d}.$$

Moreover, ii) implies

iii) There exists positive constant C_3 such that for any $\lambda \in \mathbb{R}$, $h \in]0, h_0]$ we have

$$|\sigma(\lambda, h) - (\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\lambda)| \leq C_3 r(h)h^{1-d}.$$

Proof. We assume $i)$ and we are going to prove $ii)$. It is clear that we can assume that η is bounded. Since $\sigma(\lambda, h)$ is constant outside $[a, b]$, it is sufficient to prove $ii)$ for δ bounded and λ in a compact set. As in the proof of the Tauberian theorem (see [20] or [17] for more details), the inequality $\hat{\theta}(\epsilon) \geq \delta_0$ for $|\epsilon| \leq \epsilon_0$, implies

$$|\sigma(\mu+\epsilon h, h) - \sigma(\mu - \epsilon h, h)| \leq \frac{2\pi h}{\delta_0}\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\mu), \ \ |\epsilon| \leq \epsilon_0, \ \ \forall \mu \in \mathbb{R}.$$

Exploiting $i)$ for $\eta \leq \epsilon_0 h$, we have

$$\sigma(\mu+\eta, h) - \sigma(\mu-\eta, h) \leq Cr(h)h^{1-d}.$$

On the other hand, for $\eta \geq \epsilon_0 h$ applying the above inequality for $\mu = \lambda + \eta - (2j+1)\epsilon_0 h$ at the right hand side of

$$\sigma(\mu+\eta, h) - \sigma(\mu-\eta, h) \leq \sum_{j=0}^{[\eta/\epsilon_0 h]} \Big(\sigma(\lambda+\eta-2j\epsilon_0 h, h) - \sigma(\lambda+\eta-2(j+1)\epsilon_0 h, h)\Big),$$

we obtain

$$\sigma(\mu+\eta, h) - \sigma(\mu-\eta, h) \leq C(\frac{\eta}{\epsilon_0 h}+1)r(h)h^{1-d}$$

and this implies $ii)$.

Now let us assume $ii)$ fulfilled. Then

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\lambda) = \frac{1}{2\pi h}\int_{\mathbb{R}} \hat{\theta}(\frac{\mu-\lambda}{h})d\sigma(\mu,h)$$

and this implies

$$\Big|\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\lambda)\Big| \leq \frac{C}{2\pi h}(\sigma(\lambda+h,h)-\sigma(\lambda-h,h))$$

$$+\frac{1}{2\pi h}\sum_{k=1}^{\infty}\int_{kh\leq|\mu-\lambda|<(k+1)h}\hat{\theta}(\frac{\mu-\lambda}{h})d\sigma(\mu,h).$$

Combining this with the estimate $|\hat{\theta}(\nu)| \leq C(1+|\nu|)^{-2}$ and applying $ii)$, we deduce

$$|\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\lambda)| \leq Cr(h)h^{-d} + \frac{C}{2\pi h}\sum_{k=1}^{\infty}\frac{1}{k^2}r(h)h^{1-d},$$

which yields $i)$. Here we have used that

$$|\sigma(\lambda \pm (k+1)h,h) - \sigma(\lambda \pm kh,h)| \leq Cr(h)h^{1-d}.$$

The proof of $ii) \Rightarrow iii)$ follows from the relation

$$\sigma(\lambda,h) - (\mathcal{F}_h^{-1}\theta * \sigma(.,h))(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}}(\sigma(\lambda,h)-\sigma(\lambda+\nu h,h))\hat{\theta}(\nu)d\nu,$$

where we have used that $\theta(0) = 1$. □

Remark. In the applications below we will use the estimate

$$\sigma(\lambda+\eta) - \sigma(\lambda-\eta) \leq C_2\eta r(h)h^{-d}$$

for $\frac{h}{B} \leq \eta \leq B$. Since $\sigma(\lambda,h)$ is increasing, this is equivalent to the assumption $ii)$ of Theorem 3 for $\eta \geq 0$.

Proof of Theorem 1. We assume $i)$ and we are going to prove $ii)$. Let $[E_0, E_1] \subset I_2 \subset\subset I_1 \subset\subset I_0$ be as above and let each $\mu \in I_0$ be a non critical energy level for Q, $Q^{\#}$. Choosing $\varphi \in C_0^\infty(I_1;\mathbb{R}^+)$, $\varphi = 1$ on I_2 as above, we will show that for $\frac{h}{B_2} \leq \delta \leq B_2$ we have

$$M_{\varphi,\Omega_0}(\lambda+\delta) - M_{\varphi,\Omega_0}(\lambda-\delta) = \mathcal{O}_\varphi(\delta)r(h)h^{-n^{\#}}, \quad \lambda \in [E_0-\epsilon_2, E_1+\epsilon_2]. \tag{3.6}$$

According to Theorem 3, to obtain (3.6) it is sufficient to show that

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi,\Omega_0})(\mu) = \mathcal{O}(r(h)h^{-n^{\#}}) \tag{3.7}$$

uniformly with respect to $\mu \in \mathbb{R}$. Exploiting the assumption $i)$, Theorem 3 and the Remark above, we get the estimate

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N_\varphi^{\#})(\mu) = \mathcal{O}(r(h)h^{-n^{\#}})$$

uniformly with respect to $\mu \in \mathbb{R}$. This implies (3.7) using the representation of Proposition 2.

Now let $\Omega \subset\subset \{z \in \mathbb{C} :\ \mathrm{Re}\, z > 0\}$ be a relatively compact open neighborhood of $[E_0, E_1]$ in $\{z \in \mathbb{C} :\ \mathrm{Re}\, z > 0\}$ containing Ω_0. Then taking into account (1.1), we obtain

$$\sum_{\substack{w \in \mathrm{Res}\ L \cap (\Omega \setminus \Omega_0),\\ \mathrm{Im}\, w \neq 0}} \int_{\mu-\delta}^{\mu+\delta} \frac{-\mathrm{Im}\, w}{\pi |\nu - w|^2} \varphi(\nu) d\nu \leq \frac{C\delta}{\eta_0^2} h^{-n^\#},\ \forall \delta > 0.$$

Consequently, the function

$$M_{\varphi,\,\Omega}(\mu) = \sum_{\substack{w \in \mathrm{Res}\ L \cap \Omega,\\ \mathrm{Im}\, w \neq 0}} \int_{]-\infty,\mu]} \frac{-\mathrm{Im}\, w}{\pi |\nu - w|^2} \varphi(\nu) d\nu + \sum_{w \in \mathrm{Res}\ L \cap]-\infty,\mu]} \varphi(w)$$

satisfies the estimate

$$M_{\varphi,\,\Omega}(\lambda + \delta) - M_{\varphi,\,\Omega}(\lambda - \delta) \leq C_\varphi \delta r(h) h^{-n^\#},\ \ \lambda \in [E_0 - 2\epsilon_2, E_1 + 2\epsilon_2],\ \frac{h}{C_1} \leq \delta \leq C_1$$

with a sufficiently small $\epsilon_2 > 0$. Moreover, the constant $C_\varphi > 0$ depends on η_0, Ω and C_φ is independent on h. Since φ is equal to 1 on a neighborhood of $[E_0, E_1]$, we deduce $ii)$.

The proof of the implication $ii) \Rightarrow i)$ is very similar. As in the analysis of the function $M_{\varphi,\Omega_0}(\mu)$, the estimate of $N_\varphi^\#(\lambda + \delta) - N_\varphi^\#(\lambda - \delta)$ is a consequence of the bound

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N_\varphi^\#)(\mu) = \mathcal{O}_\varphi(r(h)h^{-n^\#}), \quad \mu \in \mathbb{R}.$$

According to the representation given in Proposition 2, we have to prove that

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi,\Omega_0})(\mu) = \mathcal{O}_\varphi(r(h)h^{-n^\#}). \tag{3.8}$$

First, using the notations introduced above, notice that

$$M_{\varphi,\Omega_0}(\lambda + \delta) - M_{\varphi,\Omega_0}(\lambda - \delta) \leq M_{\varphi,\,\Omega}(\lambda + \delta) - M_{\varphi,\,\Omega}(\lambda - \delta) + C_\varphi \delta h^{-n^\#}.$$

Secondly, it is clear that

$$M_{\varphi,\,\Omega}(\lambda + \delta) - M_{\varphi,\,\Omega}(\lambda - \delta) \leq \sum_{\substack{w \in \mathrm{Res}\ L(h) \cap \Omega,\\ \mathrm{Im}\, w \neq 0}} \omega_{\mathbb{C}_-}(w, [\lambda - \delta, \lambda + \delta])$$

$$+ \#\{\mu \in \mathbb{R} : \mu \in \mathrm{sp}_{pp}(L(h)) \cap [\lambda - \delta, \lambda + \delta]\} \leq C\delta r(h) h^{-n^\#},$$

where in the second inequality we have used $ii)$. Combining these estimates, we get

$$M_{\varphi,\Omega_0}(\lambda + \delta) - M_{\varphi,\Omega_0}(\lambda - \delta) \leq C_\varphi \delta r(h) h^{-n^\#},\ \frac{h}{C_2} \leq \delta. \tag{3.9}$$

It is clear that for $0 \leq \delta \leq \frac{h}{C_2}$ the last estimate remains true if we replace C_φ by $\frac{C_\varphi}{C_2}(\delta + h)$. Thus we can apply Theorem 3 for $M_{\varphi,\Omega_0}(\lambda + \delta) - M_{\varphi,\Omega_0}(\lambda - \delta)$ and this implies (3.8). By using that φ is equal to 1 on a neighborhood of $[E_0, E_1]$, we complete the proof of $i)$. The equivalence $ii) \Leftrightarrow iii)$ is a consequence of (3.2) and (3.3). $\square$

4. Applications of Theorems 1 and 2

First we will examine the connection between the condition (1.3) and Theorems 1 and 2. We have the following.

Proposition 3. *Assume that L satisfies the assumptions* (2.1) – (2.9) *and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for Q and $Q^{\#}$. Assume that for $\lambda \in [E_0 - \epsilon_2, E_1 + \epsilon_2]$, $\epsilon_2 > 0$ and $0 < \frac{h}{B} \leq \delta \leq B$ we have*

$$\#\{w \in \mathbb{C} : \ w \in \operatorname{Res} L, \ |w - \lambda| \leq \delta\} \leq C\delta h^{-n^{\#}}. \tag{4.1}$$

Then we can apply Theorem 1 and Theorem 2 with $r(h) = \ln(1/h)$.

Proof. It is sufficient to prove that the assertion *ii*) of Theorem 1 holds with $r(h) = \ln(1/h)$. It is easy to show that (4.1) implies

$$\sum_{\substack{w\in (\operatorname{Res} L) \cap \Omega, \\ 0 < |\operatorname{Im} w| \leq A\delta}} \omega_{\mathbb{C}_-}(w, [\lambda - \delta, \lambda + \delta]) \leq \mathcal{O}(\delta) h^{-n^{\#}}, \ \frac{h}{B_2} \leq \delta \leq B_2. \tag{4.2}$$

In fact, taking into account the estimate

$$\int_\alpha^\beta \frac{|\operatorname{Im} w|}{|\nu - w|^2} d\nu \leq \pi, \ -\infty \leq \alpha < \beta \leq \infty, \tag{4.3}$$

we obtain

$$\sum_{\substack{w\in \operatorname{Res} \ L, \operatorname{Im} w \neq 0, \\ |w-\lambda| \leq 2\delta}} \int_{\lambda-\delta}^{\lambda+\delta} \frac{|\operatorname{Im} w|}{\pi |t - w|^2} dt \leq C\delta h^{-n^{\#}}. \tag{4.4}$$

On the other hand, for $|t - \lambda| \leq \delta \leq 1/2$ we get

$$\sum_{\substack{w\in \operatorname{Res} \ L, \ 0 < |\operatorname{Im} w| \leq A\delta, \\ |w-\lambda| > 2\delta}} \frac{|\operatorname{Im} w|}{|t - w|^2}$$

$$\leq \sum_{k=1}^{C \log \frac{1}{\delta}} \sum_{\substack{2^k\delta < |w-\lambda| \leq 2^{k+1}\delta, \\ |\operatorname{Im} w| \leq A\delta}} \frac{|\operatorname{Im} w|}{|t - w|^2} \leq \sum_{k=1}^{C \log \frac{1}{\delta}} \frac{C\delta(2^{k+1}\delta) h^{-n^{\#}}}{(2^k\delta)^2} \leq C h^{-n^{\#}},$$

and after an integration over the interval $[\lambda - \delta, \lambda + \delta]$ we obtain immediately (4.2) since the case $\delta > 1/2$ is trivial.

To obtain the assumption *ii*), we will show that

$$\sum_{\substack{w\in (\operatorname{Res} L) \cap \Omega, \\ \operatorname{Im} w \neq 0}} \omega_{\mathbb{C}_-}(w, [\lambda - \delta, \lambda + \delta]) \leq \mathcal{O}(\delta) \max\Big(\log \frac{1}{\delta}, 1\Big) h^{-n^{\#}}, \ \frac{h}{B_2} \leq \delta \leq B_2. \tag{4.5}$$

To see this, first we apply (4.2) with $A = 2$. Next for $|t - \lambda| \leq \delta \leq 1/2$ we have

$$\sum_{w\in (\operatorname{Res} L) \cap \Omega, \ |\operatorname{Im} w| \geq 2\delta} \frac{1}{|t - w|}$$

$$\leq \sum_{k=1}^{C \log \frac{1}{\delta}} \sum_{2^k\delta \leq |w-\lambda| \leq 2^{k+1}\delta} \frac{1}{|t - w|} \leq \sum_{k=1}^{C \log \frac{1}{\delta}} \frac{C 2^{k+1}\delta h^{-n^{\#}}}{2^k\delta} \leq C\Big(\log \frac{1}{\delta}\Big) h^{-n^{\#}}.$$

So writing

$$\frac{|\operatorname{Im} w|}{|t-w|^2} = \frac{1}{2i}\Big(\frac{1}{t-\overline{w}} - \frac{1}{t-w}\Big),$$

we obtain (4.4) for $\delta \leq 1/2$. The analysis of the case $1/2 < \delta \leq B_2$ is trivial. $\square$

Remark. There are examples, where the result of Proposition 3 is sharp. In fact consider the case $n = 1$ and let $L(h) = -h^2\Delta + V(x)$. If $V(x)$ has an absolute non-degenerate maximum at only one point α, then the analysis in [22] shows that (4.1) holds, while following the approximation of the resonances given in [5], [21], [10] the assumption *iii)* of Theorem 1 is satisfied with $r(h) = \log \frac{1}{h}$. We will discuss with more details this example in the next Section.

Next consider a classical h-pseudodifferential operator $L(h) = L$ on $L^2(\mathbb{R}^n)$ with symbol

$$l(x,\xi;h) \sim \sum_{j\geq 0} l_j(x,\xi)h^j,\ \ l_j(x,\xi) \in S_0^{-j}(<\xi>^2),$$

where we use the notations of [9] for the symbols of h-pseudodifferential operators. Assume that there is no "black box" and that $L(h)$ satisfies the conditions (2.2)-(2.9). Moreover, we suppose that there exist constants $C_1 > 0,\ \ C_0 > 0$ so that

$$l_0(x,\xi) \geq C_1|\xi|^2 - C_0,\ \ \forall (x,\xi) \in \mathbb{R}^n, \tag{4.6}$$

$l_0(x,\xi)$ being the principal symbol of L. As we have mentioned in Section 2, the symbol $l_0(x,\xi)$ may have critical points. Given a critical point $E_c \in [E_0, E_1]$, we assume that the set $\mathcal{C}$ of the critical points of $l_0(x,\xi)$ on the surface $\{(x,\xi) \in \mathbb{R}^n :\ l_0(x,\xi) = E_c\}$ is a submanifold of $T^*(\mathbb{R}^n)$ such that the Hessian of $l_0(x,\xi)$ is non degenerate on the subspace normal to $\mathcal{C}$. This implies that E_c is an isolated critical point of $l_0(x,\xi)$ and the conditions on $\mathcal{C}$ are the same as those in [6], [3]. Moreover, the assumption (2.6) shows that $\mathcal{C}$ is a finite union of connected compact sets $\mathcal{C} = \mathcal{C}_1 \cup ... \cup \mathcal{C}_N$. Let (r_j, s_j) be the signature of the Hessian of $l_0(x,\xi)$ on the subspace normal to $\mathcal{C}_j$. The codimension of $\mathcal{C}_j$ is equal to $r_j + s_j$. Notice that if L is a differential operator, the ellipticity condition (4.6) implies (see [3]) that $r_j + s_j \geq n + 1$. In order to apply Theorem 1, we need the following result of J. F. Bony.

Proposition 4 ([3]). *Under the above assumptions on the set of the critical points $\mathcal{C}$ for E in a small neighborhood of E_c, $h \in]0, h_0]$ and $h/B \leq \delta \leq B$ we have*

$$\#\{\mu \in \mathbb{R} : \mu \in \operatorname{sp}_{pp} L^{\#}(h),\ \ |\mu - E| \leq \delta\} \leq C\delta r(\delta) h^{-n},$$

where in the general case $r(\delta) = \Big(|E - E_c| + \delta\Big)^{-1/2}$. Moreover, if for all $1 \leq j \leq N$ we have $r_j + s_j \geq 2$, then $r(\delta) = |\log(\delta + |E - E_c|)|$ and if for all $1 \leq j \leq N$ we have $\max\{r_j, s_j\} \geq 2$, then $r(\delta) = 1$.

The proof in [3] is based on the estimation of the trace norm

$$\Big\| f\Big(\frac{L^{\#} - E}{\delta}\Big)\Big\|_{\mathrm{tr}}$$

for a cut-off function $f \in C_0^\infty(\mathbb{R}; [0,1]),\ \ f = 1$ on $[-1, 1]$ following the tools developed in [6]. Under the above assumptions the critical points are isolated and by using a finite covering of $[E_0, E_1]$, we obtain a global version of Proposition 4 with constants $B > 0$, $C > 0$ and $h_0 > 0$ which are uniform with respect to $E \in [E_0 - \epsilon, E_1 + \epsilon],\ \ \epsilon > 0$.

Corollary 3. *Under the above assumptions on $\mathcal{C}$, we can apply Theorems* 1 *and* 2 *for the h-pseudodifferential operator L with $r(h)$ given above.*

5. Breit-Wigner approximation near critical energy levels

In this section we assume that the critical manifold $\mathcal{C}$ has the form described in Section 4. Thus the assumption $i)$ of Theorem 1 holds and we are going to discuss the form of the sum of harmonic measures related to the resonances. Throughout this section we assume that

$$L(h) = -h^2\Delta + V(x),$$

where $V(x)$ is real valued on $\mathbb{R}^n$ and

$$|V(x)| \leq C(1+|x|)^{-n-\sigma}, \ \sigma > 0.$$

Moreover, we suppose that $V(x)$ is holomorphic in the domain

$$\{x \in \mathbb{C}^n : |\operatorname{Im} x| \leq \tan\theta_0 |\operatorname{Re} x|\} \cup \{x \in \mathbb{C} : |\operatorname{Im} x| \leq \delta_0\}$$

for $0 < \theta_0 < \pi/2$ and $\delta_0 > 0$.

Denote by $\xi(\lambda, h)$ the spectral shift function related to $L(h)$ and $L_0(h) = -h^2\Delta$. Let $l(x,\xi) = |\xi|^2 + V(x)$ be the symbol of $L(h) = L$ and let the set $\mathcal{C}$ of the critical points of $l(x,\xi)$ have the form described in the previous section.

First we will treat the case $n = 1$. An application of Corollary 3 yields the following.

Proposition 5. *Assume $n = 1$ and let the set of the critical points in $l^{-1}([E_0, E_1])$ have the form $\mathcal{C} = \cup_{i=1}^N \{(\alpha_i, 0)\}$ with $V(\alpha_i) = E_i$, $V'(\alpha_i) = 0$, $V''(\alpha_i) \neq 0$, $i = 1, ..., N$. Then for each E_i, $i = 1, 2, ..., N$, $h \in]0, h_0]$ and for $|\lambda - E_i| \leq C_1 h$, $C_2 > C_1$, we have*

$$\frac{\partial \xi}{\partial \lambda}(\lambda, h) = -\frac{1}{\pi} \sum_{\substack{|E_i - w| \leq C_2 h, \\ w \in \operatorname{Res} L(h)}} \frac{\operatorname{Im} w}{|\lambda - w|^2} + \mathcal{O}\Big(h^{-1} \log \frac{1}{h}\Big).$$

Our next purpose is to obtain an estimate of the term involving the harmonic measures. Consider the simplest case when the manifold $\mathcal{C}$ is given by a single point $\{(\alpha, 0)\}$, where

$$V(\alpha) = \max_{x \in \mathbb{R}} V(x) = E_c, \ \ V'(\alpha) = 0, \ V''(\alpha) = -\frac{1}{2\rho^2} < 0, \ \rho > 0.$$

The resonances in a disk $D(E_c, r)$ for $r > 0$ sufficiently small have the form (see [5], [21], [10])

$$w_k = E_c - i(k + \frac{1}{2})\frac{1}{\rho} h + \mathcal{O}(h^2), \ \ k \in \mathbb{N}$$

and for $|\lambda - E_c| \leq C_1 h < \frac{1}{2\rho} h$ we obtain

$$-\frac{1}{\pi} \sum_{|E_c - w_k| \leq \frac{r}{2}} \frac{\operatorname{Im} w_k}{|\lambda - w_k|^2} = -\frac{1}{\pi} \sum_{\frac{1}{2\rho} h \leq |E_c - w_k| \leq \frac{r}{2}} \frac{\operatorname{Im} w_k}{|\lambda - w_k|^2} = \frac{\rho}{\pi h} \sum_{k=1}^{C/h} \frac{1}{k} + \mathcal{O}(h^{-1})$$

$$= \frac{\rho}{\pi h} \int_1^{C/h} \frac{dx}{x} + \mathcal{O}(h^{-1}) = \frac{\rho}{\pi} h^{-1} \log \frac{1}{h} + \mathcal{O}(h^{-1}).$$

Thus applying Theorem 1 in [8] in the disk $D(E_c, r/2)$, we get for $\lambda \in \mathbb{R}$, $|\lambda - E_c| \leq C_1 h$

$$\frac{\partial \xi}{\partial \lambda}(\lambda, h) = \frac{\rho}{\pi} h^{-1} \log \frac{1}{h} + \mathcal{O}(h^{-1}) \tag{5.1}$$

and we obtain the result of Theorem 2.2 in [11] concerning the case of an unique non-degenerate maximum point.

Next assume that $\mathcal{C} = \{(\alpha_1, 0)\} \cup \{(\alpha_2, 0)\}$, $\alpha_1 < \alpha_2$, with

$$V(\alpha_i) = \max_{x \in \mathbb{R}} V(x) = E_c,\ V'(\alpha_i) = 0,\ V''(\alpha_i) = -\frac{1}{2\rho_i^2} < 0,\ \rho_i > 0,\ i = 1, 2.$$

Following the results of [10], the resonances in a disk $D(E_c, rh)$, $r > 0$, have the form

$$w_k = E_c + \frac{S_0 - (k + 1/2)\pi h + ih \log 2}{K \log h} + \mathcal{O}\Big(\frac{h}{(\log h)^2}\Big),\ k \in \mathbb{N}$$

if $E_c - \operatorname{Re} w_k = \mathcal{O}(h/\log h)$ and

$$z_k = E_c + \frac{S_0 - (k + 1/2)\pi h}{K \log h} + \mathcal{O}\Big(\frac{h}{\log h}\Big),\ \ k \in \mathbb{N}$$

in the exterior of this domain, where $S_0 \in \mathbb{R}$ and $K = \frac{1}{2}(\rho_1 + \rho_2)$.

First we are going to estimate for $|\lambda - E_c| \leq C_1 \frac{h}{\log \frac{1}{h}}$, $C_1 \leq \frac{\pi}{2K}$, the sum

$$-\frac{1}{\pi} \sum_{\substack{|E_c - z| \leq rh, \\ z \in \operatorname{Res} L(h)}} \frac{\operatorname{Im} z}{|\lambda - z|^2}$$

$$\leq \#\{z \in \operatorname{Res} L(h) :\ |E_c - z| \leq rh,\ \ |E_c - \operatorname{Re} z| < \frac{\pi h}{K \log \frac{1}{h}}\} \mathcal{O}\Big(h^{-1} \log \frac{1}{h}\Big)$$

$$+ \frac{Ch}{\log \frac{1}{h}} \sum_{k=1}^{C \log \frac{1}{h}} \sum_{\substack{\frac{k\pi h}{K \log \frac{1}{h}} \leq |E_c - \operatorname{Re} z| < \frac{(k+1)\pi h}{K \log \frac{1}{h}} \\ z \in \operatorname{Res} L(h),\ |E_c - z| \leq rh}} \frac{1}{|\lambda - \operatorname{Re} z|^2}$$

$$\leq C h^{-1} \log \frac{1}{h} \sum_{k=1}^{\infty} \frac{1}{k^2} + C_2 h^{-1} \log \frac{1}{h} = C_3 h^{-1} \log \frac{1}{h}.$$

Here we have used the fact that there are only finite number resonances z for which

$$|E_c - z| \leq rh,\ \ \frac{k\pi h}{K \log \frac{1}{h}} \leq |E_c - \operatorname{Re} z| < \frac{(k+1)\pi h}{K \log \frac{1}{h}},\ k \in \mathbb{N}.$$

By using the lower bound

$$-\operatorname{Im} z \geq C_0 \frac{h}{\log \frac{1}{h}},\ C_0 > 0,\ |E_c - z| \leq rh,\ E_c - \operatorname{Re} z = \mathcal{O}(h/\log h),\ z \in \operatorname{Res} L(h),$$

a similar argument yields

$$-\frac{1}{\pi} \sum_{\substack{|E_c - z| \leq rh, \\ z \in \operatorname{Res} L(h)}} \frac{\operatorname{Im} z}{|\lambda - z|^2} \geq\ C_4 h^{-1} \log \frac{1}{h},\ C_4 > 0.$$

On the other hand, an application of Theorem 1 in [8] yields the representation

$$\frac{\partial \xi}{\partial \lambda}(\lambda, h) = -\frac{1}{\pi} \sum_{\substack{|E_c - z| \leq r/2, \\ z \in \mathrm{Res}\, L(h)}} \frac{\operatorname{Im} z}{|\lambda - z|^2} + \mathcal{O}(h^{-1}). \tag{5.2}$$

In a recent work, Fujiié and Ramond [11] proved that for $|\lambda - E_c| \leq C_1 \frac{h}{\log \frac{1}{h}}$ we have

$$\pi \frac{\partial \xi}{\partial \lambda}(\lambda, h) = \frac{\rho_1 + \rho_2}{2} \Big(1 + \frac{\gamma}{(1 - \gamma^2)\cos^2(\sigma_i / h) + \gamma^2}\Big) h^{-1} \log \frac{1}{h} + \mathcal{O}(h^{-1}),$$

where the function $\gamma(\lambda, h)$ is holomorphic in $[E_c - \epsilon, E_c + \epsilon] + i[-Ch, Ch]$, $\epsilon > 0$, while the function $\sigma_i(\lambda, h)$ is real valued on the real axis and holomorphic in a disk $D(E_c, Ch)$. Moreover, for $\lambda \in (E_c - \delta, E_c + \delta) \subset \mathbb{R}$, $\delta > 0$, we have $0 < \gamma(\lambda, h) < 1$.

Comparing the leading terms in these representations, for $|\lambda - E_c| \leq C_1 \frac{h}{\log \frac{1}{h}}$ we get

$$-\sum_{\substack{|E_c - z| \leq r/2, \\ z \in \mathrm{Res}\, L(h)}} \frac{\operatorname{Im} z}{|\lambda - z|^2} = \frac{\rho_1 + \rho_2}{2} \Big(1 + \frac{\gamma}{(1 - \gamma^2)\cos^2(\sigma_i / h) + \gamma^2}\Big) h^{-1} \log \frac{1}{h} + \mathcal{O}(h^{-1}).$$

For γ small we have spikes at each zero of $\cos(\sigma_i / h)$, while the spikes in (3.2) are related to the real part of the resonances.

To treat the case $\lambda \in \mathbb{R}$, $C_1 \frac{h}{\log \frac{1}{h}} \leq |\lambda - E_c| \leq rh$, we apply Proposition 5. Notice that we have at most $\mathcal{O}\Big(\log \frac{1}{h}\Big)$ resonances in $D(E_c, rh)$ and the upper bound of the imaginary part of the resonances implies

$$-\sum_{\substack{|E_c - z| \leq rh, \\ z \in \mathrm{Res}\, L(h)}} \frac{\operatorname{Im} z}{|\lambda - z|^2} = -\sum_{\substack{|\lambda - z| \leq C_2 \frac{h}{\log \frac{1}{h}}, \\ |E_c - z| \leq rh,\ z \in \mathrm{Res}\, L(h)}} \frac{\operatorname{Im} z}{|\lambda - z|^2} + \mathcal{O}\Big(h^{-1} \Big(\log \frac{1}{h}\Big)^2\Big). \tag{5.3}$$

To estimate the sum at the right hand part of (5.3), we need a more precise information for the resonances $z \in \mathrm{Res}\, L(h)$ lying inside the domain $|\lambda - z| \leq C_2 \frac{h}{\log \frac{1}{h}}$.

Now consider the case $n \geq 2$. In this situation Corollary 3 yields for $|\lambda - E| \leq C_1 h$ the representation

$$\frac{\partial \xi}{\partial \lambda}(\lambda, h) = -\frac{1}{\pi} \sum_{\substack{|E - w| \leq C_2 h, \\ w \in \mathrm{Res}\, L(h)}} \frac{\operatorname{Im} w}{|\lambda - w|^2} + \mathcal{O}(h^{-n}). \tag{5.4}$$

Let us discuss the simplest case when the set K of trapping points of $L(h)$ lying in $l^{-1}(E_c)$ is given by a single point $\{(\alpha, 0)\}$ so that

$$V(\alpha) = E_c,\ \nabla_x V(\alpha) = 0.$$

Assume that the Hessian of $V(x)$ at α is non-degenerate and let $(n - d, d)$, $d \geq 1$, be the signature of this Hessian. Then the linearization of the Hamiltonian field H_l at $(\alpha, 0)$ has eigenvalues

$$\pm i\lambda_j,\ 1 \leq j \leq n - d,$$

$$\pm \lambda_j,\ n - d + 1 \leq j \leq n$$

with $\lambda_j > 0$ (see [21], [22]). Following the results in [21], [14], the resonances of $L(h)$ in a disk $D(E_c, Ch)$ admit an asymptotic representation and the condition $d \geq 1$ implies easily that

$$-\operatorname{Im} w \geq c_0 h,\ c_0 > 0,\ \forall w \in D(E_c, Ch) \cap \operatorname{Res} L(h).$$

On the other hand, the result of [21] says that the number of the resonances w lying in $D(E_c, C_2 h)$ is at most C_0 and for $|\lambda - E_c| \leq C_1 h,\ C_1 < C_2$ we obtain the estimate

$$-\frac{1}{\pi} \sum_{\substack{|E-w| \leq C_2 h, \\ w \in \operatorname{Res} L(h)}} \frac{\operatorname{Im} w}{|\lambda - w|^2} = \mathcal{O}(h^{-1}). \tag{5.5}$$

In contrast to the case $n = 1$ for $n \geq 2$ the sum of the Breit-Wigner factors is bounded by a term having a lower order than the remainder $\mathcal{O}(h^{-n})$ in (5.4). In this direction we notice the analysis of the radial case in [12] concerning the partial scattering phases $\sigma_l(\lambda, h),\ l \in \mathbb{N}$, for a potential having an absolute maximum ($d = n$). By using the asymptotics of $\frac{\partial \sigma_l}{\partial \lambda}(\lambda, h),\ \ l \in \mathbb{N}$, it seems difficult to obtain a representation for $\frac{\partial \xi}{\partial \lambda}(\lambda, h)$ like (5.4) with remainder $\mathcal{O}(h^{-n})$.

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