On the Large Time Behavior of Solutions of the Dirichlet problem for Subquadratic Viscous Hamilton-Jacobi Equations

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Abstract In this article, we are interested in the large time behavior of solutions of the Dirichlet problem for subquadratic viscous Hamilton-Jacobi Equations. In the superquadratic case, the third author has proved that these solutions can have only two different behaviors: either the solution of the evolution equation converges to the solution of the associated stationary generalized Dirichlet problem (provided that it exists) or it behaves like $-ct+\varphi(x)$ where $c \geq 0$ is a constant, often called the "ergodic constant" and φ is a solution of the so-called "ergodic problem". In the present subquadratic case, we show that the situation is slightly more complicated: if the gradient-growth in the equation is like $|Du|^m$ with $m > 3/2$, then analogous results hold as in the superquadratic case, at least if $c > 0$. But, on the contrary, if $m \leq 3/2$ or $c = 0$, then another different behavior appears since $u(x,t)+ct$ can be unbounded from below where u is the solution of the subquadratic viscous Hamilton-Jacobi Equations.

Key-words : Viscous Hamilton-Jacobi Equations, large time behavior, subquadratic case, Dirichlet problem, ergodic problem, viscosity solutions.

AMS subject classifications : 35K55, 35B40, 49L25

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Contents

1 Introduction

We are interested in this work in the behavior, when $t \to +\infty$, of the solution of the following initial-boundary value problem

$$
\begin{aligned}
u_t - \Delta u + |Du|^m &= f(x) \quad \text{in } \Omega \times (0, +\infty) && (1.1)\\
u(x, 0) &= u_0(x) \quad \text{on } \overline{\Omega} && (1.2)\\
u(x, t) &= g(x) \quad \text{on } \partial\Omega \times (0, +\infty) && (1.3)
\end{aligned}
$$

where Ω is a C^2- bounded and connected subset of $\mathbb{R}^N$, $1 < m \leq 2$ and f, u_0, g are real-valued continuous functions defined respectively on $\overline{\Omega}, \overline{\Omega}$ and $\partial\Omega$. The boundary and initial data satisfy the following compatibility condition

$$u_0(x) = g(x) \quad \text{for all } x \in \partial\Omega. \tag{1.4}$$

It is standard to show that this problem has a unique solution $u : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ and, as long as one does not need regularity properties, proofs of this fact are easy by using viscosity solutions theory (see Barles and Da Lio [4] and references therein).

In the superquadratic case ($m > 2$) the study of the asymptotic behavior has been done by the third author in [16] where it is shown that the solution u can have only two different behaviors whether the equation

$$-\Delta v + |Dv|^m = f(x) \quad \text{in } \Omega \,, \tag{1.5}$$

has *bounded* subsolutions or not. If (1.5) has a bounded subsolution, then there exists a solution u_∞ of the stationary Dirichlet problem, i.e. of (1.5) together with the generalized Dirichlet boundary condition

$$v(x) = g(x) \quad \text{on } \partial\Omega \;, \tag{1.6}$$

and $u(x,t) \to u_\infty(x)$ uniformly on $\overline{\Omega}$. This is the most expected behavior. On the other hand, it can happen that (1.5) has no bounded subsolution and, in this case, one has to introduce the *ergodic problem* with state constraint boundary conditions, namely (the reader has to keep in mind that we are here in the superquadratic case)

$$\begin{cases} -\Delta\varphi + |D\varphi|^m &= f(x) + c \quad \text{in } \Omega \;, \\ -\Delta\varphi + |D\varphi|^m &\geq f(x) + c \quad \text{on } \partial\Omega \;. \end{cases} \tag{1.7}$$

We recall that, in this type of problems, both the solution φ and the constant c (the ergodic constant) are unknown. The existence and uniqueness of solutions (c, φ) for (1.7) is studied in Lasry and Lions [14] (see also [16] for a viscosity solutions approach): the constant c is indeed unique while the solution φ is continuous up to the boundary and unique up to an additive constant.

Concluding the reference when $m > 2$, it is proved in [16] that, if (1.5) has no bounded subsolution, then $c > 0$ and the function $u(x,t) + ct$ converges uniformly on $\overline{\Omega}$ to a solution φ of (1.7) when $t \to +\infty$. In fact, in the superquadratic case, even if the boundary condition reads $u(x,t) + ct = g(x) + ct$ on $\partial\Omega$ with $g(x) + ct \to +\infty$ as $t \to +\infty$, there is a loss of boundary condition (cf. [4]) and $u(x,t) + ct$ remains bounded on $\overline{\Omega}$.

The first key difference in the subquadratic case is that there is no loss of boundary conditions and $u(x,t) + ct$ is actually equal to $g(x) + ct$ on the boundary. Formally this forces the limit of $u(x,t) + ct$ to tend to $+\infty$ on the boundary and, actually, the analogue of (1.7) is

$$\begin{cases} -\Delta\varphi + |D\varphi|^m &= f(x) + c \quad \text{in } \Omega \;, \\ \varphi(x) &\to +\infty \quad \text{when } x \to \partial\Omega \;. \end{cases} \tag{1.8}$$

This problem was also studied in [14] where it is proved that there exists a unique constant c such that (1.8) has a solution and, as in the superquadratic case, the solution is unique up to an additive constant.

Coming back to the asymptotic behavior of u, it can be thought, at first glance, that it is essentially the same as in the superquadratic case, with problem (1.7) being replaced by problem (1.8). Surprisingly this is not true for any $1 < m \leq 2$ nor for any c. Part of the explanations concerning c is that the equation

$$-\Delta\varphi + |D\varphi|^m = f(x) + \tilde{c} \quad \text{in } \Omega \;,$$

has subsolutions which are bounded on $\overline{\Omega}$ for any $\tilde{c} \geq c$ in the superquadratic case but only for $\tilde{c} > c$ in the subquadratic one.

In this article we prove the following: first, if the stationary Dirichlet problem (1.5)-(1.6) has a solution, then u converges uniformly in $\overline{\Omega}$ to this solution. Otherwise we show that necessarily $c \geq 0$ and u can have different types of behavior. Observe that a curious feature in the subquadratic case (which does not occur in the superquadratic case) is that c can be equal to 0 even if the stationary Dirichlet problem has no solution; this can be seen as a corollary of the above remark. The different behaviors of u can be described as follows.

(i) If $3/2 < m \leq 2$ and $c > 0$ then $u(x,t) + ct$ converges locally uniformly in Ω to φ where φ is a solution of (1.8).

(ii) If $1 < m \leq 3/2$ and $c > 0$ then $\dfrac{u(x,t)}{t} \to -c$ locally uniformly in Ω but it can happen that $u(x,t) + ct \to -\infty$ in Ω.

(iii) If $c = 0$, then $\dfrac{u(x,t)}{t} \to 0$ locally uniformly in Ω but it can happen that $u(x,t) \to -\infty$ in Ω.

The behaviors (ii) and (iii) are striking differences with the superquadratic case. This is related to the blow-up rate of $u(x,t)$ which we estimate in Theorem 3.2, and which is influenced by the behavior of φ (solution of (1.8)) near the boundary $\partial\Omega$ (changing according to the values of m) and eventually by the case that $c = 0$. We prove the optimality of such estimates in star-shaped domains in cases when $f + c < 0$ on $\overline{\Omega}$: in such situations, we show that it can actually happen that $u(x,t) + ct \to -\infty$.

This article is organized as follows. In Section 2, we recall the main results about the problem (1.8); these results are mainly borrowed from Lasry and Lions [14] and Porretta and Véron [15] where the precise behavior of φ near $\partial\Omega$ is described. Section 3.1 is devoted to the easy case when (1.5)-(1.6) has a solution; in such situation $u(\cdot,t)$ is bounded and converges to the (unique) solution of the stationary problem. We recall that the existence of a stationary solution corresponds, for the ergodic constant c of (1.8), to the case $c < 0$ (see Proposition 2.2). In Section 3.2 we consider the case $c \geq 0$ and we state the estimates on the blow-up rate of u proving, as a consequence, that $\frac{u(x,t)}{t} \to -c$ locally uniformly in Ω, in the whole range $1 < m \leq 2$. In the Section 3.3, we prove the result (i) above. Finally, in Section 4 we show that a similar result cannot hold in general in the range $m \geq \frac{3}{2}$ or if $c = 0$ proving that, in these cases, we have $u + ct \to -\infty$ at least in some circumstances. We leave to the Appendix the construction of sub and supersolutions, for the problem (1.1)–(1.2)–(1.3), which we use to obtain the crucial estimates on the blow-up rate of u when $c \geq 0$.

Throughout this paper, we assume that Ω is a domain of class C^2. We denote by d the signed-distance to $\partial\Omega$, which is positive in Ω, i.e. $d(x) := \inf\{|x-y| : y \in \partial\Omega\}$ if $x \in \Omega$ and negative in the complementary of Ω in $\mathbb{R}^N$. As a consequence of the regularity of $\partial\Omega$, d is a C^2- function in a neighborhood $\mathcal{W}$ of $\partial\Omega$. We also denote by ν the C^1- function defined by $\nu(x) = -Dd(x)$ in $\mathcal{W}$; if $x \in \partial\Omega$, then $\nu(x)$ is just the unit outward normal vector to $\partial\Omega$ at x.

2 Preliminary Results on the Stationary Ergodic and Dirichlet Problems

In this section, we deal with properties of the pair (c, φ) solution of (1.8). We start by recalling the main results concerning (c, φ) which were proved in [14] and [15]. Next, we provide some useful properties of c giving a relationship between the stationary ergodic problem (1.8) and the stationary Dirichlet problem (1.5)-(1.6). We end the section with an example to illustrate the possibly unsolvability of (1.5)-(1.6).

We start with

Theorem 2.1 (On the Stationary Ergodic Problem)
Assume that $1 < m \leq 2$ and that $f \in C(\overline{\Omega})$. There exists a unique constant $c \in \mathbb{R}$ such that the problem (1.8) has a solution $\varphi \in W^{2,p}_{loc}(\Omega)$ for every $p > 1$. This solution is unique up to an additive constant and satisfies the following properties

(i) If $1 < m < 2$, then

$$\varphi(x) = C^* d^{-\frac{2-m}{m-1}}(x)(1+o(1)) \quad \textit{as} \quad d(x) \to 0 \quad , \textit{ with } C^* = \frac{(m-1)^{\frac{m-2}{m-1}}}{(2-m)}. \tag{2.1}$$

(ii) If $m = 2$, then

$$\varphi(x) = |\log d(x)|(1+o(1)) \quad \textit{as} \quad d(x) \to 0. \tag{2.2}$$

(iii) There exists a constant $\Lambda > 0$ depending only on m, Ω and f such that

$$|D\varphi(x)| \leq \Lambda d^{-\frac{1}{m-1}}(x) \qquad \textit{in} \quad \Omega. \tag{2.3}$$

(iv) We have

$$\lim_{d(x)\to 0} d^{\frac{1}{m-1}}(x) D\varphi(x) = c_m\, \nu(x) \tag{2.4}$$

where $c_m = (m-1)^{-\frac{1}{m-1}}$.

As far as the proof of Theorem 2.1 is concerned, we refer the reader to the proofs of [14, Theorems I.1, IV.1 and VI.1] for (i), (ii), (iii) and to [15, Theorem 1.1-B] for (iv).

We continue with further estimates on the solutions of (1.8), which can be deduced by the previous ones.

Lemma 2.1 *Assume* $1 < m \leq 2$, $f \in W^{1,\infty}(\Omega)$ *and let* φ *be a solution of (1.8) given by Theorem 2.1. Then* $\varphi \in C^{2,\beta}_{loc}(\Omega)$ *for all* $\beta \in (0,1)$ *and*
(i) There exists a constant $K_0 > 0$ *such that, for every* $x \in \Omega$, *we have*

$$|D^2\varphi(x)| \leq K_0\, d^{-\alpha-2}(x) \quad \text{with} \quad \alpha = \frac{2-m}{m-1}\,. \tag{2.5}$$

(ii) There exists $K_1 > 0$ *and* $\sigma > 0$ *such that, for every* $x \in \Omega$ *with* $0 \leq d(x) < \sigma$

$$|D^2\varphi(x)| \leq K_1\, |D\varphi(x)|^m \tag{2.6}$$

and

$$\begin{cases} |\varphi(x)| \leq K_1\, |D\varphi(x)|^{2-m} & \text{if } 1 < m < 2, \\ \\ |\varphi(x)| \leq |\log(|D\varphi(x)|)| + K_1 & \text{if } m = 2. \end{cases} \tag{2.7}$$

Proof of Lemma 2.1. We first remark that the $C^{2,\beta}_{loc}$–regularity of the solutions of (1.8) comes from the additional assumption on f ($f \in W^{1,\infty}(\Omega)$) and a standard bootstrap argument : indeed, by Theorem 2.1, φ is in $W^{2,p}_{loc}(\Omega)$ for every $p > 1$, hence in $C^{1,\beta}_{loc}(\Omega)$ for any $\beta \in (0,1)$, and therefore a standard regularity result implies that $\varphi \in C^{2,\beta}_{loc}(\Omega)$ for all $\beta \in (0,1)$ since $|D\varphi|^m$ and f are in $C^{0,\beta}_{loc}(\Omega)$.

In order to prove (i), let $x_0 \in \Omega$, and $r = \frac{d(x_0)}{2}$. We introduce the function $\psi(x) = r^\alpha \varphi(x_0 + rx)$ and compute

$$D\psi(x) = r^{\alpha+1} D\varphi(x_0 + rx) \quad \text{and} \quad D^2\psi(x) = r^{\alpha+2} D^2\varphi(x_0 + rx).$$

Since $m(\alpha+1) = \alpha + 2$, it is easy to see that ψ is a viscosity solution of

$$-\Delta\psi + |D\psi|^m = r^{\alpha+2}(f(x_0 + rx) + c) \quad \text{for any } x \in B(0,1)\,.$$

Since $r^{\alpha+2} f$ and $|Df| r^{\alpha+2}$ are in $L^\infty(\Omega)$, we can use the interior estimates available in [14, Theorem A.1] and obtain

$$\|D\psi\|_{L^\infty(B(0,\frac{1}{2}))} \leq K$$

and then, by elliptic regularity (see e.g. [13]),

$$\|D^2\psi\|_{L^\infty(B(0,\frac{1}{2}))} \leq K'$$

which yields

$$|D^2\varphi(x_0+rx)| \le K' r^{-\alpha-2} \quad \text{ for all } x \in B(0,1/2).$$

By taking $x = 0$ and remembering the definition of r, we obtain (2.5) with $K_0 := 2^{\alpha+2}K'$. The estimate (2.6) is a consequence of (2.4) which implies

$$|D\varphi(x)| \ge K\, d^{-\frac{1}{m-1}}(x) \tag{2.8}$$

for some constant $K > 0$ and for x in a suitable neighborhood of $\partial\Omega$. Since $\frac{1}{m-1} = \frac{\alpha+2}{m}$, combining (2.8) with (2.5) we obtain

$$|D^2\varphi(x)| \le K_0\, d^{-\alpha-2}(x) = K_0 d^{-\frac{m}{m-1}}(x) \le K_0 \left(\frac{1}{K}|D\varphi(x)|\right)^m$$

hence (2.6) for any $K_1 > K_0/K^m$. As far as (2.7) is concerned, we use (2.1) and (2.8) if $m < 2$ and (2.2) and (2.8) if $m = 2$ obtaining that (2.7) holds true for some $K_1 > 0$. □

We continue by showing monotonicity and stability properties of the ergodic constant c with respect to the domain.

Proposition 2.1

(i) Let Ω' be an open bounded subset of $\mathbb{R}^N$ such that $\Omega \subseteq \Omega'$. Let c_Ω and $c_{\Omega'}$ be the ergodic constants associated to (1.8) in Ω and Ω' respectively. Then we have

$$c_\Omega \le c_{\Omega'}. \tag{2.9}$$

(ii) Let Ω' be an open bounded subset of $\mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$. The respective ergodic constants c_Ω and $c_{\Omega'}$ of Ω and Ω' satisfy

$$c_\Omega < c_{\Omega'}. \tag{2.10}$$

(iii) Moreover, the ergodic constant c depends continuously on Ω. Otherwise said, for $0 < \eta < 1$, if c_η is the ergodic constant in (1.8) set in $\Omega + B(0,\eta)$[1] then

$$c_\eta \downarrow c_\Omega \quad \text{as} \quad \eta \downarrow 0 \tag{2.11}$$

where $B(0,\eta)$ is a ball of radius η.

[1] $\Omega + B(0,\eta) := \{x \in \mathbb{R}^N : d(x) > -\eta\}$.

Proof of Proposition 2.1. (i) Let $(c_\Omega, \varphi_\Omega)$ and $(c_{\Omega'}, \varphi_{\Omega'})$ be the pair of solutions of the ergodic problem (1.8) in Ω and Ω' respectively. From what we obtained above, the constants c_Ω and $c_{\Omega'}$ are unique whereas the functions φ_Ω and $\varphi_{\Omega'}$ are unique up to a constant. We study $\max_{\overline{\Omega}}(\mu\varphi_{\Omega'} - \varphi_\Omega)$ for some $\mu \in (0,1)$ close to 1. Now, we need to show that the maximum is achieved inside Ω.

Observe that, since $\Omega \subseteq \Omega'$, it could happen that $\partial\Omega \cap \partial\Omega' \neq \emptyset$, meaning that the two domains Ω and Ω' touch at some points. Due to the behavior of $\varphi_{\Omega'}$ and φ_Ω near $\partial\Omega'$ and $\partial\Omega$ respectively, we first deal with points on $\partial\Omega$ which do not belong to $\partial\Omega'$ and next we treat common points of $\partial\Omega$ and $\partial\Omega'$.

We pick any $x_0 \in \partial\Omega$ such that $x_0 \notin \partial\Omega'$; in this case, since $\varphi_{\Omega'}$ is bounded in $\overline{\Omega} \setminus (\partial\Omega \cap \partial\Omega')$, it follows that

$$\lim_{x \to x_0} (\mu\varphi_{\Omega'} - \varphi_\Omega)(x) = \mu\varphi_{\Omega'}(x_0) - \lim_{x \to x_0} \varphi_\Omega(x) = -\infty.$$

On the other hand, if $x_0 \in \partial\Omega \cap \partial\Omega'$, using the asymptotic behavior (2.1) and that, if $d_{\partial\Omega'}$ denotes the distance to $\partial\Omega'$, $d_{\partial\Omega'}(x) \geq d(x)$ we have, when $1 < m < 2$

$$(\mu\varphi_{\Omega'} - \varphi_\Omega)(x) \leq C^*(\mu - 1)d^{-\frac{2-m}{m-1}}(x)(1 + o(1)) \quad \text{ as } x \to x_0\,,$$

hence $(\mu\varphi_{\Omega'} - \varphi_\Omega)(x) \to -\infty$ in this case too. When $m = 2$, the same conclusion holds by the use of (2.2). Therefore, in any case it follows that $(\mu\varphi_{\Omega'} - \varphi_\Omega)$ has a maximum point $x_\mu \in \Omega$. Going back to the equations solved by $\mu\varphi_{\Omega'}$ and φ_Ω, we obtain:

$$\begin{aligned} -\Delta(\mu\varphi_{\Omega'})(x_\mu) + |D(\mu\varphi_{\Omega'})(x_\mu)|^m &= -\mu\Delta\varphi_{\Omega'}(x_\mu) + \mu^m|D\varphi_{\Omega'}(x_\mu)|^m \\ &\leq \mu(-\Delta\varphi_{\Omega'}(x_\mu) + |D\varphi_{\Omega'}(x_\mu)|^m) \\ &= \mu f(x_\mu) + \mu c_{\Omega'} \end{aligned} \tag{2.12}$$

and

$$-\Delta\varphi_\Omega(x_\mu) + |D\varphi_\Omega(x_\mu)|^m = f(x_\mu) + c_\Omega. \tag{2.13}$$

By subtracting (2.13) from (2.12) and using the following properties: $D(\mu\varphi_{\Omega'})(x_\mu) = D\varphi_\Omega(x_\mu)$ and $\Delta(\mu\varphi_{\Omega'} - \varphi_\Omega)(x_\mu) \leq 0$, one gets: $c_\Omega \leq \mu c_{\Omega'} + (\mu - 1)f(x_\mu)$ and (2.9) follows by sending μ to 1.

(ii) Let Ω' be such that $\Omega \subset\subset \Omega'$; from (2.9) it follows that $c_\Omega \leq c_{\Omega'}$. Moreover, we would like to show that for this case, we have: $c_\Omega < c_{\Omega'}$; to do so, we assume the contrary by setting $c_\Omega = c_{\Omega'}$. Using the boundedness of $\varphi_{\Omega'}$ on $\overline{\Omega}$ and behavior of φ_Ω near $\partial\Omega$, it is easy to see that $\varphi_{\Omega'} - \varphi_\Omega \to -\infty$ on $\partial\Omega$, meaning that $\varphi_{\Omega'} - \varphi_\Omega$ achieves its global maximum on $\overline{\Omega}$ at some $\bar{x}$ inside Ω. On the other hand, the convexity of $p \mapsto |p|^m$ yields

$$|D\varphi_{\Omega'}|^m \geq |D\varphi_\Omega|^m + m|D\varphi_\Omega|^{m-2}D\varphi_\Omega \cdot (D(\varphi_{\Omega'} - \varphi_\Omega)).$$

With this argument and using the local bounds on $|D\varphi_\Omega|$ in (2.3), we have that $\varphi_{\Omega'} - \varphi_\Omega$ solves

$$-\Delta(\varphi_{\Omega'} - \varphi_\Omega) - C(x)|D(\varphi_{\Omega'} - \varphi_\Omega)| \leq 0 \quad \text{in } \Omega \tag{2.14}$$

for some $C(x) > 0$ which is bounded in any compact subset. Applying the Strong Maximum Principle (see [16, Lemma 2.1]) we find that $\varphi_{\Omega'} - \varphi_\Omega$ is constant in $\overline{\Omega}$ which clearly leads to a contradiction since $\varphi_{\Omega'} - \varphi_\Omega$ blows up on the boundary $\partial\Omega$, whence (2.10) holds.

(iii) Now, we turn to the proof of the continuous dependence of c and φ in Ω, namely (2.11). Let $0 < \eta < \eta' < 1$, since $\Omega + B(0,\eta)$ is a strict subset of $\Omega + B(0,\eta')$, we find from (2.10) that $c_\eta < c_{\eta'}$, meaning that the sequence $(c_\eta)_{0<\eta<1}$ decreases as η goes to 0. On the other hand, knowing that $\Omega \subset \Omega + B(0,\eta)$ for any $0 < \eta < 1$, again by (2.9), we obtain $c_\Omega < c_\eta$. Therefore, $(c_\eta)_{0<\eta<1}$ is convergent in $\mathbb{R}$ and $c_\eta \downarrow c_0 = c_\Omega$ as $\eta \to 0^+$ by using the arguments on stability of (1.8) and the uniqueness of the ergodic constant. □

Hereafter, to stress on the dependence of the initial boundary value problem (1.1)-(1.2)-(1.3) on f, u_0 and g, we denote it by $E(\Omega, f, g, u_0)$ where "E" stands for Evolution. Likewise, we denote by $S(\Omega, f, g)$, the boundary value problem (1.5)-(1.6) where "S" stands for Stationary.

Now, we link the problems (1.8) and $S(\Omega, f, g)$ by pointing out that the existence of a solution for $S(\Omega, f, g)$ depends on the ergodic constant c in (1.8).

Proposition 2.2 *Let c be the ergodic constant associated to (1.8) and let us denote by $\mathcal{S}$ the set of all $\lambda \in \mathbb{R}$ such that there exists a viscosity subsolution $\phi \in C(\overline{\Omega})$ of*

$$-\Delta\phi + |D\phi|^m \leq f + \lambda \quad \text{in } \Omega. \tag{2.15}$$

Then

$$c = \inf\ \{\lambda : \lambda \in \mathcal{S}\} \tag{2.16}$$

and the infimum in (2.16) is not attained. Moreover, a necessary and sufficient condition for $S(\Omega, f, g)$ to have a viscosity solution is that $c < 0$.

Remark 2.1 *It is worth mentioning that by a viscosity solution of $S(\Omega, f, g)$, we mean a $C(\overline{\Omega})$- function satisfying (1.5) in the viscosity sense and (1.6) pointwisely. Indeed, when $1 < m \leq 2$, there is no loss of boundary conditions as specified in [4, Propositions 3.1 and 3.2], and $S(\Omega, f, g)$ is a classical Dirichlet problem.*

Proof of Proposition 2.2. In order to prove (2.16), we first remark that $\mathcal{S} \neq \emptyset$; indeed, it is easy to see that $\|f\|_\infty \in \mathcal{S}$ because $\varphi \equiv 0$ is a subsolution of

$$-\Delta\varphi + |D\varphi|^m \leq f + \|f\|_\infty \quad \text{in } \Omega.$$

On the other hand, we pick any $\lambda \in \mathcal{S}$ and denote by $\phi_\lambda \in C(\overline{\Omega})$, the function satisfying $-\Delta\phi_\lambda + |D\phi_\lambda|^m \leq f + \lambda$ in the viscosity sense. We argue as in the proof of Proposition 2.1, by studying $\max_{\overline{\Omega}}(\phi_\lambda - \varphi)$, in order to reach the conclusion that $c \leq \lambda$. Therefore, the right hand side of (2.16), which we denote by λ^*, is well-defined and $c \leq \lambda^*$.

Now, we assume that $c < \lambda^*$, by keeping the notations of Proposition 2.1-(iii), there exists some $0 < \eta_0 \ll 1$ such that, for any $0 < \eta \leq \eta_0$, we have $c_\eta \in]c, \frac{c+\lambda^*}{2}[$. But, if φ_η solves

$$-\Delta\varphi_\eta + |D\varphi_\eta|^m = f + c_\eta \quad \text{ in } \Omega_\eta := \Omega + B(0,\eta)$$

then φ_η is in $W^{2,p}_{loc}(\Omega_\eta)$ for any $p > 1$ and therefore $\varphi_\eta \in C(\overline{\Omega})$; thus φ_η is a viscosity subsolution of the equation associated to c_η in Ω and $c_\eta \in \mathcal{S}$. This clearly is a contradiction since $c_\eta < \lambda^*$ and we conclude that $c = \lambda^*$ and then (2.16) holds.

Moreover, the infimum in (2.16) is not attained: indeed, if $\varphi_c \in C(\overline{\Omega})$ is a subsolution of the c-equation (2.15) and if φ is a solution of the ergodic problem (1.8), then the $\max_{\overline{\Omega}}(\varphi_c - \varphi)$ is achieved at some point of Ω since φ blows up on $\partial\Omega$ and applying the Strong Maximum Principle exactly as in the proof of Proposition 2.1 (ii), we find that $\varphi_c - \varphi$ is constant in Ω, a contradiction since $\varphi_c \in C(\overline{\Omega})$ and $\varphi(x) \to +\infty$ as $x \to \partial\Omega$.

Now we turn to the proof of the second part of the result. If the generalized Dirichlet problem $S(\Omega, f, g)$ has a bounded viscosity solution, then $0 \in \mathcal{S}$, and then the first part implies that necessarily $c < 0$.

Conversely, we assume that $c < 0$ and prove that $S(\Omega, f, g)$ has a unique bounded viscosity solution. To do so, we are going to apply the Perron's method (cf. [11], [8] and [9]) and in order to do it, we have to build sub and supersolution for $S(\Omega, f, g)$.

For the subsolution, since $c < 0$, we find from (2.11) (see Proposition 2.1), the existence of $0 < \eta_0 < 1$ such that $c_\eta \leq 0$ for all $\eta \leq \eta_0$ where c_η is such that there exists a function φ_η which is a viscosity solution of the ergodic problem (1.8) in $\Omega + B(0,\eta)$ for all $\eta \leq \eta_0$.

Since g is bounded on $\partial\Omega$ whereas φ_η is bounded in $\overline{\Omega}$, there exists $K > 0$ such that $\varphi_\eta - K \leq g$ on $\partial\Omega$. Therefore we have, at the same time, a subsolution of $S(\Omega, f, g)$ required in the Perron's method, but also a strict subsolution of $S(\Omega, f, g)$ which implies a comparison result for $S(\Omega, f, g)$ (See [16, Theorem 2.3]).

For the supersolution, it is easy to see that $l(x) = |x - x_0|^2 + \|g\|_\infty + 1$ is a supersolution of $S(\Omega, f, g)$ for some $x_0 \in \mathbb{R}^N$ such that $B(x_0, (\|f\|_\infty + 2N)^{1/m}) \cap \overline{\Omega} = \emptyset$. Using the (strict) subsolution $\varphi_\eta - K$ built above, the existence of the solution therefore follows by combining the comparison result and the classical Perron's method. □

We end this section by giving an example showing that the stationary Dirichlet Problem $S(\Omega, f, g)$ does not always have a solution.

Example 2.1 We fix some $R > 0$ and consider the one-dimensional equation

$$-\eta'' + |\eta'|^m = -C^m \quad \text{in} \quad (-R, R) \tag{2.17}$$

with $C > 0$. If η solves (2.17), then after some easy change of variable and computations, we find that

$$\frac{1}{C^{m-1}} \int_{\frac{\eta'(0)}{C}}^{\frac{\eta'(x)}{C}} \frac{ds}{|s|^m + 1} = x.$$

It follows that

$$C^{m-1} x = \int_{\frac{\eta'(0)}{C}}^{\frac{\eta'(x)}{C}} \frac{ds}{|s|^m + 1} \leq \int_{-\infty}^{+\infty} \frac{ds}{|s|^m + 1} := K.$$

Therefore, letting $x \to R$, we obtain :

$$C^{m-1} \leq \frac{K}{R} \tag{2.18}$$

Since $R > 0$ is fixed and $m > 1$, by choosing $C > 0$ large enough, we find that the inequality (2.18) cannot hold and we conclude that the ordinary differential equation (2.17) is not solvable for large C. Therefore, (1.5)-(1.6) is not always solvable as specified above when we considered $f := -C^m < 0$. $\square$

3 Asymptotic Behavior for the Parabolic Problem

This section is devoted to the description of the asymptotic behavior of the solution u of the initial boundary-value problem $E(\Omega, f, g, u_0)$.

3.1 Convergence to the solution of the Stationary Dirichlet Problem

We start with the case that u converges to the solution of the stationary Dirichlet problem. The main result of this subsection is the following.

Theorem 3.1 *Let $1 < m \leq 2$, $f \in C(\overline{\Omega})$, $u_0 \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Assume that $S(\Omega, f, g)$ has a unique viscosity solution which we denote by u_∞. Let u be the unique continuous viscosity solution of $E(\Omega, f, g, u_0)$. Then, as $t \to +\infty$,*

$$u(x,t) \to u_\infty(x) \quad \text{uniformly for all } x \in \overline{\Omega}. \tag{3.1}$$

Proof of Theorem 3.1. We first notice that u is uniformly bounded on $\overline{\Omega}\times[0,+\infty)$. Indeed, since u_∞ solves $S(\Omega,f,g)$, it is straighforward that u_∞ solves $E(\Omega,f,g,u_\infty)$ and by [16, Corollary 2.1], one gets

$$\|u(x,t)-u_\infty\|_\infty \le \|u_0-u_\infty\|_\infty.$$

Next, from the uniform boundedness of u obtained above, we use the half-relaxed limits method to say that the functions

$$\overline{u}(x)=\limsup_{\substack{y\to x\\ t\to+\infty}} u(y,t) \quad \text{and} \quad \underline{u}(x)=\liminf_{\substack{y\to x\\ t\to+\infty}} u(y,t)$$

are respectively subsolution and supersolution of (1.5)-(1.6). By definition of the half limits, we have $\underline{u}\le\overline{u}$ on Ω but given that the ergodic constant c is strictly negative (see Proposition 2.2), there exists a strict subsolution for $S(\Omega,f,g)$ and we can therefore apply [16, Theorem 2.3] to obtain $\overline{u}\le\underline{u}$ on $\overline{\Omega}$. It is worth noticing that since no loss of boundary condition could happen (see [4, Propositions 3.1 and 3.2]), the Dirichlet condition (1.6) is understood in the classical sense: $\underline{u}\le g$ on $\partial\Omega$ and $\overline{u}\ge g$ on $\partial\Omega$. Thus, we obtain $u_\infty=\underline{u}=\overline{u}$ on $\overline{\Omega}$, meaning that (3.1) holds.

□

3.2 Convergence of $\dfrac{u(x,t)}{t}$

In this subsection, we assume that $c\ge 0$. Indeed, due to Proposition 2.2, the case $c<0$ is already described by Theorem 3.1. The main result of this section is the following convergence of $\frac{u(x,t)}{t}$ to $-c$ which always holds true, even if at different rates.

Theorem 3.2 *Let $1<m\le 2$, $f\in W^{1,\infty}(\Omega)$, $u_0\in C(\overline{\Omega})$ and $g\in C(\partial\Omega)$. Let u be the unique continuous viscosity solution of $E(\Omega,f,g,u_0)$. Then we have*

$$\frac{u(x,t)}{t}\to -c \quad \textit{locally uniformly on } \Omega \quad \textit{as } t\to+\infty\,.$$

In particular, for any compact set $K\subset\Omega$ there exists a constant M_K such that, as $t\to+\infty$,

(i) if $c>0$ then

$$\begin{cases} \|\frac{u(x,t)}{t}+c\|_{C(K)}\le \frac{M_K}{t} & \textit{when } \frac{3}{2}<m\le 2\\ \|\frac{u(x,t)}{t}+c\|_{C(K)}\le M_K\,\frac{\log t}{t} & \textit{when } m=\frac{3}{2}\\ \|\frac{u(x,t)}{t}+c\|_{C(K)}\le \frac{M_K}{t^{\frac{m-1}{2-m}}} & \textit{when } 1<m<\frac{3}{2}\end{cases} \tag{3.2}$$

(ii) if $c = 0$ then

$$\begin{cases} \|\frac{u(x,t)}{t}\|_{C(K)} \le M_K \frac{\log t}{t} & \textit{when } m = 2 \\ \|\frac{u(x,t)}{t}\|_{C(K)} \le \frac{M_K}{t^{2-m}} & \textit{when } 1 < m < 2. \end{cases} \tag{3.3}$$

Theorem 3.2 is a direct consequence of the following estimate. For technical reasons, it will be convenient to consider the unique solution φ_0 of (1.8) satisfying $\min_{\overline{\Omega}} \varphi_0 = 0$. In this way, as said above, any solution φ of (1.8) is described as $\varphi = \varphi_0 + k$ for some constant k.

Theorem 3.3 *Let Ω be a domain of class C^2. Let $f \in W^{1,\infty}(\Omega)$. Then we have, as $t \to +\infty$*

(i) If $c > 0$ there exists a constant $M > 0$ such that

$$\begin{cases} u + ct \ge \gamma(t)\varphi_0(x - \mu(t)n(x)) - M & \textit{if } \frac{3}{2} < m \le 2, \\ u + ct \ge \gamma(t)\varphi_0(x - \mu(t)n(x)) - M \log t & \textit{if } m = \frac{3}{2}, \\ u + ct \ge \gamma(t)\varphi_0(x - \mu(t)n(x)) - M\, t^{\frac{3-2m}{2-m}} & \textit{if } 1 < m < \frac{3}{2}, \end{cases} \tag{3.4}$$

where $n(x)$ is a vector field such that $n(x) \cdot Dd(x) < 0$ and $\gamma(t)$, $\mu(t)$ are positive continuous functions such that $\gamma(t)$ is increasing and $\gamma(t) \uparrow 1$, $\mu(t)$ is decreasing and $\mu(t) \downarrow 0$ as $t \to \infty$.

(ii) If $c = 0$ there exists a constant $M > 0$ such that

$$\begin{cases} u \ge \gamma(t)\varphi_0(x - \mu(t)n(x)) - M \log t & \textit{if } m = 2, \\ u \ge \gamma(t)\varphi_0(x - \mu(t)n(x)) - M\, t^{2-m} & \textit{if } 1 < m < 2, \end{cases} \tag{3.5}$$

where $\gamma(t) \uparrow 1$, $\mu(t) \downarrow 0$ as $t \to +\infty$.

Knowing that φ_0 is nonnegative and solves (1.8), it is easy to observe that $\varphi_0 + \|u_0\|_{L^\infty(\Omega)} - ct$ is always a super-solution of (1.1)-(1.2)-(1.3). Therefore, by means of Strong Comparison Principle for $E(\Omega, f, g, u_0)$ (see [4, Theorem 3.1] or [16, Theorem 2.1]), we have for all $1 < m \le 2$,

$$u(x,t) + ct \le \varphi_0(x) + \|u_0\|_{L^\infty(\Omega)} \quad \text{in } \Omega \times [0, +\infty). \tag{3.6}$$

Theorem 3.3 provides some estimates on $u(x,t) + ct$ when $c \ge 0$ which can be used to locally bound $u + ct$ from below in order to complement (3.6).

We refer the reader to the Appendix for the proof of Theorem 3.3 and turn to the

Proof of Theorem 3.2. It follows from (3.6) that, in any compact set K, we have $\frac{u(x,t)}{t} + c \leq \frac{M_K}{t}$ for some constant M_K. On the other hand the estimate from below varies according to the values of m and whether $c = 0$ or $c > 0$. Using the estimates (3.4) and (3.5) in Theorem 3.3 and taking into account that φ_0 is locally bounded, we immediately deduce (3.2) and (3.3) and in particular that $\frac{u(x,t)}{t} \to -c$ locally uniformly as $t \to +\infty$.

$\square$

In the case that $c > 0$ and $1 < m \leq \frac{3}{2}$, or if $c = 0$, the rates of convergence given above cannot in general be improved, as we will see later (Theorem 4.1).

3.3 Convergence to the Stationary Ergodic Problem when $\frac{3}{2} < m \leq 2$ and $c > 0$

The goal of this section is to describe the asymptotic behavior of the solution u of the generalized initial boundary-value problem (1.1)-(1.2)-(1.3) in connection with the stationary ergodic problem (1.8). Our main result is the following:

Theorem 3.4 (Convergence result)
Let $f \in W^{1,\infty}(\Omega)$, $u_0 \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Let $c \in \mathbb{R}$ and $\varphi_0 \in W^{1,\infty}_{loc}(\Omega)$ be the unique viscosity solution of the ergodic problem (1.8) such that $\min_\Omega \varphi_0 = 0$. Let u be the unique continuous viscosity solution of $E(\Omega, f, g, u_0)$. Assume that $c > 0$ and $\frac{3}{2} < m \leq 2$, then we have

$$u(x,t) + ct \to \varphi_0(x) + C \quad \textit{locally uniformly in } \Omega, \quad \textit{as } t \to +\infty \tag{3.7}$$

for some constant C depending on Ω, f, c, u_0, and g.

Recalling that all solutions of problem (1.8) only differ by addition of a constant, one can rephrase (3.7) saying that $u(x,t) + ct$ converges to a solution of the ergodic problem (1.8). We choose to represent all solutions as $\varphi_0 + C$ in order to emphasize that there is precisely one constant C, depending on the data, which determines the asymptotic limit of $u + ct - \varphi_0$.

We also notice that by combining (3.6) and (3.4), it follows that

$$\gamma(t)\varphi_0(x - \mu(t)n(x)) - M \leq u(x,t) + ct \leq \varphi_0(x) + \|u_0\|_{L^\infty(\Omega)} \quad \text{in } \Omega \times [0, +\infty). \tag{3.8}$$

If $\lim\limits_{t \to +\infty} (u(x,t) + ct)$ exists, then by sending $t \to +\infty$ in (3.8), we obtain

$$\varphi_0(x) - M \leq \lim_{t \to +\infty} (u(x,t) + ct) \leq \varphi_0(x) + \|u_0\|_{L^\infty(\Omega)}$$

From the blow-up behavior of φ_0 near the boundary, it obviously follows that

$$\lim_{t\to+\infty} (u(x,t)+ct) \to +\infty \quad \text{as } x \to \partial\Omega \tag{3.9}$$

which gives the expected behavior near the boundary since there is no loss of boundary condition for all $1 < m \leq 2$.

The rest of this section is devoted to the proof of Theorem 3.4.

Hereafter, for all $(x,t) \in \overline{\Omega} \times [0,+\infty)$, we set

$$v(x,t) := u(x,t) + ct \quad \text{and} \quad w(x,t) = v(x,t) - \varphi_0(x).$$

The function $v(\cdot, \cdot + t)$ solves $E(\Omega, f+c, g+ct, v(\cdot,t))$ whereas φ_0 is a supersolution of $E(\Omega, f+c, g+ct, \varphi_0)$ since φ_0 solves (1.8). From the comparison principle, one gets, for all $x \in \overline{\Omega}$ and $s \geq t \geq 0$,

$$\max_{x\in\overline{\Omega}} w(x,s) \leq \max_{x\in\overline{\Omega}} w(x,t).$$

It follows that the function $t \mapsto m(t) := \max_{x\in\overline{\Omega}} w(x,t)$ is non-increasing. Moreover, as a by-product of (3.8), $m(t)$ is bounded. Therefore $m(t) \downarrow \overline{m}$ as $t \to +\infty$.

Now, since we want to deal with bounded functions and clearly $w(x,t) \to -\infty$ when $x \to \partial\Omega$, we choose any constant $K > |\overline{m}|$ and set

$$z(x,t) := \sup[w(x,t), -K].$$

We notice that, since $m(t)$ is bounded from below by $\overline{m}$, we still have

$$m(t) := \max_{x\in\overline{\Omega}} w(x,t) = \max_{x\in\overline{\Omega}} [\sup(w(x,t), -K)] = \max_{x\in\overline{\Omega}} z(x,t) \quad \text{for all} \quad t \geq 0,$$

and from (3.8), z is uniformly bounded on $\overline{\Omega}\times[0,+\infty)$. Moreover, since $w(x,t) \to -\infty$ as $x \to \partial\Omega$ for all $t \geq 0$, we have $z = -K$ on $\partial\Omega \times [0,+\infty)$ and we also remark that z is a viscosity solution of

$$\begin{cases} \phi_t(x,t) - \Delta\phi(x,t) + \mathcal{H}(x)\cdot D\phi(x,t) & \leq 0 \qquad \text{in } \Omega \times (0,+\infty) \\ \phi(x,t) + K & = 0 \qquad \text{on } \partial\Omega \times (0,+\infty) \end{cases} \tag{3.10}$$

where

$$\mathcal{H}(x) = m|D\varphi_0(x)|^{m-2} D\varphi_0(x). \tag{3.11}$$

Indeed, by the convexity of $p \mapsto |p|^m$, w is a subsolution of the above equation and we recall that the maximum of two subsolutions is a subsolution.

In order to have an equation with continuous coefficients, we introduce the operator

$$\mathcal{L}\phi(x,t) := d(x)\phi_t(x,t) - d(x)\Delta\phi(x,t) + d(x)\mathcal{H}(x)\cdot D\phi(x,t)\,, \tag{3.12}$$

just obtained from the previous equation by multiplying by $d(x)$. From (2.4), it is easy to note that

$$|D\varphi_0(x)|^{m-2}D\varphi_0(x) = -(m-1)^{-1}\frac{Dd(x)}{d(x)} + o\left(\frac{1}{d(x)}\right) \quad \text{as} \quad d(x)\to 0. \tag{3.13}$$

It is therefore obvious to see that $\mathcal{H}$ has a singularity on $\partial\Omega$ whereas $d(x)\mathcal{H}(x)$ can be extended as a continuous function on $\overline{\Omega}$. Using this new operator, we have

$$\mathcal{L}z \le 0 \qquad \text{in } \Omega\times(0,+\infty)\,,$$

which replaces the subsolution property in (3.10).

To complete the proof of (3.7), we first give the following local Hölder continuity of the unique solution v of $E(\Omega, f+c, g+ct, u_0)$, with respect to its x and t variables.

Proposition 3.1 *Let $1 < m \le 2$ and v be the unique continuous viscosity solution of $E(\Omega, f+c, g+ct, u_0)$. Then*

- *(i) For all $\eta > 0$ and for all $\nu \in (0,1)$, we have $v(\cdot,t) \in C^{0,\nu}_{loc}(\Omega)$ for all $t \ge \eta$. Moreover, if $K_\delta := \{y \in \Omega \,:\, d(y) \ge \delta\}$ for any $\delta > 0$, then, for all $t \ge \eta$, the $C^{0,\nu}$-norm of $v(\cdot,t)$ on K_δ depends only on ν, δ, η, $\|f\|_\infty$ and the L^∞-norm of v on $K_{\delta/2}\times[\frac{\eta}{2},+\infty)$.*
- *(ii) For any $x \in \Omega$, $v(x,\cdot) \in C^{0,\frac{\nu}{2}}_{loc}(0,+\infty)$ for any $\eta > 0$. Moreover the $C^{0,\frac{\nu}{2}}$-norm of $v(x,\cdot)$ in $[\eta,+\infty)$ depends only on ν, η, $\|f\|_\infty$ and the L^∞-norm of v on $K_{d(x)/2}\times[\eta/2,+\infty)$.*

We postpone the proof of Proposition 3.1-(i) to the Appendix and refer the reader to [3, Lemma 9.1] for the proof of Proposition 3.1-(ii).

In order to prove (3.7), we will use the following

Lemma 3.1 *$z(x,t)\to\overline{m}$ locally uniformly in Ω as $t\to+\infty$.*

Indeed, from Lemma 3.1, since $K > |\overline{m}|$, it easily follows that $w(x,t)\to\overline{m}$ locally uniformly in Ω as $t\to+\infty$, and we get (3.7) with $C=\overline{m}$.

Proof of Lemma 3.1. We split it into several parts.
1. Let $\hat{x}\in\Omega$ and $r>0$ such that $\overline{B}(\hat{x},r)\subset\Omega$. From Proposition 3.1, it follows that there exists a sequence $t_n\to+\infty$ as $n\to+\infty$ and a function $\hat{z}$ such that

$$z(x,t_n)\to\hat{z}(x) \quad \text{uniformly on } \overline{B}(\hat{x},r).$$

We define

$$z_n(x,t) := z(x, t+t_n) \quad \text{ for all } (x,t) \in \overline{\Omega} \times (-t_n, +\infty)$$

and notice that z_n is a viscosity solution of (3.10) on $\overline{\Omega} \times (-t_n, +\infty)$. From the uniform boundedness of z, we derive the one of z_n and the half-relaxed limits method implies that

$$\tilde{z}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} z_n(y,s)$$

is a viscosity subsolution of the generalized Dirichlet problem

$$\begin{cases} \mathcal{L}\tilde{z}(x,t) & \leq 0 \qquad \text{in } \Omega \times (-\infty, +\infty) \\ \min\{\mathcal{L}\tilde{z}(x,t), \tilde{z}(x,t)+K\} & \leq 0 \qquad \text{on } \partial\Omega \times (-\infty, +\infty) \end{cases} \tag{3.14}$$

2. We claim that $\max_{x\in\overline{\Omega}} \tilde{z}(x,t) = \overline{m}$. Indeed, on one hand, for all $n \in \mathbb{N}$, there exists $x_n \in \overline{\Omega}$ such that

$$z(x_n, t+t_n) = \max_{x\in\overline{\Omega}} z(x, t+t_n) = \max_{x\in\overline{\Omega}} z_n(x,t)\ .$$

From the compactness of $\overline{\Omega}$, we have, up to subsequence, $x_n \to \tilde{x}$ for some $\tilde{x} \in \overline{\Omega}$. It follows that

$$\begin{aligned} \overline{m} = \lim_{n\to+\infty} m(t+t_n) &= \lim_{n\to+\infty} [\max_{x\in\overline{\Omega}} z(x, t+t_n)] \\ &= \lim_{n\to+\infty} z_n(x_n, t) \\ &\leq \tilde{z}(\tilde{x}, t) \leq \max_{x\in\overline{\Omega}} \tilde{z}(x,t). \end{aligned}$$

On the other hand, for all $x \in \overline{\Omega}$, we have

$$\tilde{z}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n\to+\infty}} z_n(y,s) \leq \limsup_{(s,n)\to(t,+\infty)} m(s+t_n) = \overline{m}$$

and the claim is proved.

3. Let $t_0 > 0$ be fixed; as a consequence of step 2, there exists a point $x_0 \in \overline{\Omega}$ such that

$$\max_{x\in\overline{\Omega}} \tilde{z}(x, t_0) = \tilde{z}(x_0, t_0) = \overline{m}. \tag{3.15}$$

To end this proof, it is enough to prove that (for some $t_0 > 0$) x_0 lies inside Ω.

Indeed, if $x_0 \in \Omega$, then the Parabolic Strong Maximum Principle and (3.15) imply

$$\tilde{z} = \overline{m} \quad \text{ in } \Omega \times (-\infty, t_0] \tag{3.16}$$

Therefore, by taking (3.16) into account, for any $x \in \overline{B}(\hat{x}, r)$, it would follow that

$$\begin{aligned}
\overline{m} = \tilde{z}(x,0) &= \limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}} z_n(y,s)\\
&= \limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}} z(y, s+t_n)\\
&= \limsup_{\substack{(y,s)\to(x,0)\\ n\to+\infty}} [z(y,s+t_n) - z(y,t_n)] + \limsup_{\substack{y\to x\\ n\to+\infty}} z(y,t_n)\\
&\le \limsup_{(y,s)\to(x,0)} \left[\|z(y,\cdot)\|_{C^{0,\frac{\nu}{2}}_{loc}(0,+\infty)} |s|^{\frac{\nu}{2}} \right] + \hat{z}(x) = \hat{z}(x)
\end{aligned}$$

with the last inequality following from Proposition 3.1-(ii). Finally, since $\hat{z}$ is independent of the sequence $(t_n)_n$, we have obtained

$$z(x,t) \to \overline{m} \quad \text{uniformly on } \overline{B}(\hat{x}, r) \text{ as } t \to +\infty,$$

for any $\hat{x} \in \Omega$ and $r > 0$ such that $\overline{B}(\hat{x}, r) \subset \Omega$, thus proving the statement of Lemma 3.1.

4. Now, we are going to prove that, for some $t_0 > 0$, $\tilde{z}(x, t_0)$ has a maximum point x_0 in Ω. For that purpose, let us assume by contradiction that

$$x_0 \in \partial\Omega \quad \text{and} \quad \tilde{z} < \overline{m} \quad \text{in } \Omega \times (0, t_0]. \tag{3.17}$$

We pick some $0 < \delta < t_0$ and argue in the subset $\Omega^\delta := \{x \in \Omega : 0 < d(x) < \delta\}$ by introducing the function χ defined by

$$\chi(x,t) := \overline{m} + k(e^{-d(x)} - 1) - k(t - t_0) \quad \text{for all } (x,t) \in \overline{\Omega^\delta} \times [t_0 - \delta, t_0]$$

where δ and k are to be chosen later in such a way to obtain

$$\tilde{z} \le \chi \quad \text{in} \quad \overline{\Omega^\delta} \times [t_0 - \delta, t_0]. \tag{3.18}$$

(i) We start by proving that χ satisfies

$$\mathcal{L}\chi(x,t) > 0 \quad \text{in } \Omega^\delta \times (t_0 - \delta, t_0].$$

Computing we have

$$\mathcal{L}\chi(x,t) \;=\; -kd(x) + ke^{-d(x)}[d(x)\Delta d(x) - d(x) - d(x)\mathcal{H}(x) \cdot Dd(x)].$$

By using (3.13), we deduce that

$$\mathcal{L}\chi(x,t) = ke^{-d(x)}[-d(x)\, e^{d(x)} + d(x)(\Delta d(x) - 1) + m/(m-1) + o_\delta(1)]\,.$$

By the regularity of $\partial\Omega$, Δd is bounded in Ω^δ and since $d(x) < \delta$, we can choose $\delta > 0$ small enough such that

$$-d(x)\, e^{d(x)} + d(x)(\Delta d(x) - 1) + m/(m-1) + o_\delta(1) > 0\ ,$$

for all x such that $d(x) \leq \delta$. Hence we obtain

$$\mathcal{L}\chi > 0 \quad \text{on} \quad \Omega^\delta \times (t_0 - \delta, t_0]\,. \tag{3.19}$$

(ii) For all $(x,t) \in \partial\Omega \times (t_0 - \delta, t_0]$, we have $\chi(x,t) = \overline{m} - k(t - t_0) \geq \overline{m}$ and knowing that $\tilde{z} \leq \overline{m}$ in $\overline{\Omega} \times \mathbb{R}$, we easily conclude that

$$\tilde{z} \leq \chi \quad \text{ on } \partial\Omega \times (t_0 - \delta, t_0]. \tag{3.20}$$

(iii) We set $\Gamma_\delta = \{x \in \overline{\Omega} : d(x) = \delta\}$. For any $(x,t) \in \Gamma_\delta \times (t_0 - \delta, t_0]$, we have

$$\chi(x,t) = \overline{m} + k(e^{-\delta} - 1) - k(t - t_0) \geq \overline{m} + k(e^{-\delta} - 1).$$

Moreover, since $\tilde{z}(x,t) < \overline{m}$ in $\Omega \times (0, t_0]$, we have $\tilde{z}(x,t) < \overline{m}$ on $\Gamma_\delta \times [t_0 - \delta, t_0]$. We use the upper semi-continuity of $\tilde{z}$ to define

$$\eta(\delta) := \min_{\Gamma_\delta \times [t_0 - \delta, t_0]} (\overline{m} - \tilde{z})$$

and we find that $\tilde{z}(x,t) \leq \overline{m} - \eta(\delta)$ on $\Gamma_\delta \times [t_0 - \delta, t_0]$. By choosing $k > 0$ such that $k(e^{-\delta} - 1) \geq -\eta(\delta)$, that is

$$0 < k \leq \frac{\eta(\delta)}{1 - e^{-\delta}}, \tag{3.21}$$

one gets

$$\tilde{z} \leq \chi \quad \text{ on } \Gamma_\delta \times (t_0 - \delta, t_0]. \tag{3.22}$$

(iv) For all $x \in \overline{\Omega^\delta}$, we have

$$\chi(x, t_0 - \delta) = \overline{m} + k(e^{-d(x)} - 1) + k\delta \geq \overline{m} + k(e^{-\delta} - 1 + \delta) \geq \overline{m}$$

hence we deduce

$$\tilde{z} < \chi \quad \text{ on } \overline{\Omega^\delta} \times \{t_0 - \delta\}. \tag{3.23}$$

5. Now we use (3.14) in the interval $(t_0 - \delta, t_0)$, together with (3.19), (3.20), (3.22) and (3.23), so that applying the comparison principle on $\overline{\Omega^\delta} \times [t_0 - \delta, t_0]$ we finally conclude that (3.18) holds.

Since $\tilde{z}(x_0,t_0) = \overline{m} = \chi(x_0,t_0)$, it follows from (3.18) that $\tilde{z} - \chi$ achieves its global maximum on $\overline{\Omega^\delta} \times [t_0-\delta, t_0]$ at (x_0,t_0). By using the definition of the viscosity subsolution $\tilde{z}$ of (3.14), it necessarily follows that

$$\min\{\mathcal{L}\chi(x_0,t_0), \overline{m}+K\} \le 0. \tag{3.24}$$

But $\mathcal{L}\chi(x_0,t_0) = \frac{km}{m-1} > 0$ and $\overline{m}+K > 0$, hence we reach a contradiction with (3.24) and we conclude that (3.17) does not hold. Hence, we have that (3.15) holds for some $t_0 > 0$ and $x_0 \in \Omega$, and by Step 3 we deduce that (3.16) holds and the end of the proof follows.

$\square$

4 The Non-Convergence Cases: $c = 0$ or $c > 0$ and $1 < m \le 3/2$

The main result of this section is

Theorem 4.1 *Let Ω be star-shaped with respect to a point $x_0 \in \mathbb{R}^N$. Assume*[2] *that there exist constants $\delta > 0$ and $\mu_0 > 0$ such that we have, for any $r \in [1-\mu_0, 1)$:*

$$f(x) + c + \frac{|f(x_0 + r(x-x_0)) - f(x)|}{(1-r)} \le -\delta \quad \text{on} \quad \overline{\Omega}. \tag{4.1}$$

If $c = 0$ and $1 < m \le 2$, or if $c > 0$ and $1 < m \le \frac{3}{2}$, then

$$u(x,t) + ct \to -\infty \quad \text{as } t \to +\infty \quad \text{locally uniformly in } \Omega. \tag{4.2}$$

More precisely, there exist a solution φ of (1.8) and continuous functions $r(t)$, $H(t)$ such that

$$u(x,t) + ct \le r(t)^{\frac{2-m}{m-1}} \varphi(x_0 + r(t)(x-x_0)) - H(t) \tag{4.3}$$

where $r(t) \uparrow 1$ and $H(t) \to +\infty$ as $t \to +\infty$ with the following rate

(a) If $c > 0$ then

$$\begin{cases} H(t) = O(t^{\frac{3-2m}{2-m}}) & \text{if } 1 < m < \frac{3}{2}, \\ H(t) = O(\log t) & \text{if } m = \frac{3}{2}. \end{cases} \tag{4.4}$$

[2]this assumption is for instance verified when $f \in W^{1,\infty}(\Omega)$ and

$$f(x) + c + \|Df\|_\infty \operatorname{diam}(\Omega) \le -\delta$$

(b) If $c = 0$ then

$$\begin{cases} H(t) = O(t^{2-m}) & \textit{if } 1 < m < 2, \\ H(t) = O(\log t) & \textit{if } m = 2. \end{cases} \tag{4.5}$$

Let us recall that the ergodic constant c depends itself on f, so that the assumptions made in the above theorem are not obviously checked. However, such assumptions can actually happen to be true. To fix the ideas, consider the following Example which provides a specific case where condition (4.1) holds.

Remark 4.1 *Assumption (4.1) is always verified when f is a constant.*

We first consider the case that $f = 0$ and denote by c_0 the corresponding ergodic constant. One can easily deduce that $c_0 < 0$. This is a consequence of the characterization (2.16) in Proposition 2.2: indeed the constant functions are subsolutions of (2.15) with $\lambda = 0$ while Proposition 2.2 yields that the infimum in (2.16) is not attained. Consider now the case that $f = f_0$ is a constant, possibly different from zero. Of course the corresponding ergodic constant now is $c = c_0 - f_0$. For every value of f_0, (4.1) is clearly verified with any $\delta < |c_0|$.

Next, it is not difficult to construct some function f which is not constant and such that (4.1) is verified. In particular, a small perturbation (in Lipschitz norm) of the constant f_0 still verifies (4.1); indeed it is easy to check that the ergodic constant c depends continuously on f (with respect to perturbations in the sup-norm) as a consequence of formula (2.16).

Proof of Theorem 4.1. Without loss of generality, assume that $x_0 = 0$. Moreover observe that we can always replace δ with a smaller value in (4.1) without loss of generality.

In order to prove that (4.3) holds, it is sufficient to prove that $r(t)^{\frac{2-m}{m-1}}\varphi(r(t)x) - H(t)$ is a supersolution of $E(\Omega, f + c, g + ct, u_0)$, since then the estimate (4.3) will follow by an application of the comparison result. Let then φ be a solution of (1.8), we define the function φ_r on $\overline{\Omega} \times [0, +\infty)$ as follows:

$$\varphi_r(x,t) = r(t)^{\frac{2-m}{m-1}}\varphi(r(t)x)$$

where $r(t)$ will be chosen below in a way that $0 < r(t) < 1$ and $r(t) \uparrow 1$ as $t \to +\infty$. In particular, we fix

$$r(0) = 1 - \mu_0$$

so that $r(t) \in [1 - \mu_0, 1)$ and (4.1) may be applied. Moreover, by the uniqueness result for the ergodic problem, we can represent φ as $\varphi = \varphi_0 + L$ with φ_0 being the unique solution of (1.8) such that $\min_{\overline{\Omega}} \varphi_0 = 0$ and L a constant to be fixed later. In particular, note that $\varphi \geq L$ since $\varphi_0 \geq 0$ on $\overline{\Omega}$.

1. Our first step is in Ω. It is easy to see that

$$-\Delta\varphi_r + |D\varphi_r|^m = r^{\frac{m}{m-1}} \left(-\Delta\varphi + |D\varphi|^m\right)$$

hence, with the notation $A(z) = z_t - \Delta z + |Dz|^m$, we get

$$A(\varphi_r) = r(t)^{\frac{m}{m-1}}(f(r(t)x) + c) + r(t)^{\frac{2-m}{m-1}} r'(t) \left(\frac{2-m}{m-1} \frac{\varphi(r(t)x)}{r(t)} + D\varphi(r(t)x) \cdot x\right). \tag{4.6}$$

Using the result (2.4), we know that $\frac{\partial\varphi}{\partial\tau(x)} = o\left(\frac{\partial\varphi}{\partial\nu(x)}\right)$ as $x \to \partial\Omega$, where τ and ν are tangential and normal vectors. Then, since Ω is star-shaped with respect to 0 and since $D\varphi \cdot \nu > 0$, we deduce the existence of $\sigma > 0$ such that $d(y) < \sigma$ implies $D\varphi(y) \cdot y \geq 0$. Hence, since φ is smooth inside Ω, we have

$$D\varphi(y) \cdot y \geq \chi_{\{d(y)>\sigma\}} \geq -C_\sigma$$

for some constant $C_\sigma > 0$. We deduce that

$$D\varphi(r(t)x) \cdot x \geq -\frac{C_\sigma}{r(t)}.$$

In particular we have, for L large enough,

$$\frac{2-m}{m-1} \frac{\varphi(r(t)x)}{r(t)} + D\varphi(r(t)x) \cdot x \geq \frac{1}{r(t)} \left(\frac{2-m}{m-1} L - C_\sigma\right) > 0.$$

Going back to (4.6), we have, using that $r(t) < 1$,

$$\begin{aligned}
A(\varphi_r) &\geq r(t)^{\frac{m}{m-1}}(f(r(t)x) + c) \\
&= r(t)^{\frac{m}{m-1}}(f(r(t)x) - f(x)) + (r(t)^{\frac{m}{m-1}} - 1)(f(x) + c) + f(x) + c \\
&\geq -(1 - r(t))\frac{|f(r(t)x) - f(x))|}{(1 - r(t))} + (r(t)^{\frac{m}{m-1}} - 1)(f(x) + c) + f(x) + c \\
&\geq -(1 - r(t)^{\frac{m}{m-1}}) \left[\frac{|f(r(t)x) - f(x))|}{(1 - r(t))} + f(x) + c\right] + f(x) + c.
\end{aligned}$$

Now we use (4.1) and we finally obtain

$$A(\varphi_r) \geq f(x) + c + \delta\left(1 - r(t)^{\frac{m}{m-1}}\right). \tag{4.7}$$

Setting

$$\tilde{H}(t) = \int_0^t \left(1 - r(s)^{\frac{m}{m-1}}\right) ds \,,$$

it follows that

$$A(\varphi_r(x,t) - \delta\, \tilde{H}(t)) \geq f(x) + c = A(u(x,t) + ct).$$

2. Now, we consider the boundary $\partial\Omega$ and we turn to the choice of $r(t)$, which depends on the different values of m and c.

(i) When $c > 0$ and $1 < m < \frac{3}{2}$. First observe that from (2.1), when $x \in \partial\Omega$, we have that

$$\varphi(r(t)x) \geq K^*(1 - r(t))^{-\frac{2-m}{m-1}} + L \tag{4.8}$$

for some positive constant K^*. Then we choose r such that

$$1 - r(t) = \left(\frac{(c+\lambda)t + K^* \mu_0^{-\frac{2-m}{m-1}}}{K^*}\right)^{-\frac{m-1}{2-m}} \qquad \text{for all } t \geq 0$$

where $\lambda > 0$ will be determined later on. Note that $r(t)$ is increasing, $r(0) = 1 - \mu_0$, and $r(t) \uparrow 1$ as $t \to +\infty$. On the boundary, we get

$$\varphi_r(x,t) \geq (1 - \mu_0)^{\frac{2-m}{m-1}} \left((c+\lambda)t + L + K^* \mu_0^{-\frac{2-m}{m-1}}\right),$$

hence, up to choosing λ sufficiently large (only depending on c and μ_0), we have

$$\varphi_r(x,t) \geq 2ct + (1 - \mu_0)^{\frac{2-m}{m-1}} \left(L + K^* \mu_0^{-\frac{2-m}{m-1}}\right). \tag{4.9}$$

Next, since

$$\tilde{H}(t) = \int_0^t \left(1 - r(s)^{\frac{m}{m-1}}\right) ds \leq K\, t$$

for some K depending on c, K^*, μ_0, λ, and since the boundary datum g is bounded on $\partial\Omega$, we have from (4.9)

$$\varphi_r(x,t) \geq g(x) + ct + \delta\, \tilde{H}(t)$$

up to taking in (4.1) some δ eventually smaller and choosing L large enough. We conclude that

$$\varphi_r(x,t) - H(t) \geq u(x,t) + ct \qquad \text{for all } x \in \partial\Omega,\ t \geq 0\,,$$

where $H(t) = \delta\, \tilde{H}(t)$.

(ii) When $c = 0$ and $1 < m < 2$, we set

$$1 - r(t) = (\mu_0^{-\frac{1}{m-1}} + t)^{-(m-1)}$$

As before, $r(t)$ is increasing, $r(0) = 1 - \mu_0$ and $r(t) \uparrow 1$. Using (4.8) we have, for every $x \in \partial\Omega$,

$$\varphi_r(x,t) \geq r(0)^{\frac{2-m}{m-1}} \varphi(r(t)x) \geq (1-\mu_0)^{\frac{2-m}{m-1}} \left(K^*(\mu_0^{-\frac{1}{m-1}} + t)^{2-m} + L \right) ,$$

while

$$\tilde{H}(t) = \int_0^t \left(1 - r(s)^{\frac{m}{m-1}}\right) ds \leq K \int_0^t (1 - r(s))\, ds = \frac{K}{2-m} (\mu_0^{-\frac{1}{m-1}} + t)^{2-m} .$$

Therefore, choosing δ eventually smaller in (4.1), and then choosing L sufficiently large, we obtain again

$$\varphi_r(x,t) \geq g(x) + \delta \tilde{H}(t) \qquad \text{for all } x \in \partial\Omega,\ t \geq 0 \,, \tag{4.10}$$

which means that $\varphi_r(x,t) \geq u(x,t) + H(t)$ on the lateral boundary with $H(t) = \delta\, \tilde{H}(t)$.

(iii) When $c = 0$ and $m = 2$. Here recall that we have

$$\varphi(r(t)x) \geq -\log(1 - r(t)) - K^* + L$$

for some positive constant K^*. We set now

$$1 - r(t) = \frac{1}{t + \mu_0^{-1}} \,,$$

hence

$$\varphi_r(x,t) \geq \log(t + \mu_0^{-1}) - K^* + L \,.$$

Since

$$\tilde{H}(t) = \int_0^t \left(1 - r(s)^{\frac{m}{m-1}}\right) ds \leq K \int_0^t (1 - r(s))\, ds = K\, \log(t + \mu_0^{-1}) \,,$$

we get again (4.10) by choosing δ sufficiently small and L large enough.

3. Finally, at $t = 0$, we have

$$\varphi_r(x,0) = r(0)^{\frac{2-m}{m-1}} \varphi(r(0)x) = (1-\mu_0)^{\frac{2-m}{m-1}} \varphi((1-\mu_0)x) \geq L(1-\mu_0)^{\frac{2-m}{m-1}}$$

hence for L large we obtain $\varphi_r(x,0) \geq u_0(x)$ for all $x \in \overline{\Omega}$.

4. We conclude from the above comparison that $\varphi_r(x,t) \geq u(x,t) + ct + H(t)$. Now, according to the value of c and m, the choice of $r(t)$ in (i)–(iii) gives the rates (4.4) or (4.5) claimed for H. $\square$

5 Appendix

In this section, we will give the proofs of Theorem 3.3 and Proposition 3.1-(i).

Proof of Theorem 3.3.

1. Let $d(x)$ be the signed distance function, which is negative when $x \notin \overline{\Omega}$. Let us fix $\delta_0 > 0$ such that $d(x)$ is C^2 in $\{x \in \mathbb{R}^N : |d(x)| < 2\delta_0\}$ and set

$$\tilde{d}(x) = \chi(d(x)) \,,$$

where $\chi(s)$ is a smooth, nondecreasing function such that $\chi(s) = s$ for $0 < |s| < \frac{\delta_0}{2}$ and $\chi(s)$ is constant for $|s| > \delta_0$. Without loss of generality, we may have $\chi'(s) \leq 1$ for every s. Consider now the vector field

$$n_k(x) = -\int_{\mathbb{R}^N} D\tilde{d}(y)\, \rho_k(x-y)\, dy$$

where ρ_k is a standard mollifying kernel (supported in the ball $B_{\frac{1}{k}}(0)$). Recalling the definition of $\tilde{d}$, the field $n_k(x)$ is supported in a neighborhood of $\partial\Omega$ of radius $\delta_0 + \frac{1}{k}$, and we have $n_k \in C^\infty$ for k large. Moreover, using the properties of $d(x)$ and in particular that $d \in C^2$, we have

$$|n_k| \leq 1 \,, \quad |Dn_k| \leq \|D^2 d\|_\infty \,, \quad |D^2 n_k| \leq k\, \|D^2 d\|_\infty \,. \tag{5.1}$$

Clearly n_k is an approximation of the normal vector $\nu(x) = -Dd(x)$, in particular

$$|n_k(x) + D\tilde{d}(x)| \leq \frac{\|D^2 d\|_\infty}{k} \,. \tag{5.2}$$

Then we consider the function

$$v(x,t) = \gamma(t)\varphi(x - \mu(t)\, n_k(x)) \qquad \text{where } \varphi(x) = \varphi_0(x) - L.$$

Here φ_0 is the unique solution of (1.8) such that $\min \varphi_0 = 0$, and L is an additive constant to be chosen, whereas $\gamma(t)$ and $\mu(t)$ are positive functions, with values in $(0,1)$, that will be fixed later in a way that $\gamma(t) \uparrow 1$ and $\mu(t) \downarrow 0$ as $t \to +\infty$. Observe that since

$$d(x - \mu(t)\, n_k(x)) = d(x) - \mu\, Dd(x) \cdot n_k(x) + O(\mu^2),$$

by choosing $k \geq 2\|D^2 d\|_\infty$ it follows that for any $x \in \Omega$ such that $d(x) \leq \frac{\delta_0}{2}$, we have, by using (5.2) and the definition of $\tilde{d}$, that:

$$d(x) + \frac{1}{2}\mu + O(\mu^2) \leq d(x - \mu(t)\, n_k(x)) \leq d(x) + \frac{3}{2}\mu + O(\mu^2). \tag{5.3}$$

In the following, we fix k as above. Moreover, in order to have $\mu(t)$ sufficiently small, it is enough to fix $\mu(0)$ small enough, since $\mu(t)$ is decreasing. To fix the ideas, we set $\beta = \mu(0)$, and we choose β small enough so that, thanks to (5.3), we have the following:

$$\begin{aligned}&x-\mu(t)\,n_k(x)\in\Omega \qquad \forall\, x\in\Omega, t\geq 0,\\ &d(x)+\tfrac{1}{4}\mu(t)\leq d(x-\mu(t)\,n_k(x))\leq d(x)+2\mu(t) \qquad \text{in } \Omega^{\delta_0/2}\times(0,+\infty).\end{aligned} \tag{5.4}$$

Note that this choice of β only depends on δ_0 and $\|D^2d\|_\infty$, in other words only on the domain Ω, and, eventually, we are allowed to take a smaller value of β if needed later.

2. Let us compute now the equation for v. Henceforth, we denote, for any function z, $A(z) := z_t - \Delta z + |Dz|^m$, and we use the letter K to denote possibly different constants only depending on Ω, m, f, c.

Since $Dv = \gamma(t)(I-\mu(t)Dn_k(x))D\varphi(x-\mu(t)\,n_k(x))$, using (5.1) and that γ, $\mu<1$, we get

$$\begin{gathered}A(v)\leq -\gamma\Delta\varphi+\gamma^m\,(1+K\mu)^m|D\varphi|^m\\ +\gamma\,K\mu\,(|Dn_k(x)|\,|D^2\varphi|+|D\varphi|)+\gamma'(t)\varphi-\mu'(t)\,\gamma\,D\varphi\cdot n_k(x)\end{gathered}$$

where the argument of φ is $x-\mu(t)n_k(x)$. Since φ satisfies (1.8), we deduce that

$$\begin{aligned}&A(v)\leq\gamma(f(x-\mu(t)n_k(x))+c)+\gamma\,(\gamma^{m-1}(1+K\mu)^m-1)\,|D\varphi|^m\\ &+\gamma\,K\mu\,(|Dn_k(x)|\,|D^2\varphi|+|D\varphi|)+\gamma'(t)\varphi-\mu'(t)\,\gamma(t)D\varphi\cdot n_k(x).\end{aligned} \tag{5.5}$$

From the Lipschitz continuity of f, we have

$$\gamma(f(x+\mu(t)n_k(x))+c)\leq(f(x)+c)+(1-\gamma(t))\|(f+c)^-\|_{L^\infty(\Omega)}+\mu(t)\|Df\|_{L^\infty(\Omega)}.$$

Moreover, since $\varphi=\varphi_0-L$, using (2.7) for φ_0 and that $2-m<1$ we deduce that we have, in the whole range $1<m\leq 2$,

$$\varphi\leq K(1+|D\varphi|)-L \qquad \forall x\in\Omega\,,$$

for some constant $K>0$. We also use (2.6), which we can suppose to hold true in the support of $n_k(x)$ without loss of generality. Therefore, we obtain from (5.5):

$$\begin{aligned}A(v)\ \leq\ &f(x)+c+(1-\gamma(t))\|(f+c)^-\|_{L^\infty(\Omega)}+\mu(t)\|Df\|_{L^\infty(\Omega)}\\ &+\gamma\left(\gamma^{m-1}(1+K\mu)^m-1+K\mu\right)|D\varphi|^m\\ &+\gamma'(t)[K(1+|D\varphi|)-L]+\gamma(t)\,K(|\mu'(t)|+\mu(t))|D\varphi|\,,\end{aligned}$$

hence there exists a constant, still denoted by K, such that

$$
\begin{aligned}
A(v) \;\leq\; & f(x) + c + (1-\gamma(t))\|(f+c)^-\|_{L^\infty(\Omega)} + \mu(t)\|Df\|_{L^\infty(\Omega)} \\
& + \gamma\left(\gamma^{m-1} - 1 + K\mu\right)|D\varphi|^m + \gamma'(t)[K(1+|D\varphi|) - L] \\
& + \gamma(t)\, K(|\mu'(t)| + \mu(t))|D\varphi| .
\end{aligned}
$$

Here we take $L > K$ and we choose $\gamma(t)$ such that

$$
\gamma(t)^{m-1} = 1 - \lambda\mu(t) \tag{5.6}
$$

for some λ large enough. Without loss of generality, we can assume that $\mu(t)$ is small in a way that $\gamma(t) > 0$ (this amounts to ask $\beta = \mu(0) < \frac{1}{\lambda}$). Then we obtain

$$
\begin{aligned}
A(v) \;\leq\; & f(x) + c + (1-\gamma(t))\|(f+c)^-\|_{L^\infty(\Omega)} + \mu(t)\|Df\|_{L^\infty(\Omega)} \\
& + \gamma\mu\,(K-\lambda)|\,D\varphi|^m + \gamma'(t)\,K\,|D\varphi| + \gamma(t)\,K(|\mu'(t)| + \mu(t))|D\varphi|
\end{aligned}
$$

which yields, by applying Young's inequality,

$$
\begin{aligned}
A(v) \;\leq\; & f(x) + c + (1-\gamma(t))\|(f+c)^-\|_{L^\infty(\Omega)} + \mu(t)\|Df\|_{L^\infty(\Omega)} \\
& + \frac{1}{2}\gamma\mu(K-\lambda)|D\varphi|^m + K\,\gamma\,\mu\left(\frac{\gamma'(t)}{\gamma\,\mu}\right)^{\frac{m}{m-1}} + K\,\gamma\,\mu\left(\frac{|\mu'| + \mu}{\mu}\right)^{\frac{m}{m-1}}
\end{aligned}
$$

hence, choosing $\lambda > K$ we get

$$
\begin{aligned}
A(v) \;\leq\; & f(x) + c + (1-\gamma(t))\|(f+c)^-\|_{L^\infty(\Omega)} + \mu(t)\|Df\|_{L^\infty(\Omega)} \\
& + K\,\gamma\,\mu\left(\frac{\gamma'(t)}{\gamma\,\mu}\right)^{\frac{m}{m-1}} + K\,\gamma\,\mu\left(\frac{|\mu'| + \mu}{\mu}\right)^{\frac{m}{m-1}} .
\end{aligned} \tag{5.7}
$$

3. Let us consider now the boundary. Since we have from (5.4)

$$
d(x - \mu(t)n_k(x)) \geq \frac{1}{4}\mu(t) \qquad \text{for all } x \in \partial\Omega ,
$$

using the asymptotic behavior (2.1)–(2.2) for φ_0 we deduce that there exists a constant K^* such that

$$
\varphi_0(x - \mu(t)n_k(x)) \leq K^*(\mu(t)^{-\alpha} + 1) - L \qquad \text{for all } x \in \partial\Omega
$$

when $\alpha = \frac{2-m}{m-1} > 0$ (i.e. $m < 2$), while when $m = 2$ we have

$$
\varphi_0(x - \mu(t)n_k(x)) \leq -\log\mu(t) + K^* - L \qquad \text{for all } x \in \partial\Omega .
$$

We are going to take later L sufficiently large. In particular, considering $L > K^*$ we deduce that for every $x \in \partial\Omega$ and $t > 0$:

$$\begin{cases} \text{if } 1 < m < 2, & v(x,t) \le K^* \mu(t)^{-\alpha} + \gamma(0)(K^* - L) \\ \text{if } m = 2, & v(x,t) \le -\log \mu(t) + \gamma(0)(K^* - L) \end{cases} \tag{5.8}$$

Moreover, for the initial condition we have

$$v(x,0) = \gamma(0)\varphi_0(x - \mu(0)n_k(x)) - \gamma(0)L\,. \tag{5.9}$$

Let us now distinguish the choice of $\mu(t)$ according to different situations:

(i) If $c > 0$ and $1 < m < 2$, we set

$$\mu(t) = \left(\frac{ct + K^*\beta^{-\alpha}}{K^*}\right)^{-\frac{1}{\alpha}},$$

where $\alpha = \frac{2-m}{m-1}$. The value of $\beta = \mu(0)$ has been already chosen, as explained before, in order that (5.4) holds true and also $1 - \lambda\beta > 0$, this choice only depends on Ω, f, m. Note that $\mu(t)$ is a decreasing function and satisfies

$$K^*\mu(t)^{-\alpha} = ct + K^*\beta^{-\alpha}\,,$$

hence (5.8) implies

$$v(x,t) \le ct + K^*\beta^{-\alpha} + \gamma(0)(K^* - L) \quad \text{for all } x \in \partial\Omega\,.$$

Since $\gamma(0)^{m-1} = 1 - \lambda\,\mu(0) = 1 - \lambda\beta > 0$, and since g is bounded in $\partial\Omega$, up to choosing L sufficiently large we will have

$$v(x,t) \le g(x) + ct \quad \text{for all } x \in \partial\Omega. \tag{5.10}$$

Similarly, from the boundedness of u_0 on $\overline{\Omega}$, up to choosing L large enough (again depending on β), we obtain from (5.9)

$$v(x,0) \le u_0(x) \quad \text{for all } x \in \overline{\Omega}. \tag{5.11}$$

Consider now the behavior of $\mu(t)$, which implies that $\mu(t) \to 0$, and $\mu'(t) = o(\mu(t))$ as $t \to +\infty$. By definition of γ in (5.6), we have that $\gamma(t) \uparrow 1$, $1 - \gamma(t) = O(\mu(t))$ and $\gamma'(t) = O(|\mu'(t)|)$ as $t \to +\infty$. In particular we have

$$\gamma\,\mu\left(\frac{\gamma'}{\gamma\,\mu}\right)^{\frac{m}{m-1}} + \gamma\,\mu\left(\frac{|\mu'| + \mu}{\mu}\right)^{\frac{m}{m-1}} \le K\,\mu \tag{5.12}$$

hence we deduce from (5.7)

$$A(v) \leq f(x) + c + K\,\mu(t)\,\left(1 + \|(f+c)^{-}\|_{L^\infty(\Omega)} + \|Df\|_{L^\infty(\Omega)}\right)\,. \tag{5.13}$$

Therefore, if we set

$$\overline{K} = K\left(1 + \|(f+c)^{-}\|_{L^\infty(\Omega)} + \|Df\|_{L^\infty(\Omega)}\right)$$

and we define

$$w = u + ct + \overline{K}\int_0^t \mu(s)ds\,,$$

we can conclude that $A(w) \geq A(v)$. Moreover, from (5.10) and (5.11) we have that $v \leq w$ on the parabolic boundary. By the standard comparison result, it follows that

$$u(x,t) + ct \geq v(x,t) - \overline{K}\int_0^t \mu(s)ds \tag{5.14}$$

for all $(x,t) \in \overline{\Omega} \times [0,+\infty)$. Since $\mu(t) = O\left(t^{-\frac{1}{\alpha}}\right)$ as $t \to +\infty$, the conclusion of (3.4) follows according to the values of α, i.e. of m. Note in particular that $\mu(t) \in L^1(0,\infty)$ if and only if $\alpha < 1$ which corresponds to $m > \frac{3}{2}$.

(ii) If $c > 0$ and $m = 2$, we set

$$\mu(t) = \beta\, e^{-ct}\,,$$

where $\beta = \mu(0)$ is chosen as before. With this choice we have from (5.8)

$$v(x,t) \leq ct - \log\beta + \gamma(0)(K^* - L)\,,$$

and choosing L large enogh we deduce (5.10). Of course (5.11) remains true as before. Finally, in this case we have $\mu'(t) = O(\mu(t))$ as $t \to +\infty$, and then $\gamma'(t) = O(\mu(t))$ as well; therefore (5.12) still holds true. Thus we obtain again (5.13) and we conclude as before the inequality (5.14). Being μ integrable in $(0,+\infty)$, this implies (3.4) for the case $m = 2$.

(iii) If $c = 0$ and $1 < m < 2$, we set $\mu(t)$ as

$$\mu(t) = (\beta^{-(1+\alpha)} + \Lambda\, t)^{-\frac{1}{1+\alpha}}$$

where Λ will be fixed later. The initial condition $\mu(0) = \beta$ is fixed as before. We have

$$\mu'(t) = \frac{\Lambda\,\mu(t)}{\beta^{-(1+\alpha)} + \Lambda t} \leq \beta^{1+\alpha}\,\Lambda\,\mu(t)$$

hence, using also the definition of γ, we obtain (5.12) with a constant depending on Λ. We deduce then from (5.7) (recall that here $c = 0$)

$$A(v) \leq f(x) + \overline{K}\,\mu(t) + K_\Lambda\,\mu(t)\,, \tag{5.15}$$

where

$$\overline{K} = K\,\left(1 + \|(f)^-\|_{L^\infty(\Omega)} + \|Df\|_\infty\right)$$

is a constant only depending on Ω, m, f. In particular we obtain that the function

$$w = u + (\overline{K} + K_\Lambda)\int_0^t \mu(s)ds \tag{5.16}$$

satisfies $A(w) \geq A(v)$. Now we choose $\Lambda = \frac{\overline{K}\,(1+\alpha)}{K^*\,\alpha}$, in order to have

$$\begin{aligned} K^*\mu(t)^{-\alpha} &= K^*\,(\beta^{-(1+\alpha)} + \Lambda\,t)^{\frac{\alpha}{1+\alpha}} \\ &= K^*\Lambda\,\frac{\alpha}{1+\alpha}\int_0^t \mu(s)\,ds + K^*\beta^{-\alpha} \\ &= \overline{K}\int_0^t \mu(s)\,ds + K^*\beta^{-\alpha} \\ &\leq w(x,t) - u(x,t) + K^*\beta^{-\alpha}. \end{aligned}$$

Therefore from (5.8) we deduce that on the boundary

$$v(x,t) \leq w(x,t) - g(x) + K^*\beta^{-\alpha} + \gamma(0)(K^* - L) \qquad \text{for all } x \in \partial\Omega.$$

Choosing L large enough we conclude that $v(x,t) \leq w(x,t)$ for every $x \in \partial\Omega$ and $t > 0$; as before, we also have (5.11), hence by comparison we conclude that

$$u(x,t) \geq v(x,t) - (\overline{K} + K_\Lambda)\int_0^t \mu(s)ds$$

Since $\int_0^t \mu(s)\,ds = O\left(t^{\frac{\alpha}{1+\alpha}}\right)$ as $t \to +\infty$, we obtain (3.5) by definition of α.

(iv) If $c = 0$ and $m = 2$ we set

$$\mu(t) = \frac{1}{\Lambda\,t + \beta^{-1}}\,.$$

Since we have

$$|\mu'(t)| \leq \frac{\Lambda\,\mu(t)}{\Lambda\,t + \beta^{-1}} \leq \Lambda\,\beta\,\mu(t)$$

we obtain (5.15) as before, hence $A(w) \geq A(v)$ where w is defined in (5.16). Since

$$-\log \mu = \Lambda \int_0^t \mu(s)\, ds - \log \beta$$

we choose $\Lambda = \overline{K}$ and by using (5.8) we obtain on the lateral boundary:

$$\begin{aligned} v(x,t) &\leq \overline{K} \int_0^t \mu(s)\, ds - \log \beta + \gamma(0)(K^* - L) \\ &\leq w(x,t) - g(x) - \log \beta + \gamma(0)(K^* - L)\,. \end{aligned}$$

We conclude that $v \leq w$ on the boundary up to choosing L large enough and then, by comparison, we deduce that $v(x,t) \leq w(x,t)$ in $\Omega \times (0,T)$, which implies (3.5) for the case $m = 2$.

□

Now, we turn to the proof of the interior Hölder estimates on v with respect to the x-variable uniformly in $t \in (0, +\infty)$ which is based on an idea introduced by Ishii and Lions [12]. This idea has been already used for instance in Barles [1] and Barles and Souganidis [7] to show gradient estimates of viscosity solutions to quasilinear elliptic ans parabolic PDE with Lipschitz initial conditions, by Barles and Da Lio [5] to prove local Hölder estimates up to the boundary of bounded solutions to fully non linear elliptic PDE with Neumann boundary conditions and by Da Lio [10] to obtain $C^{0,\nu}$-estimates for viscosity solutions of parabolic equations with nonlinear Neumann-type boundary conditions.

Proof of Proposition 3.1-(i). We fix $\nu \in (0,1)$, $\delta > 0$, $\eta > 0$ and $x_0 \in \Omega$ such that $d(x_0) \geq \delta$. We are going to show that there exists a suitable constant C depending on ν, δ, η, $M := \sup_{K_{\frac{\delta}{2}} \times (\eta/2, +\infty)} |v|$ and the data of the problem such that, for all $y \in B(x_0, \delta/2)$ and $t_0 \geq \eta$, we have

$$v(x_0, t_0) - v(y, t_0) \leq C|x_0 - y|^{\nu}\,. \tag{5.17}$$

The property (5.17) clearly implies the $C^{0,\nu}$-estimates of $v(\cdot, t_0)$ in K_δ. Indeed, for all $x, y \in K_\delta$, if $|x - y| \geq \delta/2$ then

$$|v(x,t_0) - v(y,t_0)| \leq \frac{2^{\nu+1}M}{\delta^{\nu}}|x - y|^{\nu}.$$

But when $|x - y| < \delta/2$, we can therefore apply (5.17) with $x_0 = x$ and $y \in B(x, \delta/2)$ or with $x_0 = y$ and $x \in B(y, \delta/2)$ and finally obtain the desired estimates.

To prove (5.17), we consider the function $(x,y,t) \mapsto \Phi(x,y,t)$ defined on $\Omega\times\Omega\times(0,+\infty)$ as follows

$$\Phi(x,y,t) = v(x,t) - v(y,t) - \varphi(|x-y|) - L(|x-x_0|^2 + |t-t_0|^2)$$

where $\varphi(t) = Ct^\nu$. The constants $L>0$ and $C>0$ will be chosen in such a way that Φ is a non-positive function.

We first choose L and C in order to have

$$\Phi \leq 0 \quad \text{ on } \quad \partial\Big(\overline{B}(x_0,\delta/2)\times\overline{B}(x_0,\delta/2)\times(\eta/2,+\infty)\Big).$$

This leads to the constraints

$$L\left(\frac{\delta}{4}\right)^2 \geq 2M\ ,\ L\left(\frac{\eta}{2}\right)^2 \geq 2M \text{ and } C\left(\frac{\delta}{4}\right)^\nu \geq 2M\ .$$

With these choices of C and L, it is easily checked that $\Phi(x,y,t)\leq 0$ for any x,y,t such that $|x-y|\geq\delta/4$ or $|x-x_0|\geq\delta/4$ or $|t-t_0|\geq\eta/2$, and putting together these properties, we clearly have the desired property.

Next, L being fixed as above, we argue by contradiction, assuming that, for any constant C (satisfying the above constraints), we have

$$\mathcal{M}_{C,L} := \max_{\overline{B}(x_0,\delta/2)\times\overline{B}(x_0,\delta/2)\times[\eta/2,+\infty)} \Phi(x,y,t) > 0. \tag{5.18}$$

Let $(\bar{x},\bar{y},\bar{t}) \in \overline{B}(x_0,\delta/2)\times\overline{B}(x_0,\delta/2)\times[\eta/2,+\infty)$ be a maximum point of Φ; we have dropped the dependence of $\bar{x},\bar{y}$ and $\bar{t}$ in C for sake of simplicity. By using (5.18), it is clear that $\bar{x}\neq\bar{y}$, otherwise, we would have $\mathcal{M}_{C,L}\leq 0$. With the choices of C and L we made above, it is obvious that $(\bar{x},\bar{y},\bar{t}) \in B(x_0,\delta/2)\times B(x_0,\delta/2)\times(\eta/2,+\infty)$.

From (5.18), we get

$$C|\bar{x}-\bar{y}|^\nu + L(|\bar{x}-x_0|^2 + |\bar{t}-t_0|^2) \leq v(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) \tag{5.19}$$

which yields

$$C|\bar{x}-\bar{y}|^\nu \leq 2M \quad \text{ and } \quad L(|\bar{x}-x_0|^2 + |\bar{t}-t_0|^2) \leq 2M \tag{5.20}$$

and it follows, in particular, that $|\bar{x}-\bar{y}|\to 0$ as $C\to+\infty$ and we recall that we may assume without loss of generality that $|\bar{x}-\bar{y}|>0$ for C large enough.

We define ϕ by

$$\phi(x-y) = \varphi(|x-y|) \tag{5.21}$$

and use the arguments of [12, Proposition IV.1] to prove the existence of two $N \times N$ symmetric matrices B_1 and B_2, and $a, b \in \mathbb{R}$ such that

$$\begin{pmatrix} B_1 - 2LI & 0 \\ 0 & -B_2 \end{pmatrix} \leq \begin{pmatrix} D^2\phi(\bar{x}-\bar{y}) & -D^2\phi(\bar{x}-\bar{y}) \\ -D^2\phi(\bar{x}-\bar{y}) & D^2\phi(\bar{x}-\bar{y}) \end{pmatrix}, \tag{5.22}$$

$a - b - 2L(\bar{t} - t_0) \geq 0$ and

$$\begin{cases} a - Tr(B_1) + |p|^m - f(\bar{x}) - c \leq 0 \\ b - Tr(B_2) + |q|^m - f(\bar{y}) - c \geq 0 \end{cases} \tag{5.23}$$

where

$$p = D\phi(\bar{x}-\bar{y}) + 2L(\bar{x} - x_0) \quad \text{and} \quad q = D\phi(\bar{x}-\bar{y}).$$

For all $\xi, \zeta \in \mathbb{R}^N$, we rewrite (5.22) as

$$\langle B_1\xi, \xi\rangle - \langle B_2\zeta, \zeta\rangle \leq \langle D^2\phi(\bar{x}-\bar{y})(\xi - \zeta), \xi - \zeta\rangle + 2L|\xi|^2. \tag{5.24}$$

Let $(e_i)_{1\leq i\leq N-1}$ be a familly of $(N-1)$ vectors in $\mathbb{R}^N$ such that $(e_1, e_2, ..., e_{N-1}, \frac{q}{|q|})$ is an orthonormal basis of $\mathbb{R}^N$. Plugging successively $\xi = \zeta = e_i$ for all $i = 1, ..., N-1$ and $\xi = -\zeta = \frac{q}{|q|}$ in (5.24) and by adding all the inequalities obtained, we obtain:

$$\begin{aligned} Tr(B_1 - B_2) &= \sum_{i=1}^{N-1}\langle (B_1 - B_2)e_i, e_i\rangle + \langle (B_1 - B_2)\frac{q}{|q|}, \frac{q}{|q|}\rangle \\ &\leq 2NL + 4\frac{\langle D^2\phi(\bar{x}-\bar{y})q, q\rangle}{|q|^2}. \end{aligned} \tag{5.25}$$

Going back to the form of ϕ, we set $\chi(z) = |z|$ and obtain

$$\langle D^2\phi(\bar{x}-\bar{y})q, q\rangle = \varphi'\langle D^2\chi(\bar{x}-\bar{y})q, q\rangle + \varphi''\langle (D\chi(\bar{x}-\bar{y}) \otimes D\chi(\bar{x}-\bar{y}))q, q\rangle. \tag{5.26}$$

But, knowing that $D\chi(z) = \frac{z}{|z|}$, it follows that $|D\chi(z)|^2 = 1$ which yields by differentiation $2D^2\chi(z)D\chi(z) = 0$ and we obtain $D^2\chi(\bar{x}-\bar{y})q = 0$ by taking $z = \bar{x} - \bar{y}$. Therefore, we use (5.26) and find that(5.25) becomes

$$Tr(B_1 - B_2) \leq 2NL + 4\varphi''\frac{\langle (q \otimes q)q, q\rangle}{|q|^2} = 2NL + 4\varphi''$$

Finally, we obtain

$$Tr(B_1 - B_2) \leq 2LN + 4C\nu(\nu - 1)|\bar{x} - \bar{y}|^{\nu-2}. \tag{5.27}$$

By subtracting the two inequalities in (5.23) and using (5.27), we have

$$2L(\bar{t}-t_0)+4C\nu(1-\nu)|\bar{x}-\bar{y}|^{\nu-2}\leq |q|^m-|p|^m+f(\bar{x})-f(\bar{y})+2LN\ .$$

Then the convexity of $p\mapsto |p|^m$ yields $|p|^m\geq |q|^m+m|q|^{m-2}q\cdot(p-q)$ and therefore

$$|q|^m-|p|^m\leq m|q|^{m-1}|p-q|\leq mL\delta|q|^{m-1}\ ,$$

since $|\bar{x}-x_0|\leq\delta/2$.

Plugging this estimate in the above inequality gives

$$2L(\bar{t}-t_0)+4C\nu(1-\nu)|\bar{x}-\bar{y}|^{\nu-2}\leq mL\delta|q|^{m-1}+2\|f\|_\infty+2LN\ ,$$

and therefore

$$2L(\bar{t}-t_0)+4C\nu(1-\nu)|\bar{x}-\bar{y}|^{\nu-2}\leq mL\delta(\nu C|\bar{x}-\bar{y}|^{(\nu-1)})^{m-1}+2\|f\|_\infty+2LN\ ,$$

Hence, using that $2L|\bar{t}-t_0|\leq 2L^{1/2}(2M)^{1/2}$, if $\tilde{K}:=2L^{1/2}(2M)^{1/2}+2\|f\|_\infty+2LN$, we finally have

$$4\nu(1-\nu)\leq\tilde{K}\frac{|\bar{x}-\bar{y}|^{2-\nu}}{C}+mL\delta\nu^{m-1}C^{m-2}|\bar{x}-\bar{y}|^{(\nu-1)(m-2)+1}\ .$$

But $m-2\leq 0$ and $(\nu-1)(m-2)+1\geq 1$ since $\nu-1\leq 0$ as well; from (5.20), we know that $|\bar{x}-\bar{y}|\to 0$ when C tends to infinity. Therefore it is clear that this inequality cannot hold for C large enough, thus contradicting (5.18). Then, if C is large enough, the estimate (5.17) holds true. Examining this proof, we find that C depends on ν, δ, η, $M:=\sup_{K_{\frac{\delta}{2}}\times(\eta/2,+\infty)}|v|$, $\|f\|_\infty$ and the constants m, N.

□

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