Length-Type Phase Transitions and Extendable Shift Maps on Generalized Countable Markov Shifts

Webinar Brazil-France THERMOGAMAS

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November, 2025

Joint work with Thiago Raszeja (PUC-Rio, Brazil) and
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Supported by CAPES and CNPq.

Summary

Some recent references

- C^*-algebra and groupoids aspects of this project:
  Ruy Exel's presentation at ICM 2018 YouTube channel, '**Conformal and DLR measures on Markov subshifts with infinitely many states**', at ICM 2018 Youtube Channel.
- R. Bissacot, R. Exel, R. Frausino, T. R. **Thermodynamic Formalism for Generalized Markov Shifts on Infinitely Many States**. ArXiv 1808.00765 (2022). (89 pages)
- R. Bissacot, I. Diaz-Granados and T. R. 'Extendable Shift Maps and Weighted Endomorphisms on Generalized Countable Markov Shifts.', (37 pages)(2025) arXiv:2506.07487.

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Markov shifts

Markov shifts

I =countable alphabet. ($\mathbb{N}$ or a finite set)

A = transitive $\{0,1\}$-matrix indexed by $I \times I$.

Admissible words: $\omega \in I^N$, allowed by the matrix:

$$A(\omega_i, \omega_{i+1}) = 1, \quad \text{for all } i = 0, 1, \ldots, N-2$$

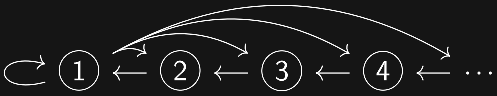
Markov shifts:

$$\Sigma_A := \{x \in I^{\mathbb{N}_0} : A(x_i, x_{i+1}) = 1, i \in \mathbb{N}_0\}.$$

Transitivity: for every $a, b \in I$ there exists an admissible word ω s.t $a\omega b$ is admissible.

Markov shifts: Examples

Full shift: $A(i,j) = 1$ for every $i, j \in I$.
Renewal shift: $I = \mathbb{N}$, $A(1,n) = A(n+1,n) = 1$ for every $n \in \mathbb{N}$ and $A(i,j) = 0$ in the rest of the matrix entries.
infinite 1's at the first row.



$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

C^*-algebras, Cuntz-Krieger and Exel-Laca algebras

Cuntz-Krieger algebras

Consider a family $\{S_i\}_{i=1}^n$ of partial isometries $(S_i S_i^* S_i = S_i)$ which satisfies the relations

$$\sum_{j=1}^{n} S_j S_j^* = 1 \quad \text{and} \quad S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^*.$$

Definition

The Cuntz-Krieger algebra $\mathscr{O}_A$ is the universal C*-algebra generated by a family of partial isometries $\{S_i\}_{i=1}^n$ which satisfies the relations above.

J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains. Inventiones Mathematicae, (1980).

Cuntz-Krieger algebras

The Cuntz-Krieger algebras encode the Markov shift space:

- for each $\omega = \omega_0 \cdots \omega_{k-1}$,

$$S_\omega := S_{\omega_0} \cdots S_{\omega_{k-1}} \neq 0 \iff \omega \text{ is admissible;}$$

- the projections $S_\omega S_\omega^*$ pairwise commutes and hence they generate a commutative C^*-sub-algebra $\mathscr{D}_A \subseteq \mathscr{O}_A$ s.t.

$$\mathscr{D}_A \simeq C(\Sigma_A).$$

Big question in operator algebras at this point:

How to construct the Cuntz-Krieger algebra for infinitely many symbols?

Exel-Laca algebras - Countable Alphabet case

Consider a countably infinite transition matrix A and the universal **unital** C^*-algebra $\widetilde{\mathscr{O}}_A$ generated by a family of partial isometries $\{S_j : j \in \mathbb{N}\}$ which satisfies the relations below:

$(EL1)$ $S_i^* S_i$ and $S_j^* S_j$ commute for every $i, j \in \mathbb{N}$;

$(EL2)$ $S_i^* S_j = 0$ whenever $i \neq j$;

$(EL3)$ $(S_i^* S_i) S_j = A(i,j) S_j$ for all $i, j \in \mathbb{N}$;

$(EL4)$ for every pair X, Y of finite subsets of $\mathbb{N}$ such that the quantity

$$A(X,Y,j) := \prod_{x \in X} A(x,j) \prod_{y \in Y} (1 - A(y,j)), j \in \mathbb{N}$$

is non-zero only for a finite number of j's, we have

$$\left(\prod_{x \in X} S_x^* S_x \right) \left(\prod_{y \in Y} (1 - S_y^* S_y) \right) = \sum_{j \in \mathbb{N}} A(X,Y,j) S_j S_j^*.$$

Generalized countable Markov shifts X_A

Definition

The Exel-Laca algebra $\mathcal{O}_A$ is the C*-subalgebra of $\widetilde{\mathcal{O}_A}$ generated the a family of partial isometries $\{S_i\}_{i \in \mathbb{N}}$.

- the analogous commutative C^*-sub-algebra generated by projections

$$\mathscr{D}_A \simeq C(X_A) \text{ or } C_0(X_A).$$

Main facts about X_A:

- X_A is always locally compact.
- Σ_A and Y_A are dense in X_A.
- Σ_A locally compact $\implies X_A = \Sigma_A$.
- $X_A = \Sigma_A \sqcup Y_A$.
- In many important cases X_A is compact.

When X_A is compact, is a compactification of Σ_A.

A faithful representation of $\mathcal{O}_A$

Consider $\Pi : \mathcal{O}_A \to \mathfrak{B}(\ell^2(\Sigma_A))$ defined as $\Pi(S_i) := T_i$ as follows:

Definition

Let Σ_A be a transitive Markov shift. For each $i \in \mathbb{N}$,

$$T_i(\delta_\omega) = \begin{cases} \delta_{i\omega}, & \text{if } \omega \in \sigma([i]), \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad T_i^*(\delta_\omega) = \begin{cases} \delta_{\sigma(\omega)}, & \text{if } \omega \in [i], \\ 0, & \text{otherwise;} \end{cases}$$

$\{\delta_\omega\}_{\omega \in \Sigma_A}$ is the canonical basis of $\ell^2(\Sigma_A)$. And the projections $P_i = T_i T_i^*$ and $Q_i = T_i^* T_i$.

$$P_i(\delta_\omega) = \begin{cases} \delta_\omega \text{ if } \omega \in [i], \\ 0 \text{ otherwise;} \end{cases} \quad \text{and} \quad Q_i(\delta_\omega) = \begin{cases} \delta_\omega \text{ if } \omega \in \sigma([i]), \\ 0 \text{ otherwise.} \end{cases}$$

Common properties - Cuntz-Krieger and Exel-Laca

- MASA (Maximal Abelian SubAlgebra)

$$\mathscr{D}_A := \overline{\operatorname{span}\left\{ T_\alpha \left(\prod_{i\in F} Q_i\right) T_\alpha^* : \begin{array}{l} F \text{ finite}; \alpha \text{ finite admissible word}; \\ F \neq \emptyset \text{ or } \alpha \text{ is not the empty word} \end{array} \right\}}.$$

By Gelfand theorem: $\mathscr{D}_A \simeq C_0(X)$, with $X \simeq \widehat{\mathscr{D}_A}$.

$$I_A := \{(\alpha, F) \in \mathbb{F}_+ \times \mathbb{P}_f(\mathbb{N}) : \alpha \text{ finite admissible word}; F \neq \emptyset \text{ or } \alpha \neq e\}.$$

$$\mathscr{G}_A := \left\{ e_{\gamma,F} := T_\alpha \prod_{i\in F} Q_i T_\alpha^* \in \mathfrak{B}(\ell^2(\Sigma_A)) : (\gamma, F) \in I_A \right\},$$

$$e_{\beta,F}(\delta_\omega) = \begin{cases} \delta_\omega, & \text{if } \omega \in [\beta] \text{ and } A(i, \omega_{|\beta|}) = 1 \text{ for all } i \in F; \\ 0, & \text{otherwise.} \end{cases}$$

Graphical Representation of X_A

Space of 4 rules

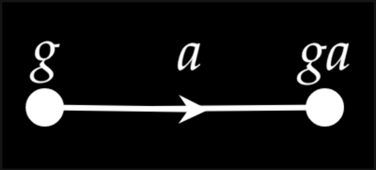
Geometric view of the configurations, similar to classical cases.

$\mathbb{F}$ = free group generated by the alphabet $\mathbb{N}$.

$\{0,1\}^{\mathbb{F}}$ (product topology)

Cayley Tree:

- vertices = elements of $\mathbb{F}$ (reduced form).
- oriented edges are labelled by the symbols (products).



Space of 4 rules

Subspace of 4 rules $\Omega_A \subset \{0,1\}^{\mathbb{F}}$ (compact):

(R1) $\xi_e = 1$;

(R2) ξ is connected.

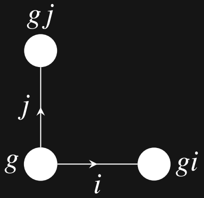
(R1) + (R2): given $g = g_0 \cdots g_p \in \mathbb{F}$ (reduced),

$$\xi_g = 1 \implies \xi_w = 1,$$

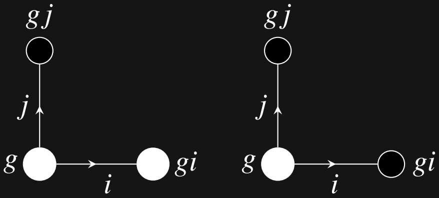
for every $w = g_0 \cdots g_k$ subword of g.

Space of 4 rules

(R3) for every $g \in \mathbb{F}$, if $\xi_g = 1$, then there exists at most one $i \in \mathbb{N}$ s.t. $\xi_{gi} = 1$;



Forbidden filling

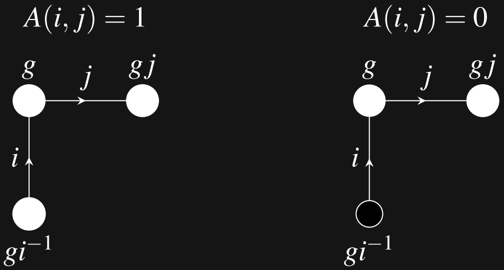


Allowed fillings

Space of 4 rules

(R4) given $g \in \mathbb{F}$ and $j \in \mathbb{N}$, if $\xi_g = \xi_{gj} = 1$, then, for every $i \in \mathbb{N}$,

$$\xi_{gi^{-1}} = 1 \iff A(i,j) = 1.$$



Generalized Markov shift spaces

Canonical inclusion of $\mathscr{I} : \Sigma_A \to \Omega_A$:

$$x \mapsto \xi,$$

where ξ is the unique configuration in Ω_A s.t.

$$\xi_\alpha = 1$$

for every α subword of x.

Generalized Markov shift space: $X_A := \overline{\mathscr{I}(\Sigma_A)}$ (when X_A is compact)

- X_A is always locally compact;
- when X_A is not compact, its Alexandrov' compactification $\widetilde{X}_A$ adds the extra point φ_0, which is filled only in e;

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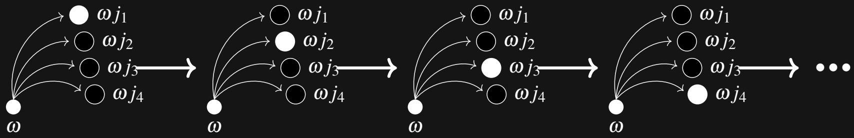
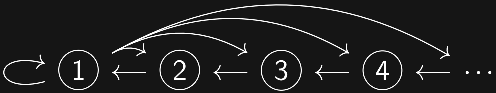
Generalized Markov shift spaces - generating Y_A

$X_A = Y_A \sqcup \Sigma_A$.



Concrete example de X_A

Renewal



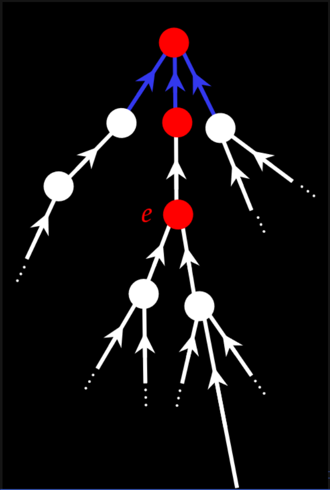
$$X_A = \Sigma_A \sqcup Y_A$$

- X_A is compact;
- 1 is the unique infinite emitter;

$$Y_A = \{\text{words ending in } 1\} \cup \{\text{empty word}\}.$$

Generalized Markov shifts - stems and roots

Realizing X_A as a space of configurations inside of $\{0,1\}^{\mathbb{F}}$.
Bijections: for Σ_A, $\xi \leftrightarrow \kappa(\xi)$.
For Y_A, $\xi \leftrightarrow (\kappa(\xi), R_\xi)$.

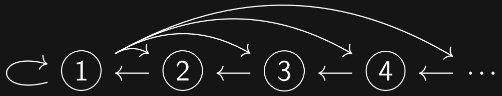


Generalized renewal shift - the empty word

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Concrete example X_A

Renewal



$$X_A = \Sigma_A \sqcup Y_A$$

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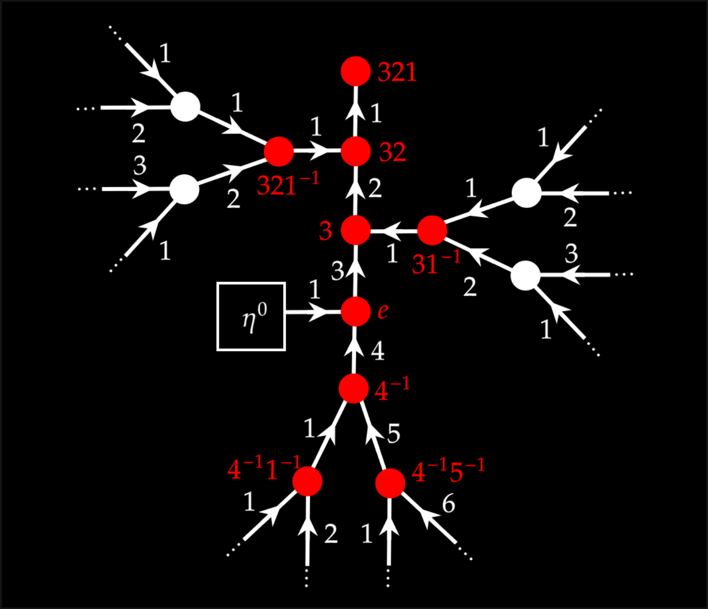
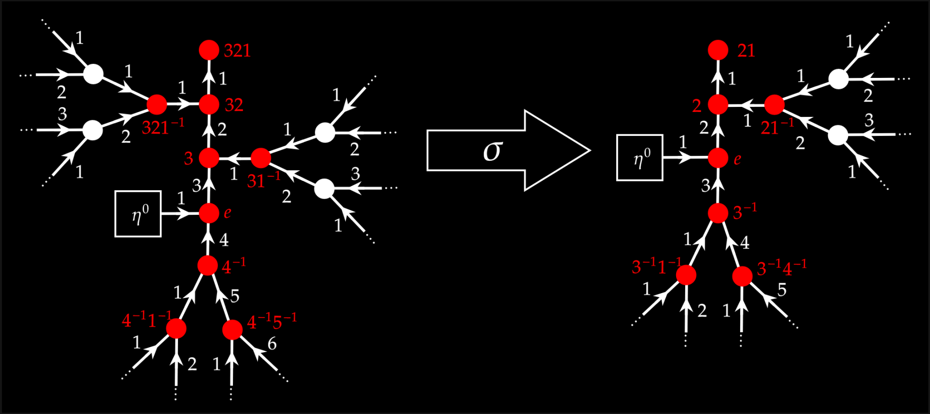
Generalized Markov shift spaces: Y_A



Generalized Markov shift spaces: dynamics

$\sigma : U \to X_A$, partially defined.



Extendable shift maps

Endomorphisms on C*-algebras

Proposition

$\alpha : \mathfrak{A} \to \mathfrak{A}$ $*$-endomorphisms of $\mathfrak{A}$ a unital commutative C*-algebra. There is $\varphi : \Delta \to X$, a uniquely determined partial continuous map on X algebra's spectrum defined on $\Delta \subseteq X$ subset clopen such that

$$\widehat{\alpha(a)}(x) = \begin{cases} \widehat{a}(\varphi(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \qquad a \in \mathfrak{A}. \tag{1}$$

Moreover, $\Delta = X$ iff α preserves the unit of $\mathfrak{A}$.

- B. K. Kwaśniewski and A.V. Lebedev. *Variational principles for spectral radius of weighted endomorphisms of $C(X, D)$*. Trans. Amer. Math. Soc, 2659-2698, **373**, (2020).

Extendable shift maps

Definition

A transition matrix, we define $\alpha_0 : \mathscr{G}_A \cup \{0\} \to \mathscr{B}(\ell^2(\Sigma_A))$ given by

$$\alpha_0(e_{\gamma,F})\delta_\omega = \sum_{i\in\mathbb{N}} A(i,\omega_1)e_{i\gamma,F}\delta_\omega = e_{\omega_0\gamma,F}\delta_\omega \tag{2}$$

and $\alpha_0(0) = 0$, where $\{\delta_\omega\}_{\omega\in\Sigma_A}$ is the canonical basis of $\ell^2(\Sigma_A)$.

$$e_{\beta,F}(\delta_\omega) = \begin{cases} \delta_\omega, & \text{if } \omega \in [\beta] \text{ and } A(i,\omega_{|\beta|}) = 1 \text{ for all } i \in F; \\ 0, & \text{otherwise.} \end{cases}$$

Extendable shift maps

Theorem (RB, I. Díaz-Granados, T. Raszeja (2025))

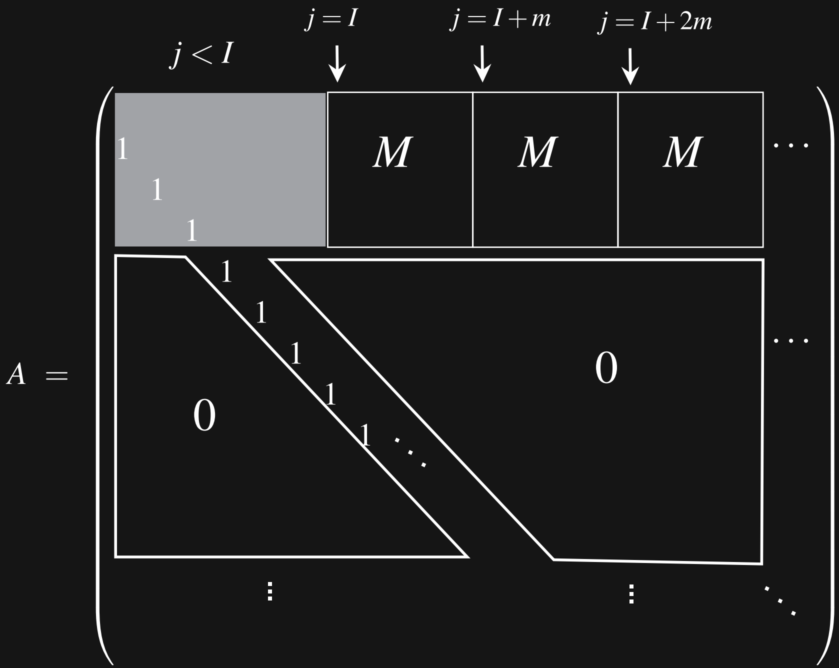
A an irreducible transition matrix for which the Exel-Laca algebra $\mathscr{O}_A$ is unital. The following statements are equivalent:

1. $\alpha_0(\mathscr{G}_A) \subseteq \mathscr{D}_A$;
2. *There exists a $*$-endomorphism α on $\mathscr{D}_A$ which extends α_0.*
3. $\sigma : \Sigma_A \sqcup F_A \to X_A$ *is continuously extended in all* X_A,

where σ is dual to α, that is, $\widehat{\alpha(a)} = \widehat{a} \circ \sigma$ for every $a \in \mathscr{D}_A$.

Corollary (RB, I. Díaz-Granados, T. Raszeja (2025))

If the $*$-endomorphism $\alpha : \mathscr{D}_A \to \mathscr{D}_A$ that extends α_0 exists, then it preserves unity and A is necessarily column-finite.


j = I
j = I + m
j = I + 2m
j < I
A =
1
1
1
M
M
M
1
1
1
1
1
0
0

Periodic renewal class

(i) A is irreducible;

(ii) X_A is compact;

(iii) $A(i+1,i) = 1$ for every $i \in \mathbb{N}$;

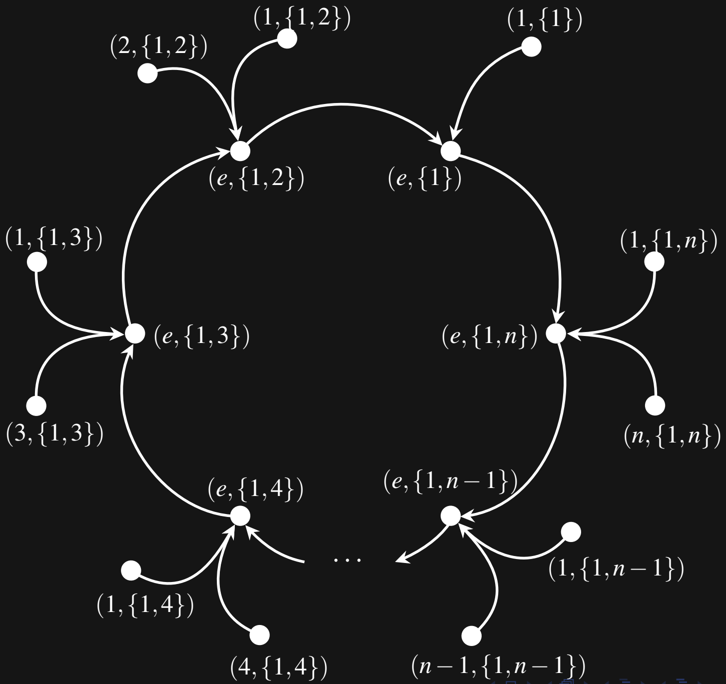
(iv) the set of infinite emitters is finite, where we define

$$n_e := \max\{j \in \mathbb{N} : j \text{ is an infinite emitter}\};$$

(v) there exists an $n_e \times m$ $\{0,1\}$-matrix M repeating;

(vii) the columns of M are pairwise distinct;

$(viii)$ For every $i > n_e$, $A(i,k) = 1$ if and only if $k = i-1$.


(1, {1, 2})
(2, {1, 2})
(1, {1})
(e, {1, 2})
(e, {1})
(1, {1, 3})
(1, {1, n})
(e, {1, 3})
(e, {1, n})
(3, {1, 3})
(n, {1, n})
(e, {1, n − 1})
(e, {1, 4})
. . .
(1, {1, n − 1})
(1, {1, 4})
(4, {1, 4})
(n − 1, {1, n − 1})

Dual endomorphisms to the shift map

When σ can be continuously extended to the whole space X_A.
Then, via the Gelfand transformation $\pi(a) = \widehat{a}$ defined as $\pi(a)(\eta) = \eta(a)$.
There exists a unique α $*$-endomorphism $\alpha : \mathscr{D}_A \to \mathscr{D}_A$ such that

$$\begin{array}{ccc} \mathscr{D}_A & \xrightarrow{\quad \alpha \quad} & \mathscr{D}_A \\ {\scriptstyle \pi^{-1}} \Big\uparrow & & \Big\downarrow {\scriptstyle \pi} \\ C(X_A) & \xrightarrow{\quad \Theta \quad} & C(X_A) \end{array}$$

where $\Theta(f) = f \circ \sigma$.

Spectral Radius of Weighted Endomorphisms

Theorem (B. K. Kwaśniewski and A.V. Lebedev.)

Let $\alpha : \mathfrak{A} \to \mathfrak{A}$ be an endomorphism of a unital commutative C^*-algebra $\mathfrak{A}$ which preserves the unity. Let (X, σ) be the dynamical system dual to $(\mathfrak{A}, \alpha)$. Then, for any $a \in \mathfrak{A}$, the spectral radius of the weighted endomorphism $a\alpha : \mathfrak{A} \to \mathfrak{A}$ is given by

$$r(a\alpha) = \max_{\mu \in Inv(X,\sigma)} \exp \int_X \ln |\hat{a}(x)|\, d\mu = \max_{\mu \in Erg(X,\sigma)} \exp \int_X \ln |\hat{a}(x)|\, d\mu, \qquad (3)$$

where $\ln(0) = -\infty$, $\exp(-\infty) = 0$, *and* $a\alpha = a\alpha(\cdot) \in \mathfrak{B}(\mathfrak{A})$.

- B. K. Kwaśniewski and A.V. Lebedev. *Variational principles for spectral radius of weighted endomorphisms of $C(X, D)$*. Trans. Amer. Math. Soc, 2659-2698, **373**, (2020).

Spectral Radius of Weighted Endomorphisms

Proposition (RB, I. Díaz-Granados, T. Raszeja (2025))

Let A be an irreducible transition matrix s.t. σ extends continuously to X_A. Given $a \in \mathscr{D}_A$, the spectral radius of the weighted endomorphism $a\alpha$ is given by

$$r(a\alpha) = \max_{\mu \in \mathsf{Inv}(\Sigma_A \cup E_A, \sigma)} \exp \int_{\Sigma_A \cup E_A} \ln |\widehat{a}| d\mu;$$

Corollary (RB, I. Díaz-Granados, T. Raszeja (2025))

Let A be an irreducible transition matrix s.t. σ extends continuously to X_A. Given $f \in C(X_A)$, the spectral radius $r(f\Theta)$ satisfies

$$r(f\Theta) = \max_{\mu \in \mathit{Inv}(\Sigma_A \cup E_A, \sigma)} \exp \int_{\Sigma_A \cup E_A} \ln |f| d\mu.$$

Length-Type Phase Transitions

Distortion and regularity on the potential

Potential: $F : U \to \mathbb{R}$.

n-th variation:

$$\text{Var}_n F := \sup_{\substack{\omega \in \mathbb{F}_+ \text{ admissible}, \\ |\omega| = n}} \{ |f(x) - f(y)| : x, y \in C_\omega \},$$

where $C_\omega = \{\xi \in X_A : \xi_\omega = 1\}$.

n-th distortion of F: $\text{Var}_n F_n$.
Potential with uniformly bounded distortion: $\sup_{n \in \mathbb{N}} \text{Var}_n F_n < \infty$

Example: Walters' potentials s.t. $\text{Var}_1 F < \infty$.

Thermodynamic Formalism on X_A: Pressures

Consider Σ_A transitive, $a \in \mathbb{N}$.

$F_n(x) = \sum_{i=0}^{n-1} F(\sigma^i(x)$ is the Birkhoff's sum.

Partition function: $Z_n(F,[a]) := \sum_{\sigma^n x = x} e^{F_n(x)} \mathbb{1}_{[a]}(x)$.

Gurevich pressure (Walters' potentials):

$$P_G(F) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n(F,[a]).$$

Gurevich entropy: $h_G := P_G(0)$.

For $x \in X_A$, the partition function and the pressure at x are

$$Z_n(\beta F, x) := \sum_{\sigma^n(y)=x} e^{\beta F_n(y)} \quad \text{and} \quad P(\beta F, x) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\beta F, x).$$

M. Denker and M. Yuri, *Conformal families of measures for general iterated function systems*. Contemporary Mathematics, **631**, 93-108, (2015).

Pressures

Theorem (R.B., R. Exel, R. Frausino, T.Raszeja)

Let $F : U \to \mathbb{R}$ be a potential with uniformly bounded distortion. Suppose that X_A is compact and s-compact. Then, for every $x \in X_A$, we have

$$P(\beta F, x) = P_G(\beta F|_{\Sigma_A}).$$

The Gurevich pressure is also the natural definition of pressure on X_A.

Example: a complete description of eigenmeasures

Theorem (R.B., R. Exel, R. Frausino, T.Raszeja)

Let A be the renewal shift transition matrix and X_A its generalized Markov shift space. Consider the potential $F : U \to \mathbb{R}$ given by

$$F(x) = \log(x_0) - \log(x_0 + 1).$$

Then, for every $\beta > 0$, there exists a unique eigenmeasure associated to the eigenvalue $\lambda_\beta = e^{P_G(\beta F)}$. Moreover, there is critical value $\beta_c > 0$, which is the solution for $\zeta(\beta_c) = 2$ such that

(i) if $\beta > \beta_c$, then the eigenmeasure lives on Y_A;

(ii) if $\beta \leq \beta_c$, then the eigenmeasure lives on Σ_A.

Thank You!