Decoupled probability measures on shift spaces and large deviations

THERMOGAMAS webinar

November 25, 2025

Noé Cuneo
LPSM, Université Paris Cité

Based on joint work with Tristan Benoist, Vojkan Jakšić, Claude-Alain Pillet, Renaud Raquépas and Armen Shirikyan

LPSM

Université Paris Cité

Contents

Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - $\varphi =$ left shift, i.e. $(\varphi(x))_i = x_{i+1}$

Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - φ = left shift, i.e. $(\varphi(x))_i = x_{i+1}$
- $\mathbb{P}$ is a φ-invariant probability measure on Ω
  - Marginals: $\mathbb{P}_n(x_1, x_2, \ldots, x_n)$

Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - $\varphi =$ left shift, i.e. $(\varphi(x))_i = x_{i+1}$

- $\mathbb{P}$ is a φ-invariant probability measure on Ω
  - Marginals: $\mathbb{P}_n(x_1, x_2, \ldots, x_n)$
  - $\mathbb{P}$ subject to **decoupling assumptions** (see below)

Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - $\varphi =$ left shift, i.e. $(\varphi(x))_i = x_{i+1}$
- $\mathbb{P}$ is a φ-invariant probability measure on Ω
  - Marginals: $\mathbb{P}_n(x_1, x_2, \ldots, x_n)$
  - $\mathbb{P}$ subject to **decoupling assumptions** (see below)
- Examples:
  - **Bernoulli (I.I.D.)**: $\mathbb{P}_n(x_1, \ldots, x_n) = p(x_1)p(x_2) \cdots p(x_n)$

Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - $\varphi =$ left shift, i.e. $(\varphi(x))_i = x_{i+1}$
- $\mathbb{P}$ is a φ-invariant probability measure on Ω
  - Marginals: $\mathbb{P}_n(x_1, x_2, \ldots, x_n)$
  - $\mathbb{P}$ subject to **decoupling assumptions** (see below)
- Examples:
  - **Bernoulli (I.I.D.)**: $\mathbb{P}_n(x_1, \ldots, x_n) = p(x_1)p(x_2)\cdots p(x_n)$
  - **Markov**: $\mathbb{P}_n(x_1, \ldots, x_n) = \pi(x_1) P_{x_1,x_2} P_{x_2,x_3} \cdots P_{x_{n-1},x_n}$

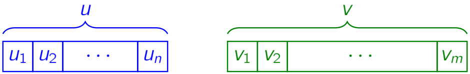
Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - $\varphi =$ left shift, i.e. $(\varphi(x))_i = x_{i+1}$
- $\mathbb{P}$ is a φ-invariant probability measure on Ω
  - Marginals: $\mathbb{P}_n(x_1, x_2, \ldots, x_n)$
  - $\mathbb{P}$ subject to **decoupling assumptions** (see below)
- Examples:
  - **Bernoulli (I.I.D.)**: $\mathbb{P}_n(x_1, \ldots, x_n) = p(x_1)p(x_2)\cdots p(x_n)$
  - **Markov**: $\mathbb{P}_n(x_1, \ldots, x_n) = \pi(x_1)P_{x_1,x_2}P_{x_2,x_3}\cdots P_{x_{n-1},x_n}$
  - **Gibbs measures** (in various senses)

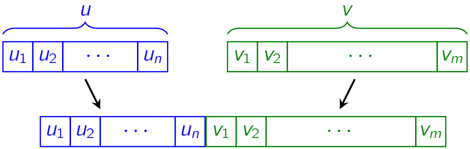
Setup

- Shift space:
  - $\mathcal{A}$ finite alphabet
  - $\Omega = \mathcal{A}^{\mathbb{N}}$. Notation: $x = x_1 x_2 x_3 \cdots \in \Omega$
  - $\varphi =$ left shift, i.e. $(\varphi(x))_i = x_{i+1}$
- $\mathbb{P}$ is a φ-invariant probability measure on Ω
  - Marginals: $\mathbb{P}_n(x_1, x_2, \ldots, x_n)$
  - $\mathbb{P}$ subject to **decoupling assumptions** (see below)
- Examples:
  - **Bernoulli (I.I.D.)**: $\mathbb{P}_n(x_1, \ldots, x_n) = p(x_1)p(x_2)\cdots p(x_n)$
  - **Markov**: $\mathbb{P}_n(x_1, \ldots, x_n) = \pi(x_1) P_{x_1,x_2} P_{x_2,x_3} \cdots P_{x_{n-1},x_n}$
  - **Gibbs measures** (in various senses)
  - **Non-Gibbsian measures**

Decoupling assumptions



Decoupling assumptions

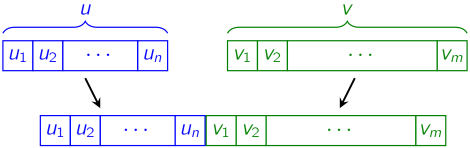


$$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$$

vs

$$\mathbb{P}_{n+m}(uv)$$

Decoupling assumptions



$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$

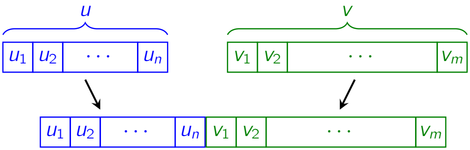
vs

$\mathbb{P}_{n+m}(uv)$

Weakly dependent measures (Lewis, Pfister, Sullivan '95). $\exists C_n = e^{o(n)}$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

$$C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v) \leq \mathbb{P}_{n+m}(uv) \leq C_n\mathbb{P}_n(u)\mathbb{P}_m(v)$$

Decoupling assumptions



$$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$$

vs

$$\mathbb{P}_{n+m}(uv)$$

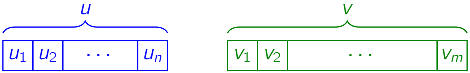
Weakly dependent measures (Lewis, Pfister, Sullivan '95). $\exists C_n = e^{o(n)}$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

$$C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v) \leq \mathbb{P}_{n+m}(uv) \leq C_n\mathbb{P}_n(u)\mathbb{P}_m(v)$$

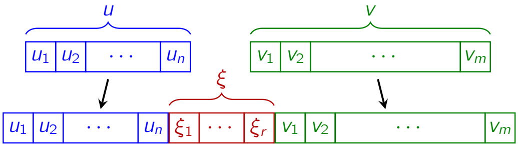
Limitation: Implies that $\operatorname{supp}\mathbb{P} = \Omega$

- For Markov chains: $P_{ij} > 0 \ \forall i, j$
- For statistical mechanics: no hard-core interaction

Decoupling assumptions (continued)



Decoupling assumptions (continued)

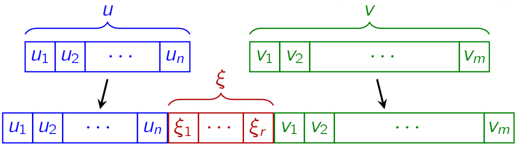


$$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$$

vs

$$\mathbb{P}_{n+|\xi|+m}(u\xi v)$$

Decoupling assumptions (continued)



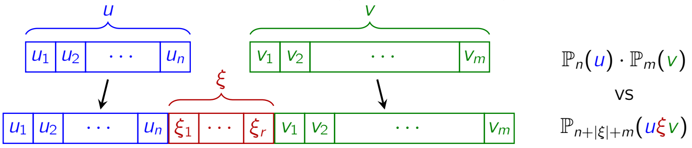
$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$

vs

$\mathbb{P}_{n+|\xi|+m}(u\xi v)$

Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

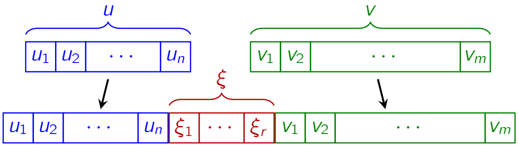
Decoupling assumptions (continued)



Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

- $\forall \xi \in \mathcal{A}^{\tau_n}$, $\mathbb{P}_{n+|\xi|+m}(u\xi v) \leq C_n \mathbb{P}_n(u)\mathbb{P}_m(v)$

Decoupling assumptions (continued)



$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$

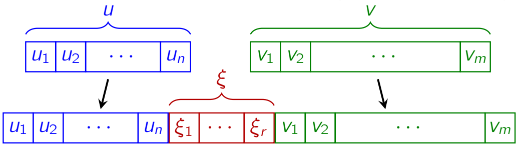
vs

$\mathbb{P}_{n+|\xi|+m}(u\xi v)$

Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

- $\forall \xi \in \mathcal{A}^{\tau_n}$, $\mathbb{P}_{n+|\xi|+m}(u\xi v) \leq C_n \mathbb{P}_n(u)\mathbb{P}_m(v)$
- $\exists \xi = \xi_{u,v} \in \mathcal{A}^{\tau_n}$ such that $\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1} \mathbb{P}_n(u)\mathbb{P}_m(v)$

Decoupling assumptions (continued)



$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$

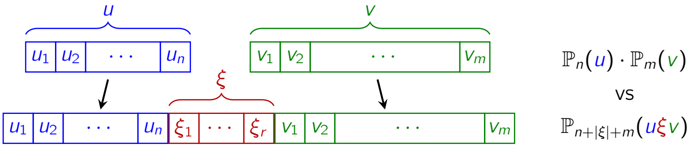
vs

$\mathbb{P}_{n+|\xi|+m}(u\xi v)$

Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

- $\forall \xi \in \mathcal{A}^{\tau_n}$, $\mathbb{P}_{n+|\xi|+m}(u\xi v) \leq C_n \mathbb{P}_n(u)\mathbb{P}_m(v)$
- $\exists \xi = \xi_{u,v} \in \mathcal{A}^{\tau_n}$ such that $\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$
  NB: may have, e.g., $\mathbb{P}_n(x_1, \dots, x_n) \sim e^{-n^2}$, highly non-Gibbsian!

Decoupling assumptions (continued)

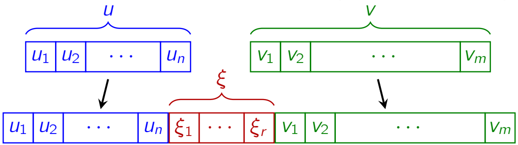


Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

- $\forall \xi \in \mathcal{A}^{\tau_n}$, $\mathbb{P}_{n+|\xi|+m}(u\xi v) \leq C_n \mathbb{P}_n(u)\mathbb{P}_m(v)$
- $\exists \xi = \xi_{u,v} \in \mathcal{A}^{\tau_n}$ such that $\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$
  NB: may have, e.g., $\mathbb{P}_n(x_1, \ldots, x_n) \sim e^{-n^2}$, highly non-Gibbsian!

Limitation: $\operatorname{supp} \mathbb{P}$ topologically mixing. For MC: irreducible and <u>aperiodic</u>

Decoupling assumptions (continued)



$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$

vs

$\mathbb{P}_{n+|\xi|+m}(u\xi v)$

Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

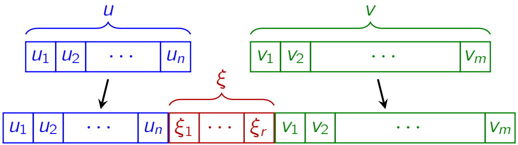
- $\forall \xi \in \mathcal{A}^{\tau_n}$, $\mathbb{P}_{n+|\xi|+m}(u\xi v) \leq C_n \mathbb{P}_n(u)\mathbb{P}_m(v)$
- $\exists \xi = \xi_{u,v} \in \mathcal{A}^{\tau_n}$ such that $\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$
  NB: may have, e.g., $\mathbb{P}_n(x_1, \dots, x_n) \sim e^{-n^2}$, highly non-Gibbsian!

Limitation: $\operatorname{supp}\mathbb{P}$ topologically mixing. For MC: irreducible and aperiodic

Selective lower decoupling (CJPS '19). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$, $\exists \xi = \xi_{u,v}$ with $0 \leq |\xi| \leq \tau_n$ such that

$$\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$$

Decoupling assumptions (continued)



$\mathbb{P}_n(u) \cdot \mathbb{P}_m(v)$

vs

$\mathbb{P}_{n+|\xi|+m}(u\xi v)$

Asymptotically decoupled measures (Pfister '02). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$,

- $\forall \xi \in \mathcal{A}^{\tau_n}$, $\mathbb{P}_{n+|\xi|+m}(u\xi v) \leq C_n \mathbb{P}_n(u)\mathbb{P}_m(v)$
- $\exists \xi = \xi_{u,v} \in \mathcal{A}^{\tau_n}$ such that $\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$
  NB: may have, e.g., $\mathbb{P}_n(x_1, \dots, x_n) \sim e^{-n^2}$, highly non-Gibbsian!

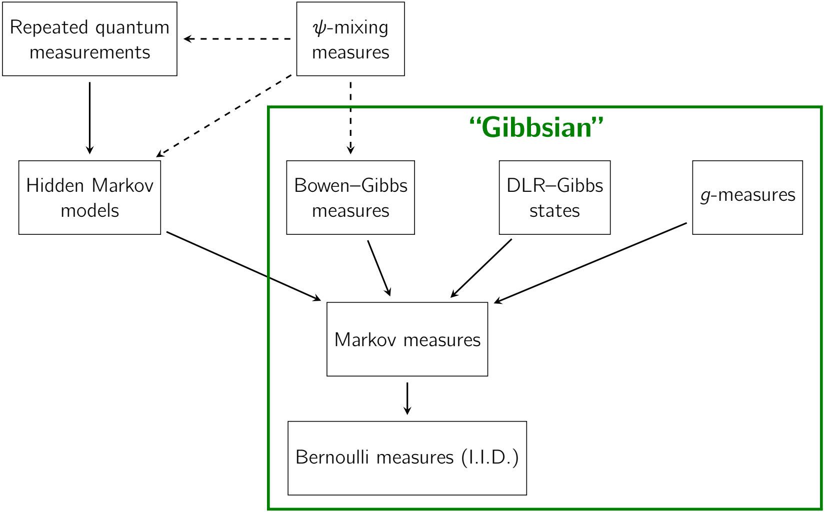
Limitation: $\operatorname{supp}\mathbb{P}$ topologically mixing. For MC: irreducible and aperiodic

Selective lower decoupling (CJPS '19). $\exists C_n = e^{o(n)}, \tau_n = o(n)$ such that $\forall n, m \in \mathbb{N}$, $u \in \mathcal{A}^n$, $v \in \mathcal{A}^m$, $\exists \xi = \xi_{u,v}$ with $0 \leq |\xi| \leq \tau_n$ such that

$$\mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$$

For Markov chains: only irreducibility required

Standard families of measures



Selective lower dec.: $\forall u, v, \ \exists \xi, \ \mathbb{P}_{n+|\xi|+m}(u\xi v) \geq C_n^{-1}\mathbb{P}_n(u)\mathbb{P}_m(v)$

Large deviations

Decoupling assumptions allow one to prove Large Deviation Principles for

$$\textbf{Time averages:} \quad \frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x))$$

Large deviations

Decoupling assumptions allow one to prove Large Deviation Principles for

Time averages: $$\frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x))$$

Empirical measures: $$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

Large deviations

Decoupling assumptions allow one to prove Large Deviation Principles for

Time averages: $$\frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x))$$

Empirical measures: $$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

Return times: $$\frac{1}{n}\ln R_n(x)$$

Large deviations

Decoupling assumptions allow one to prove Large Deviation Principles for

Time averages: $\frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x))$

Empirical measures: $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$

Return times: $\frac{1}{n}\ln R_n(x)$

Log-probability: $\frac{1}{n}\ln \mathbb{P}_n(x_1, \ldots, x_n)$

Large deviations

Decoupling assumptions allow one to prove Large Deviation Principles for

Time averages: $$\frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i(x))$$

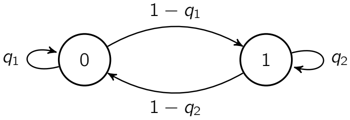
Empirical measures: $$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{\varphi^i(x)}$$

Return times: $$\frac{1}{n}\ln R_n(x)$$

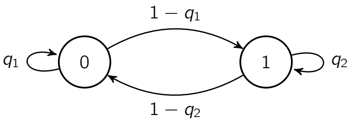
Log-probability: $$\frac{1}{n}\ln \mathbb{P}_n(x_1,\dots,x_n)$$

Entropy production: $$\frac{1}{n}\sigma_n(x), \quad \sigma_n(x) = \ln\frac{\mathbb{P}_n(x_1,\dots,x_n)}{\mathbb{P}_n(x_n,\dots,x_1)}$$

"Keep–Switch" quantum instrument [BCJP21]



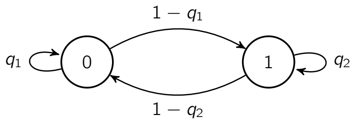
"Keep–Switch" quantum instrument [BCJP21]



$F($00110110010110010110...$)$

$=$KSKSSKSKSSSKSKSSSKSK...

"Keep–Switch" quantum instrument [BCJP21]

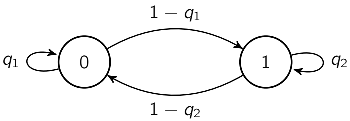


$F(00110110010110010110 0\ldots)$

$= \mathtt{KSKSSKSKSSSKSKSSSKSK}\ldots$

Let $\mathcal{A} = \{K, S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P} = \mathbb{Q} \circ F$, where $\mathbb{Q}$ = above MC

“Keep–Switch” quantum instrument [BCJP21]

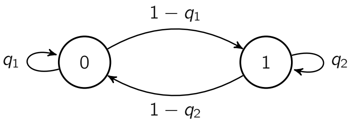


F(00110110010110010110 0...)
$=$KSKSSKSKSSSKSKSSSKSK...

Let $\mathcal{A} = \{K, S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P} = \mathbb{Q} \circ F$, where $\mathbb{Q}$ = above MC

- If $q_2 = 1 - q_1$: Blackwell–Furstenberg–Walters–van den Berg measure

"Keep–Switch" quantum instrument [BCJP21]

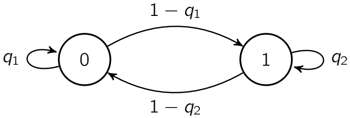


$F(00110110010110010110 0\ldots)$
$=KSKSSKSKSSSKSKSSSKSK\ldots$

Let $\mathcal{A} = \{K, S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P} = \mathbb{Q} \circ F$, where $\mathbb{Q}$ = above MC

- If $q_2 = 1 - q_1$: Blackwell–Furstenberg–Walters–van den Berg measure
- $\mathbb{P}$ is **non-Gibbsian**

“Keep–Switch” quantum instrument [BCJP21]

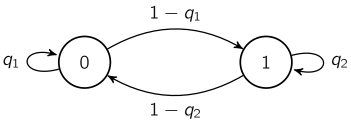


$F(00110110010110010110\ldots)$
$=$KSKSSKSKSSSKSKSSSKSK…

Let $\mathcal{A}=\{K,S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P}=\mathbb{Q}\circ F$, where $\mathbb{Q}=$ above MC

- If $q_2=1-q_1$: Blackwell–Furstenberg–Walters–van den Berg measure
- $\mathbb{P}$ is **non-Gibbsian**
- $\frac{1}{n}\sigma_n\to s_*$ a.s. (here actually $\sigma_n(x)=\ln\frac{\mathbb{P}_n(x_1,\ldots,x_n)}{\mathbb{P}_n(\overline{x}_n,\ldots,\overline{x}_1)}$)

“Keep–Switch” quantum instrument [BCJP21]



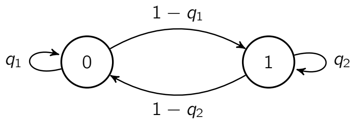
$F(00110110010110010110 0\ldots)$

$= \text{KSKSSKSKSSSKSKSSSKSK}\ldots$

Let $\mathcal{A} = \{K, S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P} = \mathbb{Q} \circ F$, where $\mathbb{Q}$ = above MC

- If $q_2 = 1 - q_1$: Blackwell–Furstenberg–Walters–van den Berg measure
- $\mathbb{P}$ is **non-Gibbsian**
- $\frac{1}{n}\sigma_n \to s_*$ a.s. (here actually $\sigma_n(x) = \ln \frac{\mathbb{P}_n(x_1,\ldots,x_n)}{\mathbb{P}_n(\bar{x}_n,\ldots,\bar{x}_1)}$)
- **Non-Gaussian** CLT: $(\sigma_n - ns_*)/\sqrt{n}$

"Keep–Switch" quantum instrument [BCJP21]

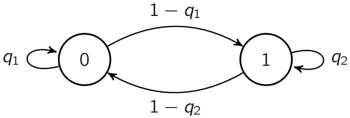


$F(\color{blue}{00}1\color{red}{10}11001011001011 00\ldots)$
$=\color{blue}{K}SK\color{red}{S}SKSKSSSKSKSSSKSK\ldots$

Let $\mathcal{A} = \{K, S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P} = \mathbb{Q} \circ F$, where $\mathbb{Q}$ = above MC

- If $q_2 = 1 - q_1$: Blackwell–Furstenberg–Walters–van den Berg measure
- $\mathbb{P}$ is **non-Gibbsian**
- $\frac{1}{n}\sigma_n \to s_*$ a.s. (here actually $\sigma_n(x) = \ln \frac{\mathbb{P}_n(x_1,\ldots,x_n)}{\mathbb{P}_n(\bar{x}_n,\ldots,\bar{x}_1)}$)
- **Non-Gaussian** CLT: $(\sigma_n - ns_*)/\sqrt{n} \xrightarrow{law} Z_1 - |Z_2|$

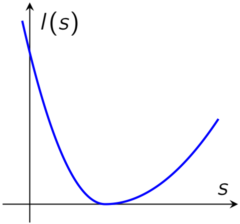
"Keep–Switch" quantum instrument [BCJP21]



$F(00110110010110010110 0\ldots)$
$=$KSKSSKSKSSSKSKSSSKSK...

Let $\mathcal{A} = \{K, S\}$ and define $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ by $\mathbb{P} = \mathbb{Q} \circ F$, where $\mathbb{Q}$ = above MC

- If $q_2 = 1 - q_1$: Blackwell–Furstenberg–Walters–van den Berg measure
- $\mathbb{P}$ is **non-Gibbsian**
- $\frac{1}{n}\sigma_n \to s_*$ a.s. (here actually $\sigma_n(x) = \ln \frac{\mathbb{P}_n(x_1,\ldots,x_n)}{\mathbb{P}_n(\overline{x}_n,\ldots,\overline{x}_1)}$)
- **Non-Gaussian** CLT: $(\sigma_n - ns_*)/\sqrt{n} \xrightarrow{law} Z_1 - |Z_2|$
- Rate function for $(\frac{1}{n}\sigma_n)_{n\geq 1}$ differentiable once but not twice



Thank you!

References

[LPS95] **Entropy, concentration of probability and conditional limit theorems**, J. T. Lewis, C.-É. Pfister, and W. G. Sullivan. *Markov Proc. Relat. Fields*, 1(3):319–386, 1995.

[Pfi02] **Thermodynamical aspects of classical lattice systems**, C.-É. Pfister. *In and Out of Equilibrium: Probability with a Physics Flavor*, volume 51 of *Prog. Probab.*, pages 393–472. Birkhäuser, Boston, 2002.

[CJPS19] **Large deviations and Fluctuation Theorem for selectively decoupled measures on shift spaces**, N. Cuneo, V. Jakšić, C.-A. Pillet, and A. Shirikyan. *Rev. Math. Phys.*, 31(10):1950036–1–549, 2019.

[BCJP21] **On entropy production of repeated quantum measurements II. Examples**, T. Benoist, N. Cuneo, V. Jakšić, and C.-A. Pillet. *J. Stat. Phys.*, 182(3):1–71, 2021.

[CR24] **Large deviations of return times and related entropy estimators on shift spaces**, N. Cuneo and R. Raquépas. *Commun. Math. Phys.*, 405(6): Paper No. 135, 1–69, 2024.