Regularity of rotation numbers of cocycles

On joint works with A. Gorodetski and P. Duarte

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THERMOGAMAS
Brazil–France web seminar
March 31st, 2026

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Hence, we can define $\rho(f) = \tau(\widetilde{f}) \mod 1 \in \mathbb{R}/\mathbb{Z}$.

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Then we have a map

$$F : X \times \mathbb{S}^1 \to X \times \mathbb{S}^1, \quad F(x, y) = (Tx, f_x(y)),$$

a skew product or a cocycle. Its iterations

$$F^n(x, y) = (T^n x, (\underbrace{f_{T^{n-1}x} \circ \cdots \circ f_{Tx} \circ f_x}_{f_{n,x}})(y))$$

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satisfy the cocycle relation

$$f_{n+m,x} = f_{m,T^n x} \circ f_{n,x}.$$

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for μ-a.e. $x \in X$.

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In general, $\tau(\widetilde{F}) \mod 1$ is no longer well-defined. Sometimes, yes (if we have a preferential choice of a lift).

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Proof.

Both rotation numbers get shifted by $\int_X k(x)\, d\mu(x)$, that cancels out. □

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Dynamically defined potentials: $T : (X, \mu) \to (X, \mu)$ ergodic measure-preserving and invertible, $x_0 \in X$ generic, $\varphi : X \to \mathbb{R}$,

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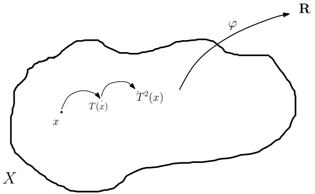

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this corresponds to the iterations of a $\mathbb{SL}(2,\mathbb{R})$-cocycle

$$(x, v) \mapsto (Tx, A_{x,E}(v)), \quad A_{x,E} := \begin{pmatrix} E - \varphi(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

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How can we see it from the dynamical point of view?

Log-Hölder property

Consider a parameter-dependent cocycle,

$$F_E : X \times \mathbb{S}^1 \to X \times \mathbb{S}^1, \quad (x, y) \mapsto (Tx, f_{x,E}(y)) :$$

- *For* $M_x := \log \max \left(2, \|f_{x,E}\|_{C^1(\mathbb{S}^1)}\right)$, *we have*

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Then the rotation number $\tau(E)$ *satisfies log-Hölder regularity estimate: for some* C *and for all sufficiently close* E_1, E_2 *one has*

$$|\tau(E_1) - \tau(E_2)| \leq \frac{C}{\log |E_1 - E_2|^{-1}}.$$

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- After just being a full circle ahead: again $+C_{(E)} \cdot \Delta E$.
- How many multiplications by M_x is required to reach 1 from $C_{(E)} \cdot \Delta E$?

Craig-Simon Theorem

Theorem (Craig-Simon, 1983)

For any ergodic discrete Schrödinger operator, if the function $\varphi : X \to \mathbb{R}^1$ *is such that*

$$\int_X \log(1 + |\varphi(x)|) d\mu(x) < \infty,$$

then the integrated density of states $N(E)$ *is log-Hölder continuous, i.e. for any compact* $J \subset \mathbb{R}$*, for some* $C > 0$ *and any* $E_1, E_2 \in J$ *with* $|E_1 - E_2| \leq 1/2$ *one has*

$$|N(E_1) - N(E_2)| \leq C \left(\log |E_1 - E_2|^{-1}\right)^{-1}.$$

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Lemma

If at least one of two measures ν_1, ν_2 is f-invariant, then

$$\rho(f) = \int_{\mathbb{S}^1} \nu_1([x, f(x)))\, d\nu_2(x).$$

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Lemma

$$\rho(E_2) - \rho(E_1) = \int_{\mathbb{S}^1} \widetilde{\nu}_{E_1}\left([\widetilde{f}_{E_1}(x), \widetilde{f}_{E_2}(x))\right) \, d\nu_{E_2}(x).$$

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(!Assume monotonicity or take signs into account.)

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Theorem (P. Duarte, A. Gorodetski, VK, 2024)

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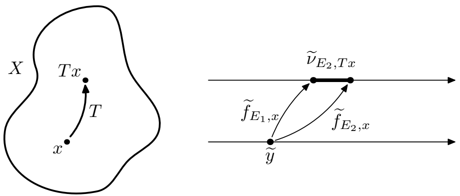

Sketch of the proof: translating measures

Definition

For probability measures m_1, m_2 on $\mathbb{R}$ and ν on $\mathbb{S}^1$, let

$$\Phi_\nu(m_1, m_2) = \int_{\mathbb{R}} [m_1((-\infty, y]) - m_2((-\infty, y])] \, d\widetilde{\nu}(y),$$

where $\widetilde{\nu}$ is a (infinite, 1-periodic) Radon measure that is the lift of ν on $\mathbb{R}$.

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Remark

$$\Phi_\nu(m_1, m_2) = \mathbb{E}\Phi_\nu(\xi_1, \xi_2),$$

where $\xi_i \sim m_i$ and

$$\Phi_\nu(a, b) = \begin{cases} \widetilde{\nu}([a, b)), & a < b, \\ 0, & a = b, \\ -\widetilde{\nu}([b, a)), & a > b, \end{cases}$$

Translating measures: an illustration

Remark

$\Phi_\nu(m_1, m_2)$ is the $\widetilde{\nu}$-measured average (signed) translation cost between m_1 and m_2,

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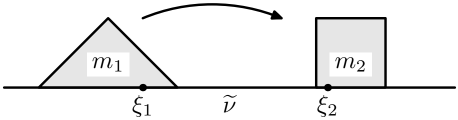

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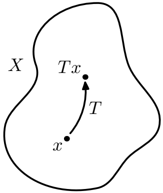
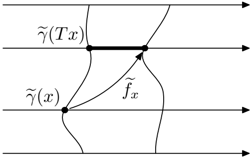
Remark

If ν_1' is a measure that is supported on a graph of a map $\gamma : X \to \mathbb{R}$, then $\mathcal{T}(\widetilde{F}; \nu_1, \nu_2)$ is the average $\widetilde{\nu}_{2,x}$-measured shift between the graph of γ and its $\widetilde{F}$-image.





Translation for $\widetilde{F}$: the key proposition

Definition

Let ν_1' be any probability measure on $X \times \mathbb{R}$ that projects to ν_1 on $X \times \mathbb{S}^1$. Define

$$\mathcal{T}(\widetilde{F}; \nu_1, \nu_2) := \int_X \Phi_{\nu_{2,Tx}}(\nu'_{1,Tx}, (\widetilde{f}_x)_* \nu'_{1,x}) \, d\mu(x).$$

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Proposition

If at least one of the measures ν_1, ν_2, projecting to μ, is F-invariant, then

$$\rho(\widetilde{F}) = \mathcal{T}(\widetilde{F}; \nu_1, \nu_2).$$

Plausibility check

Remark

- *If ν_1 is F-invariant, for every $x \in X$ the measures $\nu'_{1,Tx}$ and $(\widetilde{f}_x)_* \nu'_{1,x}$ project to the same measure $\nu_{1,Tx}$ on the circle, hence the $\widetilde{\nu}_{2,Tx}$-transport cost between them does not depend on the choice of ν_2.*

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- *If ν_2 is F-invariant, shifting a part of $\nu_{1,x}$ on the fiber over $x \in X$ shifts its image over $Tx \in X$ by the same $\widetilde{\nu}_2$-distance, thus not changing the integral in the definition of $\mathcal{T}(\widetilde{F}; \nu_1, \nu_2)$.*

Deducing the main result

Proof of the main theorem.

$$\rho(\widetilde{F}_{E_2}) - \rho(\widetilde{F}_{E_1}) = \mathcal{T}(\widetilde{F}_{E_2}; \nu_{E_1}, \nu_{E_2}) - \mathcal{T}(\widetilde{F}_{E_1}; \nu_{E_1}, \nu_{E_2})$$

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The same argument applies with the roles of ν_{E_1} and ν_{E_2} interchanged. □

Key proposition: sketch of the proof

- A change of variable by applying $\widetilde{F}$ everywhere implies

$$\mathcal{T}(\widetilde{F}, \nu_1, \nu_2) = \mathcal{T}(\widetilde{F}, F_*\nu_1, F_*\nu_2).$$

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- How much $\widetilde{F}^{m+n}$ shifts ν_1: first apply $\widetilde{F}^m$, then $\widetilde{F}^n$:

$$\mathcal{T}(\widetilde{F}^{m+n}, \nu_1, \nu_2) = \mathcal{T}(\widetilde{F}^m, \nu_1, \nu_2) + \mathcal{T}(\widetilde{F}^n, F_*^m \nu_1, \nu_2).$$

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- Finally, $\frac{1}{n}\mathcal{T}(\widetilde{F}^n, \nu_1, \nu_2) \to \rho(\widetilde{F})$.

Random dynamical systems

For the independent iterations (Bernoulli measure):

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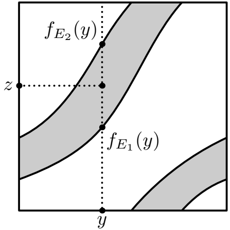

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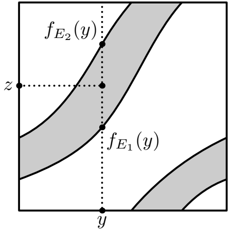

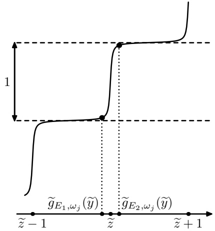

Random dynamics point of view

$$\widetilde{f}_{w_n,E_2} \circ \cdots \circ \widetilde{f}_{w_1,E_2}(y_0) - \widetilde{f}_{w_n,E_1} \circ \cdots \circ \widetilde{f}_{w_1,E_1}(y_0) =$$

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$$\sum_{j=1}^{n} \Big(\widetilde{f}_{w_n,E_2} \circ \cdots \circ \widetilde{f}_{w_{j+1},E_2} \circ {\color{red}\widetilde{f}_{w_j,E_2}} \circ \underbrace{f_{w_{j-1},E_1} \circ \widetilde{f}_{w_1,E_1}(y_0)}_{y} -$$

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Almost all summands are either almost 0, or almost 1, depending on whether for the corresponding random image y, distributed w.r.t. $\nu^+_{E_1}$, shifting from $\widetilde{f}_{w_j,E_1}(y)$ to $\widetilde{f}_{w_j,E_2}(y)$ makes it jump over the repulsion point z for the rest of the composition, that is distributed w.r.t. $\nu^-_{E_2}$.

Regularity of the rotation number

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Corollary

The rotation number is at least as regular (from the point of view of the modulus of continuity) as the stationary measures (Hölder, log-Hölder, Lipschitz; anything below C^1).

Regularity of a stationary measure

Let M be a smooth closed Riemannian manifold, and take a measure μ on $\mathrm{Diff}^1(M)$ such that

$$\exists \gamma > 0 : \quad \int \|f^{\pm 1}\|^{\gamma}_{\mathrm{Diff}^1(M)} \, d\mu(f) < \infty.$$

$$\exists C, \alpha > 0 : \quad \forall x \in M, \quad \forall r > 0 \quad \nu(B_r(x)) < Cr^{\alpha}.$$

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Texts

- Anton Gorodetski, Victor Kleptsyn, Log Hölder continuity of the rotation number, *Ergodic Theory and Dynamical Systems*, **45**:12 (2025), pp. 3749–3759; preprint arXiv:2410.15462
- Pedro Duarte, Anton Gorodetski, Victor Kleptsyn, The fibered rotation number, preprint arXiv:2512.00195
- Anton Gorodetski, Victor Kleptsyn, Grigorii Monakov, Hölder regularity of stationary measures, *Invent. math.* (2025); preprint arXiv:2209.12342.

Thank you for your attention!