Characterization of foliations via disintegration maps

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- **Goal:** To advertise the viewpoint of tackling disintegration of measures taking into consideration structures of probability spaces.

- The study of disintegration of measures considering intrinsic structures of the base space allows a unified view of geometry and dynamics, at both global and local levels.

- Using elements of Optimal Transport, which provide a geometric framework to compare conditional measures, one can track how these measures evolve along the base space.

- It reveals regularity, rigidity, and transport properties that a purely measure-theoretic approach does not capture.

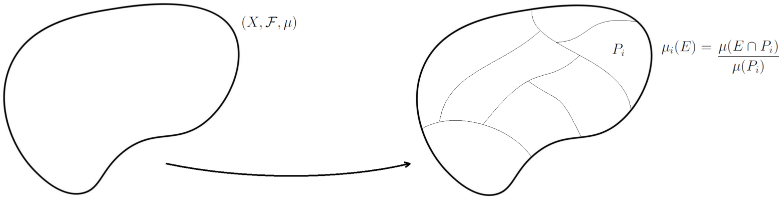
- New approach for analysing the relationship between the supports of conditional measures and their geometric arrangement in the space of probabilities.

R. Possobon and C. S. Rodrigues. **Geometric properties of disintegration of measures.** Ergodic Theory and Dynamical Systems, 45, 1619–1648, 2025.

F. Münch, R. Possobon and C. S. Rodrigues. **Characterization of foliations via disintegration maps.** Accepted in: Calculus of Variations and Partial Differential Equations.

Disintegration of measures

The disintegration of a measure over a partition of the space on which it is defined is a way to rewrite this measure as a combination of probability measures, which are concentrated on the elements of the partition.



Partition into a finite number of measurable subsets $P_1, \ldots, P_n$ with $\mu(P_i) > 0$, $i = 1, \ldots, n$.

$$\mu(E) = \sum_{i=1}^{n} \mu_i(E)\mu(P_i) = \sum_{i=1}^{n} \mu(P_i) \frac{\mu(E \cap P_i)}{\mu(P_i)}.$$

Definition: Disintegration

- $(X, \mathcal{F}, \mu)$ probability space;
- $\mathcal{P}$ a partition of X into measurable subsets;
- ϱ natural projection $X \ni x \mapsto P \in \mathcal{P}$ which contains x;
- $B \subset \mathcal{P}$ is measurable $\iff$ $\varrho^{-1}(B)$ is a measurable subset of X.
- The family $\hat{\mathcal{B}}(\mathcal{P})$ of measurable subsets is a σ-algebra on $\mathcal{P}$;
- $\hat{\mu}(B) = \varrho_* \mu(B) := \mu \circ \varrho^{-1}(B)$ for every $B \in \hat{\mathcal{B}}(\mathcal{P})$.

A **disintegration** of μ with respect to $\mathcal{P}$ into conditional measures is a family $\{\mu_P\}_{P \in \mathcal{P}}$ of probabilities on X, such that, for every $E \in \mathcal{F}$,

1. $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;
2. $P \mapsto \mu_P(E)$ is measurable;
3. $\mu(E) = \int \mu_P(E) \, d\hat{\mu}(P)$.

Theorem A

- X and Y separable locally compact metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu = \pi_* \mu \in \mathcal{M}_+(Y)$.

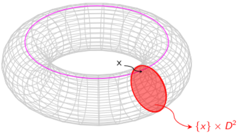
There are measures $\{\mu_y\}$, such that,

1. μ_y is concentrated in $\pi^{-1}(y)$ for ν-almost every $y \in Y$;
2. μ_y is a probability in X for ν-almost every $y \in Y$;
3. $y \mapsto \mu_y(A)$ is Borel map;
4. $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$ for every measurable set $A \subset X$.

Uniqueness: If $\mu(A) = \int_Y \eta_y(A)\, d\lambda(y)$, η_y is concentrated on $\pi^{-1}(y)$, then $\lambda \ll \nu$ and $\eta_y \ll \mu_y$ for ν-almost every $y \in Y$.

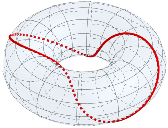
$$S^1 \times D^2$$

$$\pi : S^1 \times D^2 \to S^1 \text{ s.t. } \pi(x, y) = x$$



$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$

$$\varphi_t = (x + t, y + 2t) \mod 1$$



$$\mathbb{T}^2/\sim \cong S^1$$

$$\pi : \mathbb{T}^2 \to S^1 \text{ s.t. } \pi(x, y) = y - 2x \mod 1$$

Disintegration maps

[- , Rodrigues] Definition: Disintegration maps

- X and Y locally compact separable complete metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu = \pi_* \mu \in \mathcal{M}_+(Y)$;
- $\{\mu_y\}$ a disintegration of μ w.r.t. ν.

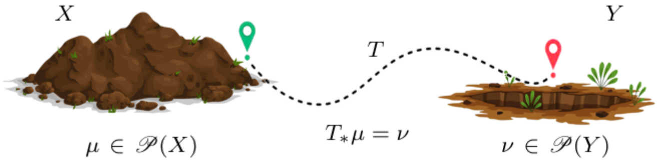
The **disintegration map** of μ w.r.t. ν is

$$\begin{aligned} f : Y &\to (\mathscr{P}_p(X), W_p) \\ y &\mapsto \mu_y \end{aligned}$$

such that $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$ for every measurable subset A of X.

Optimal Transport

Logistic problem by Gaspar Monge in 1781. **Main idea:** to transport masses from a given location to another one at minimal cost.



Monge Problem: X, Y Radon spaces, $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$, $c : X \times Y \to [0, \infty]$ fixed Borel cost function, minimize

$$T \mapsto \int_X c(x, T(x)) \, d\mu$$

among all maps $T : X \to Y$ such that $T_*\mu = \nu$.

- The maps T fulfilling $T_*\mu = \nu$ are called **transport maps**.
- A transport map between two measures may not exist.

- What if the mass is allowed to be split?
- **Transport plan**: $\gamma \in \mathscr{P}(X \times Y)$ such that $d\gamma(x, y)$ is the amount of mass transferred from x to y.
- The transport plan must ensure that for every point, the outgoing mass matches $d\mu$ and the incoming mass matches $d\nu$: γ admits μ and ν as **marginals** on X and Y. In other words, $(\mathrm{proj}_X)_*\gamma = \mu$, $(\mathrm{proj}_Y)_*\gamma = \nu$.
- **Admissible set**: $\Pi(\mu, \nu) := \{\gamma \in \mathscr{P}(X \times Y) : \gamma \text{ has } \mu \text{ and } \nu \text{ as its marginals}\}$
- $\Pi(\mu, \nu)$ is non empty, since the tensor product $\mu \times \nu$ lies in $\Pi(\mu, \nu)$ (any piece of sand, regardless of its location, is distributed over the entire hole, proportionally to the depth).

Monge-Kantorovich Problem: X, Y Radon spaces, $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$, and $c : X \times Y \to [0, \infty]$ fixed Borel cost function, minimize

$$\gamma \mapsto \int_{X \times Y} c(x, y)\ d\gamma(x, y)$$

among all measures $\gamma \in \mathscr{P}(X \times Y)$ with marginals μ and ν.

From this problem one can derive natural distance functions on spaces of probability measures, by choosing the cost function as a power of the distance.

Definition: Wasserstein distance

Let (X, d) be a separable complete metric space. Consider μ and ν probabilities on X and $p \in [1, \infty)$. The **Wasserstein distance** of order p between μ and ν is

$$W_p(\mu, \nu) := \left(\inf_{\gamma \in \Pi(\mu,\nu)} \int d(x_1, x_2)^p \; d\gamma(x_1, x_2) \right)^{\frac{1}{p}}.$$

Definition: Wasserstein space

The **Wasserstein space** of order p is, for $\tilde{x} \in X$ arbitrary, given by

$$\mathscr{P}_p(X) := \left\{ \mu \in \mathscr{P}(X) : \int d(x, \tilde{x})^p \mu(dx) < +\infty \right\}.$$

- Main interest: $p = 1$ and $p = 2$. W_1 is easier to bound, while W_2 captures geometric characteristics more effectively.

- W_p metrizes weak convergence in $\mathscr{P}_p(X)$: if $\{\mu_k\}$ is a sequence in $\mathscr{P}_p(X)$ and $\mu \in \mathscr{P}_p(X)$, then μ_k converges weakly to μ is equivalent to $W_p(\mu_k, \mu) \longrightarrow 0$.

- μ_k converges weakly to μ if $\int \varphi \, d\mu_k \longrightarrow \int \varphi \, d\mu$ for φ bounded continuous.

Disintegration maps

[- , Rodrigues] Definition: Disintegration maps

- X and Y locally compact separable complete metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu = \pi_* \mu \in \mathcal{M}_+(Y)$;
- $\{\mu_y\}$ a disintegration of μ w.r.t. ν.

The **disintegration map** of μ w.r.t. ν is

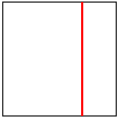
$$f : Y \to (\mathscr{P}_p(X), W_p)$$
$$y \mapsto \mu_y$$

such that $\mu(A) = \int_Y \mu_y(A)\ d\nu(y)$ for every measurable subset A of X.

Example:

- $[0, 1] \times [0, 1]$;
- $\mu = \mathrm{Leb}^2$ the two-dimensional Lebesgue measure on $[0, 1] \times [0, 1]$;
- $\pi(x, y) = x$ the projection onto the first coordinate;
- $\nu = \mathrm{Leb}^1$ the one-dimensional Lebesgue measure.

The disintegration $\{\mu_x\}_{x \in [0,1]}$ of μ w.r.t. ν is given by $\mu_x = \mathrm{Leb}^1$ on $\{x\} \times [0, 1]$.



$$\mu(A) = \int_{[0,1]} \mu_x(A)\, d\nu(x) = \int_0^1 \mathrm{Length}\Big(A \cap (\{x\} \times [0, 1])\Big)\, dx$$

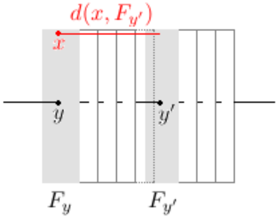
where $\mathrm{Length}(A \cap (\{x\} \times [0, 1])) := \mathrm{Leb}^1(y \in [0, 1] \ : \ (x, y) \in A)$, for any measurable $A \subseteq [0, 1] \times [0, 1]$.

In this case, the disintegration map is given by

$$\begin{aligned} f : [0,1] &\to (\mathscr{P}_p([0,1]\times[0,1]), W_p) \\ x &\mapsto \mu_x, \end{aligned}$$

where μ_x is Leb^1 on $\{x\} \times [0,1]$. This is a basic example that fits into the broader framework of metric foliations, which is the primary focus of our study.

- (X, d) metric space;
- **Foliation:** partition of X into closed subsets;
- **Metric foliation:** if $d(F, F') = d(x, F')$ for every $F, F' \in \mathcal{F}$, $x \in F$, where $d(F, F') = \inf\{d(x, x') : x \in F, x' \in F'\}$ and $d(x, F') = d(\{x\}, F')$;


$d(x, F_{y'})$
x
y
y'
F_y
$F_{y'}$

- $x \sim x' \iff \exists\, F \in \mathcal{F}$ such that $x, x' \in F$;
- $X^* := X/\sim$;
- $\pi : X \to X^*$ quotient map;
- $d^*(y, y') := d(\pi^{-1}(y), \pi^{-1}(y'))$ for $y, y' \in X^*$.

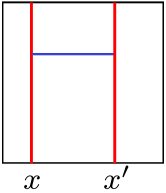
[- , Rodrigues] Definition: p-metric measure foliation

Let $\mathcal{F}$ be a metric foliation of (X, d, μ). $\mathcal{F}$ is a ***p*-metric measure foliation** if $\pi_*\mu$ is locally finite Borel measure on X^* and

$$W_p(\mu_y, \mu_{y'}) = d(\pi^{-1}(y), \pi^{-1}(y'))$$

for any $y, y' \in Y$, where $\{\mu_y\}$ is a disintegration of μ with respect to $\pi_*\mu$.

In the previous example:



Natural coupling pairs each (x, y) with (x', y) for $y \in [0, 1]$, yielding a constant transport cost of $|x - x'|$.

This coupling is optimal, and then $W_p(\mu_x, \mu_{x'}) = |x - x'|$, for all $p \geq 1$. Then, we have a p-metric measure foliation.

[- , Rodrigues] Proposition

Consider $\mathcal{F}$ a metric foliation of X, X^* the quotient space, and $\pi : X \to X^*$ the quotient map. If $\mathcal{F}$ is a p-metric measure foliation of X, then the disintegration map of μ with respect to $\nu = \pi_* \mu$ is weak continuous.

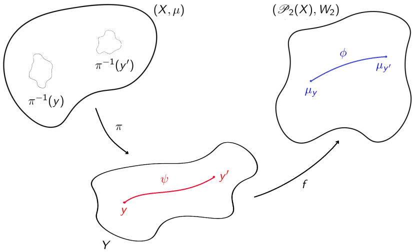
[- , Rodrigues] Proposition

Let X and Y be locally compact complete separable metric spaces, and consider $\mu \in \mathcal{M}_+(X)$. If a Borel map $\pi : X \to Y$ is such that $\nu := \pi_*\mu$ is a Borel measure, then, the disintegration map $f : Y \to (\mathscr{P}_p(X), W_p)$ of μ with respect to ν is nearly weak continuous, for $p \in [1, \infty)$.

[- , Rodrigues] Proposition

Let X and Y be locally compact complete separable metric spaces, $\mu \in \mathcal{M}_+(X)$, $\pi : X \to Y$ a Borel map, $\nu := \pi_*\mu$, and f the disintegration map of μ with respect to ν. If π is bijective and π^{-1} continuous, then f is weak continuous.

Paths of measures and absolute continuity from disintegration maps



[- , Rodrigues] Theorem B

f continuous and Y is path connected; given two points $y, y' \in Y$:
(i) $\exists$ a path of conditional measures on $\mathscr{P}_2(X)$, given by f, connecting μ_y, $\mu_{y'}$.
(ii) X smooth compact Riemannian manifold, $\mu \ll \text{vol}$, $\mu(\pi^{-1}(y)) > 0$ for ν-a.e. y, f minimizing invariant, and either $\mu_y \ll \text{vol}$ or $\mu_{y'} \ll \text{vol} \implies \mu_{y_t} \ll \text{vol}$;
(iii) X smooth compact Riemannian manifold, $\{\pi^{-1}(y)\}_{y \in Y}$ is a 2-metric measure foliation of X, and either $\mu_y \ll \text{vol}|_{F_y}$ or $\mu_{y'} \ll \text{vol}|_{F_{y'}} \implies \mu_{y_t} \ll \text{vol}|_{F_{y_t}}$.

On foliations and disintegration maps

- Support of conditional measures $\longleftrightarrow$ relative position of the conditional measures in the Wasserstein space;
- An approach that allows us to study this relation via disintegration maps.

- X and Y locally compact complete separable metric spaces;
- $\pi : X \to Y$ Borel map;
- $\mu \in \mathcal{M}_+(X)$;
- $\nu := \pi_* \mu$;
- $\{\mu_y\}$ a disintegration of μ w.r.t. ν given by Theorem A;
- f the disintegration map.

[Münch, - , Rodrigues] Definition: A derivative for disintegration maps

For $y', y'' \in Y$, we write

$$\rho(y', y'') := d(\pi^{-1}(y'), \pi^{-1}(y'')),$$

and we define a notion of derivative of f by

$$|\nabla f(y)|_p := \lim_{\varepsilon \to 0} \sup_{\substack{\rho(y,y') \leq \varepsilon \\ \rho(y,y'') \leq \varepsilon \\ y' \neq y''}} \frac{W_p(\mu_{y'}, \mu_{y''})}{\rho(y', y'')}$$

for $p \in [1, \infty)$, where $\mu_{y'} = f(y')$ and $\mu_{y''} = f(y'')$ are conditional measures.

[Münch, - , Rodrigues] Definition: p-energy

The p-energy of f is given by

$$\mathcal{E}_p(f) := \|\nabla f\|_{\infty,p} = \sup_{y \in Y} |\nabla f(y)|_p.$$

[Münch, - , Rodrigues] Theorem C

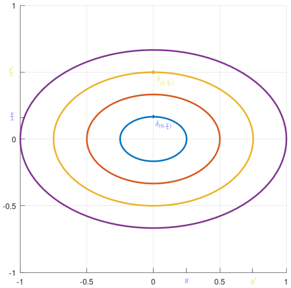
- (X, d_X) geodesic locally compact complete separable metric space;
- Y compact complete separable metric space;
- $\pi : X \to Y$ Borel map;

- $\mu \in \mathcal{M}_+(X)$;
- $\nu := \pi_* \mu$;
- $\{\mu_y\}_{y \in Y}$ disintegration of μ w.r.t. ν such that $\mathrm{supp}(\mu_y) = \pi^{-1}(y)$;
- f the respective disintegration map.

Then, $\mathcal{E}_p(f) = 1$ if and only if $\{\pi^{-1}(y)\}$ defines a p-metric measure foliation on X.

Example (full support):

- $X = \mathbb{R}^2$;
- $Y = [0, \infty)$;
- $\pi : X \to Y$ such that $\pi^{-1}(y) = \{(x_1, x_2) : x_1^2 + \lambda^2 x_2^2 = y^2\}$;
- $\mu \in \mathcal{M}_+(X)$ such that the disintegration of μ w.r.t. ν is $\mu_y = \delta_{(0, \frac{y}{\lambda})}$;
- Then, $d(\pi^{-1}(y), \pi^{-1}(y')) = \left| \frac{y}{\lambda} - \frac{y'}{\lambda} \right| = W_p(\mu_y, \mu_{y'})$, for every $y, y' \in Y$.

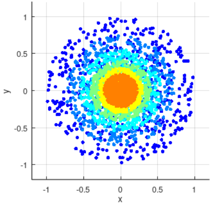


Example: Dynamical systems and the evolution of foliations in time.

- Flow φ_t on X;
- $X = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 \leq R^2\}$, $R > 0$;
- $Y = (0, R]$;
- $\mu \in \mathscr{P}(X)$ with full support and uniformly distributed;
- Foliation $\mathcal{F}$ of X such that $F_y = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = y^2\}$ is a leaf, for every $y \in Y$;
- $\pi : X \to Y$ such that $\pi((x_1, x_2)) = \sqrt{x_1^2 + x_2^2} = y$;
- f_0 the disintegration map of μ with respect to ν;
- Evolution of μ by φ_t: $\mu_t = (\varphi_t)_* \mu$.

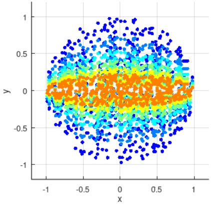
Spiral flow to the origin

$$\begin{cases} x(t) = r_0 e^{-\alpha t} \cos(\theta_0 + \omega t) \\ y(t) = r_0 e^{-\alpha t} \sin(\theta_0 + \omega t) \end{cases}$$



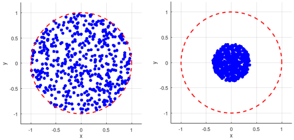
Vertical compression

$$\begin{cases} x(t) = x_0 \\ y(t) = y_0 e^{-\alpha t} \end{cases}$$

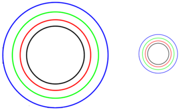


Spiral flow to the origin

- Radial contraction combined with angular rotation;



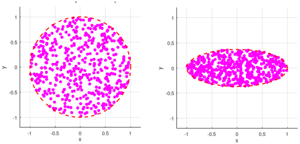
- The circular leaves remain circles with reduced radius;



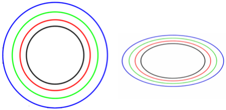
- Since μ_r on C_r is pushed forward to $\mu_{r(t)}$ on $C_{re^{-\alpha t}}$, the disintegration structure is preserved by the flow, and $\mathcal{E}_p(f_t) = 1$ for every t.

Vertical compression

- Compression effect;



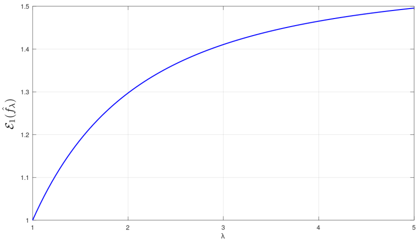
- Leaves are non longer circular;



- $F_y = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + \lambda^2 x_2^2 = y^2\}$ is a leaf, for every $y \in Y$;
- $\mathcal{E}_p$ is sensitive to deformations in the foliation.

The energy for $p = 1$ is the ratio between the length of the circle with radius y and the length of the ellipse with major axis equal to y:

$$\mathcal{E}_1(\hat{f}_\lambda) = \frac{2\pi y}{\mathcal{L}_y}.$$



Work in progress

- Study of dynamics and geometry in Wasserstein spaces;
- Differential calculus for disintegration maps; investigation of quasi-conformal holonomies and their relationship with the regularity of measures; application of these concepts to foliated dynamical systems and Ergodic Theory;
- Characterization of geodesics; modeling of stochastic processes as flows in Wasserstein space;
- Analysis of convergence and stability of generative algorithms.

Thank you!