Around notions of non-determinism in group actions

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Motivation: ϵ-asymptotic points

In this talk, we consider a topological dynamical system (X, G).

- X is a metric compact space.
- G is a group of homeomorphisms.
- d will denote a metric on X, ρ a metric on G.

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Theorem (Schwartzmann, 1952)

Let $(X, \mathbb{Z})$ be an infinite topological dynamical system. Then for any $\epsilon > 0$ there exist points $x \neq y$ with $d(T^n x, T^n y) \leq \epsilon$ for all $n > 0$.

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Boyle and Lind extended this type of results for $\mathbb{Z}^d$.

Theorem (Boyle-Lind, 1997)

Let $(X, \mathbb{Z}^d)$ be an infinite topological dynamical system. Then for any $\epsilon > 0$ there exist points $x \neq y$ and a half-plane H such $d(T^n x, T^n y) \leq \epsilon$ for all $n \in H$.

Groups and Horofunctions

Let G be a countable group with a proper, right-invariant metric ρ.

Busemann/Gromov Compactification

Embed G into the space of continuous functions $C(G)$ via:

$$b_g(x) = \rho(g, x) - \rho(g, 1_G)$$

- **Border:** $\partial G = \overline{b(G)} \setminus b(G)$.
- **Horofunctions:** Elements $h \in \partial G$.
- **Horoballs:** Generalizing the notion of half-spaces, the horoball associated with $h \in \partial G$ is:

$$H = \{x \in G \mid h(x) < 0\}$$

Non-determinism in Topological Dynamics

Consider a topological dynamical system (X, G). For $\varepsilon > 0$:

Definition (Deterministic Horofunction)

A horofunction $h \in \partial G$ is *ε-deterministic* if $d(T^g x, T^g y) \leq \varepsilon$ for all $g \in \{h < 0\}$ implies $x = y$.

Definition (Non-deterministic Horofunction)

A horofunction h is *ε-non-deterministic* if there exists an (h, ε)-asymptotic pair $x \neq y$ such that $d(T^g x, T^g y) \leq \varepsilon$ for all $g \in \{h < 0\}$.

We denote the collection of all ε-non-deterministic horofunctions as $\mathrm{ND}_\varepsilon(X)$.

Existence I: The Robinson Crusoe Theorem

Does a non-deterministic horofunction always exist?

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Theorem (Robinson Crusoe Theorem, D. Petite, Maass (2024))

Let (X, G) and (Y, G) be topological dynamical systems, and $\pi : X \to Y$ a factor map. Then either:

1. *π is bounded-to-1, or*
2. *For any $\varepsilon > 0$, there exists a horofunction $h \in \partial G$ and an (h, ε)-asymptotic pair (x, y) such that $\pi(x) = \pi(y)$.*

Corollary: If X is an infinite system (taking Y as trivial), it must have at least one non-deterministic horofunction. So $\mathrm{ND}_\varepsilon(X)$ is nonempty.

Existence II: Directed Existence

Can we force a non-deterministic horofunction to exist while in pointing into a "specific direction"?

Definition (Half-horoboundary associated with k)

For $k \in G$, the half-horoboundary associated with k is:

$$S(k) = \{h \in \partial G \mid h(k) \geq 0\}$$

Theorem (Directed Existence, D. Petite, Maass (2024); Bitar, D., Petite (2026))

Let $\pi : X \to Y$ be a factor map that is not bounded-to-1. Let $k \in G$ be an element of infinite order.

Then, for any $\varepsilon > 0$, there exists a non-deterministic horofunction $h \in S(k)$.

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Then, for any $\varepsilon > 0$, there exists a non-deterministic horofunction $h \in S(k)$.

We can always find a non-expansive horoball that does not contain k.

The Closedness of $\mathrm{ND}_\varepsilon(X)$

Boyle & Lind proved there is a half-space valid for all $\varepsilon > 0$. It is used that $\mathrm{ND}_\varepsilon(X)$ is closed.

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Theorem (Bitar, D., Petite (2026))

If G has:

1. *Uniformly quasi-subadditive horofunctions*
   There exists $R > 0$ such that

   $$h(gg') \leq h(g) + h(g') + R$$

   for all $h \in \partial G$ and all $g, g' \in G$.
2. *No lower bound for horofunctions* $\inf(h(x))) = -\infty$ *for all* $h \in \partial G$.

Then $\mathrm{ND}_\varepsilon(X)$ *is a closed set in* ∂G.

When closedness holds, the nested intersection $\mathrm{ND}(X) = \bigcap_{\varepsilon>0} \mathrm{ND}_\varepsilon(X)$ is a non-empty compact set.

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Theorem (Bitar, D., Petite (2026))

Let G be a group with a symmetric horoball space. Suppose G is torsion-free OR satisfies the horoball shift-property. Then, for every $\varepsilon > 0$:

$$\bigcap_{h \in \mathrm{ND}_\varepsilon(X)} \{h < 0\} = \emptyset$$

Application: Topological Minimal Self-Joinings

We apply previous notions to Topological Minimal Self-Joinings (TMSJ).

Definition

A system (X, G) has *n-fold TMSJ* if for any n points on different $Z(G)$-orbits, their joint orbit is dense in X^n.

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Theorem (King (1990))

No system $(X, \mathbb{Z})$ *has 4-fold TMSJ.*

We can limit say about other groups.

Theorem

No system $(X, \mathbb{Z}^d)$ *has* $2(d+1)$*-fold TMSJs.*

Theorem

No system $(X, H_{2d+1}(\mathbb{Z}))$ *has* $(2d+4)$*-fold TMSJs.*

Merci!
Obrigado!