(Joint) Typical Periodic Optimization

Oliver Jenkinson

Queen Mary University of London

12 May 2026

(Joint) Typical Periodic Optimization

Papers:

1. “Typical periodic optimization for dynamical systems: Symbolic dynamics”

With: Wen HUANG, Leiye XU, Yiwei ZHANG

Invent. Math., 2026.

2. “ Joint typical periodic optimization”

With: Zelai HAO, Yinying HUANG, Zhiqiang LI

arXiv:2502.12269

3. “Joint typical periodic optimization: Systems with stable hyperbolicity”

With: Zelai HAO, Yinying HUANG, Zhiqiang LI

arXiv:2605.01550

Background

- Optimization
- Periodic Optimization
- Typical Periodic Optimization (TPO)
- Joint Typical Periodic Optimization (Joint TPO)

Ergodic Optimization

Setup. $T : X \to X$ continuous, X compact metric space.

$$\mathcal{M}(X,T) = \{T\text{-invariant Borel probability measures on } X\}.$$

For a continuous function $f : X \to \mathbb{R}$, define the maximum ergodic average

$$Q(f) = Q(T,f) := \max_{\mu \in \mathcal{M}(X,T)} \int f \, d\mu.$$

$\mu \in \mathcal{M}(X,T)$ is maximizing (for f) if $\displaystyle\int f \, d\mu = Q(f)$.

The Ergodic Optimization Problem

Question: What do maximizing measures look like?

Periodic Optimization

Suppose T has **periodic orbits** $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots$

Each periodic $\mathcal{O}$ gives **invariant measure**

$$\mu_{\mathcal{O}} := |\mathcal{O}|^{-1} \sum_{x \in \mathcal{O}} \delta_x$$

Definition

$f : X \to \mathbb{R}$ has the **periodic optimization** property if there exists a periodic $\mathcal{O}$ with

$$\int f \, d\mu_{\mathcal{O}} > \int f \, d\mu \quad \text{for all } \mu \in \mathcal{M}(X,T) \setminus \{\mu_{\mathcal{O}}\}.$$

i.e. **Unique** maximizing measure supported on single **periodic** orbit.

Is this rare? **Or common?** (as we vary f)

What is “Typical”?

Uniqueness of maximizing measure is typical (in reasonable function spaces)

Mid-1990s: Numerical experiments for ‘**chaotic**’ T (Conze & Guivarc’h, Hunt & Ott, J, Bousch). **Periodic** optimization seems to be **typical**.

Hunt–Ott Conjecture (1996)

For typical chaotic $T : X \to X$, and typical families of functions $f_\theta : X \to \mathbb{R}$, the f_θ-maximizing measure is **periodic** for **Lebesgue almost every** θ

Progress:
Bochi & Zhang (2016), **Ding, Li & Zhang** (2024), **Gao, Shen & Zhang** (2025).

Yuan–Hunt Conjecture: Typical Periodic Optimization

Choose a *suitable* **Banach space** of real-valued functions: $\mathrm{Lip}(X)$ (**Lipschitz**) or $C^1(X)$

($C^0(X)$ is **not** a suitable Banach space: here periodic optimization is **atypical**)

Yuan–Hunt Conjecture (1999)

Let T be an **expanding** map, or an **Axiom A** diffeomorphism.
There exists an **open and dense** subset P of the function space, such that every $f \in P$ has the **periodic optimization** property.

NB: Open & Dense $\Rightarrow$ Residual $\iff$ Generic

Definition

$T : X \to X$ has the **Typical Periodic Optimization (TPO)** property if there exists an **open and dense** subset P such that every $f \in P$ has the periodic optimization property.

Towards the Yuan–Hunt Conjecture

Yuan–Hunt Conjecture (1999)

Let T be an expanding map, or an Axiom A diffeomorphism.
There exists an **open and dense** subset P of the function space, such that every $f \in P$ has the **periodic optimization** property.

Compare: Mañé Conjecture in Lagrangian dynamics

Progress includes:

- **Contreras, Lopes & Thieullen (2001)**: 'little Lipschitz' space
- **Bousch (2001)**: Space of Walters functions
- **Bressaud & Quas (2007)**: Rapid periodic orbit approximation
- **Morris (2008)**: Generic zero entropy optimization
- **Quas & Siefken (2012)**: Super-continuous functions

Resolution of Yuan–Hunt Conjecture

Theorem (Contreras: arXiv 2013, Invent. Math. 2016)

Every Lipschitz, **expanding**, open mapping $T : X \to X$ has TPO (for $\mathrm{Lip}(X)$).

New approach from **Huang, Lian, Ma, Xu & Zhang** (HLMXZ) fully resolves it:

Theorem (HLMXZ: arXiv 2019, JEMS 2027)

Every Lipschitz, **uniformly hyperbolic** map has TPO (for both $\mathrm{Lip}(X)$ and $C^1(X)$).

Theorem (Li & Zhang: Math. Ann. 2025)

TPO holds for **expanding Thurston** maps.

Proof strategy for Uniformly Hyperbolic TPO (à la HLMXZ)

- Banach space $\mathrm{Lip}(X)$.
- $P(X,T) := \mathit{interior}(\{f \in \mathrm{Lip}(X) : f \text{ has the periodic optimization property}\})$

Step 1: Mañé cohomology lemma (more classical)

Each $f \in \mathrm{Lip}(X)$ has $\varphi \in \mathrm{Lip}(X)$ with $f + \varphi - \varphi \circ T \leq Q(f)$
(Compare: Livsic's Theorem)
Work with this 'corrected' function, which has the same maximizing measure(s).

Step 2: Refined closing lemma(s) (Novelty)

Closed invariant sets can be approximated *rapidly* (**Bressaud–Quas**) by *well-proportioned* (**Huang-Lian-Ma-Xu-Zhang**) periodic orbits $\mathcal{O}$.

Step 3: Perturbation

$f - \varepsilon d(\cdot, \mathcal{O}) \in P(X,T)$. So the **open** set $P(X,T)$ is **dense**. So TPO holds.

Broader Conjectures: TPO / TLC

In the spirit of the 1996 Hunt–Ott Conjecture:

Typical Periodic Optimization (TPO) Conjecture

If T is suitably hyperbolic, and V is a space of suitably regular functions, then TPO holds (i.e. an open dense subset in V where periodic optimization holds).

Beyond periodic orbits:

Bochi Conjecture (Typical Low Complexity)

Suppose $T : X \to X$ is chaotic. Then for typical regular functions $f : X \to \mathbb{R}$, the maximizing measure has low complexity.

This talk: TPO beyond Uniform Hyperbolicity

How widespread is the TPO phenomenon?

- Need lots of periodic orbits
- TPO for systems with **hyperbolic-like** behaviour?
- **Symbolic dynamics** (shift spaces **beyond SFTs** = uniformly hyperbolic)
- **Beta-transformations** $T_\beta(x) = \beta x \pmod 1,\ \beta > 1$
- Other systems with some hyperbolicity

Typical Periodic Optimization: Symbolic Dynamics

Joint work with:

Wen Huang, Leiye Xu, Yiwei Zhang

(Anhui: University of Science and Technology of China
Anhui University of Science and Technology)

(Joint) Typical Periodic Optimization

Papers:

1. "Typical periodic optimization for dynamical systems: Symbolic dynamics"

With: Wen HUANG, Leiye XU, Yiwei ZHANG

Invent. Math., 2026.

2. " Joint typical periodic optimization"

With: Zelai HAO, Yinying HUANG, Zhiqiang LI

arXiv:2502.12269

3. "Joint typical periodic optimization: Systems with stable hyperbolicity"

With: Zelai HAO, Yinying HUANG, Zhiqiang LI

arXiv:2605.01550

Finite alphabet $\mathcal{A}$

X a **shift space**: closed, shift-invariant subset of set of all sequences $\mathcal{A}^{\mathbb{N}}$ (or $\mathcal{A}^{\mathbb{Z}}$)

$T = \sigma$ the **shift map** $(\sigma(x))_n = x_{n+1}$

Banach space $\mathrm{Lip}(X)$: **Lipschitz** functions $X \to \mathbb{R}$

Distance function d on X a standard ultra-metric ($\alpha \in (0,1)$):

$$d(x,y) := \alpha^{N(x,y)},$$

where

$$N(x,y) := \inf\{i \in \mathbb{N} : x_i \neq y_i\} - 1$$

TPO for Sofic Shifts

Subshifts of Finite Type (SFTs)

TPO already known, by Contreras (1-sided SFTs are open expanding), HLMXZ (2-sided)
SFTs are the ‘uniformly hyperbolic’ subshifts

Sofic shifts

Sofic shift spaces are factors of SFTs

Example: **Even shift** (even length runs of 0’s between 1’s)

No Mañé cohomology lemma (in general): so SFT proof method fails

Theorem (Sofic TPO)

Every **sofic** shift has TPO.

Markov Boundary $\partial_M X$

- For **any** shift space X, the **Markov boundary**, denoted $\boldsymbol{\partial_M X}$, is a (canonical) **subshift** of X (possibly $\emptyset$), introduced by Klaus Thomsen (2006)
- For $w \in \mathbb{W}(X)$ (the *language* of X):

$$\textbf{Predecessor set:} \quad P_X(w) := \{u \in \mathbb{W}(X) : uw \in \mathbb{W}(X)\}$$

$$\textbf{Follower set:} \quad F_X(w) := \{u \in \mathbb{W}(X) : wu \in \mathbb{W}(X)\}\,.$$

- $\mathbb{W}_\infty(X) := \big\{w \in \mathbb{W}(X) : |\{F_X(uw) : u \in P_X(w)\}| = \infty\big\}$, set of X-allowed words with **infinitely many distinct follower sets**
- $\partial_M X := \{x \in X : w \in \mathbb{W}_\infty(X)$ for **all** words w in $x\}$

Markov Boundary & Ergodic Optimization

Well known characterisation of sofic shifts

$$X \text{ is sofic} \iff \{F_X(w) : w \in \mathbb{W}(X)\} \text{ is finite}$$

Consequently:

Thomsen's characterisation of sofic shifts

$$X \text{ is sofic} \iff \partial_M X = \emptyset$$

The Markov boundary is a **potential obstacle** to Typical Periodic Optimization.

Definition (Boundary-optimized functions)

$$\mathbf{Lip}_{\boldsymbol{\partial}}(\boldsymbol{X}) := \{f \in \mathrm{Lip}(X) : f \text{ has a maximizing measure with support in } \partial_M X\}$$

Structural Theorem

Theorem (Structural Theorem)

For any shift space X, the **union** of the **open** sets $\mathit{interior}(\mathrm{Lip}_\partial(X))$ and

$$\mathit{interior}(\{f \in \mathrm{Lip}(X) : f \text{ has the periodic optimization property}\})$$

is **dense** in $\mathrm{Lip}(X)$.

Proof idea for Structural Theorem:

$f \notin \mathrm{Lip}_\partial(X) \Rightarrow$ some maximizing measure supported by some **SFT** in X; **Baire category argument** + **TPO for SFTs** $\Rightarrow$ open-dense periodic optimization in $\mathrm{Lip}(X) \setminus \mathrm{Lip}_\partial(X)$

Proof that Sofic $\Rightarrow$ TPO

Sofic $\iff \partial_M X = \emptyset \iff \mathrm{Lip}_\partial(X) = \emptyset \implies \mathit{interior}(\mathrm{Lip}_\partial(X)) = \emptyset$
$\implies \mathit{interior}(\{f \in \mathrm{Lip}(X) : f$ has the periodic optimization property$\})$ is dense in $\mathrm{Lip}(X) \iff$ TPO.

Gap shifts

Definition

For $S \subseteq \mathbb{N}_0$, the S**-gap shift** on $\{0,1\}$: **run lengths** of 0's lie in S

$$\cdots \; 1 \; \underbrace{00000}_{l_1 \in S} \; 1 \; \underbrace{000}_{l_2 \in S} \; 1 \; \underbrace{0000}_{l_3 \in S} \; 1 \; \underbrace{000000}_{l_4 \in S} \; 1 \; \cdots$$

Flavours:

- SFT: e.g. **Golden mean** shift: $S = \mathbb{N} = \{1, 2, 3 \ldots\}$
- Sofic: e.g. **Even shift**: $S = \{0, 2, 4, \ldots\}$
- Non-sofic: e.g. $S = \{n^2 : n \geq 0\}$

TPO via descent to the Markov boundary (or beyond...)

The Markov boundary is the only potential obstacle to TPO:

Descent Theorem

If $\partial_M X$ has TPO, then X has TPO.

Theorem

Every **gap shift** has TPO.

Proof:

X non-sofic $\Rightarrow \partial_M X = \{0^\infty\}$ (only the 0^n have infinitely many follower sets) $\Rightarrow$ TPO ✓

Descend further:

Theorem (2-step descent)

$\partial_M(\partial_M X)$ has TPO $\implies \partial_M X$ has TPO $\implies X$ has TPO.

Eventually Sofic shifts

Definition

A shift space is **eventually sofic** (of level n) if the n-th **iterated Markov boundary** $\partial_M^n X$ is sofic (and non-empty).

- **Level 0**: Sofic shifts
- **Level 1**: X non-sofic, $\partial_M X$ sofic. e.g. Non-sofic gap shifts
- **Level 2**: X non-sofic, $\partial_M X$ non-sofic, $\partial_M^2 X = \partial_M(\partial_M X)$ sofic

$\vdots$

- **Level n**: Can be constructed (e.g. via repeated 'interspersion' of shift spaces)

Theorem

Every **eventually sofic** shift has TPO.

Other shift spaces with TPO: Fragile Shifts

Definition

A (non-sofic) shift X is **fragile** if the Markov boundary $\partial_M X \neq \emptyset$, but

$$\mathrm{Lip}_\partial(X) = \{f \in \mathrm{Lip}(X) : f \text{ has a maximizing measure with support in } \partial_M X\}$$

has **empty interior**.

Fragile shift spaces **do exist** (e.g. 'interspersion' of minimal non-uniquely ergodic shift).

Theorem (Structural Theorem)

For any shift space X, the union of the **open** sets $\mathit{interior}(\mathrm{Lip}_\partial(X))$ and

$$\mathit{interior}(\{f \in \mathrm{Lip}(X) : f \text{ has the periodic optimization property}\})$$

is **dense** in $\mathrm{Lip}(X)$.

Corollary (Fragile $\Rightarrow$ TPO)

Every (eventually) fragile shift space has TPO.

Is TPO a ‘universal’ property in symbolic dynamics?

Which shift spaces do *not* have TPO?

- Shift spaces without periodic orbits
- Minimal (non-periodic) subshifts
- If **periodic measures** are **not dense** in the set of **ergodic measures**

A shift space without TPO (despite dense periodic measures)

Question on Universality

Is $\{\text{periodic measures}\}$ being weak* dense in $\mathcal{M}(X, \sigma)$ a **sufficient condition** for TPO?

Answer: **No** - there are explicit counterexamples.

Theorem/Example (Failure of TPO)

Morse sequence $\omega := 0110100110010110\ldots$

Morse shift $Z :=$ shift orbit closure of ω (minimal, **uniquely ergodic** (μ_Z), zero-entropy)

$\theta_n :=$ length-2^n prefix of ω ('special prefixes')

Introduce a third symbol **2** and let X be the **smallest subshift containing** all sequences of form $\theta_{n_1}\mathbf{2}\,\theta_{n_2}\mathbf{2}\,\theta_{n_3}\mathbf{2}\cdots$ ('interspersion')

X has **periodic measures dense**, but **not TPO**: non-periodic μ_Z robustly maximizing

Joint Typical Periodic Optimization

Joint work with:

Zelai Hao, Yinying Huang, Zhiqiang Li

(Peking University)

(Joint) Typical Periodic Optimization

Papers:

1. "Typical periodic optimization for dynamical systems: Symbolic dynamics"

With: Wen HUANG, Leiye XU, Yiwei ZHANG

Invent. Math., 2026.

2. " Joint typical periodic optimization"

With: Zelai HAO, Yinying HUANG, Zhiqiang LI

arXiv:2502.12269

3. "Joint typical periodic optimization: Systems with stable hyperbolicity"

With: Zelai HAO, Yinying HUANG, Zhiqiang LI

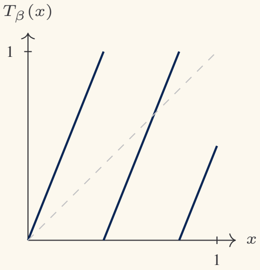
arXiv:2605.01550

Beta-Transformations

For $\beta > 1$, the **beta-transformation** is

$$T_\beta : [0,1] \to [0,1], \qquad T_\beta(x) = \beta x \pmod 1.$$

(Rényi, Parry)



Link with **beta-expansions**:

$$\varepsilon_1/\beta + \varepsilon_2/\beta^2 + \varepsilon_3/\beta^3 + \cdots$$

T_β is **discontinuous** (challenge for ergodic optimization). Periodic measures weak* dense

Question

Do the T_β have TPO?

TPO for Beta-Transformations

Fix T_β, and **Lipschitz** function space $\mathcal{F} = \mathrm{Lip}([0,1])$

Theorem (TPO: generic β)

For a **residual** (topologically generic) set of $\beta > 1, \quad T_\beta$ has TPO.

Theorem (TPO: almost every β)

For **Lebesgue almost every** $\beta > 1, \quad T_\beta$ has TPO.

Follows from: TPO holds for T_β if $\overline{\{T_\beta^n(1) : n \geq 0\}}$ is **not minimal**, true for generic β

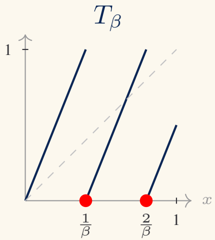
These theorems are optimal in the sense that:

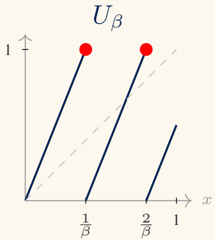
Observation

Countable dense set of values $\beta > 1$ such that T_β does **not have TPO**

Upper beta-transformation U_β

- These **'non-TPO parameters'** correspond to **simple beta-numbers** β
- $\mathcal{M}([0,1],T_\beta)$ non-compact, because 1 has no T_β-preimage
- Ghost 'periodic orbit' in $\overline{\mathcal{M}([0,1],T_\beta)}$: 1 is non-periodic since $1 \notin T_\beta([0,1])$
- Can *correct* via **upper beta-transformation** U_β: whenever $T_\beta(x)=0$ then let $U_\beta(x)=1$ (except that $U_\beta(0)=0$).
- U_β-orbit of **1 is periodic**, $\mathcal{M}([0,1],U_\beta)$ compact, $\boldsymbol{U_\beta}$ **has TPO**





TPO for *all* β ?

- TPO an open question for other parameters β
- Those where $\overline{\{T_\beta^n(1) : n \geq 0\}}$ **is minimal**

Open Question

Does $U_\beta : [0,1] \to [0,1]$ have TPO for **every** parameter $\beta > 1$?

Open Question (Symbolic version)

Does **every beta-shift** (subshift modelling T_β) have TPO?

Resistant to methods of 'TPO: Symbolic dynamics' paper

Joint Typical Periodic Optimization

Irrespective of the Open Question, since beta-transformations form a **parametrised family**:

- Let $\mathcal{F} = \mathrm{Lip}([0,1])$
- Let $\mathcal{T} = (1, \infty)$

Question

In the **product space** $\mathcal{T} \times \mathcal{F}$, what does a typical (**open & dense**) maximizing measure look like?

Yang–Hunt–Ott (2000)

Numerical experiments for the *Lorenz equations*:
"enlarging the **meaning of typicality** to be with respect to *both* variation of system parameters and performance function parameters."

Joint Typical Periodic Optimization: General Definition

Setup.

$\mathcal{T}$ a topological space of maps $T : X \to X$

$\mathcal{F}$ a Banach space of functions $f : X \to \mathbb{R}$

Definition

$\mathcal{T} \times \mathcal{F}$ has **Joint TPO** (Joint Typical Periodic Optimization) if there exists an **open dense** subset of $\mathcal{T} \times \mathcal{F}$ consisting of (T, f) with the **periodic optimization** property.

Topology of product spaces:

- Joint TPO $\not\Rightarrow$ TPO in each fibre $\{T\} \times \mathcal{F}$
- TPO in each fibre $\not\Rightarrow$ Joint TPO

Joint TPO: Beta-Transformations

Periodic optimization **is typical** for the **family** of beta-transformations ($\mathcal{F} = \mathrm{Lip}([0,1])$):

Theorem (Joint TPO for beta-transformations)

There is an **open dense** subset of pairs $(\beta, f) \in (1, \infty) \times \mathrm{Lip}([0,1])$ such that (T_β, f) has the periodic optimization property.

Despite:

- TPO is known to fail for infinitely many T_β
- TPO currently unknown for infinitely many other T_β

Proof ideas (TPO for generic & a.e. β **/ Joint TPO for beta-transformations)**:

- Beta-shifts are **monotone increasing** with β
- **Mañé cohomology lemma** for beta-transformations (one-sided continuity)
- **Joint Perturbation** theorem

Joint TPO: Uniform Hyperbolicity - Expanding maps

Question

If $\mathcal{T}$ consists of **uniformly hyperbolic** systems (each of which has **TPO** for the function space $\mathcal{F}$), does **Joint TPO** hold for $\mathcal{T} \times \mathcal{F}$?

Expanding Maps (in the sense of Ruelle):

- X a compact locally connected metric space
- $\mathcal{E}(X)$ the space of **open Lipschitz distance-expanding** maps on X

Theorem (Joint TPO: Expanding Maps)

$\mathcal{E}(X) \times \mathrm{Lip}(X)$ has **Joint TPO**

$+$ **'Effective' Version**, of the form: If (T, f) satisfies $d_{\mathrm{Lip}}(T_0, T) < \varepsilon$ and $\|f - f_0\| < \delta$, then (T, f) has the periodic optimization property.

Joint TPO: Uniform Hyperbolicity - Anosov diffeomorphisms

- M - compact Riemannian manifold
- $\mathcal{A}(M)$ - space of C^1 Anosov diffeos on M, equipped with C^1 topology

Theorem (Joint TPO: Anosov Diffeomorphisms)

$\mathcal{A}(M) \times \mathrm{Lip}(M)$ has Joint TPO

Theorem (Joint TPO: Anosov Diffeomorphisms - C^1 functions)

$\mathcal{A}(M) \times C^1(M)$ has Joint TPO

Joint TPO: Systems with Stable Hyperbolicity

$\mathcal{T}$ = a space of Lipschitz self-maps of compact metric space X

$\mathrm{Lip}(X)$ = space of real-valued Lipschitz functions

Definition ($\mathcal{T}$-stable hyperbolicity)

$T \in \mathcal{T}$ is said to be $\mathcal{T}$**-stably hyperbolic** if it enjoys:

- Version of **Ω-stability**
- **Robust hyperbolic estimates** (hyperbolicity constants: **uniform** over some $\mathcal{T}$-neighbourhood of T): if $S^m(x) \approx S^m(y)$ for $0 \le m \le n$, then
$$d(S^m(x), S^m(y)) \le K\,\lambda^{-\min\{m,n-m\}}(d(x,y) + d(S^n(x), S^n(y))).$$
- **Mañé cohomology lemma** (with uniform control of Lipschitz constant of sub-action)

Applies to various 'hyperbolic-like' $\mathcal{T}$, but **not** to $\mathcal{T}$ = {**beta-transformations** T_β}

Joint TPO: Applications of Stable Hyperbolicity

Theorem (Joint TPO: Axiom A diffeos with no cycles)

$\mathcal{A}^r(M)$ = space of C^r **Axiom A diffeos** with the **no-cycle** property on a compact smooth Riemannian manifold M, $r \in \mathbb{N}$. Then $\mathcal{A}^r(M) \times \mathrm{Lip}(M)$ has **Joint TPO**.

Theorem (Joint TPO: Hyperbolic Rational Maps)

$\mathcal{HR}^m$ = space of degree-m **hyperbolic rational maps** on $\widehat{\mathbb{C}}$, $m \geq 2$. Then $\mathcal{HR}^m \times \mathrm{Lip}(\widehat{\mathbb{C}})$ has **Joint TPO**.

Theorem (Joint TPO: Real Quadratic Polynomials)

$\mathcal{Q}$ = family of **real quadratic polynomials** $T(z) = z^2 + c$, $c \in [-2, \frac{1}{4}]$, on $\widehat{\mathbb{C}}$. Then $\mathcal{Q} \times \mathrm{Lip}(\widehat{\mathbb{C}})$ has **Joint TPO**.

Joint TPO: Maps of the Interval/Circle

- $M = [0, 1]$, or the circle
- $C^r(M)$ = space of C^r maps on M, $r \in \mathbb{N}$.

Theorem (Joint TPO: C^r One-Dimensional Maps)

$C^r(M) \times \mathrm{Lip}(M)$ has **Joint TPO**.

- Logistic family $T_a : [0, 1] \to [0, 1]$, $T_a(x) = ax(1 - x)$, $a \in [0, 4]$,

Theorem (Joint TPO: Logistic Family)

$\{T_a : a \in [0, 4]\} \times \mathrm{Lip}([0, 1])$ has **Joint TPO**.

(Joint) Typical Periodic Optimization: References

1. "TPO for dynamical systems: Symbolic dynamics"
DOI: 10.1007/s00222-026-01411-x

2. " Joint typical periodic optimization"
arXiv: 2502.12269

3. "Joint TPO: systems with stable hyperbolicity"
arXiv: 2605.01550