Affine Interval Exchange Transformations and Wandering Intervals

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19/05/2026

1 Goals

2 Interval Exchange Transformations

3 Symbolic Dynamics

4 Main result

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2. Interval Exchange Transformations
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Main result

Theorem [Bressaud-Hubert-Maass]

Let $T(\lambda, \pi)$ be a self-similar interval exchange transformation and R the associated matrix obtained by Rauzy induction. Let θ_1 be the Perron-Frobenius eigenvalue of R. Assume that R has an eigenvalue θ_2 such that:

1) θ_2 is a conjugate of θ_1;
2) θ_2 is a real number;
3) $1 < \theta_2 (< \theta_1)$.

Then there exists an affine interval exchange transformation f with wandering intervals that is semi-conjugated to $T(\lambda, \pi)$.

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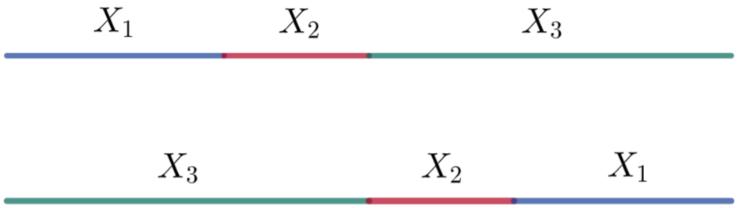
Definition

Let $X \subset \mathbb{R}$ be an interval and $\mathcal{A} = \{1, 2, \ldots, r\}$, in which $r \in \mathbb{N} = \{0, 1, \ldots\}$, $r \geq 2$. Consider $\{X_j, j \in \mathcal{A}\}$ to be a partition of X, where each X_j is a subinterval of the form $X_j = [a_j, b_j[$.

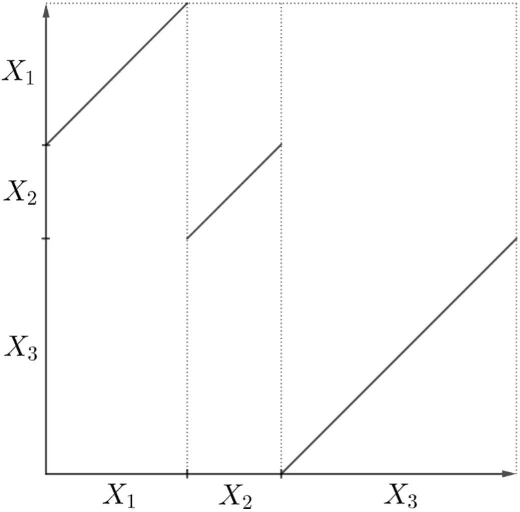
Definition

An **interval exchange transformation** (IET) is a bijective map $T : X \longrightarrow X$ that translates each subinterval X_j.

Diagram



Graph



Important results

Theorem [Masur-Veech]

Almost every interval exchange transformation is uniquely ergodic.

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Theorem [Avila-Forni]

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Representation

An interval exchange transformation can be determined by a vector that describes the lengths of the subintervals and a combination in their order.

Example

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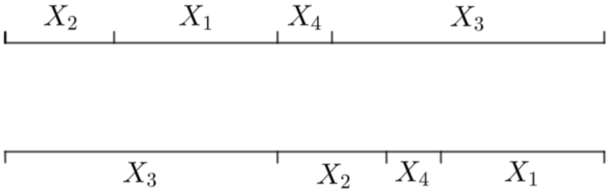


Is given by $T(\lambda, \pi)$, in which

$$\lambda = (3, 2, 5, 1) \text{ and } \pi = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

Reducibility

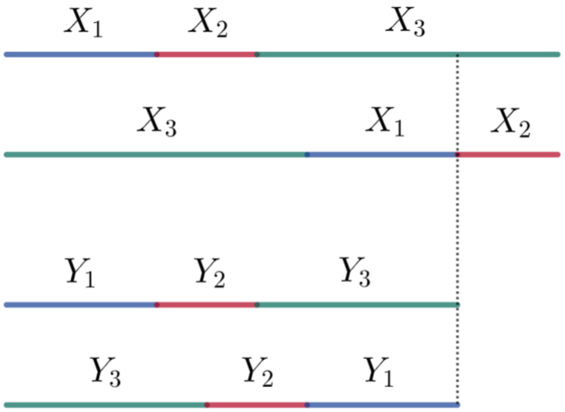
Definition

We say that $\pi = (\pi_0, \pi_1)$ is **reducible** if there exists $k \in \{1, 2, \dots, r-1\}$ such that

$$\pi_1 \circ \pi_0^{-1}(\{1, 2, \dots, k\}) = \{1, 2, \dots, k\}.$$

Otherwise, we say that π is **irreducible**.
In the last example, π is irreducible.

Rauzy Induction



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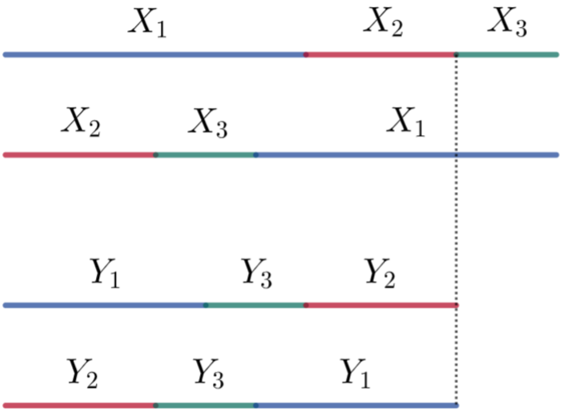

Rauzy diagram

Example: For three intervals, there are only three irreducible permutations and all of them are connected by one graph describing the iterations of the Rauzy induction.

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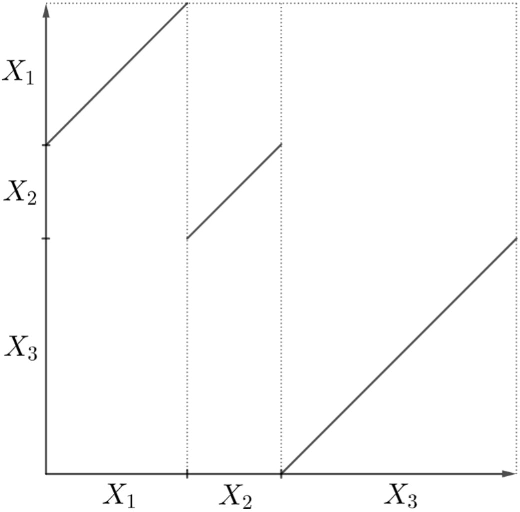
Wandering intervals

Let N be a finite subinterval of either $\mathbb{R}$ or the circle S^1. Let $f\colon N \to N$ be a piecewise continuous map and let $J \subset N$ be an interval. We say that J is a **wandering interval** for the map f if:

(i) $J, f(J), f^2(J), \ldots$ are pairwise disjoint intervals;

(ii) the ω-limit set of J is infinite.

Affine interval exchange transformations



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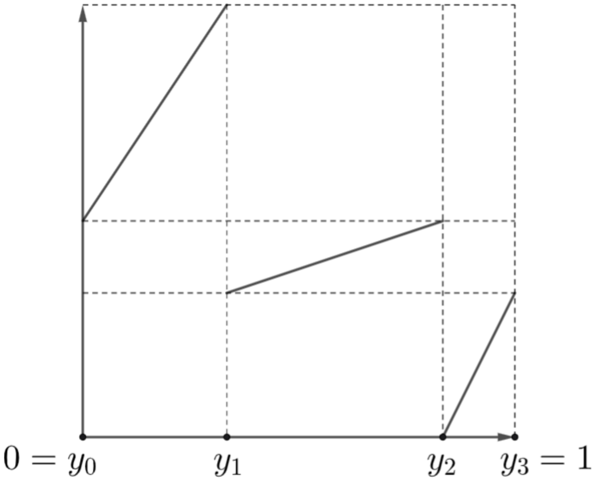

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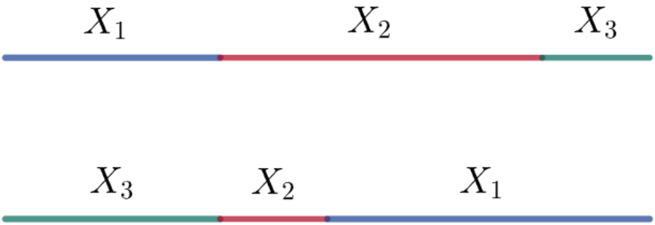

Affine interval exchange transformations

Definition

A bijective map $T : [a, b) \to [a, b)$ is an **affine interval exchange transformation** (AIET) if there exists a finite sequence $a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = b$ such that for all $i \in \{0, 1, \dots, m-1\}$, $T|_{[y_i, y_{i+1})}$ is continuously differentiable, and its derivative is identically equal to a constant $\beta_i > 0$. When T is discontinuous exactly at the points $y_1, y_2, \dots, y_{m-1}$, we associate to T the pair of vectors

$$(y, \tilde{\gamma}) = ((y_1, y_2, \dots, y_m), (\gamma_1, \gamma_2, \dots, \gamma_m)),$$

in which $\gamma_i = \log \beta_i$.

Affine interval exchange transformations

Problem

Are there affine interval exchange transformations with wandering intervals?

Solutions

Levitt - 1983

Non-uniquely ergodic AIET with wandering intervals.

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Improvements on these results.

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Bressaud-Hubert-Maass - 2010

Combinatoric conditions for the existence of these AIETs.

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Words and Substitutions

Definitions

Let $n \in \mathbb{N}$ and let $\mathcal{A} = \{w_0, \ldots, w_n\}$ be a finite set.

(i) We call $\mathcal{A}$ an **alphabet**, and each w_i a **letter** of the alphabet $\mathcal{A}$;

(ii) A **finite word** (respectively, an **infinite word**) over $\mathcal{A}$ is a finite (respectively, infinite) sequence of letters from $\mathcal{A}$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$, and the set of all infinite words by $\mathcal{A}^{\mathbb{N}}$;

(iii) We call **shift** the map $S : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ given by: $S(a_0 a_1 \cdots) = a_1 a_2 \cdots$.

Words and Substitutions

We can also define $\mathcal{A}^{\mathbb{Z}}$ as the set of bi-infinite words, that is,

$$\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots\}.$$

We also call $w \in \mathcal{A}^{\mathbb{Z}}$ a dotted word.

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We also call $w \in \mathcal{A}^{\mathbb{Z}}$ a dotted word.
The main advantage of working in $\mathcal{A}^{\mathbb{Z}}$ is that the shift map $S\big((x_i)_{i\in\mathbb{Z}}\big) = (x_{i+1})_{i\in\mathbb{Z}}$ is a bijection on $\mathcal{A}^{\mathbb{Z}}$.

Words and Substitutions

A **substitution** is a map $\sigma : \mathcal{A} \longrightarrow \mathcal{A}^*$ that associates to each letter $a \in \mathcal{A}$ a word $\sigma(a) \in \mathcal{A}^*$. We can extend a substitution σ to the set of finite (respectively infinite) words by concatenation, that is, $\sigma(w_0 w_1 \cdots w_n) = \sigma(w_0)\sigma(w_1)\cdots\sigma(w_n)$ (respectively to the infinite case).

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Example: Consider the alphabet $\mathcal{A} = \{0, 1\}$. The Fibonacci substitution σ is defined by $\sigma(0) = 01$ and $\sigma(1) = 0$.

Symbolic Dynamical Systems

For every substitution σ defined over an alphabet $\mathcal{A}$, it is possible to associate a matrix M_σ with entries $l_{ij} = L_i(\sigma(j)), i, j \in \mathcal{A}$, in which $L_i(\sigma(j))$ represents the number of times that i appears in $\sigma(j)$. We say that M_σ is the **matrix associated with** σ.

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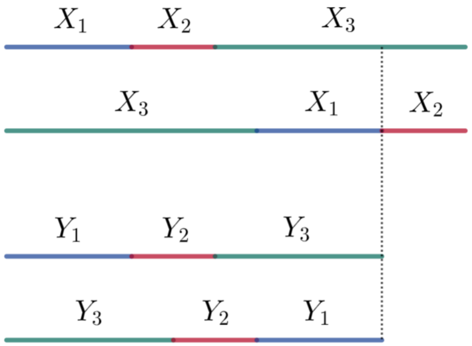
Substitution associated with Rauzy Induction



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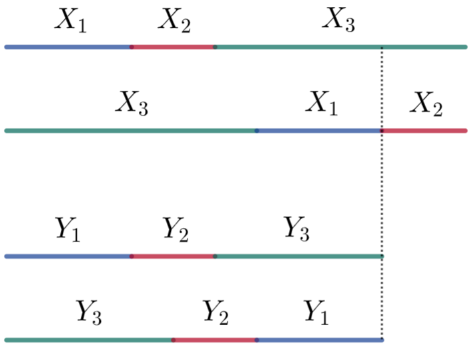

$$\sigma(1) = 1, \quad \sigma(2) = 23, \quad \sigma(3) = 3.$$

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Consider the loop 10100 that starts at the center of the graph above. By composing the substitutions along this loop, we obtain a substitution σ given by

$$\sigma(1) = 1213;\ \sigma(2) = 12213;\ \sigma(3) = 13,$$

and its associated matrix:

$$M_\sigma = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Self-similarity

Definition

We say that an IET $T(\lambda, \pi)$ is **self-similar** if there exists a finite loop in the Rauzy diagram of T such that λ is a Perron-Frobenius eigenvector of the matrix associated to the corresponding substitution.

Main Theorem

Theorem [Bressaud-Hubert-Maass]

Let $T(\lambda, \pi)$ be a self-similar interval exchange transformation and R the associated matrix obtained by Rauzy induction. Let θ_1 be the Perron-Frobenius eigenvalue of R. Assume that R has an eigenvalue θ_2 such that:

1) θ_2 is a conjugate of θ_1;
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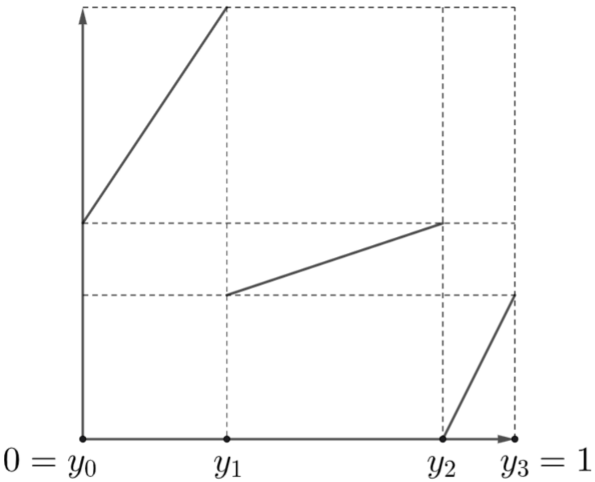
Idea of the proof



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$$\sum_{n \geq 1} \exp(-\gamma_{x_0} - \cdots - \gamma_{x_{n-1}}) < \infty \quad \text{and} \quad \sum_{n \geq 1} \exp(\gamma_{x_{-n}} + \cdots + \gamma_{x_{-1}}) < \infty$$

in which γ_i is the log of the slope of the corresponding affine part of the AIET.

Idea of the proof

Lemma

Let $\eta\colon \mathbb{Q}[\theta_1] \to \mathbb{Q}[\theta_2]$ be the field homomorphism that sends θ_1 to θ_2. The vector

$$\gamma = \eta(\lambda) = (\eta(\lambda(a)) : a \in \mathcal{A})^t$$

is an eigenvector of M associated with the eigenvalue θ_2.

Idea of the proof

Consider X_σ to be the symbolic dynamical system associated with σ. Then, for every $x \in X_\sigma$, $x = (x_i)_{i\in\mathbb{Z}}$ we define:

Definition

(i) $\gamma_0(x) = 0,$

(ii) $\gamma_n(x) = \sum_{i=0}^{n-1} \gamma(x_i) \quad \text{for } n > 0,$

(iii) $\gamma_n(x) = -\sum_{i=n}^{-1} \gamma(x_i) \quad \text{for } n < 0.$

Idea of the proof

Definition

A point $x \in X_\sigma$ is said to be **minimal for** γ if
$\gamma_n(x) \geq 0 \quad$ for all $n \in \mathbb{Z}$. The set of points that are minimal for γ is denoted by $\mathcal{M}_\sigma(\gamma)$.

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Proposition

$M_\sigma(\gamma)$ is a finite non-empty set.

Idea of the proof

Proposition

Let $x \in X_\sigma$ be a minimal point for γ. Then,

$$\sum_{n\geq 1} \exp(-\gamma(x_0 \cdots x_{n-1})) < \infty \quad \text{and} \quad \sum_{n\geq 1} \exp(\gamma(x_{-n} \cdots x_{-1})) < \infty.$$

Idea of the proof

Consider a minimal point $x \in X_\sigma$ and

$$K = \sum_{n \geq 1} \exp(\gamma(x_{-n} \cdots x_{-1})) + 1 + \sum_{n \geq 1} \exp(-\gamma(x_0 \cdots x_{n-1})) < \infty.$$

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There exists a point $t \in [0, 1)$ such that x is the coding of t by $T(\lambda, \pi)$.

Idea of the proof

Define the probability measure μ_t on $[0,1)$ by

$$\mu_t = \frac{1}{K}\left(\sum_{n\geq 1}\exp(\gamma(x_{-n}\cdots x_{-1}))\delta_{T^{-n}(t)} + \delta_t + \sum_{n\geq 1}\exp(-\gamma(x_0\cdots x_{n-1}))\delta_{T^n(t)}\right).$$

Idea of the proof

Lemma

For everey Borel set $I \subset [0, 1)$

$$\mu_t(T(I)) = \sum_{i=1}^{r} e^{-\gamma_i} \mu_t(I \cap [a_{i-1}, a_i)).$$

Idea of the proof

Define $g : [0, 1) \to [0, 1)$ by $g(s) = \mu_t([0, s])$. This function is non-decreasing, right continuous and has left limits. Let $i \in \mathcal{A}$, denote $a_i' = T(a_i)$ and define

$$b_i = \lim_{a \to a_i^-} g(a) \quad \text{and} \quad b_i' = \lim_{a' \to (a_i')^-} g(a')$$

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.

Then at interval $[b_{i-1}, b_i)$ define linearly the AIET f with image $[b'_{i-1}, b'_i)$. The slope vector of f is $w = (e^{-\gamma_1}, \ldots, e^{-\gamma_r})$. Indeed,

$$\frac{b'_i - b'_{i-1}}{b_i - b_{i-1}} = \frac{\mu_t([a'_{i-1}, a'_i))}{\mu_t([a_{i-1}, a_i))} = e^{-\gamma_i},$$

where the last equality follows from the last Lemma.

Idea of the proof

Let $h : [0,1) \to [0,1)$ be the map defined by:

$$\begin{cases} h(v) = u, & \text{if} \quad g(u) = v; \\ h(v) = u, & \text{if} \quad \lim_{w \to u^-} g(w) \leq v \leq g(u). \end{cases}$$

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Clearly h is surjective, continuous and non-decreasing. Since μ_t has atoms, then h is not injective.

Idea of the proof

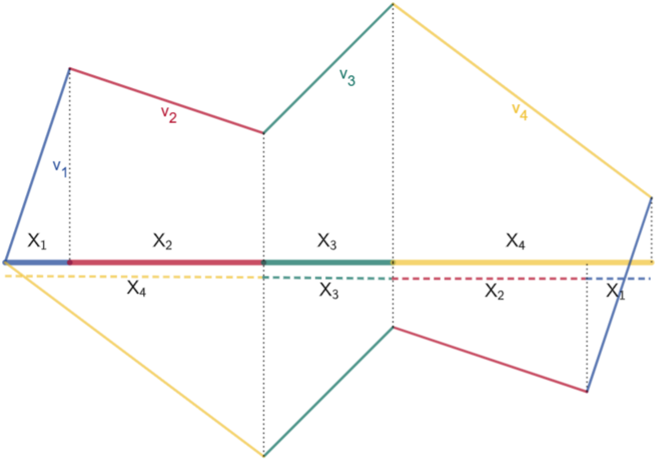
Lemma

The map h defines a semi-conjugacy between the AIET f and T. Moreover,

$$I = \left(\lim_{s \to t^-} g(s), g(t) \right]$$

is a wandering interval for f.

Generalization



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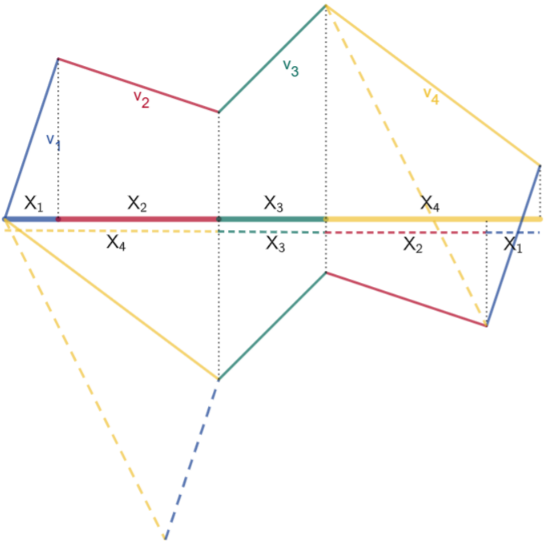

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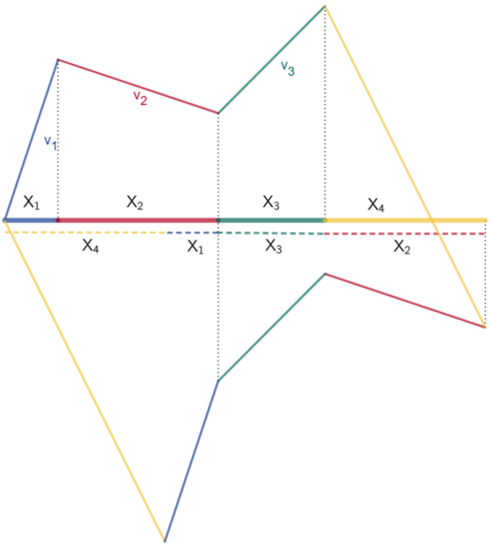

Generalization

Theorem [Marmi-Moussa-Yoccoz]

For almost every interval exchange transformation T_0 whose associated surface has genus $g \geq 2$, there exists an affine interval exchange transformation T that is semi-conjugate to T_0 and has a wandering interval.

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Thank you!