Conformal measures for intermittent maps

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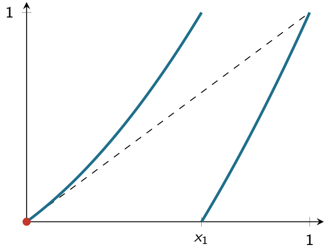
Intermittent maps and the model map

An intermittent map is an interval map with a neutral fixed point ($|Df(p)| = 1$): the boundary of uniform expansion (Pomeau–Manneville, 1980).

Model: fix $\alpha > 0$ and

$$f(x) = \begin{cases} x(1+x^\alpha) & x(1+x^\alpha) \leq 1, \\ x(1+x^\alpha) - 1 & \text{else.} \end{cases}$$

- $f(0) = 0,\ f'(0) = 1$ (neutral);
- one discontinuity x_1, $f(x_1) = 1$; first branch a diffeo;
- $f^{-1}(0) = \{0\}$; f top. exact on $(0, 1]$.



tangency at 0: Tangency at 0: where the difficulty begins.

Lack of uniform expansion

Uniformly expanding ($\inf |Df| > 1$): the thermodynamic formalism is classical: unique equilibrium states, real-analytic pressure, spectral gap, exponential decay of correlations, no phase transitions
Neutral point $Df(0) = 1$: the transfer operator may fail quasi-compactness; the atom δ_0 coexists with a.c. measures; phase transitions appear (e.g. $-t \log |Df|$).

Thermodynamic formalism: pressure and Gibbs states

Write $\mathcal{M} := \mathcal{M}([0,1], f)$ for the f-invariant Borel probabilities, $h_\mu(f)$ for entropy, $\chi_\mu = \int \log |Df| d\mu$ for the Lyapunov exponent, and $S_n\varphi = \sum_{j=0}^{n-1} \varphi \circ f^j$.

For a continuous potential φ the topological pressure is

$$P(\varphi) := \sup_{\mu \in \mathcal{M}} \left\{ h_\mu(f) + \int \varphi d\mu \right\}.$$

A measure attaining the supremum is an equilibrium (or Gibbs) state.

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Two distinguished invariant measures

- δ_0: the atom at the neutral point, always invariant, with $\int \varphi \delta_0 = \varphi(0)$ and $h_{\delta_0}(f) = 0$.
- the "expanding" Gibbs state $\mu \neq \delta_0$, with $\chi_\mu > 0$.

Conformal measures

Definition

A Borel measure μ is *g-conformal* if, for every Borel set A on which f is injective,

$$\mu(f(A)) = \int_A g \, d\mu.$$

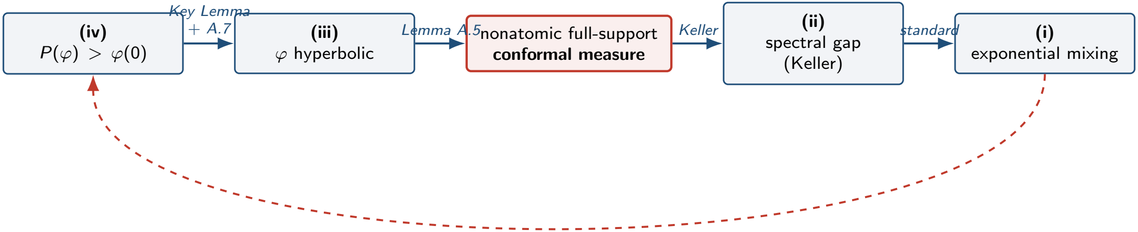
For thermodynamic formalism we take $g = \exp\big(P(\varphi) - \varphi\big)$.

Key observation:

Since $f^{-1}(0) = \{0\}$, the atom δ_0 is $\exp(P(\varphi) - \varphi)$-conformal whenever $P(\varphi) = \varphi(0)$.

When does f admit a nonatomic conformal measure of full support?

Equivalences: (with J. Rivera-Letelier, in preparation 2026)



The main Theorem (-, J. Rivera-Letelier 2026):

For $\varphi \in C^{\gamma}([0,1],\mathbb{R})$, the following are equivalent:

1. there is an exponentially mixing Gibbs state with support containing at least two points;
2. the transfer operator $\mathscr{L}_{\varphi}$ acts with a spectral gap on a Keller space;
3. φ is hyperbolic: $\exists\, n \geq 1 :\ \sup_{x\in[0,1]} \frac{1}{n} S_n\varphi(x) < P(\varphi)$;
4. $\boxed{P(\varphi) > \varphi(0)}$.

Consequences

The Gibbs state is unique, the conformal probability is unique, and for every mixing rate ρ,

$$P(\varphi) - \varphi(0) \ \geq \ -\log \rho.$$

Constructing conformal measures: Patterson–Sullivan method

Idea (Patterson 1976 for Fuchsian groups; Sullivan 1983 for rational maps): build a conformal measure as a weak* limit of normalized sums over iterated preimages.

Fix a base point x_0 and a continuous ψ. The iterated preimages pressure is

$$P_{x_0}(\psi) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in f^{-n}(x_0)} \exp\big(S_n\psi(x)\big).$$

Lemma (Iterated preimages pressure = pressure)

For a topologically exact map (on its Julia set) and a continuous potential φ,

$$\sup_{x_0} P_{x_0}(\varphi) = P(\varphi).$$

Patterson's sublemma

Sublemma: Let (a_n) have transition parameter c ($c = \limsup_{n\to\infty} a_n/n$ finite). Then there exist $b_n > 0$ with

$$\lim_n \frac{b_n}{b_{n+1}} = 1, \qquad \sum_n b_n \exp(a_n - ns) \begin{cases} < \infty & s > c, \\ = \infty & s \leq c. \end{cases}$$

Put $a_n := \log \sum_{x \in f^{-n}(x_0)} \exp(S_n\psi(x))$, with transition parameter $c = \limsup a_n/n = P_{x_0}(\psi)$.

The slowly varying weights b_n (with $b_n/b_{n+1} \to 1$) make the Dirichlet-type series diverge exactly at $s = c$.

The Patterson–Sullivan measures

Put $a_n := \log \sum_{x \in f^{-n}(x_0)} \exp(S_n\psi(x))$, with transition parameter $c = \limsup a_n/n = P_{x_0}(\psi)$.

With $P_0 := P_{x_0}(\psi)$ and (b_n) from the sublemma, set for $s > P_0$

$$M_s := \sum_{n\geq 1} b_n \exp(a_n - ns), \qquad m_s := \frac{1}{M_s} \sum_{n\geq 1} \sum_{y \in f^{-n}(x_0)} b_n \exp\big(S_n\psi(y) - ns\big)\, \delta_y.$$

Each m_s is a Borel probability measure. The sublemma gives

$$\lim_n \frac{b_n}{b_{n+1}} = 1, \qquad \lim_{s \to P_0^+} M_s = +\infty, \qquad \lim_{s \to P_0^+} \Delta(s) = 0,$$

where $\Delta(s)$ is an explicit conformality defect controlled by $\left|1 - \frac{b_{n+1}}{b_n} e^{P_0 - s}\right|$ and the boundary term $b_1 e^{P_0 - s}$.

weak-$*$ limits of m_s are "mostly conformal" measures.

The Keller space

The spectral gap of $\mathscr{L}_\varphi$ is obtained on a Keller space $H^{1,\gamma}(\mu)$ [Kel85]: a Banach space of functions *built on a reference conformal measure* μ [LRL14a].

The space needs the reference measure to be...

- conformal $\big(\exp(P(\varphi) - \varphi)\big)$: gives $\mathscr{L}_\varphi$ a clean action with leading eigenvalue $e^{P(\varphi)}$;
- nonatomic: the very *norm* of $H^{1,\gamma}(\mu)$ is defined through μ
- full support: together with topological exactness it makes the leading eigenvalue *simple* (the spectral gap).

Goal: a nonatomic, full-support, $\exp(P(\varphi) - \varphi)$-conformal measure.

Lemma: the dichotomy

Lemma (A.)

For every continuous φ *there is an* $\exp(P(\varphi) - \varphi)$*-conformal probability measure* μ *for* f *satisfying exactly one of:*

1. $\mu(\{0\}) = 0$ *and* $\operatorname{supp}(\mu) = [0, 1]$*;*
2. $\mu = \delta_0$ *and* $\Theta(x_1) < \infty$*.*

If φ *is hyperbolic for* f*, then case* (1) *holds.*

Keller's theory

Suppose φ is hyperbolic. Put $g = \exp(\varphi - P(\varphi)) \in C^\gamma$. Two ingredients:

- hyperbolicity gives n with $\sup_{[0,1]} \prod_{j=0}^{n-1} g \circ f^j < 1$;
- Lemma A.5 gives a nonatomic $\exp(P(\varphi) - \varphi)$-conformal measure μ with $\mathrm{supp}(u) = [0,1]$.

These verify Keller's hypotheses for the transfer operator on the Keller space of bounded $(1/\gamma)$-variation built on the conformal measure: $(1/\gamma)$-variation, and μ is conformal; product estimate. Keller's theorems then give quasi-compactness and a spectral gap; topological exactness upgrades it to a simple leading eigenvalue.

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