Transportation of measures along the stable leaves

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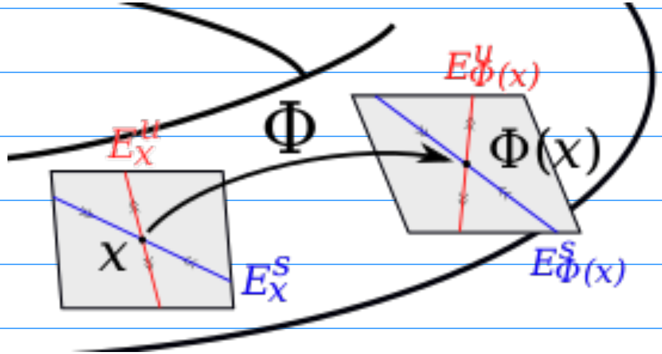
Anosov diffeomorphisms

M a closed manifold (e.g. torus)

$\Phi : M \to M$ a transitive $C^{1,\alpha}$ Anosov diffeomorphism:

at each $x \in M$, $T_x M = E^s_x \oplus E^u_x$

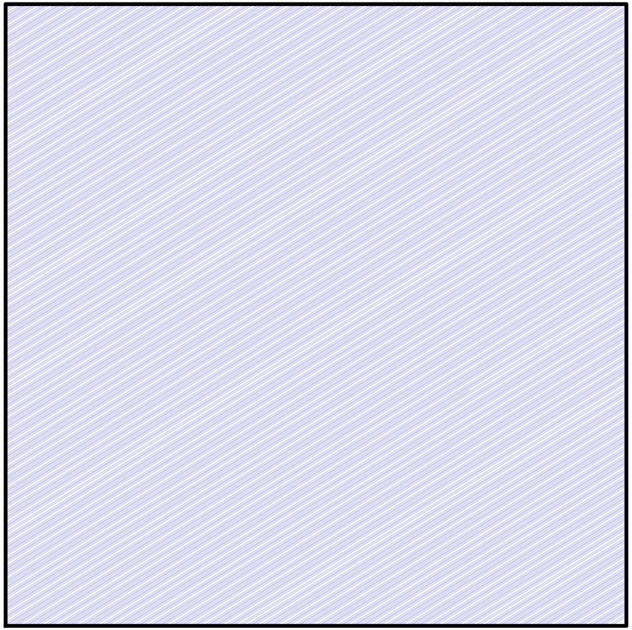


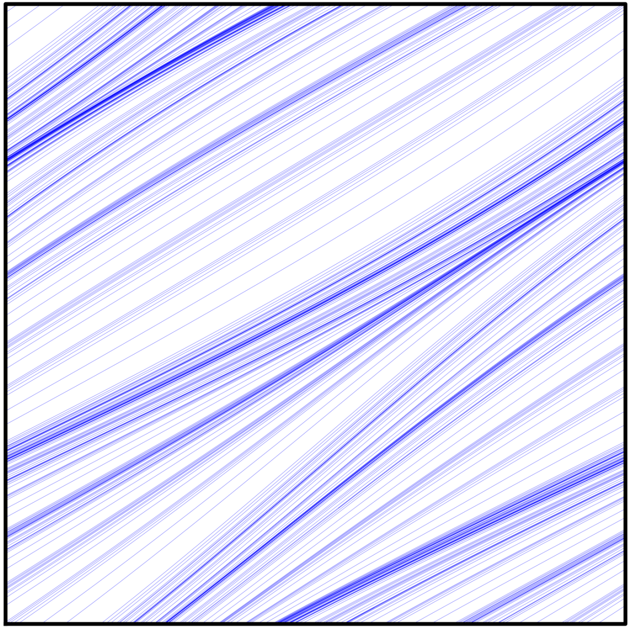
$$\left.\begin{array}{l} E^s = (E^s_x)_{x \in M} \\ E^u = (E^u_x)_{x \in M} \end{array}\right\} \Phi\text{- invariant}$$

$$\| D_x \Phi|_{E^s_x} \| , \; \| D_x \Phi^{-1}|_{E^u_x} \| \leq \lambda < 1$$

(for some λ & some metric)

Then E^s and E^u integrate in the stable/unstable foliations W^s, W^u



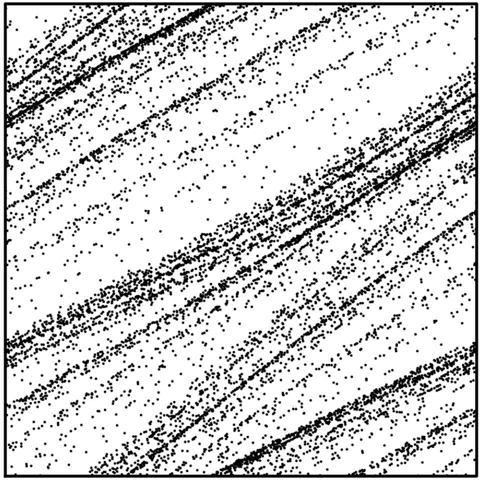


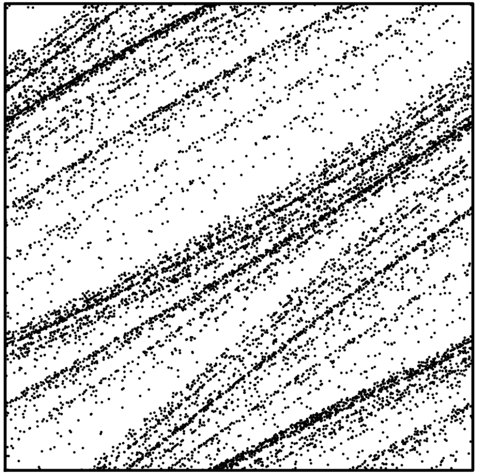
SRB measures

Φ may preserve the volume, but often does not have an absolutely continuous invariant probability (ACIP).

$\Phi^{15}_* \mathrm{Leb}$

$\simeq \Phi^{50}_* \mathrm{Leb}$



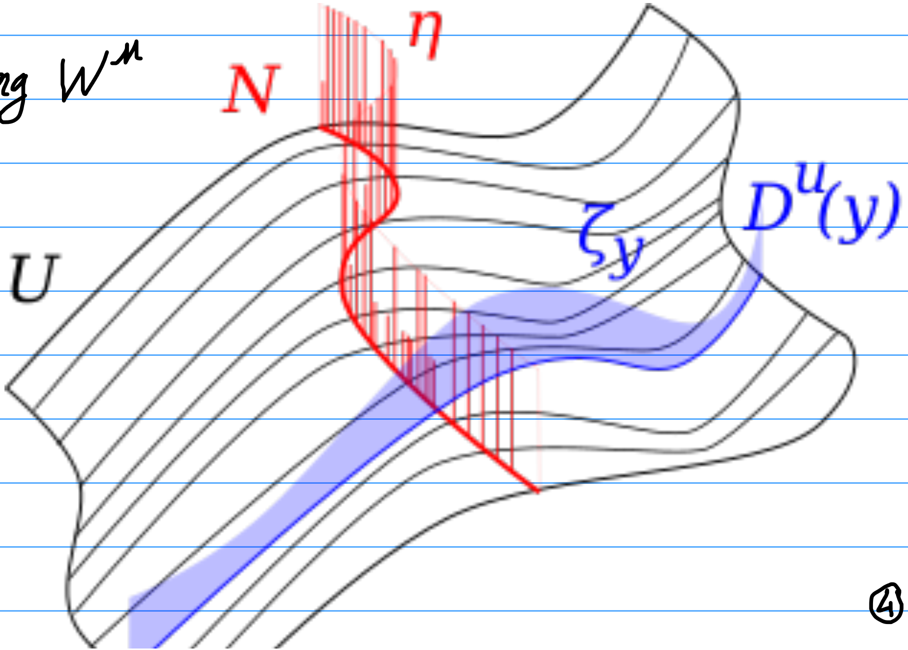


$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{\Phi^k x}$$

x Leb-typical

A Φ-invariant probability μ is:

- **physical** when $\frac{1}{n}\sum \delta_{\Phi^k x} \longrightarrow \mu$ for all x in a Leb>0 set
- **SRB** when it is A.C. along W^u



local chart U transversal N to W^u

$$U = \bigcup_{y\in N} \underbrace{D^u(y)}_{\subset W^u(y)}$$

$$\mu|_U = \int_N \zeta_y \, d\eta(y)$$

ζ_y: abs. cont.

For Anosov (and many other hyperbolic-type dynamics):

- physical $\Leftarrow$ SRB
- existence and uniqueness
- decay of correlations

Our goal: "statistical stability"

Almost-Theorem (Boukhecham-K.)

$$\boxed{\begin{array}{l} \mu_0 \text{ the SRB measure of } \Phi. \\ \text{There exist } \cdot \text{ a neighborhood } V \text{ of } \Phi \text{ in the } C^{1,\alpha} \text{ topology} \\ \qquad \cdot\ \beta>0,\ C>0 \\ \text{s.t. for all } \Phi_1 \in V \text{ with SRB measure } \mu_1, \\ \qquad \mathrm{Wass}(\mu_0, \mu_1) \leqslant C \cdot \|\Phi - \Phi_1\|_\infty^\beta \end{array}}$$

Optimal transportation

A way to turn the metric d on M into a metric W on $\mathcal{P}(M) = \{\text{proba.}\}$

$\mu, \nu \in \mathcal{P}(M)$

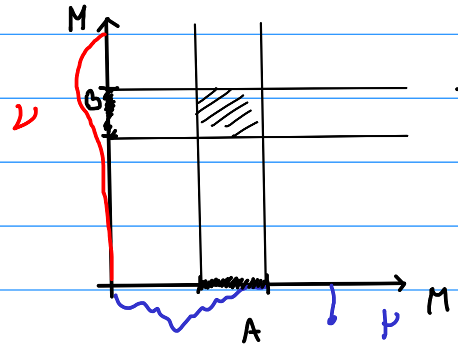
A transport plan is $\gamma \in \mathcal{P}(M \times M)$ s.t.

$$\forall A, B \subset M \text{ (Borel)} \qquad \gamma(A \times M) = \mu(A) \quad \& \quad \gamma(M \times B) = \nu(B)$$

Probabilistic p.o.v. : if $(X,Y) \sim \gamma$

then $X \sim \mu$

$Y \sim \nu$



$\gamma(A \times B)$: amount of mass moved from A to B.

$\Gamma(\mu,\nu) = \{\text{transp. plans}\}$

Examples

- $\mu \otimes \nu \in \Gamma(\mu,\nu)$ (independant coupling)
- $(Id, Id)_* \mu \in \Gamma(\mu,\mu)$ i.e. $\gamma(A \times B) = \mu(A \cap B)$ (identity coupling)
- $\Gamma(\mu, \delta_x)$ is reduced to one element $\gamma(A \times B) = \mu(A) \cdot \mathbb{1}(B \ni x)$
- $\Gamma\left(\frac{1}{p}\sum_{i=1}^{p} \delta_{x_i}, \frac{1}{q}\sum_{j=1}^{q} \delta_{y_j}\right) \simeq \{$bi-stochastic $p \times q$ matrices$\}$

The Wasserstein metric is

$$\mathrm{Wass}(\mu,\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{M \times M} d(x,y)\, d\gamma(x,y)$$

inf = min ; metrizes weak-* topology ; $W(\mu,\nu) = \max_{f \text{ 1-Lip}} \left| \int_M f\, d\mu - \int_M f\, d\nu \right|$

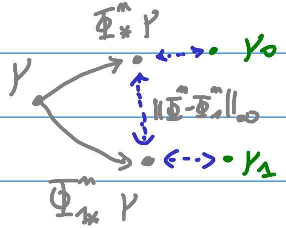
Principle of the proof of the almost-theorem

For any Φ_1 $C^{1,\alpha}$-close from Φ, find a measure μ s.t.

$$\begin{cases} \Phi^n_{*}\mu \to \mu_0 \\ \& \\ \Phi^n_{1*}\mu \to \mu_1 \end{cases}$$

exponentially fast in W

with cst. and exponent uniform in Φ_1



Possible choice: $\mu = \mathrm{Leb}$; $\mu = \mu_1$

Reduced to find a set R_Φ of measures $\begin{cases} \text{being exp attracted to SRB} \\ \text{containing Leb or } \mu_1 \end{cases}$

Bressaud-Liverani 2002: we can take $R_\Phi = \{ f \cdot \mathrm{Leb} \mid \|f\|_\infty < \infty$ and f Hölder along $W^u \}$

The coupling method

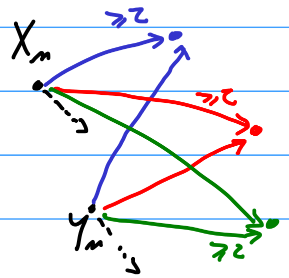
Perron-Frobenius theorem for Markov chains: $(X_n)_{n\in\mathbb{N}}$ M.C. on $[\![1,p]\!]$ s.t.

$$\exists \tau>0, \quad \forall i,j : \quad \mathbb{P}(X_{n+1}=j \mid X_n=i) \geq \tau$$

Then $\exists!$ stationary measure μ and $X_n \xrightarrow{\mathcal{L}} \mu$ exponentially fast (in T.V.)

Proof: For any (X_n) and Y_0 there exist a pair of realizations $(\tilde{X}_n, \tilde{Y}_n)$ s.t. $\forall n$:

- $\mathbb{P}(\tilde{X}_{n+1} = \tilde{Y}_{n+1} \mid \tilde{X}_n = \tilde{Y}_n) = 1$
- $\mathbb{P}(\tilde{X}_{n+1} = \tilde{Y}_{n+1} \mid \tilde{X}_n \neq \tilde{Y}_n) \geq \tau$
- $\tilde{X}_n \sim X_n$

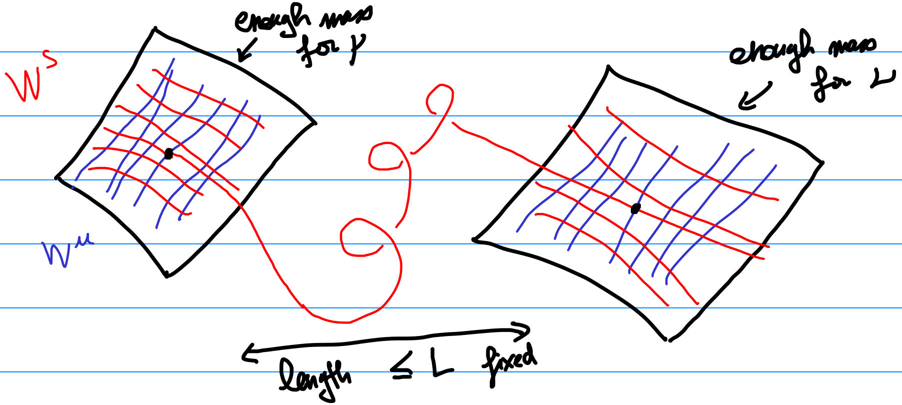


Then $\mathbb{P}(\tilde{X}_n \neq \tilde{Y}_n) \leq (1-\tau)^n$

Apply this with $Y_0 \sim \mu$ any stationary measure.

Coupling along the stable leaves

μ, ν "good" measures to be coupled / γ s.t. $(\Phi^n, \Phi^n)_* \gamma$ has small cost



Assuming μ, ν "regular along W^u", we can couple a mass $\tau > 0$ along paths of length $\leq L$ of W^s. Then applying Φ_* will regularize the remainders $\rightsquigarrow$ iterate.