where the horizontal arrows are respectively induced by relative Frobenius on X and $\mathbb{A}^d_k$, and the vertical arrows are field extensions of degree m. The bottom arrow is a field extension of degree $[k(T_1, ..., T_d) : k(T_1^p, ..., T_d^p)] = p^d$. Therefore the top arrow have degree p^d. This achieves the proof. □

Exercises

2.1. Let Y be a Noetherian scheme. Show that any Y-scheme X of finite type is Noetherian. Moreover, if Y is of finite dimension, then so is X.

2.2. Show that any open immersion into a locally Noetherian scheme is a morphism of finite type.

2.3. An *immersion* of schemes is a morphism which is a closed immersion followed by an open immersion[2].

(a) Let $f : X \to Y$ be a morphism which is an open immersion followed by a closed immersion. Show that f is an immersion.

(b) Show that a quasi-compact immersion $X \to Y$ (e.g., when if Y is locally Noetherian) can be decomposed to an open immersion followed by a closed immersion. Use the scheme-theoretic closure of f (Exercise 2.3.17).

(c) Show that the composition of two immersions is an immersion.

(d) Show that $f : X \to Y$ is an immersion if and only if f is a topological immersion (i.e. $f : X \to f(X)$ is a homeomorphism) and if for all $x \in X$, the canonical homomorphism $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective.

2.4. Let X, Y be schemes over a locally Noetherian scheme S, with Y of finite type over S. Let $x \in X$. Show that for any morphism of S-schemes $f_x : \operatorname{Spec} \mathcal{O}_{X,x} \to Y$, there exist an open subset $U \ni x$ of X and a morphism of S-schemes $f : U \to Y$ such that $f_x = f \circ i_x$, where $i_x : \operatorname{Spec} \mathcal{O}_{X,x} \to U$ is the canonical morphism (in other words, the morphism f_x extends to an open neighborhood of x).

2.5. Let S be a locally Noetherian scheme. Let X, Y be S-schemes of finite type. Let us fix $s \in S$. Let $\varphi : X \times_S \operatorname{Spec} \mathcal{O}_{S,s} \to Y \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ be a morphism of S-schemes. Show that there exist an open set $U \ni s$ and a morphism of S-schemes $f : X \times_S U \to Y \times_S U$ such that φ is obtained from f by the base change $\operatorname{Spec} \mathcal{O}_{S,s} \to U$. If φ is an isomorphism, show that there exists such an f which is moreover an isomorphism.

2.6. Let $f : X \to Y$ be a morphism of integral schemes. Let ξ be the generic point of X. We say that f is *birational* if $f_\xi^{\#} : K(Y) \to K(X)$ is an

[2]This definition (following [42], I.4.1.3) differs from that in the first edition (where immersions are morphisms as in (a)). Both definitions coincide for quasi-compact morphisms. But in general morphisms as in (a) are not stable by composition.