Lemma 1.29. *Let A be a ring, $B = A[T_0, \dots, T_d]$, and $X = \operatorname{Proj} B$. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, the direct sum $\oplus_{n \geq 0} \mathcal{F}(n)(X)$ is naturally a graded B-module, and we have*

$$(\oplus_{n \geq 0} \mathcal{F}(n)(X))^{\sim} \simeq \mathcal{F}.$$

Proof Let $M = \oplus_{n \geq 0} \mathcal{F}(n)(X)$. For any $b \in B_m = \mathcal{O}_X(m)(X)$ and for any $t \in \mathcal{F}(n)(X)$, the tensor product $b \otimes t \in \mathcal{O}_X(m)(X) \otimes \mathcal{F}(n)(X)$ is naturally sent to an element of $(\mathcal{O}_X(m) \otimes \mathcal{F}(n))(X) = \mathcal{F}(n+m)(X)$. This defines the structure of a graded B-module on M.

Let $f \in B$ be one of the elements T_i. Let $U = D_+(f)$. Let us show that we have a canonical isomorphism $\varphi : M_{(f)} \simeq \mathcal{F}(U)$. Let $f^{-n}t \in M_{(f)}$ with $t \in \mathcal{F}(n)(X)$. Then $t|_U \in \mathcal{F}(n)(U) = f^n \otimes \mathcal{F}(U)$. There exists a unique $s \in \mathcal{F}(U)$ such that $t|_U = f^n \otimes s$, because f^n is a basis of $\mathcal{O}_X(n)(U)$ over $\mathcal{O}_X(U)$. Let us set $\varphi(f^{-n}t) = s$. It follows from Lemma 1.25(b) that φ is surjective. If $s = 0$, then $t|_U = 0$. Hence there exists an $m \geq 1$ such that $f^m \otimes t = 0 \in \mathcal{O}_X(m)(X) \otimes \mathcal{F}(n)(X)$ (Lemma 1.25(a)). It follows that $f^m t = 0$ in M. Hence $f^{-n}t = 0$ and φ is bijective. When we let f vary, this defines an isomorphism $M^{\sim} \simeq \mathcal{F}$. □

Proposition 1.30. *Let A be a ring, $B = A[T_0, \dots, T_d]$, and $X = \operatorname{Proj} B$. Then any closed subscheme Z of X is of the form $\operatorname{Proj} B/I$ for a homogeneous ideal I of B. In particular, any projective scheme over A is isomorphic to $\operatorname{Proj} C$, where C is a homogeneous A-algebra.*

Proof Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_X$ which defines the closed subscheme Z. Then $\mathcal{I}$ is quasi-coherent by Proposition 1.15. Let us set $I = \oplus_{n \geq 0} \mathcal{I}(n)(X)$. The canonical homomorphism $\mathcal{I}(n) \to \mathcal{O}_X(n)$ is injective because it is injective at every point $x \in X$ owing to the fact that $\mathcal{O}_X(n)_x \simeq \mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{X,x}$. Therefore I is an ideal of $\oplus_{n \geq 0} B_n = B$ (Lemma 1.22). By the lemma above, we have the equality of sheaves of ideals $\widetilde{I} = \mathcal{I}$. It is now easy to see that $V(\widetilde{I}) = \operatorname{Proj} B/I$ as schemes. This proves the proposition. □

To conclude, we are going to consider the relation between morphisms to a projective space and invertible sheaves. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \mathcal{L}(X)$. Then the multiplication by s induces an isomorphism

$$\mathcal{O}_X|_{X_s} \xrightarrow{\cdot s} s\mathcal{O}_X|_{X_s} = \mathcal{L}|_{X_s}$$

because s_x is a basis of $\mathcal{L}_x$ over $\mathcal{O}_{X,x}$ for every $x \in X_s$. In particular, for every $t \in \mathcal{L}(X)$, we can, without ambiguity, write t/s as an element of $\mathcal{L}(X_s)$.

Proposition 1.31. *Let $Y = \operatorname{Proj} A[T_0, \dots, T_d]$ be a projective space over a ring A, and let X be a scheme over A.*

(a) *Let $f : X \to Y$ be a morphism of A-schemes. Then $f^*\mathcal{O}_Y(1)$ is an invertible sheaf on X, generated by $d+1$ of its global sections.*

(b) *Conversely, for any invertible sheaf $\mathcal{L}$ on X generated by $d+1$ global sections $s_0, \dots, s_d$, there exists a morphism $f : X \to Y$ and an isomorphism $\rho : \mathcal{L} \simeq f^*\mathcal{O}_Y(1)$ such that $\rho(X)(s_i) = f^*T_i$. Moreover, $f(X)$ is* *contained in a hyperplane $V_+(F)$ (i.e. $F \neq 0$ is homogeneous of degree 1) if and only if $\{s_0, \dots, s_d\}$ is not free over A.*

(c) *Suppose that $\mathcal{L}$ is very ample for $X \to \operatorname{Spec} A$. Let $t_0, \dots, t_n \in \mathcal{L}(X)$ be sections given by an immersion $\theta : X \to \mathbb{P}^n_A$. If $s_0, \dots, s_d \in \mathcal{L}(X)$ generate a sub-A-module containing the t_i (e.g. if the s_i generate $\mathcal{L}(X)$), then the morphism f defined in* (b) *is an immersion (closed if X is proper over A).*

Proof (a) By the computations of Example 1.19, $\mathcal{O}_Y(1)$ is generated by $d+1$ global sections $T_0, \dots, T_d$. These sections induce global sections $s_0, \dots, s_d$ of $f^*\mathcal{O}_Y(1)$ by the canonical map $\mathcal{O}_Y(1)(Y) \to f^*\mathcal{O}_Y(1)(X)$. Let $x \in X$, let $y = f(x)$. Then we have

$$(f^*\mathcal{O}_Y(1))_x = \mathcal{O}_Y(1)_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} = \sum_i (T_i)_y \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} = \sum_i (s_i)_x \mathcal{O}_{X,x}.$$

Hence $f^*\mathcal{O}_Y(1)$ is generated by the global sections $s_0, \dots, s_d$.

(b) The open subsets X_{s_i} cover X. For every $i \leq d$, let us consider the morphism $f_i : X_{s_i} \to D_+(T_i)$ corresponding to the ring homomorphism

$$\varphi_i : \mathcal{O}_Y(D_+(T_i)) \to \mathcal{O}_X(X_{s_i}), \quad T_j/T_i \mapsto s_j/s_i \in \mathcal{O}_X(X_{s_i}).$$

It is clear that the morphisms f_i glue to a morphism $f : X \to Y$. For every $i \leq d$, we have an isomorphism

$$\mathcal{L}(X_i) = s_i\mathcal{O}_X(X_{s_i}) \to f^*\mathcal{O}_Y(1)(D_+(T_i)) = T_i \otimes \mathcal{O}_X(X_{s_i}), \quad s_i \mapsto T_i \otimes 1$$

by Proposition 1.14(b) and the description in Example 1.19. They glue together to define an isomorphism $\mathcal{L} \simeq f^*\mathcal{O}_Y(1)$. We see easily by construction that this isomorphism maps s_i to f^*T_i.

Set-theoretically, we can describe f as follows: if $x \in X_z$, $z \in \operatorname{Spec} A$, is a rational point and if $e \in \mathcal{L}_x$ is a basis, then $f(x) = (\lambda_0, ..., \lambda_d) \in \mathbb{P}^d(k(z))$ where λ_i is the image of $(s_i)_x/e \in \mathcal{O}_{X,x}$ in $k(x) = k(z)$.

Let $a_0, \dots, a_d \in A$ be such that $a_0 s_0 + \cdots + a_d s_d = 0$. Let $F = a_0 T_0 + \cdots + a_d T_d$. Then $\varphi_i(F/T_i) = 0$ for all i. Hence $f(X) \subseteq V_+(F)$. The converse is proved in the same way.

(c) The sections $s_0, \dots, s_d, t_0, \dots, t_n$ generate $\mathcal{L}$. Let $g : X \to \mathbb{P}^{d+n+1}_A$ be the morphism associated to this family of global sections. Let us first show that g is an immersion. As $t_0, \dots, t_n$ generate $\mathcal{L}$, the image of g is contained in $\mathbb{P}^{d+n+1}_A \setminus V_+(T_{d+1}, \dots, T_{d+n+1})$ (where t_i corresponds to g^*T_{d+1+i}). Let

$$p : \mathbb{P}^{d+n+1}_A \setminus V_+(T_{d+1}, \dots, T_{d+n+1}) \to \mathbb{P}^n_A$$

be the projection to the last $(n+1)$ coordinates. Then $\theta = p \circ g$. As $\mathbb{P}^{d+n+1}_A$ is a separated scheme, p is separated (Proposition 3.3.9(a) and (e)). Applying Lemma 3.3.15 to immersions, we see that g is an immersion. If X is proper over A, then g is a closed immersion (Proposition 3.3.16(e)).

Now for all $0 \leq i \leq n$, write $t_i = a_{i,0}s_0 + \cdots + a_{i,d}s_d$ for some $a_{i,0}, \dots, a_{i,d} \in A$ and let $F_i = T_{d+i+1} - a_{i,0}T_0 - \cdots - a_{i,d}T_d$. By similar computations as in (b), we see that the image of g is contained in $V_+(F_0, \dots, F_n)$. Moreover, the projection $q : V_+(F_0, \dots, F_n) \to \mathbb{P}^d_A$ to the first $(d+1)$ coordinates is an isomorphism and satisfies $f = q \circ g$. Therefore f is an immersion, closed if X is proper over A. □

Remark 1.32. Let us fix a set of sections $\{s_0, \dots, s_d\}$ of $\mathcal{L}(X)$ which generates $\mathcal{L}$. Then the morphism f as in (b) is unique and is compatible with base change $A \to B$ is a natural way. Moreover, let us consider an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible sheaves on X, and let t_i be the image of s_i under $\varphi(X)$. By examining the proof above, we easily see that the morphism $X \to Y$ associated to $\mathcal{L}'$ and to the sections $t_0, \dots, t_d$ is identical to f.

A more flexible notion than that of a very ample sheaf is the following.

Definition 1.33. Let X be a scheme which is neither quasi-compact and separated or Noetherian. Let $\mathcal{L}$ be an invertible sheaf on X. We say that $\mathcal{L}$ is *ample* if for any finitely generated quasi-coherent sheaf $\mathcal{F}$ on X, there exists an integer $n_0 \geq 1$ such that for every $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections. Theorem 1.27 says that a very ample sheaf (on a projective scheme over an affine scheme) is ample. Note that the notion of ample sheaf is an absolute notion (independent of a base scheme).

Theorem 1.34. *Let $f : X \to \operatorname{Spec} A$ be a morphism of finite type to an affine scheme $\operatorname{Spec} A$. Let us suppose that X is Noetherian or separated. Let $\mathcal{L}$ be an ample sheaf on X. Then there exists an $m \geq 1$ such that $\mathcal{L}^m$ is very ample for f.*

Proof Let $s_1, \dots, s_r \in \mathcal{L}^n(X)$ as given by Lemma 1.35(a) below. We have $\mathcal{O}_X(X_{s_i}) = A[f_{ij}]_j$ with a finite number of $f_{ij} \in \mathcal{O}_X(X_{s_i})$ (Definition 3.2.1). By Lemma 1.25, there exists an $m \geq 1$ such that $s_i^m \otimes f_{ij}$ is the restriction to X_{s_i} of a section $s_{ij} \in \mathcal{L}^{nm}(X)$. We can choose the same m for all of the f_{ij}. The sections $\{s_i^m, s_{ij}\}_{i,j}$ of $\mathcal{L}^{nm}$ generate it, because the s_i^m generate $\mathcal{L}^{nm}$ owing to the fact that $X = \cup_i X_{s_i}$. Let $\pi : X \to \operatorname{Proj} A[S_i, S_{ij}]_{i,j}$ be the morphism from X to a projective space associated to the sections s_i, s_{ij} (Proposition 1.31). Let $U_i = D_+(S_i)$. Then $X_{s_i} = \pi^{-1}(U_i)$ and $\mathcal{O}(U_i) \to \mathcal{O}_X(X_{s_i})$ is surjective because it sends S_{ij}/S_i to f_{ij}. It follows that π induces a closed immersion from X into $U := \cup_i U_i$. Hence π is an immersion (Exercise 3.2.3). This proves that $\mathcal{L}^{nr}$ is very ample for f. □

Lemma 1.35. *Let X be a Noetherian or separated and quasi-compact scheme and $\mathcal{L}$ an invertible sheaf on X.*

- (a') (Extension of finitely generated quasi-coherent sheaves) *Let U be as above. Let $\mathcal{F}$ be a quasi-coherent sheaf over X and let $\mathcal{G}$ be a finitely generated quasi-coherent subsheaf of $\mathcal{F}|_U$. Then there exists a finitely generated quasi-coherent subsheaf $\mathcal{G}'$ of $\mathcal{F}$ such that $\mathcal{G} = \mathcal{G}'|_U$.*
- (a) *The sheaf $\mathcal{L}$ is ample if and only if there exist $n \geq 1$ and $s_1, \dots, s_r \in \mathcal{L}^n(X)$ such that X_{s_i} is affine for every i and that $X = \cup_{1 \leq i \leq r} X_{s_i}$.*
- (b) *The restriction of an ample sheaf $\mathcal{L}$ to any quasi-compact open subset of X is ample.*

Proof (a') Suppose X is affine. Let $M = \{s \in \mathcal{F}(X) \mid s|_U \in \mathcal{G}(U)\}$ and $\overline{\mathcal{G}} = M^\sim$. Let us first show that $\overline{\mathcal{G}}|_U = \mathcal{G}$. Let $D(f)$ be a principal open subset of X contained in U. Then inside $\mathcal{F}(D(f))$ we have

$$\overline{\mathcal{G}}(D(f)) = M_f \subseteq \mathcal{G}(U)_f \subseteq \mathcal{G}(D(f)).$$

Conversely, let $v \in \mathcal{G}(D(f))$. By Proposition 1.6, there exists $t \in \mathcal{G}(U)$ and $r \geq 1$ such that $v = t|_{D(f)}/f^r$. As $v \in \mathcal{F}(D(f))$, it can be written as $v = s/f^q$ for some $s \in \mathcal{F}(X)$ and $q \geq 1$. The equality $s/f^q = t/f^r$ in $\mathcal{F}(D(f))$ implies that $(f^k s)|_U \in \mathcal{O}_X(U)t \subseteq \mathcal{G}(U)$ for some $k \geq 1$ and $v = (f^k s)/f^{k+q} \in \overline{\mathcal{G}}(D(h))$ with $f^k s \in M$. Hence $\overline{\mathcal{G}}|_U = \mathcal{G}$. As M generates $\mathcal{G}$ on U and U is quasi-compact, a finitely generated submodule N of M will generate $\mathcal{G}$ on U. We can take $\mathcal{G}' = N^{\sim}$.

In general, we can write $X = U \cup V_1 \cup \cdots \cup V_n$ with $n \geq 1$ and V_i open and affine. Then $U \cup V_1$ is either Noetherian or quasi-compact and separated, and $U \cap V_1$ is quasi-compact. By the above, there exists a finitely generated quasi-coherent sheaf $\mathcal{G}_1$ on V_1 such that $\mathcal{G}_1|_{U\cap V_1} = \mathcal{G}|_{U\cap V_1}$. Glueing $\mathcal{G}$ and $\mathcal{G}_1$ then gives a finitely generated quasi-coherent sheaf $\mathcal{G}'_1$ on $U \cup V_1$ which restricts to $\mathcal{G}$ on U. We then repeat the same process to extend the sheaf $\mathcal{G}'_1$ on $U \cup V_1$ to a finitely generated quasi-coherent subsheaf $\mathcal{G}'_2$ of $\mathcal{F}|_{(U\cup V_1)\cup V_2}$ etc.

(a) First suppose the existence of $s_1, \dots, s_r \in \mathcal{L}^n(X)$ as stated. Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on X. Let $f_1, \dots, f_q$ be generators of $\mathcal{F}(X_{s_1})$. By Lemma 1.25, there exists an $m_0 \geq 1$ such that $f_j \otimes s_1^m|_{X_{s_1}}$ lifts to a section in $(\mathcal{F} \otimes \mathcal{L}^{nm})(X)$ for every $j \leq q$ and all $m \geq m_0$. As s_1^m is a basis of $\mathcal{L}^{nm}|_{X_{s_1}}$, $(\mathcal{F} \otimes \mathcal{L}^{nm})(X_{s_1})$ is generated by the $f_j \otimes s_1^m|_{X_{s_1}}$. Hence, at the points of X_{s_1}, $\mathcal{F} \otimes \mathcal{L}^{nm}$ is generated by its global sections. Therefore, by applying the result to the other s_i, we see that $\mathcal{F} \otimes \mathcal{L}^{nm}$ is generated by its global sections for every sufficiently large m. Now apply the same result to $\mathcal{F} \otimes \mathcal{L}, \dots, \mathcal{F} \otimes \mathcal{L}^{n-1}$, then we see that for every m bigger or equal to some m_1, $\mathcal{F} \otimes \mathcal{L}^{k+nm}$ is generated by its global sections for $k = 0, 1, \dots, n-1$. Therefore $\mathcal{F} \otimes \mathcal{L}^m$ is generated by its global sections for all $m \geq nm_1$. Hence $\mathcal{L}$ is ample.

Suppose conversely that $\mathcal{L}$ is ample. Notice that sections s_i as above may not exist in $\mathcal{L}(X)$ because, for instance, it would imply that $\mathcal{L}$ is generated by its global sections. This is reason why we have to consider powers of $\mathcal{L}$.

Let x be a point of X. Let us show that there exist an $n = n(x)$ and a section $t \in \mathcal{L}^n(X)$ such that X_t is an affine neighborhood of x. Let U be an affine open neighborhood of x such that $\mathcal{L}|_U$ is free, and let $\mathcal{J}$ be a quasi-coherent sheaf of ideals of $\mathcal{O}_X$ such that $V(\mathcal{J}) = X \setminus U$. We want to show that there exists an $n \geq 1$ such that $\mathcal{J} \otimes \mathcal{L}^n$ is generated by its global sections at x. If X is Noetherian, then this follows from the ampleness hypothesis because $\mathcal{J}$ is coherent. In general, we notice that $\mathcal{J}|_U = \mathcal{O}_U$ is finitely generated on U. By (a'), there exists a finitely generated quasi-coherent subsheaf $\mathcal{J}'$ of $\mathcal{J}$ such that $\mathcal{J}'|_U = \mathcal{J}|_U$. So there exists $n \geq 1$ such that $\mathcal{J}' \otimes \mathcal{L}^n$ is generated by its global sections. Let $t \in (\mathcal{J}' \otimes \mathcal{L}^n)(X)$ be a section whose image in $(\mathcal{J}' \otimes \mathcal{L}^n)_x = \mathcal{L}^n_x$ is a basis. As $\mathcal{L}^n$ is flat, we can identify canonically $\mathcal{J}' \otimes \mathcal{L}^n$ with $\mathcal{J}'\mathcal{L}^n$ and consider t as a global section of $\mathcal{L}^n$. Then $x \in X_t$. For any $y \in X \setminus U = V(\mathcal{J})$, we have $t_y \in \mathcal{J}'_y\mathcal{L}^n_y \subseteq \mathfrak{m}_y\mathcal{L}^n_y$, so $y \notin X_t$ and therefore $X_t \subseteq U$. Let us write $\mathcal{L}|_U = e\mathcal{O}_U$ and $t|_U = eh$ with $h \in \mathcal{O}_X(U)$. Then $X_t = D_U(h)$. This implies that X_t is affine.

As X is quasi-compact, it is covered by a finite number of affine open subsets $X_{t_1}, \dots, X_{t_r}$ with $t_i \in \mathcal{L}^{n_i}(X)$. Let n be a common multiple of the n_i. Then

$s_i := t_i^{\otimes n/n_i} \in \mathcal{L}^n(X)$, $X_{s_i} = X_{t_i}$, and we have the secitons s_i as required.

(b) Let V be a quasi-compact open subset of X. In the second part of the proof of (a), we saw that every point of V has an affine open neighborhood of the form X_s for a global section s of some positive power of $\mathcal{L}$, and the same reasoning shows that for some $n \geq 1$, V is covered by a finite number of affine open subsets X_{s_i}, $s_i \in \mathcal{L}^n(X)$. As $X_{s_i} \subseteq V$, we have $V_{s_i|V} = X_{s_i}$. By (a) $\mathcal{L}|_V$ is ample. □

Corollary 1.36. *Let* $X \to \operatorname{Spec} A$ *be a morphism as in* Theorem 1.34. *Then* $X \to \operatorname{Spec} A$ *is quasi-projective if and only if there exists an ample sheaf on* X.

Proof Let us suppose $X \to \operatorname{Spec} A$ is quasi-projective. Then X is an open subscheme of a projective scheme Y over A. There exists an ample sheaf on Y by Theorem 1.27. The restriction of this ample sheaf to X is an ample sheaf on X by Lemma 1.35(b). The converse is an immediate consequence of Theorem 1.34 by the definition of a very ampe sheaf for $X \to \operatorname{Spec} A$. □

Proposition 1.37. *Let* $f : X \to Y$ *be a proper morphism of locally Noetherian schemes. Let* $\mathcal{L}$ *be an invertible sheaf on* X. *Let us fix a point* $y \in Y$ *and let* $\varphi : X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y} \to X$ *be the canonical morphism. Then the following properties are true.*

(a) *If* $\varphi^*\mathcal{L}$ *is generated by its global sections, then there exists an open neighborhood* V *of* y *such that* $\mathcal{L}|_{f^{-1}(V)}$ *is generated by its global sections.*

(b) *If* $\varphi^*\mathcal{L}$ *is ample, then there exists an affine open neighborhood* V *of* y *such that* $\mathcal{L}|_{f^{-1}(V)}$ *is ample.*

(See also Corollary 3.24 and Remark 3.25.)

Proof We can suppose $Y = \operatorname{Spec} A$ is affine.

(a) Let $Z = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}$. As $A \to \mathcal{O}_{Y,y}$ is flat, in a way similar to Proposition 3.1.24, we show that $\varphi^*\mathcal{L}(Z) = \mathcal{L}(X) \otimes_A \mathcal{O}_{Y,y}$. So $\varphi^*\mathcal{L}(Z)$ is generated by the canonical image of $\mathcal{L}(X)$. Let $x \in f^{-1}(y) = X_y = Z_y$. Then the canonical homomorphisms $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,x}$ and $\mathcal{L}_x \to (\varphi^*\mathcal{L})_x$ are isomorphisms. As $(\varphi^*\mathcal{L})_x$ is generated by the image of $\varphi^*\mathcal{L}(Z)$, $\mathcal{L}_x$ is generated by the image of $\mathcal{L}(X)$. We therefore see that $\mathcal{L}$ is generated by its global sections at the points $x \in f^{-1}(y)$. The set of points where $\mathcal{L}$ is not generated by its global sections is a closed subset F (its complementary is the union of the X_s, $s \in \mathcal{L}(X)$) that does not meet $f^{-1}(y)$. Let $V = Y \setminus f(F)$. Then $\mathcal{L}|_{f^{-1}(V)}$ is generated by its global sections.

(b) If necessary replacing $\mathcal{L}$ by a tensor power, we can suppose $\varphi^*\mathcal{L}$ is very ample and generated by its global sections. By (a), we can suppose that $\mathcal{L}$ is generated by its global sections. As A is Noetherian, $\mathcal{L}(X)$ is finitely generated over A (Remark 3.3). Let $s_0, \dots, s_d \in \mathcal{L}(X)$ be a system of generators. Let $g : X \to \mathbb{P}^d_Y$ be the morphism associated to these sections and let h be the induced morphism from X to the scheme-theoretic closure of $g(X)$ in $\mathbb{P}^n_Y$. As $\rho^*\mathcal{L}$ is very ample and $s_0, \dots, s_d$ generate $\rho^*\mathcal{L}(Z)$, by Proposition 1.31(c), h is an isomorphism above $\operatorname{Spec} \mathcal{O}_{Y,y}$. Hence h is an isomorphism above an affine open subscheme $V \ni y$ (Exercise 3.2.5) This shows that $\mathcal{L}|_{f^{-1}(V)}$ is very ample. □