NON-COMMUTATIVE DYNAMICAL SYSTEMS
APPLICATIONS TO RIGIDITY OF CHARACTERS, IRS'S, URS'S

In this series of lectures, we aim to investigate various rigidity phenomena for lattices in higher rank semi-simple Lie groups. This is based on joint works with Cyril Houdayer, Uri Bader and Jesse Peterson [BH19, BBHP20].

The starting point of the discussion is Margulis normal subgroup theorem, that asserts, roughly speaking, that if a Lie group is simple, and has real rank at least 2, then its lattices are also simple, if we ignore finite index normal subgroups. Then it has been realized by Stuck and Zimmer [SZ92] that his method of proof, which is of a measurable nature, could be applied to classify IRS's. Analogous results for irreducible lattices in more general product groups have been obtained by Bader-Shalom [BS04].

More recently, Creutz-Peterson [CP13], and Peterson [Pe14] managed to push Margulis techniques to classify all the *characters* of such lattices, encompassing all the above results. Parallel to this recent work, a new conceptual approach to the so-called *C*-simplicity problem* has been discovered, by Kalantar-Kennedy [KK14] and Breuillard-Kalantar-Kennedy-Ozawa [BKKO14]. This induced a multitude of subsequent works, from which it became clear that C*-dynamics and ergodic theory could be combined with Margulis techniques to go deeper in the study of characters and unitary representations of lattices in Lie groups. This is the approach we used in [BH19] to give a more natural proof of Peterson's character rigidity theorem and obtain new, rather mysterious, results about unitary representations of these lattices. A particularly inspiring paper for us was [HK17], which discusses stationary C*-dynamics.

To invite to active reading (and to keep the text to a reasonable length), the whole text is full of gaps, presented as exercises.

1. C*-dynamics and application to characters

1.1. Characters and their GNS representation.

The main rigidity phenomenon that we will prove is about positive definite functions, and more general characters.

Definition 1.1. A *positive definite* function on a (discrete) group Γ is a function $\phi : \Gamma \to \mathbb{C}$ such that for every finite set $F \subset \Gamma$, the matrix $(\phi(g^{-1}h))_{g,h\in F}$ is positive semi-definite. We say that ϕ is a *character* if moreover $\phi(1) = 1$ and ϕ is conjugation invariant.

Example 1.2. Constant functions are characters, the Dirac function δ_e is a character, called the *regular character*. More interestingly:

- If $\Lambda < \Gamma$ is a subgroup, then the function $\mathbf{1}_\Lambda$ is positive definite. It is a character if and only if Λ is normal in Γ;
- Given a non-singular action $\Gamma \curvearrowright (X, \mu)$, the function $g \mapsto \mu(\mathrm{Fix}(g))$ is positive definite on Γ. If μ is Γ-invariant, it is a character.

Exercise 1.3. If $\pi : \Gamma \to \mathcal{U}(H)$ is a unitary representation, and ξ is a vector in H, then the function $g \mapsto \langle \pi(g)\xi, \xi\rangle$ is positive definite.

The GNS construction tells us that every positive definite function arises this way. More precisely, given a positive definite function ϕ, we may extend ϕ linearly on the group ring $\mathbb{C}\Gamma$, and consider the positive semi-definite hermitian form $(x, y) \in (\mathbb{C}\Gamma)^2 \mapsto \phi(y^*x) \in \mathbb{C}$. Denote by H_ϕ the Hilbert space obtained by moding out by the kernel of this hermitian form and completing w.r.t. the corresponding scalar product. Denote by $\xi_\phi \in H_\phi$ the image in H_ϕ of the unit $1 \in \mathbb{C}\Gamma$. Note that $\|\xi_\phi\| = \phi(1)$.

For every $g \in \Gamma$, we may then consider the left multiplication linear transformation $x \in \mathbb{C}\Gamma \mapsto gx \in \mathbb{C}\Gamma$. This map clearly preserves the hermitian form and thus defines a unitary operator $\pi_\phi(g)$ of H_ϕ. One checks that the map $\pi_\phi : \Gamma \to \mathcal{U}(H_\phi)$ is in fact a group homomorphism. Moreover, we have

$$\langle \pi_\phi(g)\xi_\phi, \xi_\phi\rangle = \phi(g), \text{ for every } g \in \Gamma.$$

1.2. **C*-dynamical systems.** In the non-commutative world, the category of compact spaces is replaced by that of C*-algebras.

Definition 1.4. A *C*-algebra* is a normed algebra $(A, \|\cdot\|)$ over the field of complex numbers, with an anti-linear involution $* : A \to A$ satisfying:

- A is complete as a normed space and $\|ab\| \leq \|a\|\|b\|$, for all $a, b \in A$ (Banach algebra axioms)
- $*$ is isometric, and satisfies $\|a^*a\| = \|a\|^2$, for all $a \in A$ (C*-axiom).

A *morphism of C*-algebras*, or *C*-morphism*, is an algebra homomorphism $\pi : A \to B$ which preserves the *-operation. Such an algebraic morphism is automatically contractive for the norms: $\|\pi(a)\| \leq \|a\|$ for all $a \in A$. We denote by $\mathrm{Aut}(A)$ the group of all automorphisms of A, endowed with the topology of pointwise norm convergence.

Exercise 1.5. Admitting the above fact on C*-morphisms, check that the image of a C*-morphism is again a C*-algebra, and that an injective C*-morphism is an isometry.

In fact, to truly correspond to compact spaces, our C*-algebras will always be assumed to be unital, i.e. they have a unit 1.

Definition 1.6. A *state* on A is a linear functional $\phi : A \to \mathbb{C}$ such that $\phi(1) = 1$ and $\phi(a^*a) \geq 0$, for every $a \in A$. The set $\mathcal{S}(A)$ of all states on A is a compact space for the weak-* topology.

Example 1.7. Assume that X is a compact space. Then the algebra $C(X)$ of continuous complex valued functions over X, endowed with pointwise operations, the sup-norm and the pointwise complex conjugation as a *-operation is a C*-algebra. Its product is commutative.

Conversely, any *commutative* C*-algebra A arises this way. The space X is canonically attached to A, and can be recovered as the space of all C*-morphisms $A \to \mathbb{C}$. This is explicitly given by the so-called *Gelfand transform*. In particular, the category of compact spaces is equivalent to that of unital C*-algebras.

By Riesz representation theorem, states on $C(X)$ correspond to Borel probability measures on X.

Example 1.8. Given a Hilbert space H, the algebra $B(H)$ of all continuous linear operators on H, with composition of operators as its product law, adjoint as its *-operation and operator norm, is a C*-algebra. The unit is the identity operator. Note that any vector $\xi \in H$ gives rise to a state $T \in B(H) \mapsto \langle T\xi, \xi\rangle \in \mathbb{C}$ on $B(H)$, and hence on any of its C*-subalgebras.

Example 1.9. Any *-invariant subalgebra of $B(H)$ which is normed closed in of course a C*-algebra as well. For example if $\pi : G \to \mathcal{U}(H)$ is a unitary representation of a locally compact group G, then the norm closure of $\mathrm{span}(\pi(G))$ is a C*-algebra, denoted by $C^*_\pi(G)$.

Definition 1.10. An *action* of a locally compact group G on a C*-algebra A is a continuous group homomorphism $\sigma : g \in G \mapsto \sigma_g \in \mathrm{Aut}(A)$. A state ϕ on A is called *G-invariant* if $\phi \circ \sigma_g = \phi$ for every $g \in G$. In the sequel we will use the notation $g\phi$ for $\phi \circ \sigma_g^{-1}$.

Example 1.11. Consider a lcsc group G and a unitary representation $\pi : G \to \mathcal{U}(H)$ on the Hilbert space H. Then G acts on $B(H)$ by unitary conjugation. The existence of an invariant state is a property of the representation, known as Bekka's *amenability*.

Also interesting is the sub-C*-system $G \curvearrowright C^*_\pi(G)$. Check that any invariant state ϕ on $C^*_\pi(G)$ is a *trace*, in the sense that $\phi(ab) = \phi(ba)$ for every $a, b \in C^*_\pi(G)$.

When G is discrete and $\pi = \lambda$ is the regular representation $G \to \mathcal{U}(\ell^2 G)$, given by $\lambda_g(\delta_h) = \delta_{gh}$, for every $g, h \in G$, there is always a trace on $C^*_\lambda(G)$ (for every discrete G), given by the vector state $a \mapsto \langle T\delta_e, \delta_e\rangle$. Note that this state is not invariant on $B(\ell^2 G)$ in general.

In this language, the famous unique trace property is naturally phrased as a unique ergodicity property.

Definition 1.12. Two states ϕ, ψ on a C*-algebra A are called *singular* if $\|\phi - \psi\| = 2$. In this case, we write $\phi \perp \psi$.

Exercise 1.13. Check that $\phi \perp \psi$ if and only if there exists a sequence of elements $a_n \in A$, such that $0 \leq a_n \leq 1$, and $\lim_n \phi(a_n) = 0$ and $\lim_n \psi(a_n) = 1$.

Lemma 1.14. *Take a C*-algebra A with a unitary $u \in \mathcal{U}(A)$ and a state $\phi \in \mathcal{S}(A)$. Denote by $\phi_u \in \mathcal{S}(A)$ the state such that $\phi_u(a) = \phi(u^*au)$, for all $a \in A$. If $\phi \perp \phi_u$, then $\phi(u) = 0$.*

Proof. By assumption, we may find a sequence $a_n \in A$, such that $0 \leq a_n \leq 1$, and

$$\lim_n \phi(a_n) = 0 \quad \text{and} \quad \lim_n \phi(u^*a_nu) = 1.$$

By Cauchy-Schwarz inequality, we find

$$\lim_n |\phi(a_nu)| = \lim_n |\phi(a_n)^{1/2}a_n^{1/2}u)| \leq \lim_n \phi(a_n)^{1/2}\phi(u^*a_nu)^{1/2} = 0.$$

A similar computation gives that $\lim_n |\phi(a_nu) - \phi(u)| = \lim_n |\phi(u(u^*a_nu - 1))| = 0$. This gives $\phi(u) = \lim_n \phi_b(a_nu) = 0$, as claimed. □

1.3. **Main theorem and consequences.** Our main theorem is about actions of lattices in higher rank Lie groups. We will use repeatedly specific notation relative to this setting:

- G will denote a connected semi-simple Lie group of (real) rank at least 2, with finite center;
- Γ will be an irreducible lattice in G;
- P will be a minimal parabolic subgroup in G; K a maximal compact subgroup.
- μ will be a probability measure on G which is absolutely continuous with respect to the Haar measure, with a continuous, compactly supported Radon-Nykodym derivative. We also assume that μ is left K-invariant.

In the case where $G = \mathrm{SL}_d(\mathbb{R})$, we may choose P as the subgroup of upper triangular matrices and $K = \mathrm{SO}(d)$. Some facts we will use about P:

- P is co-compact in G,
- K acts transitively on G/P,
- G/P, endowed with its unique K-invariant probability measure ν_P realizes the Poisson boundary of μ. See Section 2.3 for a recap on Poisson boundaries.

Note that the last item above implies that G/P is *amenable* as a G-space and hence also as a Γ-space. In particular, for every separable convex compact space $\mathcal{C}$ on which G (or Γ) acts continuously by affine transformations, there exists a measurable G-map (or Γ-map) $G/P \to \mathcal{C}$.

By a theorem of Furstenberg, there exists a probability measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ such that $(G/P, \nu_P)$ is also the Poisson boundary of (Γ, μ_0).

We keep this notation for the rest of this section.

Theorem 1.15 (Main dynamical theorem)**.** *Take a separable Γ-C*-algebra A and a measurable Γ-equivariant map $\theta : G/P \to \mathcal{S}(A)$, which is extremal with these properties. Then either θ is essentially constant or for every $g \in \Gamma \setminus \mathcal{Z}(\Gamma)$, for almost every $b \in G/P$, we have $\theta_{gb} \perp \theta_b$.*

Let us assume for simplicity that G and Γ have trivial center, and derive some corollaries in this case.

Corollary 1.16 (Peterson)**.** *Let ϕ be a character of Γ. Then either ϕ is the regular character or the GNS representation π_ϕ is amenable in the sense of Bekka, i.e. there exists a Γ-invariant state on $B(H_\phi)$.*

Proof. Simply denote by (H, π, ξ) the GNS triple associated with ϕ. Assume that ϕ not amenable, so that there is no Γ-invariant state on $B(H)$.

Exercise. Check then that there exists a separable C*-subalgebra $A \subset B(H)$ which contains $\pi(\Gamma)$ such that there is no Γ-invariant state on A.

By Theorem 1.15, for every extremal measurable Γ-map $\theta : G/P \to \mathcal{S}(A)$, for every non-trivial $g \in \Gamma$, for almost every $b \in G/P$, we have $\theta_{gb} \perp \theta_b$. Indeed θ cannot be essentially constant because this would imply that its essential range consists of a single Γ-invariant state on A. Note that $\theta_{gb} = g\theta_b = \theta_b \circ \mathrm{Ad}(\pi(g)^*)$. So Lemma 1.14 implies that $\theta_b(\pi(g)) = 0$. By Krein-Millman's theorem, every measurable Γ-map $\theta : G/P \to \mathcal{S}(A)$ lies in the closed convex hull of extremal ones. So the formula $\theta_b \circ \pi = \delta_e$, for almost every b still holds when θ is not assumed to be extremal. Now we consider the convex set

$$\mathcal{C} := \{\psi \in \mathcal{S}(A) \mid \psi \circ \pi_\phi = \phi\}.$$

It is a compact set for the weak-* topology, which is Γ-invariant. By amenability of G/P, we may find a measurable Γ-map $\theta : G/P \to \mathcal{C}$. By the observations above we find $\theta \circ \pi = \delta_e = \phi$ for almost every $b \in B$, which implies that ϕ is the regular character. □

Exercise 1.17. In the case where Γ has property (T), derive that every extremal point in the space of characters of Γ is either the regular character, or is almost periodic, in the sense that its GNS representation is finite dimensional.

Applying the above corollary to the subgroup characters and IRS characters described in Example 1.2 gives the following corollaries.

Corollary 1.18. *Assume that Γ has property (T).*

(1) Every non-trivial normal subgroup of Γ has finite index in Γ (Margulis);
(2) every ergodic pmp action of Γ on a probability space is either essentially transitive or essentially free (Stuck-Zimmer).

Corollary 1.19. *Every minimal action of Γ on a compact space is either topologically free or fixes a probability measure. In particular, if Γ has property (T), then any of its URS's is finite.*

Proof. Consider a minimal $\Gamma \curvearrowright X$ on the compact space X. Denote by $A = C(X)$, and choose an extremal Γ-equivariant measurable map $\theta : G/P \to \mathrm{Prob}(X)$. Note that the barycenter measure $\nu := \mathrm{Bar}(\theta_*(\nu_P)) \in \mathrm{Prob}(X)$ is quasi-invariant. By minimality, we find that ν has full support.

If θ is essentially constant, then ν is equal to the essential value of θ and is Γ-invariant. Otherwise, Theorem 1.15 implies that $\theta_b \perp g\theta_b$, for every non-trivial $g \in \Gamma$, for ν-almost every $b \in B$. In particular $\theta_b(\mathrm{Fix}(g)) = 0$, where $\mathrm{Fix}(g)$ is the fixed point set of g in X. Integrating this over $b \in B$, we find that $\nu(\mathrm{Fix}(g)) = 0$. Since the support of ν is the whole of X, this forces $\mathrm{Fix}(g)$ to have empty interior. □

Corollary 1.20. *Take a unitary representation π of Γ. The following facts hold true:*

(1) $C^*_\pi(\Gamma)$ admits a trace.
(2) Assume that Γ has property (T). Either π contains a finite dimensional invariant subspace, or weakly contains the regular representation of Γ. The later means that the map $\pi(g) \mapsto \lambda(g)$ extends to a C-morphism $C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma)$.*

Proof. (1) We view $A := C^*_\pi(\Gamma)$ as a Γ-C*-algebra for the conjugation action. Take an extremal measurable Γ-map $\theta : G/P \to \mathcal{S}(A)$. By Theorem 1.15, either θ is essentially constant, and its image is Γ-invariant, or $\theta_b \perp g\theta_b$, for every non-trivial $g \in \Gamma$, for almost every $b \in G/P$. By Lemma 1.14, the later implies that $\theta_b(\pi(g)) = 0$. In this case $\theta_b \circ \pi = \delta_e$ for almost every b. Since A is generated by $\pi(\Gamma)$ and δ_e is conjugation invariant, we find that in this case, θ_b is Γ-invariant, almost surely. This contradicts the assumption that $\theta_b \perp g\theta_b$.

(2) Denoting by τ a trace on $C^*_\pi(\Gamma)$, we find that $\phi = \tau \circ \pi$ is a character. We note that the GNS representation of ϕ is of the form $\pi_\tau \circ \pi$, where $\pi_\tau : A \to B(H_\tau)$ is the GNS representation of the trace τ (we didn't mention it, but the GNS construction also works in the framework of states on C*-algebras).

Exercise. This implies that π_τ is weakly contained in π.

We apply Corollary 1.16 to ϕ. If ϕ is the regular character, then π_ϕ is the regular representation and thus π weakly contains the regular representation. Otherwise, if π_ϕ is an amenable representation, and this can be checked to imply that π itself must be amenable. But an amenable representation of a property (T) group must have a finite dimensional invariant subspace. □

Exercise 1.21. Figure out what the above corollaries become when we allow G to have finite center.

In fact, the same corollaries hold true for more general S-adic lattices, $\mathrm{PSL}_d(\mathbb{Z}[1/p])$, $d \geq 3$, although we cannot prove Theorem 1.15 in this case.

2. Non-commutative measurable dynamics

2.1. **Von Neumann algebras and actions.**

Definition 2.1. Given a Hilbert space H, we define the weak topology on $B(H)$ as the initial topology for the family of semi-norms $T \in B(H) \mapsto |\langle T\xi, \eta\rangle|$, as ξ, η vary in H. So a sequence T_n converges to $T \in B(H)$ if and only if $\lim_n \langle (T_n - T)\xi, \eta\rangle = 0$, for all $\xi, \eta \in H$. This topology is coarser than the norm topology.

A *von Neumann algebra* is a C*-subalgebra of some $B(H)$ in which it is weakly closed. This notion doesn't seem intrinsic as it seems to involve the embedding in some $B(H)$. Nevertheless, we have a well defined category by defining a *vN-morphism* from M to N as a C*-morphism $M \to N$ which is weakly continuous. It turns out that the image of such a morphism is automatically weakly closed in N, hence a von Neumann subalgebra.

A state on a von Neumann algebra is called *normal* if it is weakly continuous.

Although we will stay on the surface in our arguments about von Neumann algebras, we quote the famous bi-commutant theorem of von Neumann. By definition the *commutant* of a set of operators $\mathcal{S} \subset B(H)$ is defined as

$$\mathcal{S}' := \{T \in B(H) \mid ST = TS, \text{ for all } S \in \mathcal{S}\}.$$

Note that a commutant is always weakly closed, and von Neumann's theorem gives the converse.

Theorem 2.2. *A unital *-sublagebra $A \subset B(H)$ is weakly closed if and only if it is equal to its bi-commutant: $A = (A')'$. In particular the weak closure of a *-subalgebra of $B(H)$ is its bi-commutant.*

Example 2.3. If (X, μ) is a measure space, then $L^\infty(X, \mu)$ can be viewed as a subalgebra of $B(L^2(X, \mu))$, and it can be checked to be a von Neumann subalgebra. Indeed, one can prove that it is equal to its own commutant. In fact, any commutative von Neumann algebra is isomorphic with some $L^\infty(X, \mu)$, and again, there is an equivalence of categories between commutative von Neumann algebras and measure spaces.

We define a *projection* in a von Neumann algebra M as an element $p \in M$ such that $p = p^* = p^2$ (i.e. $p \in B(H)$ is just an orthogonal projection onto a closed subspace of H). If x is a self adjoint operator in M, then all its spectral projections are in M. In particular, the set of projections in M spans a norm dense subset of M. If p is a projection in M, then pMp is a *-algebra, acting naturally on pH. In fact it is a von Neumann algebra in $B(pH)$, called a *corner* of M

Definition 2.4. If ϕ is a normal state on a von Neumann algebra M, then there exists a (unique) smallest projection $p \in M$ such that $\phi(p) = 1$. It is called the *support projection* of ϕ. We say that ϕ is *faithful* if $p = 1$.

We endow the group $\mathrm{Aut}(M)$ of all vN-automorphisms of a von Neumann algebra M with the topology on pointwise norm convergence on the space of normal states of M. More precisely, a net $\alpha_n \in \mathrm{Aut}(M)$ converges to α if $\lim_n \|\phi \circ \alpha_n - \phi \circ \alpha\| = 0$ for every normal state ϕ on M. If M is acting on a separable Hilbert space, $\mathrm{Aut}(M)$ is a Polish group for this topology.

Definition 2.5. A *vN-action* of a lcsc group G on a von Neumann algebra M is a continuous morphism $G \to \mathrm{Aut}(M)$. We say that the action is ergodic if the only G-invariant elements in M are the scalar multiples of the identity operator 1.

Note that a vN-continuous action $G \curvearrowright M$ is *not* C*-continuous in general. Nevertheless, a convolution argument implies that M always admits a strongly dense C*-subalgebra A which is globally invariant and on which the action is norm continuous. The C*-action $G \curvearrowright A$ is an instance of what we call a compact model defined below.

2.2. **Non-singular actions.**

Definition 2.6. Given a separable C*-algebra A, we define its universal representation (π_u, H_u) of A as the direct sum of all the (equivalence classes of) representations of A on separable Hilbert spaces. We call $\pi_u(A)''$ the *enveloping von Neumann algebra* of A, and denote it by A^{**}.

By construction, every C*-morphism from A into a von Neumann algebra M extends (uniquely) to a vN-morphism $A^{**} \to M$. Thanks to the GNS construction, it follows that every state on A extends (uniquely) to a normal state on A^{**}, and more generally every continuous linear functional on A extends to a weakly continuous linear functional on A^{**}. In fact this extension property allows to canonically identify A^{**} with the bidual of A, as operator spaces. This justifies our notation.

From the universal property of A^{**}, any C*-action $G \curvearrowright A$ of a lcsc group G gives rise to an action on A^{**}. However, this action is **not** continuous in general.

Definition 2.7. The support of a state ϕ on a C*-algebra A is by definition the support projection of the normal extension of ϕ to A^{**}, in the sense of Definition 2.4. Two states ϕ, ψ are called *equivalent*, denoted by $\phi \sim \psi$ if they have the same support.

Exercise 2.8. Check that two states are singular if and only if their support projections are orthogonal.

Remark 2.9 (Caution)**.** It can be checked that the notion of singular (resp. equivalent) states coincides with the notions of singular (resp. equivalent) measures in the commutative case. For this reason, the support of a state in the above sense must differ from the notion of support of a measure. It is true that the support of a measure $\mu \in \mathrm{Prob}(X)$ gives a projection p in $C(X)^{**}$ (because any Borel function on X can be viewed as an element in $C(X)^{**}$), which satisfies $\int_X p \, \mathrm{d}\mu = 1$, but it is not equal to the support projection of μ in the above sense. The reason is that the bidual of $C(X)$ is more complicated than just the C*-algebra of bounded Borel functions on X.

Remark 2.10. Sometimes, the notions of equivalent and singular states on C*-algebra are formulated in terms of the *central* support z of ϕ in A^{**}. This makes sense because zA^{**} identifies with the GNS von Neumann algebra associated with ϕ. But unfortunately the normal extension of ϕ on its GNS von Neumann algebra is not always faithful, and we prefer to work in the faithful setting.

Definition 2.11. Take a C*-action $G \curvearrowright A$ and a state $\phi \in \mathcal{S}(A)$. We say that ϕ is *non-singular*, or *quasi-invariant* if $\phi \sim g\phi$ for every $g \in G$. Equivalently, this means that the support projection p of ϕ in A^{**} is G-invariant.

Classical automatic continuity statements imply that if G a lcsc group acting continuously on a separable C*-algebra A, with a non-singular state ϕ, then the corresponding action $G \curvearrowright pA^{**}p$ is a (continuous) vN-action.

From now on we will only deal with separable C-algebras and von Neumann algebras acting on separable Hilbert spaces.*

Definition 2.12. Given a C*-action $G \curvearrowright A$, with a non-singular state $\phi \in \mathcal{S}(A)$, we call the vN-action $G \curvearrowright M := pA^{**}p$ the *vN-envelope*[1] of (A, ϕ). In this case, we say that $G \curvearrowright A$ is a *compact model* of the action $G \curvearrowright M$.

2.3. Stationary dynamics.

Definition 2.13. Let G be an lcsc group and $\mu \in \mathrm{Prob}(G)$. Consider a action $G \curvearrowright A$ on a C*-algebra A. Define the convolution operator $T_\mu : \mathcal{S}(A) \to \mathcal{S}(A)$, by $T_\mu(\phi) := \int_G g\phi \, \mathrm{d}\mu(g)$. This is a continuous affine transformation of $\mathcal{S}(A)$, hence by Kakutani's theorem, it admits fixed points, called *μ-stationary states*. The subset of all μ-stationary states on A is a closed convex subset of $\mathcal{S}(A)$.

We will always assume that the support of μ generates a dense sub-semigroup of G. We say that μ is *generating*.

Note that for every μ-stationary state ϕ on the C*-algebra A, we may define the *Poisson transform* $\mathcal{P}_\phi : A \to L^\infty(G)$, by the formula $\mathcal{P}_\phi(a)(g) = (g\phi)(a)$. The fact that ϕ is μ-stationary implies that $\mathcal{P}_\phi(a)$ ranges into the space of right-μ-Harmonic functions on G, denoted by $\mathcal{H}_\mu(G)$.

Recall that given a probability measure $\mu \in \mathrm{Prob}(G)$, there is a measure space (B, ν), on which G acts and that encodes the behavior at infinity of the μ-random walk on G. The G-space (B, ν) is called the *Poisson boundary of* G and can be defined as the unique G-space (B, ν) such that

[1] In view of Remark 2.10, it would be more appropriate to call it the *faithful vN-enveloppe*. But we will keep the lighter notation in this text.

- ν is μ-stationary and
- The Poisson transform: $\mathcal{P}_\nu : L^\infty(B,\nu) \to \mathcal{H}_\mu(G)$ is an onto isometry.

Given a G-C*-algebra A with a μ-stationary state ϕ, we obtain a positive linear map $\mathcal{P}_\nu^{-1}\mathcal{P}_\phi : A \to L^\infty(B,\nu)$, which is G-equivariant.

Exercise 2.14. Any positive map $E : A \to L^\infty(B,\nu)$ corresponds to a unique measurable map $\theta : B \to \mathcal{S}(A)$, such that $E(a)(b) = \theta_b(a)$, and vice versa (this relies on the separability of A). In particular E is G-equivariant if and only if θ is G-equivariant.

Thanks to the above exercise, any μ-stationary state ϕ on A comes with a uniquely defined (modulo null sets) G-equivariant measurable map $\theta : B \to \mathcal{S}(A)$, where $G \curvearrowright (B,\nu)$ is the Poisson boundary of (G,μ). Conversely, any such boundary map gives a μ-stationary state $\phi = \mathrm{Bar}(\theta_*\nu)$, the barycenter of the push-forward of ν. These two constructions are inverse of each other.

Definition 2.15. In the above context, the states $\theta(b)$, $b \in B$, are called the *conditional states* attached to the μ-stationary state ϕ, and are denoted by ϕ_b.

Proposition 2.16. *Take a C*-action $G \curvearrowright A$ and a generating measure $\mu \in \mathrm{Prob}(G)$. Any μ-stationary state ϕ on A is non-singular, and the following are equivalent:*

(i) ϕ is extremal as a μ-stationary state on A;
(ii) The corresponding measurable G-map $\theta : B \to \mathcal{S}(A)$ is extremal;
(iii) The vN-envelope $G \curvearrowright M$ is ergodic.

Proof. We only prove the non-singularity statement in the case where G is discrete (countable). The proof of the equivalence $(i) - (iii)$ is somewhat similar to the well-known commutative proof but it requires some knowledge on operator algebras. It is essentially given in [BBHP20, Proposition 2.8].

Denote by $p \in A^{**}$ the support projection of the μ-stationary state ϕ. Then we have

$$1 = \phi(p) = \int_G (g\phi)(p)\,\mathrm{d}\mu(g).$$

Since $(g\phi)(p) \in [0,1]$ for every $g \in G$, the above equality implies that for every g in the support of μ, $\sigma_g^{-1}(p) = p$. Since the support of μ generates G, this shows that p is indeed G-invariant. ☐

2.4. **A measurable version of Theorem 1.15.** The above tools allow to switch between the von Neumann language and the C*-language. Using these tools one can prove that Theorem 1.15 is equivalent to the following theorem on von Neumann dynamical systems.

Theorem 2.17. *Use the notation $\Gamma < G$, $\mu \in \mathrm{Prob}(G)$, $\mu_0 \in \mathrm{Prob}(\Gamma)$ given in Section 1.3. For every ergodic vN-action $\Gamma \curvearrowright M$ on a separable von Neumann algebra with a faithful μ_0-stationary normal state ϕ on M, one of the following happens:*

- *Either ϕ is Γ-invariant;*
- *Or for some compact model A of M, for every $g \in \Gamma \setminus \mathcal{Z}(\Gamma)$, the conditional states of $\phi_{|A}$ at b and at gb are singular, for almost every $b \in B$.*

In the above theorem one can show that the singularity condition appearing in the second bullet point is in fact independent of the choice of the compact model. This is based on [BBHP20, Proposition 4.10].

Although this version is less naturally phrases as the C*-version Theorem 1.15, the von Neumann algebraic setting has the advantage to allow induction from Γ-actions to G-action, which is by nature, a measurable technique.

3. Proof of the singularity theorem: reduction to G-actions

In the rest of these notes, we discuss the proof of Theorem 2.17. There are essentially two cases: the case where G is simple, and the case where Γ is an irreducible lattice in a product of at least two groups. The simple case is dealt with in [BH19], and relies on a non-commutative version of a Theorem of Nevo and Zimmer. The product case is treated in [BBHP20], it relies on irreducibility methods originating in work of Bader-Shalom [BS04]. In both cases we use induction to get to a vN-action of G.

3.1. **Induction and stationary actions.** Due to the measurable nature of von Neumann algebras and the measurable relationship between a lcsc group G and a lattice Γ in it, it is possible to induce a vN-action of Γ to a vN-action of G. The construction goes as follows.

Take a lattice $\Gamma < G$, and a Γ-action on a von Neumann algebra M. We denote by σ this action. We consider the algebra $\mathcal{M}$ of L^∞-functions $f : G \to M$, up to null sets, which are Γ-equivariant for the right Γ-action on G, in the sense that

$$f(g\gamma^{-1}) = \sigma_\gamma(f(g)), \text{ for almost every } g \in G, \text{ and every } \gamma \in \Gamma.$$

For people more familiar with von Neumann algebras, this can be realized as the fixed point algebra in $L^\infty(G) \overline{\otimes} M$ under the diagonal action $\rho \otimes \sigma$, where $\rho : \Gamma \curvearrowright L^\infty(G)$ is the right translation action. In this presentation, it is seen to be a von Neumann algebra acting on $L^2(G) \otimes H$, where $M \subset B(H)$.

Definition 3.1. The *induced action* of $\Gamma \curvearrowright M$ to G, is the action of G on $\mathcal{M}$ given by left translation: $\widetilde{\sigma}_g(f) : h \mapsto f(g^{-1}h)$, for every $g \in G$, $f \in \mathcal{M}$.

Exercise 3.2. Check that if f is equivariant as above, so is $\lambda_g(f)$. Check that this map is implemented by conjugation within $B(L^2(G) \otimes H)$ by the unitaries $\lambda_g \otimes \mathrm{id}$, where λ is the left regular representation of G on its L^2-space. Thus $\widetilde{\sigma}$ defines a vN-action $G \curvearrowright \mathcal{M}$.

Exercise 3.3. Check that the induced G-action is ergodic if and only if the initial Γ-action is ergodic.

In this process of induction, we would like to keep track of the advantageous μ_0-stationary state ϕ on M. This is possible in our setting precisely because we choose $\mu_0 \in \mathrm{Prob}(\Gamma)$ such that there exists $\mu \in \mathrm{Prob}(G)$ such that the Poisson boundary of (Γ, μ) coincides with that of (G, μ'), as Γ-spaces.

Lemma 3.4. *Assume that $\mu_0 \in \mathrm{Prob}(\Gamma)$ and $\mu \in \mathrm{Prob}(G)$ have the same Poisson boundary as Γ-spaces. Then for any vN-action $\Gamma \curvearrowright M$, with a normal μ_0-stationary state ϕ on M, the induced action $G \curvearrowright \mathcal{M}$ admits a normal μ-stationary state $\widetilde{\phi}$.*

Proof. Denote by (B, ν) the common Poisson boundary of (Γ, μ_0) and (G, μ). As we have explained in Section 2.3, the μ_0-stationary state ϕ on M gives rise to a Γ-equivariant, positive unital map $E := \mathcal{P}_\nu^{-1}\mathcal{P}_\phi : M \to \mathcal{H}_\mu(\Gamma) \to L^\infty(B, \nu)$.

Now given $f \in \mathcal{M}$, viewed as a Γ-equivariant function $f : G \to M$, the function $E \circ f : G \to L^\infty(B)$ is still Γ-equivariant. Moreover, since $L^\infty(B)$ is not only a Γ-algebra, but a G-algebra, we may twist this function $E \circ f$ to get a Γ-invariant function $g \in G \mapsto \sigma_g(E(f(g))) \in L^\infty(B, \nu)$. Viewing this function as an L^∞-function on G/Γ, we may consider its average with respect to the unique Γ-invariant probability measure λ_Γ on G/Γ:

$$\theta(f) := \int_{G/\Gamma} \sigma_g(E(f(g)))\,\mathrm{d}\lambda_\Gamma(g\Gamma) \in L^\infty(B, \nu).$$

Exercise. Using G-invariance of the measure λ_Γ, check that the map $\theta : \mathcal{M} \to L^\infty(B, \nu)$ obtained this way is G-equivariant.

Since E is G-equivariant and ν is a μ-stationary measure, we find that $\phi_\nu \circ \theta$ is μ-stationary, where ϕ_ν is the state on $L^\infty(B, \nu)$ given by ν-integration. This is the desired state $\widetilde{\phi}$. □

In the above construction, one can prove that $\widetilde{\phi}$ is G-invariant if and only if ϕ is Γ-invariant, that $\widetilde{\phi}$ is faithful on $\mathcal{M}$ if and only if ϕ is faithful.

3.2. **Simple case: Non-commutative Nevo-Zimmer theorem.** In the late 90's, early 2000's, Nevo and Zimmer wrote a series of papers regarding stationary actions of higher rank (semi-)simple Lie groups, see [NZ97, NZ00]. One of their culminating results is the following one.

Theorem 3.5 (Nevo-Zimmer)**.** *Use the notation of Section 1.3, and assume that every simple factor of G has rank at least 2. Let (X,ν) be an ergodic G-space where ν is μ-stationary.*

Then either the measure ν is G-invariant, or there exists a G-equivariant measurable map $(X,\nu) \to (G/Q,\nu_Q)$, where Q is a proper parabolic subgroup of G containing P, and $\nu_Q \in \mathrm{Prob}(G/Q)$ is the unique K-invariant measure.

In [BH19], we extend this result to the non-commutative setting.

Theorem 3.6. *Use the notation of Section 1.3, and assume that every simple factor of G has rank at least 2. Let $\mathcal{M}$ be an ergodic G-von Neumann algebra with a faithful normal μ-stationary state $\widetilde{\phi}$.*

Then either $\widetilde{\phi}$ is G-invariant, or there exists a G-equivariant C-homomorphism $C(G/Q) \to \mathcal{M}$, where Q is a proper parabolic subgroup of G containing P.*

At first glance, there are two approaches to prove the above theorem, from that of Nevo and Zimmer: we could either try to reproduce their proof in this extended setting, or we could try to do complementary work to get to a situation where we can just apply their result. In fact, the proof uses both strategies: assuming that ϕ' is not G-invariant, we find a suitable G-invariant abelian vN-subalgebra $A \subset M$ on which ϕ' is still not G-invariant, and we apply the classical result of Nevo and Zimmer in this commutative situation. But finding such an A in general is a very hard task, which is achieved in this case by mimicking parts of Nevo and Zimmer's argument.

Using stationary induction, we are actually able to push this general result to Γ-actions.

Corollary 3.7. *Use the notation of Section 1.3, and assume that every simple factor of G has rank at least 2. Let M be a Γ-von Neumann algebra with a faithful normal μ_0-stationary state ϕ.*

Then either ϕ is Γ-invariant, or there exists a Γ-equivariant C-homomorphism $C(G/Q) \to M$, where Q is a proper parabolic subgroup of G containing P.*

The advantage of this result is that the dynamics of Γ on homogeneous spaces G/Q is well understood. In particular, in the above setting, the conditional states of ϕ on the image of $C(G/Q)$ are the conditional states of the unique μ-stationary measure ν_Q, and hence are point measures. More precisely, recalling that the Poisson algebra of (Γ,μ) is $(G/P,\nu_P)$, one verifies that the conditional measure at $b = gP$, is the Dirac measure at the image point gQ. Since $\Gamma/\mathcal{Z}(\Gamma)$ acts essentially freely on G/Q, we conclude in this case that for every $g \in \Gamma \setminus \mathcal{Z}(\Gamma)$, for almost every $hP \in G/P$, the conditional states ϕ_{ghP} and ϕ_{hP} are singular. This is how one derives Theorem 2.17 in the simple case.

The rest of these notes is devoted to explain how one proves Theorem 2.17 in the case where G has at least two simple factors, and possibly rank one factors.

3.3. **Product case: the continuity algebra.** In the general semi-simple case, the possible presence of rank one factors rules out the possibility to use a strategy as in the simple case above: we cannot hope to prove a result for G actions that would imply alone something interesting for any lattice. The irreducibility assumption has to play a role. In fact, this assumption will be used to find a closer relationship between the induced G-algebra and the initial Γ-algebra.

For simplicity we assume from now on that G has only two simple factors, $G = G_1 \times G_2$ and that Γ is an irreducible lattice in G. This means that the projection of Γ in both G_1 and G_2 are dense.

We take a Furstenberg measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ and a measure $\mu \in \mathrm{Prob}(G)$ as in Section 1.3. We moreover assume as we may that μ is a product $\mu = \mu_1 \otimes \mu_2$, where $\mu_i \in \mathrm{Prob}(G_i)$ is absolutely continuous with respect to the Haar measure, with a continuous, compactly supported Radon-Nykodym derivative.

In this setting, we can prove the following result, which is the technical heart of [BBHP20]. This can be viewed as an extension of Creutz-Peterson work on the G-algebra to the stationary setting.

Theorem 3.8. *Let M be a Γ-von Neumann algebra with a faithful normal μ-stationary state. Denote by $\mathcal{M}$ the induced G-algebra, with its μ'-stationary state. For $i \in \{1,2\}$, we have a Γ-equivariant von Neumann embedding θ_i from the fixed point algebra $\mathcal{M}^{G_i}$ into M, such that $\phi \circ \theta = \phi'$.*

This theorem will be combined with the following lemma.

Lemma 3.9. *Let N be a G von Neumann algebra with a faithful normal μ'-stationary state ψ. Recall that we chose $\mu' = \mu_1 \times \mu_2$. Then ψ is μ_1 and μ_2-stationary. Moreover given distinct indices $\{i,j\} = \{1,2\}$, if ψ is not G_j-invariant, then its restriction to the fixed point algebra N^{G_i} is not G_j-invariant.*

Proof. The first fact follows from its commutative analogue. More precisely, it is known that the μ'-stationary measure on the Poisson boundary B is also μ_1 and μ_2-stationary. Now, recall that a μ'-stationary state ψ comes with a positive unital G-equivariant map $E : M \to L^\infty(B,\nu)$, and that $\psi = \nu \circ E$. This formula clearly implies that ψ is indeed μ_1 and μ_2 stationary.

To prove the moreover part, we only need to construct a G_j-equivariant map $F : N \to N^{G_i}$ which is ψ-preserving, i.e. such that $\psi = \phi \circ F$. We define F as follows. Define the convolution map $T_i : N \to N$ by the formula $T_i(x) = \int_{G_i} \sigma_g^{-1}(x)\, \mathrm{d}\mu_i(g)$, $x \in N$. This map is naturally G_j-equivariant and ψ-preserving, because ψ is μ_i-stationary.

We choose a free ultrafilter ω on $\mathbb{N}$, and using the fact that the weak topology on the unit ball of N is compact, we may define F as the weak limit:

$$F(x) = \lim_{n\to\omega} \frac{1}{n} \sum_{k=1}^{n} T_i^k(x), \text{ for all } x \in N.$$

Then F defined this way is indeed ψ-preserving and G_j-equivariant. We only need to prove that it ranges into N^{G_i}. But one easily checks that the range of F is precisely the fixed point set of T_i. Let us then check that this fixed point set is exactly N^{G_i}.

Of course a G_i-invariant element in N is T_i-invariant. Conversely, take $x \in N$ such that $T_i(x) = x$. Using the formula $\|y\|_\psi^2 = \psi(y^*y)$, for all $y \in N$, we compute

$$\int_{G_i} \|\sigma_g^{-1}(x) - x\|_\psi^2 \,\mathrm{d}\mu_i(g) = (\mu_i * \psi)(x^*x) - \psi(T_i(x^*)x) - \psi(xT_i(x)) + \psi(x^*x).$$

Since ψ is μ_i-stationary and $T_i(x) = x$ (and thus also $T_i(x^*) = x^*$), all of the four terms above are equal, and the resulting quantity is 0. Since ψ is faithful, this shows that $\sigma_g(x) = x$ for μ_i-almost every $g \in G_i$, and classical considerations from the fact that μ_i is admissible on G_i imply that x is in fact G_i-invariant. This concludes our proof. ☐

Corollary 3.10. *Let M be a Γ-von Neumann algebra with a μ_0-stationary normal faithful state ϕ. If ϕ is not Γ-invariant, then there exists an index $i \in \{1,2\}$ and a subalgebra M_i of M with the following properties:*

- *the Γ-action on M_i extends to a G-action which factors through a continuous G_i-action, via the projection map $p_i : G \to G_i$;*
- *The restriction of ϕ to M_i is not Γ-invariant, and hence it is not G_i-invariant.*

Proof. Since ϕ is not Γ-invariant, the induced μ_0-stationary state $\widetilde{\phi}$ is not G-invariant on $\mathcal{M}$. Hence it is not G_j-invariant for some index $j = 1,2$. By Lemma 3.9, we find that $\widetilde{\phi}$ is not G_j-invariant on the fixed point algebra $\mathcal{M}^{G_i}$. Now we can just apply Theorem 3.8 to get the conclusion. ☐

4. Mautner property and singularity for G-actions

In this last lecture, we use the notation of Section 1.3, but there is no lattice, and the Lie group G is simple.

Theorem 4.1. *Consider an ergodic G-action on a separable von Neumann algebra M with a normal faithful μ-stationary state ϕ.*

Then either ϕ is G-invariant, or for every compact model $A \subset M$, the conditional states $\phi_b \in \mathcal{S}(A)$, $b \in G/P$, satisfy: for every $g \in G \setminus \mathcal{Z}(G)$, for almost every $b \in B$, $\phi_{gb} \perp \phi_b$.

Exercise 4.2. Combine Theorem 4.1 with Corollary 3.10 to prove Theorem 2.17.

In fact the above statement is best proved in the C*-algebraic framework. As we explained, we can use compact models and vN-envelopes to switch between the two settings.

Theorem 4.3. *Keep the above setting. Consider a G-action on a C*-algebra A with an extremal μ-stationary state ϕ.*

Then either ϕ is G-invariant, or the conditional states $\phi_b \in \mathcal{S}(A)$, $b \in G/P$, satisfy: for every $g \in G \setminus \mathcal{Z}(G)$, for almost every $b \in B$, $\phi_{gb} \perp \phi_b$.

4.1. Mautner phenomenon.

Denote by $T \subset P$ a maximal split torus. In the case of $G = \mathrm{SL}_d(\mathbb{R})$, we may choose for T the subgroup of diagonal matrices.

Lemma 4.4. *For any continuous action of P by isometries on a metric space (E, d), any T-invariant vector is P-invariant. In particular for every $g \in G$, any vector invariant under $gPg^{-1} \cap P$ is P-invariant.*

Proof. Since this statement is familiar to experts on Lie groups, we will write the proof for non-experts, in the case where $G = \mathrm{SL}_d(\mathbb{R})$. Fix a vector $v \in E$ which is fixed under T. We want to show that v is fixed by any group element of the form $g_{i,j}(\lambda) = \mathrm{id} + \lambda E_{i,j}$, where $i < j$ and $\lambda \in \mathbb{R}$. Fix such i, j, λ, and consider the diagonal matrices $t_n \in T$, such that

$$(t_n)_{k,k} = \begin{cases} n & \text{if } k = i \\ 1/n & \text{if } k = j \\ 1 & \text{otherwise} \end{cases}$$

Then we have that $t_n^{-1} g_{i,j}(\lambda) t_n = g_{i,j}(\lambda/n^2)$, which converges to id as $n \to \infty$. Since v is T invariant, we find

$$d(v, g_{i,j}(\lambda)v) = \lim_n d(t_n v, g_{i,j}(\lambda) t_n v) = \lim_n d(v, t_n^{-1} g_{i,j}(\lambda) t_n v) = 0.$$

Thus, v is fixed by $g_{i,j}(\lambda)$ for every i, j, λ, as desired.

Let us now discuss the second part of the statement. It is based on the fact that any two conjugates P and gPg^{-1} contain a common conjugate of T. This fact is related to the Bruhat decomposition, and in our SL_d example, it can be viewed in terms of flags. We define a *flag* on $\mathbb{R}^d$ as an increasing (strictly) sequence of nonzero subspaces $E_1 \subset E_2 \subset \cdots \subset E_n$ of $\mathbb{R}^d$. A full flag is by definition a maximal flag, i.e. $n = d$, and $\dim(E_i) = i$ for every i. Since G acts linearly on $\mathbb{R}^d$, it acts on the set of all full flags of $\mathbb{R}^d$, this action is transitive, and P is the stabilizer group of exactly one full flag.

Exercise. Assume that $(E_1, \ldots, E_d)$ and $(E'_1, \ldots, E'_d)$ are two full flags of $\mathbb{R}^d$.

a) Prove that there exists a basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ and a permutation $\alpha \in \mathcal{S}_d$ such that $E_i = \mathrm{span}(e_1, \ldots, e_i)$ and $E'_i = \mathrm{span}(e_{\alpha(1)}, \ldots, e_{\alpha(i)})$, for every $1 \le i \le d$.
b) Deduce that the torus T_0 consisting of matrices that are diagonal in the basis $(e_1, \ldots, e_d)$ fixes both flags.
c) Conclude that for every $g \in G$, there exists $h \in P$ such that $hTh^{-1} \subset P \cap gPg^{-1}$, and complete the proof of the lemma. □

4.2. P-invariant states.

Lemma 4.5. *Keep the notation G, P as above. Let A be a G-C^*-algebra with an extremal P-invariant state ψ. Then for every $g \in G$, we have either $g\psi = \psi$ or $g\psi \perp \psi$.*

Proof. We want to use Mautner phenomenon, and the fact that the action of P on the dual A^* is by isometries. However, this action is not norm continuous in general, precisely because the action $P \curvearrowright A^{**}$ is not a continuous vN-action. But we mentioned that if p denotes the support projection of ψ, then the action $P \curvearrowright pA^{**}p$ is continuous. In fact this is also true if instead of p we consider the central support z of ψ, which is by definition the support of the restriction of ψ to $\mathcal{Z}(A^{**})$. Note that $p \leq z$. It will be more convenient for us to work with $z \in \mathcal{Z}(A^{**})$. Denote by zA^* the set of continuous linear functionals α on A whose normal extension to A^{**} satisfies

$$\alpha(x) = \alpha(zx), \text{ for every } x \in A.$$

Then it can be checked that the action $P \curvearrowright zA^*$ is norm continuous (and isometric).

Fix now $g \in G$, and consider the element $\psi_g \in A^*$ such that $\psi_g(x) = \psi(\sigma_g(z)x)$, for every $x \in A$. Since by definition $\psi \in zA^*$, we have $\psi_g(x) = \psi(z\sigma_g(z)x) = \psi(\sigma_g(z)zx) = \psi_g(zx)$, and thus $\psi_g \in zA^*$. Since $\sigma_g(z)$ is gPg^{-1}-invariant and ψ is P-invariant, ψ_g is fixed by $P \cap gPg^{-1}$. By Lemma 4.4, ψ_g must actually be P-invariant. Moreover, since $z \in \mathcal{Z}(A^{**})$, we have

$$\psi_g(x^*x) = \psi(x^*\sigma_g(z)x) \leq \psi(x^*x), \text{ for all } x \in A.$$

So ψ_g is a positive linear functional which is P-invariant and dominated by ψ. Since ψ is extremal, this implies that ψ_g is proportional to ψ. Two cases may occur: $\psi_g = 0$ or $\psi_g \neq 0$.

Exercise. Check that the central support of ψ_g is $z\sigma_g(z)$.

If $\psi_g = 0$, then $z\sigma_g(z) = 0$ and in particular p and $\sigma_g(p)$ are orthogonal, which means $\psi \perp g\psi$. Otherwise, ψ and ψ_g have the same central support: $z = z\sigma_g(z)$. This implies that $z \leq \sigma_g(z)$.

Note that $\psi \perp g\psi$ if and only if $\psi \perp g^{-1}\psi$, and $\psi = g\psi$ if and only if $\psi = g^{-1}\psi$. So assuming that ψ is not singular to $g\psi$, the above argument applied to g and g^{-1} gives that in fact $z = \sigma_g(z)$. In this case, ψ may be viewed as a linear functional in $\sigma_g(z)A^{**}$, on which gPg^{-1} acts continuously, and in this case ψ is $P \cap gPg^{-1}$-invariant. Applying Lemma 4.4, we find that actually ψ must be gPg^{-1}-invariant.

But we can do better: once we know that $\sigma_g(z) = z$, we find that $\sigma_g^k(z) = z$ for every $k \in \mathbb{Z}$. Then the same argument as above applied to g^k implies that ψ is in fact invariant under g^kPg^{-k}. We conclude that ψ is fixed by the group Q defined as the closure of $\bigvee_{k\in\mathbb{Z}} g^kPg^{-k}$. Now we use a little bit of structure of the parabolics: it is known that every closed group containing P is equal to its normalizer. Since Q is normalized by g, we find that $g \in Q$, and thus $g\psi = \psi$, as desired. □

4.3. Conclusion.

Lemma 4.6. *Let G be a connected simple Lie group, and $H < G$ a proper closed subgroup. Endow G/H with its unique invariant measure class. Then for every $g \in G \backslash \mathcal{Z}(G)$, for almost every $x \in G/H$, $gx \neq x$.*

Proof. Take $g \in G \setminus \mathcal{Z}(G)$. We need to show that the fixed point set $\mathrm{Fix}(g)$ has measure 0 in G/H. Note that $\mathrm{Fix}(g)$ is a submanifold of G/H, and it is a proper subset. Indeed otherwise g would be contained in every conjugate of H, but the intersection $\bigcap_{y\in G} yHy^{-1}$ is a proper normal closed subgroup of G, hence contained in $\mathcal{Z}(G)$ by simplicity. This would force $g \in \mathcal{Z}(G)$, which we excluded.

So $\mathrm{Fix}(g)$ is a proper subvariety of G/H. Since G/H is connected, each connected component of $\mathrm{Fix}(g)$ must have smaller dimension than G/H. So $\mathrm{Fix}(g)$ is a null set in G/H. □

We can now prove Theorem 4.3, from which all of our main results follow.

Proof of Theorem 4.3. As we observed, the data of the μ-stationary state ϕ is the same as the data of the measurable G-map $G/P \to \mathcal{S}(A)$. Such a G-map θ can be modified on a null set if necessary, to assume that it is truly G-equivariant: $\theta_{gb} = g\theta_b$ for every $g \in G$, $b \in B$. In turn, the data of this map is equivalent to the data of $\psi := \theta_P \in \mathcal{S}(A)$, which is a P-invariant state. Note that these changes of points of view are affine and continuous. Since ϕ is extremal, we thus find that the corresponding P-invariant state ψ is extremal. Denote by Q, the stabilizer of ψ in G. From lemma 4.5 we have that $g\psi \perp h\psi$ whenever gQ and hQ are distinct in G/Q.

If ϕ is not G-invariant, then Q is a proper closed subgroup of G. By Lemma 4.6, for every $g \in G\backslash\mathcal{Z}(G)$, for almost every $b = hP \in G/P$, we then have $ghQ \neq hQ$. Further, $\theta_b = h\psi$ is singular with respect to $\theta_{gb} = gh\psi$. This is the desired dichotomy. $\square$

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