Exposé ANR TRECOS

Contrôlabilité à zéro de l'équation de la chaleur perturbée par un terme non local

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Null-controllability of the nonlocal heat equation

- $T > 0$: control time,
- $\Omega \subset\subset \mathbb{R}^N$: spatial domain,
- $\omega \subset \Omega$: open control set,
- $K \in L^\infty((0,T) \times \Omega \times \Omega)$: kernel.

$$\begin{cases} \partial_t y - \Delta y + \int_\Omega K(t,x,\xi) y(t,\xi) d\xi = h 1_\omega & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(NonLocal)}$$

Question: Is the equation (NonLocal) small-time null-controllable?
For every $T > 0$, $y_0 \in L^2(\Omega)$, does there exists $h \in L^2((0,T) \times \omega)$ such that the solution y of (NonLocal) satisfies $y(T,\cdot) = 0$?

When $K \equiv 0$, the heat equation is small-time null-controllable (Lebeau, Robbiano, Fursikov, Imanuvilov, 1995-96).

Observability inequality

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \int_\Omega K(t,x,\xi)\varphi(t,\xi)d\xi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases} \tag{Adjoint}$$

Proposition (H.U.M.)

Null-controllability of (NonLocal) *at time* $T > 0 \Leftrightarrow$ *there exists* $C > 0$ *such that for every* $\varphi_T \in L^2(\Omega)$, *the solution* φ *of* (Adjoint) *satisfies*

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \int_0^T \|\varphi(t,\cdot)\|^2_{L^2(\omega)} dt. \tag{Observability}$$

Global Carleman estimates with lower order perturbations

$Q_T = (0, T) \times \Omega$.

Proposition (Fursikov, Imanuvilov, 1996)

For $s \geq s_0$, $\varphi \in C^2(\overline{Q_T})$, $\varphi \equiv 0$ on Σ_T,

$$s^3 \int_{Q_T} e^{-2s\alpha} \zeta^3 |\varphi|^2 \leq C \left(\int_{Q_T} e^{-2s\alpha} |-\partial_t \varphi - \Delta \varphi|^2 + s^3 \int_{(0,T)\times\omega} e^{-2s\alpha} \zeta^3 |\varphi|^2 \right).$$

For φ, solution of (Adjoint), we have

$$s^3 \int_{Q_T} e^{-2s\alpha} \zeta^3 |\varphi|^2 \leq C \Bigg(\int_{Q_T} e^{-2s\alpha} \left| \int_{\Omega} K(t,x,\xi)\varphi(t,\xi) d\xi \right|^2 dt dx + s^3 \int_{(0,T)\times\omega} e^{-2s\alpha} \zeta^3 |\varphi|^2 \Bigg).$$

One **cannot absorb the nonlocal term** by the left hand side, taking s sufficiently large.

A counterexample to unique continuation

$(e_k, \lambda_k)_{k\geq 1}$ such that $-\Delta e_k = \lambda_k e_k$, $e_k \in H^2 \cap H_0^1(\Omega)$.
$\varphi \neq 0 \in C_c^\infty(\Omega)$, such that $\varphi \equiv 0$ in ω, $\varphi = \sum_{k\geq 1} \varphi_k e_k(x)$ and $\sum_{k\geq 1} \lambda_k \varphi_k^2 = 1$.

$$p := -\Delta\varphi \text{ i.e. } p := \sum_{k\geq 1} \lambda_k \varphi_k e_k(x).$$

In particular, we have

$$\int_\Omega p\varphi = \sum_{k\geq 1} \lambda_k \varphi_k^2 = 1,$$

so defining $K(x,\xi) = -p(x)p(\xi)$, we have

$$-\Delta\varphi + \int_\Omega K(x,\xi)\varphi(\xi)d\xi = -\Delta\varphi - p(x) = -\Delta\varphi + \Delta\varphi = 0.$$

Recalling $\varphi \equiv 0$ in ω, the unique continuation does not hold.

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Analyticity assumptions on the kernel

$K = K(x, \xi)$,

$$\begin{cases} -\partial_t \varphi - \Delta\varphi + \int_\Omega K(x,\xi)\varphi(t,\xi)d\xi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(Adjoint)}$$

Theorem (Fernandez-Cara, Lü, Zuazua, 2016)

Assume that

- $x \mapsto \int_\Omega K(x,\xi)f(\xi)d\xi$ *is analytic for every* $f \in L^2(\Omega)$,
- $K(x,\xi) = \sum_{m,j\geq 1} k_{mj} e_m(x) e_j(\xi)$ *with* $\sum_{m,j\geq 1} \lambda_j^{-1}\lambda_m^{-1} e^{2C\sqrt{\lambda_m}} |k_{mj}|^2 < +\infty$,

then

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \int_0^T \|\varphi(t,\cdot)\|^2_{L^2(\omega)} dt. \qquad \text{(Observability)}$$

Proof. Compactness-uniqueness argument.

Extension to parabolic systems by Lissy, Zuazua (2018).

Exponential decay in time of the kernel

$K = K(t, x, \xi)$,

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \int_\Omega K(t,x,\xi)\varphi(t,\xi)d\xi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(Adjoint)}$$

Theorem (Biccari, Hernandez-Santamaria, 2019)

Assume that

- $\sup_{(t,x)\in Q_T} \exp\left(\frac{C}{t(T-t)}\right) \int_\Omega |K(t,x,\xi)| d\xi < +\infty$,

then

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \int_0^T \|\varphi(t,\cdot)\|^2_{L^2(\omega)} dt. \qquad \text{(Observability)}$$

Proof. Carleman estimates.

A one-dimensional result

$N = 1$, $K = K(x, \xi) = \alpha(x)\beta(\xi)$.

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \alpha(x) \int_\Omega \beta(\xi)\varphi(t,\xi)d\xi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(Adjoint)}$$

Theorem (Micu, Takahashi, 2018)

Assume that

- *α is not identically zero in ω,*

then

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \int_0^T \|\varphi(t,\cdot)\|^2_{L^2(\omega)}\, dt. \qquad \text{(Observability)}$$

Proof. Spectral analysis and biorthogonal techniques.

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Nonlocal semilinear heat equation

$$\begin{cases} \partial_t y - \Delta y \\ = f\left(y(t,x), \fint_\Omega y(t,\xi)\,\mathrm{d}\xi\right) + h 1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(NonLocalSL)}$$

Theorem (Hernandez-Santamaria, Le Balc'h, 2020)

Take $a, b \in \mathbb{R},\ g_1 \in C_c^\infty(\mathbb{R}^2),\ g_2 \in C_c^\infty(\mathbb{R})$. *Assume that*

$$f(u,v) = au + bv + g_1(u,v)u^2 + g_2(u)uv.$$

Then, (NonLocalSL) *is small-time locally null-controllable:*
For every $T > 0$, *there exists* $\delta > 0$, *such that for every* $y_0 \in H_0^1(\Omega)$ *satisfying* $\|y_0\|_{H_0^1(\Omega)} \leq \delta$, *there exists* $h \in L^2((0,T)\times\omega)$ *such that the (unique) solution* y *of* (NonLocalSL) *verifies* $y(T,\cdot) = 0$.

Remarks on the first main result

$$\begin{cases} \partial_t y - \Delta y \\ = f\left(y(t,x), \fint_\Omega y(t,\xi)\,\mathrm{d}\xi\right) + h1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases} \qquad \text{(NonLocalSL)}$$

with $f(u,v) = au + bv + g_1(u,v)u^2 + g_2(u)uv$.
(NonLocalSL) is small-time locally null-controllable in $H_0^1(\Omega)$.

- $f(0,0) = 0$ ensures that 0 is a stationary state.
- f is globally Lipschitz so (NonLocalSL) is well-posed (Hilhorst, Rodrigues, 2000).
- If $g_1 = g_2 = 0$ then f is linear ($\Rightarrow$ global null-controllability). Extension of Micu, Takahashi's result to constant kernels to the multi-dimensional case.
- Asymmetry between the local variable y and the nonlocal variable $\fint y$.
- Due to regularizing effect, one can prove the result a small-time local null-controllability result in $L^2(\Omega)$.
- Ex: $y(t,x)$ the density at time t, depending on a physiological parameter $x \in \Omega$, B the birth rate and $-\fint y$ the death rate, $f(y, \fint y) = \chi(y)y(B - \fint y)$.

Shadow reaction-diffusion system

Consider the 2×2 reaction-diffusion system

$$\begin{cases} \partial_t u - \Delta u = f(u,v) + h1_\omega & \text{in } (0,T)\times\Omega, \\ \tau\partial_t v - \sigma\Delta v = u - v & \text{in } (0,T)\times\Omega, \\ u = \frac{\partial v}{\partial n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0,v_0) & \text{in } \Omega, \end{cases} \tag{System}$$

where τ, σ are parameters in $(0,+\infty)$.

Theorem (Hernandez-Santamaria, Le Balc'h, 2020)

(System) *is small-time locally null-controllable, uniformly with respect to* $\tau \to 0$, $\sigma \to +\infty$*:*

For every $T > 0$*, there exists* $C,\ \delta > 0$ *such that for any* $\tau \in (0,1)$, $\sigma \in (1,+\infty)$, $(u_0,v_0) \in H_0^1(\Omega) \times H^1(\Omega)$ *such that* $\|(u_0,v_0)\|_{H_0^1(\Omega)\times H^1(\Omega)} \leq \delta$*, there exists* $h_{\tau,\sigma} \in L^2((0,T)\times\omega)$ *verifying*

$$\|h_{\tau,\sigma}\|_{L^2((0,T)\times\omega)} \leq C,$$

such that the unique solution $(u,v)_{\tau,\sigma}$ *of* (System) *satisfies* $(u,v)_{\tau,\sigma}(T,\cdot) = 0$.

Remarks on the second main result

$$\begin{cases} \partial_t u - \Delta u = f(u,v) + h 1_\omega & \text{in } (0,T)\times\Omega, \\ \tau \partial_t v - \sigma \Delta v = u - v & \text{in } (0,T)\times\Omega, \\ u = \frac{\partial v}{\partial n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0, v_0) & \text{in } \Omega. \end{cases} \quad \text{(System)}$$

(System) is small-time locally null-controllable in $H_0^1(\Omega) \times H^1(\Omega)$, uniformly with respect to $\tau \to 0$ and $\sigma \to \infty$.

- h directly controls u in the first equation, u indirectly controls v in the second equation through the coupling term u.
- Main difficulty: uniformly with respect to $\tau \to 0$ and $\sigma \to \infty$.
- Same result holds with homogeneous Dirichlet boundary conditions on u, v.
- Homogeneous Neumann boundary conditions on u, v?

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Strategy

$$\begin{cases} \partial_t u - \Delta u = f(u,v) + h 1_\omega & \text{in } (0,T)\times\Omega, \\ \tau \partial_t v - \sigma \Delta v = u - v & \text{in } (0,T)\times\Omega, \\ u = \frac{\partial v}{\partial n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0, v_0) & \text{in } \Omega. \end{cases} \qquad \text{(System)}$$

- Uniform null-controllability of (System) with respect to $\tau \to 0$, $\sigma \to +\infty$.
  - Linearization of (System) around $(0,0)$.
  - Uniform null-controllability of the linearized system thanks to an uniform observability inequality for the adjoint system.
  - (Uniform) source term method by Liu, Takahashi, Tucsnak (2013).
- In the limit $\tau \to 0$, $\sigma \to \infty$,

$$(u_{\tau,\sigma}, v_{\tau,\sigma}, h_{\tau,\sigma}) \to (y, \fint_\Omega y, h) \text{ solution of}$$

$$\partial_t y - \Delta y = f\left(y(t,x), \fint_\Omega y(t,\xi)\, d\xi \right) + h 1_\omega.$$

This comes from an adaptation of Hilhorst, Rodrigues' arguments.

Uniform null-controllability for the linearized system

$$\begin{cases} \partial_t u - \Delta u = au + bv + h1_\omega & \text{in } (0,T)\times\Omega, \\ \tau\partial_t v - \sigma\Delta v = cu + dv & \text{in } (0,T)\times\Omega, \\ u = \dfrac{\partial v}{\partial n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0,v_0) & \text{in } \Omega. \end{cases} \tag{SystemL}$$

Proposition (Hernandez-Santamaria, Le Balc'h, 2020)

Let $(a,b,c,d) \in \mathbb{R}^2 \times \mathbb{R}^* \times (-\infty, 0)$. For every $T \in (0,1)$, $\tau \in (0,1)$, $\sigma \in (1,+\infty)$, $(u_0,v_0) \in L^2(\Omega)^2$, there exists $h \in L^2((0,T)\times\omega)$

$$\|h\|_{L^2((0,T)\times\omega)} \leq C\exp\left(\frac{C}{T}\right)\left(\|u_0\|_{L^2(\Omega)} + \sqrt{\tau}\,\|v_0\|_{L^2(\Omega)}\right), \tag{Cost}$$

such that the solution (u,v) of (SystemL) *verifies $(u,v)(T,\cdot) = 0$.*

- $c \neq 0$ is a necessary condition.
- Uniform null-controllability, with respect to the parameter $\sigma \in (1,+\infty)$, has already been proved by the Hernandez-Santamaria, Zuazua (2019).

Uniform observability inequality for the adjoint system

Proposition (Hernandez-Santamaria, Le Balc'h, 2020)

Let $(a, b, c, d) \in \mathbb{R}^2 \times \mathbb{R}^* \times (-\infty, 0)$. *For every* $T > 0$, $(\tau, \sigma) \in (0, 1) \times (1, +\infty)$, $(\phi_T, \psi_T) \in L^2(\Omega)^2$, *we have*

$$\|\phi(0,\cdot)\|_{L^2(\Omega)}^2 + \tau \|\psi(0,\cdot)\|_{L^2(\Omega)}^2 \leq C \exp\left(\frac{C}{T}\right) \iint_{\omega \times (0,T)} |\phi(t,x)|^2 \mathrm{d}t\, \mathrm{d}x. \quad \text{(Obs)}$$

where (ϕ, ψ) *is the solution of the adjoint system*

$$\begin{cases} -\partial_t \phi - \Delta \phi = a\phi + c\psi & \text{in } (0,T) \times \Omega, \\ -\tau \partial_t \psi - \sigma \Delta \psi = b\phi + d\psi & \text{in } (0,T) \times \Omega, \\ \phi = \dfrac{\partial \psi}{\partial n} = 0, & \text{on } (0,T) \times \partial\Omega, \\ (\phi, \psi)(T, \cdot) = (\phi_T, \psi_T) & \text{in } \Omega. \end{cases} \quad \text{(Adjoint)}$$

Carleman estimate for a 4×4 reaction-diffusion system

Following Chaves-Silva, Guerrero, Puel (2014),

$$\begin{cases} w := -\frac{\tau}{\sigma}\partial_t \phi - \Delta\phi - \frac{d}{\sigma}\phi & \text{in } Q_T, \\ -\partial_t w - \Delta w - aw = \frac{cb}{\sigma}\phi & \text{in } Q_T, \\ -\partial_t \phi - \Delta \phi = a\phi + c\psi & \text{in } Q_T, \\ -\tau\partial_t \psi - \sigma\Delta\psi = b\phi + d\psi & \text{in } Q_T, \\ w = \phi = \frac{\partial \psi}{\partial n} = 0 & \text{on } \Sigma_T, \\ (w,\phi,\psi)(T,\cdot) = (-\Delta\phi_T - \frac{d}{\sigma}\phi_T, \phi_T, \psi_T) & \text{in } \Omega. \end{cases} \quad \text{(Adjoint)}$$

- Carleman inequality for the second, third and fourth equations of (Adjoint) $\Rightarrow$ local terms in w, ϕ and ψ.
- Using the first equation in (Adjoint), local estimate of w.
- Using the third equation in (Adjoint), local estimate of ψ in terms of local integrals of the variables ϕ and $\partial_t \phi$ and some lower order terms.
- Estimate the local term of $\partial_t \phi$ by means of weighted estimates $\Rightarrow$ local integral of ϕ and several lower order terms. Use of $d < 0$ and $\phi = 0$ on Σ_T.
- Combining the different estimates.

Energy estimates for the reaction-diffusion system

$$\begin{cases} \partial_t u - \Delta u = au + bv + F & \text{in } (0,T)\times\Omega, \\ \tau\partial_t v - \sigma\Delta v = u - v & \text{in } (0,T)\times\Omega, \\ u = \dfrac{\partial v}{\partial n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0,v_0) & \text{in } \Omega. \end{cases}$$

We have

$$\begin{aligned} &\|u\|_{C([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))} + \sqrt{\tau}\,\|v\|_{C([0,T];L^2(\Omega))} + \|v\|_{L^2(0,T;H^1(\Omega))} \\ &\leq C\left(\|u_0\|_{L^2(\Omega)} + \sqrt{\tau}\,\|v_0\|_{L^2(\Omega)} + \|F\|_{L^2((0,T)\times\Omega)}\right). \end{aligned}$$

Proof. Multiply the first equation by u, the second equation by v, integrate in Ω,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2 \leq C\|u(t,\cdot)\|_{L^2}^2 + \|F(t,\cdot)\|_{L^2}^2 + \frac{1}{4}\|v(t,\cdot)\|_{L^2}^2, \\ &\frac{\tau}{2}\frac{d}{dt}\|v(t,\cdot)\|_{L^2}^2 + \sigma\|\nabla v(t,\cdot)\|_{L^2}^2 + \|v(t,\cdot)\|_{L^2}^2 \leq C\|u(t,\cdot)\|_{L^2}^2 + \frac{1}{4}\|v(t,\cdot)\|_{L^2}^2. \end{aligned}$$

Sum, integrate in time then Gronwall's estimate.

Uniform source term method

$$\begin{cases} \partial_t u - \Delta u = au + bv + S + h1_\omega & \text{in } (0,T)\times\Omega, \\ \tau\partial_t v - \sigma\Delta v = cu + dv & \text{in } (0,T)\times\Omega, \\ u = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0,v_0) & \text{in } \Omega. \end{cases} \qquad \text{(SystemS)}$$

Proposition

For every $S \in \mathcal{S}$ and $(u_0, v_0) \in H_0^1(\Omega) \times H^1(\Omega)$, there exists an unique control h of minimal norm in $\mathcal{H}$, such that the solution (u, v) of (SystemS) satisfies

$$\frac{(u,v)}{\rho} \in H^1(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2) \cap C([0,T];H_0^1(\Omega)\times H^1(\Omega)).$$

Moreover, the following estimate holds

$$\begin{aligned} &\|u/\rho\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))\cap C([0,T];H_0^1(\Omega))} \\ &+ \sqrt{\tau}\,\|v/\rho\|_{H^1(0,T;L^2(\Omega))\cap C([0,T];H^1(\Omega))} + \|v/\rho\|_{L^2(0,T;H^2(\Omega))} + \|h\|_{\mathcal{H}} \\ &\leq C\exp(C/T)\left(\|u_0\|_{H^1(\Omega)} + \sqrt{\tau}\,\|v_0\|_{H^1(\Omega)} + \|S\|_{\mathcal{S}}\right). \end{aligned}$$

From the 2×2 system to the nonlocal equation

$$\begin{cases} \partial_t u - \Delta u = f\,(u, v) + h 1_\omega \\ \tau \partial_t v - \sigma \Delta v = u - v \\ u = \frac{\partial v}{\partial n} = 0. \end{cases}$$

In the limit $\tau \to 0$, $\sigma \to \infty$,

$$\begin{cases} \partial_t y - \Delta y = f(y, \fint y) + h 1_\omega, \\ y = 0. \end{cases}$$

Proposition

Let $(u_0, v_0) \in H_0^1(\Omega) \times H^1(\Omega)$. Assume that $h_{\tau,\sigma} \rightharpoonup h$ in $L^2((0,T) \times \omega)$. Then, up to a subsequence, as $\tau \to 0$, $\sigma \to \infty$,

$$\begin{aligned} \partial_t u_{\tau,\sigma} &\rightharpoonup \partial_t y \text{ in } L^2(Q_T), \\ u_{\tau,\sigma} &\rightharpoonup^* y \text{ in } L^\infty(0,T; H_0^1(\Omega)), \\ v_{\tau,\sigma} &\to \fint_\Omega y \text{ in } L^2(0,T; H^1(\Omega)). \end{aligned}$$

Proof of the asymptotic behavior - 1

Energy estimate and maximal regularity give

$$\|u_{\tau,\sigma}\|_{C([0,T];H_0^1(\Omega))} + \|\partial_t u_{\tau,\sigma}\|_{L^2(0,T;L^2(\Omega))} + \|v_{\tau,\sigma}\|_{L^2(Q_T)} + \sqrt{\sigma}\,\|\nabla v_{\tau,\sigma}\|_{L^2(Q_T)} \leq C.$$

So, in the limt $\tau \to 0$, $\sigma \to \infty$,

$$u_{\tau\sigma} \rightharpoonup^* u \text{ in } L^\infty(0,T;H_0^1(\Omega)),\ \partial_t u_{\tau,\sigma} \rightharpoonup \partial_t u \text{ in } L^2(Q_T),$$

$$v_{\tau\sigma} \rightharpoonup v \text{ in } L^2(0,T;H^1(\Omega)),\ |\nabla v_{\tau\sigma}|_{L^2(Q_T)} \to 0 \ \Rightarrow \left\| v_{\tau,\sigma} - \fint_\Omega v_{\tau\sigma} \right\|_{L^2(Q_T)} \to 0.$$

Integrating in space the second equation, we find

$$\tau \frac{d}{dt} \fint_\Omega v_{\tau\sigma}(t,x)dx = \fint_\Omega u_{\tau\sigma}(t,x)dx - \fint_\Omega v_{\tau\sigma}(t,x)dx \text{ in } (0,T).$$

So setting $\xi(t) = \fint_\Omega v_{\tau\sigma}(t,x)dx$, $\zeta(t) = \fint_\Omega u_{\tau\sigma}(t,x)dx$, we have

$$\tau\dot{\xi}(t) = \zeta(t) - \xi(t).$$

We have

$$\|\zeta - \xi\|_{L^2(0,T)} \leq C\sqrt{\tau}.$$

Proof of the asymptotic behavior - 2

We multiply $\tau\dot{\xi}(t) = \zeta(t) - \xi(t)$ by $|\xi|^{p-2}\xi$

$$\frac{\tau}{p}|\xi(t)|^p - \frac{\tau}{p}|\xi(0)|^p + \int_0^T |\xi(s)|^p \, \mathrm{d}s = \int_0^T \zeta(s)|\xi(s)|^{p-2}\xi(s)\, \mathrm{d}s.$$

Young's inequality with the conjugate exponents $p, p/(p-1)$

$$\frac{\tau}{p}|\xi(t)|^p + \frac{1}{p}\int_0^T |\xi(s)|^p \, \mathrm{d}s \leq \frac{1}{p}\int_0^T |\zeta(s)|^p \, \mathrm{d}s + \frac{\tau}{p}|\xi(0)|^p.$$

So we get $\|\xi\|_{L^p((0,T))} \leq \|\zeta\|_{L^p((0,T))} + \tau^{1/p}|\xi(0)|$ then

$$\|\xi\|_{L^\infty(0,T)} \leq |\xi(0)| + \|\zeta\|_{L^\infty(0,T)}.$$

Integrating by parts give

$$\int_0^T (\zeta(t) - \xi(t))^2 \, dt = \tau \int_0^T \dot{\xi}\,(\zeta(t) - \xi(t))\, dt$$
$$\leq C_{0,T}\tau + \tau \left\|\dot{\zeta}\right\|_{L^1(0,T)} \|\xi\|_{L^\infty(0,T)}.$$

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Nonlocal diffusion and nonlocal coupling

$$\begin{cases} \partial_t y - d(\fint_\Omega y)\Delta y = f(y, \fint_\Omega y) + h 1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases} \qquad \text{(NonLocal)}$$

d, f are smooth and $0 < m \leq d(s) \leq M$, $f(0,0) = 0$.

- **Question.** Local null-controllability of (NonLocal)?

See for instance Clark, Fernandez-Cara, Limaco, Medeiros for $f = 0$.

Extension of the shadow system method

$K = K(t, \xi)$,

$$\begin{cases} \partial_t y - \Delta y = \int_\Omega K(t,\xi) y(t,\xi) d\xi + h 1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \qquad \text{(NonLocal)}$$

Assume that $K \neq 0$ in $(t_1, t_2) \times \omega$,

$$\begin{cases} \partial_t u - \Delta u = v + h 1_\omega & \text{in } (0,T)\times\Omega, \\ \tau \partial_t v - \sigma \Delta v = K(t,x) u - v & \text{in } (0,T)\times\Omega, \\ u = \dfrac{\partial v}{\partial n} = 0, & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,\cdot) = (u_0, v_0) & \text{in } \Omega. \end{cases} \qquad \text{(SystemL)}$$

- **Question:** Uniform null-controllability of (SystemL) with respect to $\tau \to 0$ and $\sigma \to \infty$?
- **Question:** (SystemL) $\to$ (NonLocal) as $\tau \to 0$, $\sigma \to \infty$?

Coupled parabolic systems with nonlocal terms

$K = K \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Consider the linear reaction-diffusion system

$$\begin{cases} \partial_t U - \Delta U = K \int_\Omega U(t,\xi)d\xi + Bh1_\omega & \text{in } (0,T)\times\Omega, \\ U = 0 & \text{on } (0,T)\times\partial\Omega, \\ U(0,\cdot) = U_0 & \text{in } \Omega. \end{cases} \qquad \text{(NonLocal)}$$

Open problem.
Null-controllability of (NonLocal) $\Leftrightarrow$ rank$(B, KB, \ldots, K^{n-1}B) = n$.

Example:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - \Delta u_2 = K \int_\Omega u_1(t,\xi)d\xi & \text{in } (0,T)\times\Omega. \end{cases}$$

Question: Uniform null-controllability of

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h1_\omega & \text{in } (0,T)\times\Omega, \\ \tau\partial_t \tilde{u}_1 - \sigma\Delta\tilde{u}_1 = u_1 - \tilde{u}_1 & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - \Delta u_2 = K\tilde{u}_1 & \text{in } (0,T)\times\Omega. \end{cases}$$

with respect to $\tau \to 0$, $\sigma \to \infty$?

Other questions

- Other nonlinear systems.
- Other boundary conditions.
- Carleman estimates to deal with nonlocal terms.
- Boundary control system with nonlocal terms.
- Systems with nonlocal terms of higher order, i.e.

$$\partial_t y - \Delta y = \int_\Omega K(t,x,\xi)\Delta y(t,\xi)d\xi + h1_\omega.$$