Control with state constraints

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ANR TRECOS

General Problem

Let us consider the finite dimensional control problem:

$$\dot{y} = Ay + Bu, \qquad (\star)$$

with $y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$.
It is well-known that if this system is controllable (i.e. rk $\left(B, AB, \ldots, A^{n-1}B\right) = n$), then for every $y^0, y^1 \in \mathbb{R}^n$ and every $T > 0$, there exists a control u steering the solution of $(\star)$ from y^0 to y^1 in time T.

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Let us now add the constraint $y(t) \in C$ with C a subset of $\mathbb{R}^n$ of nonempty interior.

Question

1. Given $y^0 \in C$, what is the set of reachable points $y^1 \in C$?
2. If y^1 can be reached from y^0, can it be done in arbitrarily small time?

First definitions

Remark

If $\operatorname{rk} B = n$, then for every $y^0, y^1 \in C$ and every $T > 0$, the solution of $(\star)$ can be steered from y^0 to y^1 in time T, and $y(t) \in C$ for every $t \in [0, T]$.

In the sequel, we will assume $m = 1$ (and $B = b \in \mathbb{R}^n$).

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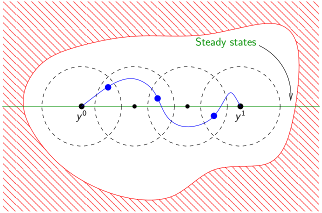
Definition

A point $\bar{y} \in \mathbb{R}^n$ is a steady state if there exists $\bar{u} \in \mathbb{R}^m$ such that $A\bar{y} + b\bar{u} = 0$.

Proposition (Controllability between steady points)

Assume that y^0 and y^1 are two steady states and assume for every $\tau \in [0, 1]$, $(1-\tau)y^0 + \tau y^1$ is in the interior of C. Then there exists a time $T > 0$ large enough such that y^0 can be steered to y^1 in time T.

Proof: Small time local controllability, and compactness.



Minimal controllability time

Let us then define:

$$T_C(y^0,y^1) = \inf\left\{ T>0,\ \exists u \in L^\infty(0,T),\ \middle|\ \begin{array}{l} y(T;u;y^0)=y^1, \\ y(t;u;y^0)\in C \quad (t\in[0,T]) \end{array} \right\},$$

with $y(t;u;y^0)$ the solution of : $\dot{y} = Ay + bu, \quad y(0)=y^0$.

Proposition

1. *There do not exist controls* $u \in L^\infty(0, T_C(y^0,y^1))$ *steering* y^0 *to* y^1 *in time* $T_C(y^0,y^1)$*;*
2. $T_C(y^0,y^1) = T_{\overline{C}}(y^0,y^1) \longrightarrow$ *we assume C closed;*
3. *For* $M>0$*, set :*

$$T_C^M(y^0,y^1) = \inf\left\{ T>0,\ \exists u \in L^\infty(0,T),\ \middle|\ \begin{array}{l} |u(t)|\leqslant M \quad (t\in[0,T]), \\ y(T;u;y^0)=y^1, \\ y(t;u;y^0)\in C \quad (t\in[0,T]) \end{array} \right\}.$$

Then,

$$\lim_{M\to\infty} T_C^M(y^0,y^1) = T_C(y^0,y^1).$$

Brunovsky form

We can rewrite the system under Brunovsky form, i.e., there exists $Q \in M_n(\mathbb{R})$ and $K \in \mathbb{R}^n$ such that $z = Qy$ and $v = K^\top y + u$ satisfies:

$$\dot{z} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} v = \mathbf{A}_n z + \mathbf{b}_n v. \qquad (\star')$$

The constraint $x \in C$ becomes $z \in QC$ and steady states of $(\star')$ are:

$$\gamma \mathrm{e}_n = \begin{pmatrix} 0 & \cdots & 0 & \gamma \end{pmatrix}^\top, \qquad (\gamma \in \mathbb{R}).$$

Set γ^0 and γ^1 such that $\gamma^i \mathrm{e}_n = Qy^i$.
We have:

$$T_C(y^0, y^1; A, b) = T_{QC}(\gamma^0 \mathrm{e}_n, \gamma^1 \mathrm{e}_n; \mathbf{A}_n, \mathbf{b}_n).$$

Goh transformation

Proposition

Define,

$$\tilde{T} = \inf \left\{ T \geqslant 0,\ \exists z_1 \in L^\infty(0,T), \;\middle|\; \begin{array}{l} (z_1, z_2, \ldots, z_n)^\top \in QC, \\ (z_2, \ldots, z_n)^\top(T) = \gamma^1 \mathrm{e}_{n-1} \end{array} \right\}.$$

Then,

$$\tilde{T} \leqslant T_{QC}(\gamma^0 \mathrm{e}_n, \gamma^1 \mathrm{e}_n; \mathbf{A}_n, \mathbf{b}_n) = T_C(x^0, x^1; A, b).$$

In addition, if C is convex, then

$$\tilde{T} = T_{QC}(\gamma^0 \mathrm{e}_n, \gamma^1 \mathrm{e}_n; \mathbf{A}_n, \mathbf{b}_n) = T_C(x^0, x^1; A, b).$$

Observe that $\tilde{z} = (z_2, \ldots, z_n)^\top \in \mathbb{R}^{n-1}$ solves:

$$\dot{\tilde{z}} = \mathbf{A}_{n-1} \tilde{z} + \mathbf{b}_{n-1} z_1, \quad \tilde{z}(0) = \gamma^0 \mathrm{e}_{n-1},$$

i.e., z^1 is seen as a control.

Consequences in dimension two

In dimension two, we end up with the reduced system:

$$\dot{z}_2 = z_1, \qquad z_2(0) = \gamma^0,$$

with the mixed state and control constraint: $(z_1(t), z_2(t))^\top \in QC$.
For every $z_2 \in \mathbb{R}$, let us define:

$$\varphi_+(z_2) = \begin{cases} 0 & \text{if } (0, z_2)^\top \notin QC, \\ \sup \{z_1 \in \mathbb{R}_+,\ [0, z_1] \times \{z_2\} \subset QC\} & \text{otherwise} \end{cases} \qquad \text{and}$$

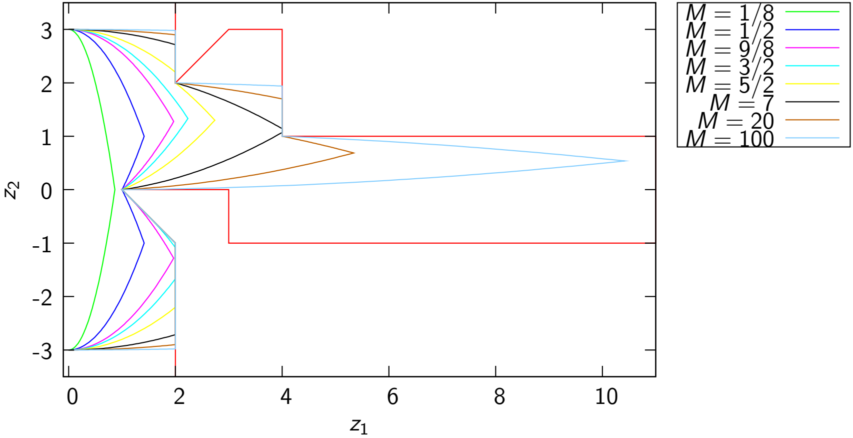
$$\varphi_-(z_2) = \begin{cases} 0 & \text{if } (0, z_2)^\top \notin QC, \\ \sup \{z_1 \in \mathbb{R}_+,\ [-z_1, 0] \times \{z_2\} \subset QC\} & \text{otherwise.} \end{cases}$$

Proposition

Assume QC is simply connected. Then, we have:

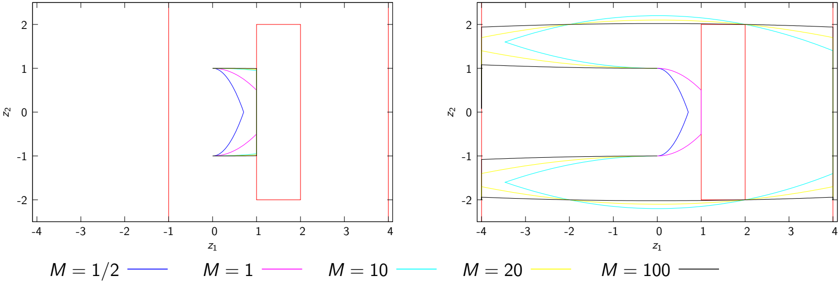
$$T_{QC}(\gamma^0 \mathrm{e}_2, \gamma^1 \mathrm{e}_2; \mathbf{A}_2, \mathbf{b}_2) = \begin{cases} \displaystyle\int_{\gamma^1}^{\gamma^0} \frac{\mathrm{d}\zeta_2}{\varphi_-(\zeta_2)} & \text{if } \gamma^1 \leqslant \gamma^0, \\ \displaystyle\int_{\gamma^0}^{\gamma^1} \frac{\mathrm{d}\zeta_2}{\varphi_+(\zeta_2)} & \text{if } \gamma^1 \geqslant \gamma^0. \end{cases}$$

Example in dimension two



Plots of the state trajectories steering $-3e_2$ to $3e_2$ with particular QC and with additional constraints $|v(t)| \leqslant M$ (for $M \in \left\{\frac{1}{8}, \frac{1}{2}, \frac{9}{8}, \frac{3}{2}, \frac{5}{2}, 7, 20, 100\right\}$).

$\mathbb{R}^2 \setminus C$ has to be a connected domain



Plots of the time optimal state trajectories steering $-\mathrm{e}_2$ to e_2 with different non simply connected state and with control constraint $|v(t)| \leqslant M$, for $M \in \left\{\frac{1}{2}, 1, 10, 20, 100\right\}$.

Case of a linear state constraint

Consider the state constraint

$$C = \{y \in \mathbb{R}^n \mid \ell_1 y_1 + \cdots + \ell_n y_n \geqslant 0\}.$$

Assume also that the system $(\star)$ is already in Brunovsky form, with $B = \mathbf{b}_n = \mathrm{e}_1 \in \mathbb{R}^n$.
Let us define $i_0 = \min\{i \in \{1, \dots, n\} \mid \ell_i \neq 0\}$, and assume that $\ell_{i_0} = 1$.
Using Goh transformation, we obtain $\tilde{T} = T_C(\gamma^0 \mathrm{e}_n, \gamma^1 \mathrm{e}_n; \mathbf{A}_n, \mathbf{b}_n)$, with

$$\tilde{T} = \inf \left\{ T > 0 \mid \exists y_{i_o} \in L^\infty(0,T), \left| \begin{array}{l} y_{i_0}(t) + \ell_{i_0+1} y_{i_0+1}(t) + \cdots + \ell_n y_n(t) \geqslant 0, \\ y_i(0) = y_i(T) = 0 \quad (i \in \{i_0+1, \dots, n\}), \\ y_n(0) = \gamma^0, \quad y_n(T) = \gamma^1, \\ \dot{y}_i = y_{i-1} \quad (i \in \{i_0+1, \dots, n\}). \end{array} \right. \right\}.$$

Let us then define the control $u = y_{i_0} + \ell_{i_0+1} y_{i_0+1} + \cdots + \ell_n y_n \geqslant 0$, i.e.,

$$y_{i_0} = u - \ell_{i_0+1} y_{i_0+1} - \cdots - \ell_n y_n, \qquad \text{with } u \geqslant 0.$$

We are then reduced to:

$$\inf \quad T \quad \left| \begin{array}{l} T \geqslant 0, \\ u \in L^\infty(0,T), \ u \geqslant 0, \\ y(0) = y^0, \quad y(T) = y^1, \\ \dot{y} = Ay + bu. \end{array} \right.$$

Positive minimal time

Consider the system $(\star)$ and assume that the pair (A, b) is controllable. We define

$$Acc_+(T) = \left\{ \int_0^T e^{(T-t)A} bu(t)\,\mathrm{d}t,\ u \in L^\infty(0,T),\ u \geqslant 0 \right\} \qquad (T > 0).$$

Lemma

$Acc_+(T)$ is a convex cone with vertex 0, and $Acc_+(T) \subsetneq \mathbb{R}^n$.

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Proof:

- $Acc_+(T)$ is a convex cone with vertex 0 for every $T > 0$.
- $Acc_+(T)$ contains a nonempty open set.
- $(-Acc_+(T)) \cap Acc_+(T) = \{0\}$ for $T > 0$ small enough.
  By contradiction, $\forall T > 0,\ \exists y^1 \neq 0$ s.t. $y^1 \in (-Acc_+(T)) \cap Acc_+(T)$.
  $\Rightarrow$ $\exists u^+ \geqslant 0$ and $u^- \geqslant 0$ s.t. $y^1 = \int_0^T e^{(T-t)A} bu^+(t)\,\mathrm{d}t = -\int_0^T e^{(T-t)A} bu^-(t)\,\mathrm{d}t$.
  $\Rightarrow$ $u = u^+ + u^- \geqslant 0$, $u \neq 0$ and $0 = \int_0^T e^{(T-t)A} bu(t)\,\mathrm{d}t$.
  $\Rightarrow$ $0 < |b|^2 \int_0^T u(t)\,\mathrm{d}t = b^\top \int_0^T \left(e^{(T-t)A} - I_n\right) bu(t)\,\mathrm{d}t$
  $\leqslant \sup_{t\in[0,T]} \left(b^\top \left(e^{(T-t)A} - I_n\right) b\right) \int_0^T u(t)\,\mathrm{d}t.$
  But, $\lim_{T\to 0} \sup_{t\in[0,T]} b^\top \left(e^{(T-t)A} - I_n\right) b = 0$, leading to a contradiction.

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Lemma

$Acc_+(T)$ is a convex cone with vertex 0, and $Acc_+(T) \subsetneq \mathbb{R}^n$.

Define,

$$\underline{T}(y^0, y^1) = \inf \Big\{ T > 0 \ \Big|\ \exists u \in L^\infty(0,T) \text{ s.t. } u \geqslant 0 \text{ and } y^1 - e^{TA} y^0 = \int_0^T e^{(T-t)A} bu(t)\,\mathrm{d}t \Big\}.$$

Consequence

For every $y^0 \in \mathbb{R}^n$, there exist $y^1 \in \mathbb{R}^n$ such that $\underline{T}(y^0, y^1) > 0$.

Some accessibility conditions

Lets us define the set of *positive steady states*,

$$\mathcal{S}_+^* = \{\bar{y} \in \mathbb{R}^n \mid \exists \bar{u} \in \mathbb{R}_+^* \text{ s.t. } A\bar{y} + b\bar{u} = 0\}.$$

Proposition

We have $\underline{T}(y^0, y^1) < \infty$ if one of the following condition is satisfied.

- $y^0, y^1 \in \mathcal{S}_+^*$;
- $\Re\sigma(A) \subset \mathbb{R}_-^*$ *and* $y^1 \in \mathcal{S}_+^*$.

Existence of a minimal time control

Proposition

If $\sigma(A) \cap \mathbb{R} \neq \emptyset$*, and* $\underline{T}(y^0, y^1) < \infty$*, then there exist a nonnegative control* $\underline{u} \in \mathcal{M}([0, \underline{T}(y^0, y^1)])$ *steering* y^0 *to* y^1 *in time* $\underline{T}(y^0, y^1)$*.*

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Proof: Let $\underline{T} = \underline{T}(y^0, y^1)$.

- $\exists (T_n)_{n\in\mathbb{N}}$, s.t. $\lim_{n\to\infty} T_n = \underline{T}$, $T_0 \geqslant T_1 \geqslant \cdots \geqslant T_n \geqslant \cdots \geqslant \underline{T}$ and $\forall n \in \mathbb{N}$, $\exists u_n \in L^\infty(0, T_n)$ s.t. $u_n \geqslant 0$ and

$$y^1 - e^{T_n A} y^0 = \int_0^{T_n} e^{(T_n - t)A} b u_n(t) \, \mathrm{d}t.$$

- Let $\varphi \in \mathbb{R}^n$, $|\varphi| = 1$ and $\varphi^\top A = \lambda \varphi^\top$, for $\lambda \in \mathbb{R}$. We have
  - $\varphi^\top b \neq 0$;
  - $\varphi^\top \left(y^1 - e^{\lambda T_n} y^0 \right) = \varphi^\top \int_0^{T_n} e^{\lambda(T_n - t)} b u_n(t) \, \mathrm{d}t$. Hence,

$$e^{-|\lambda| T_0} \|u_n\|_{L^1(0, T_n)} \leqslant \frac{|\varphi^\top y^1| + e^{|\lambda| T_0} |\varphi^\top y^0|}{|\varphi^\top b|}.$$

  - $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in L^1, hence, vaguely convergent to some $\underline{u} \in \mathcal{M}([0, \underline{T}])$, and we have $y^1 - e^{-\underline{T} A} y^0 = \int_0^{\underline{T}} e^{(\underline{T} - t)A} b \, \mathrm{d}\underline{u}(t)$.

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If $\sigma(A) \cap \mathbb{R} \neq \emptyset$, and $\underline{T}(y^0, y^1) < \infty$, then there exist a nonnegative control $\underline{u} \in \mathcal{M}([0, \underline{T}(y^0, y^1)])$ steering y^0 to y^1 in time $\underline{T}(y^0, y^1)$.

We then define,

$$\mathcal{T}(y^0, y^1) = \inf \{ T \geqslant 0 \mid \exists \underline{u} \in \mathcal{M}([0, T]) \text{ s.t. } \underline{u} \geqslant 0 \text{ and } y^1 - e^{TA} y^0 = \int_0^T e^{(T-t)A} b \, \mathrm{d}\underline{u}(t) \right\}.$$

Obviously, we have,

$$0 \leqslant \underline{T}(y^0, y^1) \leqslant \mathcal{T}(y^0, y^1).$$

Proposition

If $y^1 \in \mathcal{S}_+^*$, then $\underline{T}(y^0, y^1) = \mathcal{T}(y^0, y^1)$.

Minimal time control with Radon measures I

Theorem

Assume that $\underline{T}(y^0, y^1) < \infty$, *and let* $\underline{T} = \underline{T}(y^0, y^1)$.

- *If there exist* $\underline{u} \in \mathcal{M}([0, \underline{T}])$ *steering* y^0 *to* y^1 *in time* $\underline{T}$, *then there exist* $N \in \mathbb{N}$, $m_1, \dots, m_N \geqslant 0$ *and* $\tau_1, \dots, \tau_N \in [0, \underline{T}]$ *such that* $\underline{u} = \sum_{i=1}^N m_i \delta_{\tau_i}$, *i.e.,*

$$y^1 - e^{\underline{T}A} y^0 = \sum_{i=1}^{N} m_i e^{(\underline{T}-\tau_i)A} b. \qquad (\star)$$

- *If* $\sigma(A) \cap \mathbb{R} \neq \emptyset$, *then there exists* $\underline{u} \in \mathcal{M}([0, \underline{T}])$, $u \geqslant 0$, *steering* y^0 *to* y^1 *in time* $\underline{T}$.
- *If* $\sigma(A) \subset \mathbb{R}$, *then there exists* $\underline{u} \in \mathcal{M}([0, \underline{T}])$, $u \geqslant 0$, *steering* y^0 *to* y^1 *in time* $\underline{T}$, *and* $(\star)$ *holds with* $N \leqslant \lfloor (n+1)/2 \rfloor$. *Furthermore, this control* $\underline{u}$ *is unique.*

Minimal time control with Radon measures II

Proof (guide lines):

- Use a time rescaling[1,2,3].
- Apply Pontryagin maximum principle.

For $\sigma(A) \subset \mathbb{R}$,

- Count the maximal number of Dirac masses (see Lee Markus[4]).
- Uniqueness follows from the fact that $\left\{e^{t_1 A}B, \dots, e^{t_n A}B\right\}$ is a free family in $\mathbb{R}^n$ as soon as the t_i's are two by two distinct.

[1] R. W. Rishel. "An extended Pontryagin principle for control systems whose control laws contain measures". *J. SIAM Control Ser. A* 3 (1965)

[2] A. Bressan and F. Rampazzo. "On differential systems with vector-valued impulsive controls". *Boll. Un. Mat. Ital. B (7)* 2.3 (1988)

[3] G. Dal Maso and F. Rampazzo. "On systems of ordinary differential equations with measures as controls". *Differential Integral Equations* 4.4 (1991)

[4] E. B. Lee and L. Markus. *Foundations of optimal control theory*. 1967

Time rescaling I

Define $\upsilon(t) = \underline{u}([0, t])$, $\varsigma(0) = 0$ and $\varsigma(t) = t + \upsilon(t)$.
We set $\mathcal{T}$ the set of jumps times. For every $t \in [0, T]$,

- if $s = \varsigma(t)$, with $t \in [0, T] \setminus \mathcal{T}$, we set $\tau(s) = t$ and $\gamma(s) = \upsilon(t)$;
- if $s \in [\varsigma(t^-), \varsigma(t^+)]$, with $t \in \mathcal{T}$, we set $\tau(s) = t$ and $\gamma(s) = \upsilon(t^-) + \frac{\upsilon(t^+)-\upsilon(t^-)}{\varsigma(t^+)-\varsigma(t^-)}(s - \varsigma(t^-)) = s - t^-$.

This leads to the reparametrized system

$$\dot{z}(s) = \tau'(s)Az(s) + B\gamma'(s).$$

Noticing that $\gamma'(s) = 1 - \tau'(s)$, $\tau'(s) \in [0, 1]$ and setting $w(s) = \tau'(s)$, we obtain

$$\dot{z}(s) = w(s)Az(s) + B(1 - w(s)) \qquad (s \in [0, \varsigma(T)]),$$

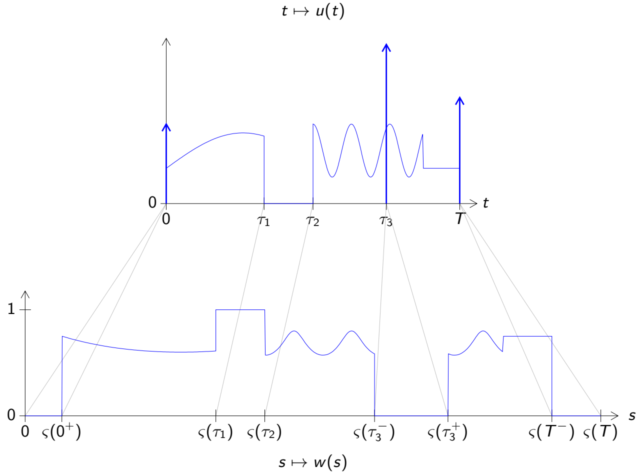
with

$$T = \int_0^{\varsigma(T)} w(s)\,\mathrm{d}s \quad \text{and} \quad y(t) = z(\varsigma(t))$$

and $w(s) \in [0, 1]$ is the *new* control.

Time rescaling II

$$\dot{y} = Ay + Bu$$



$$\dot{z} = wAz + B(1 - w)$$

Time rescaling III

The minimal time control problem becomes

$$\min \int_0^S w(s)\,\mathrm{d}s$$

$$\left|\begin{array}{l} S \geqslant 0, \\ w(s) \in [0,1] \qquad (s \in [0,S]), \\ z(S) = y^1, \text{ with } z \text{ the solution of} \\ \qquad \dot{z} = wAz + B(1-w), \\ \qquad \text{with initial condition } z(0) = y^0. \end{array}\right.$$

We can now apply the classical Pontryagin maximum principle to obtain the result.

Heat equation I

Consider the 1D heat equation

$$\begin{aligned}
\dot{y}(t,x) &= \partial_x^2 y(t,x) && (t>0, x\in(0,1)),\\
y(t,0) &= 0 && (t>0),\\
y(t,1) &= u(t) && (t>0),\\
y(0,x) &= y^0(x) && (x\in(0,1)).
\end{aligned}$$

For this system, we have $\mathcal{S}_+^* = \{x\in[0,1]\mapsto \bar{u}x,\ \bar{u}\in\mathbb{R}_+^*\}$.

Remark

If $y^0 \geqslant 0$ (and $u \geqslant 0$), the comparison principle ensures that $y(t,x)\geqslant 0$.

Heat equation II

Question

Given $y^0 \in L^2(0,1)$ and $y^1 \in \mathcal{S}_+^*$.

1. Is there a nonnegative control steering y^0 to y^1 in some time $T > 0$?
2. Do we have a positive minimal controllability time?
3. Is there some control in the minimal time?

As for the finite dimensional case, we can prove:

Proposition

$\underline{T}(y^0, y^1) < \infty$ and there exist a control $\underline{u} \in \mathcal{M}([0, \underline{T}(y^0, y^1)])$, $\underline{u} \geqslant 0$, steering y^0 to y^1 in time $\underline{T}(y^0, y^1)$.

Question

Is the control $\underline{u}$ a sum of Dirac masses?

Finite dimension approximation I

Let us define $(-\lambda_n)_{n\in\mathbb{N}^*}$ the eigenvalues of $-\partial_x^2$ and $(\varphi_n)_{n\in\mathbb{N}^*}$ the associated normalized eigenvectors. Recall that $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$.

We then have $y(t,x) = \sum_{n=1}^{\infty} y_n(t)\varphi_n(x)$, where y_n is solution of

$$\dot{y}_n = -\lambda_n y_n + \gamma_n u, \qquad y_n(0) = \int_0^1 y^0(x)\varphi_n(x)\,\mathrm{d}x = y_n^0$$

with $\gamma_n = -\partial_x\varphi_n(1) \neq 0$.
The goal is to find T and $u \geqslant 0$ such that,

$$y_n(T) = \int_0^1 y^1(x)\varphi_n(x)\,\mathrm{d}x = y_n^1 \qquad (n \in \mathbb{N}^*).$$

Finite dimension approximation II

For every $N \in \mathbb{N}^*$, we define,

$$\underline{T}_N(y^0, y^1) = \inf \left\{ T \geqslant 0 \ \middle| \ \exists \underline{u} \in \mathcal{M}([0,T]) \text{ s.t. } Y_N^1 - e^{TA_N} Y_n^0 = \int_0^T e^{(T-t)A_N} b_N \, \mathrm{d}\underline{u}(t) \right\},$$

with $Y_N^i = \left(y_1^i, \cdots, y_N^i\right)^\top \in \mathbb{R}^N$, $A_N = \operatorname{diag}(-\lambda_1, \cdots, -\lambda_N) \in \mathbb{R}^{N\times N}$ and $B_N = (\gamma_1, \cdots, \gamma_N)^\top$.
We have

$$\underline{T}_N(y^0, y^1) \leqslant \underline{T}(y^0, y^1).$$

Furthermore, there exist, $N_0 \in \mathbb{N}$ ($N_0 \leqslant \lfloor (N+1)/2 \rfloor$), $0 \leqslant \tau_1^N < \cdots < \tau_{N_0}^N \leqslant \underline{T}_N$ and $m_1^N, \ldots, n_{N_0}^N > 0$ such that

$$y_n^1 - e^{-\lambda_n \underline{T}_N} y_n^0 = \sum_{i=1}^{N_0} e^{-\lambda_n(\underline{T}_N - \tau_i^N)} \gamma_n m_i^N \qquad (n \in \{1, \ldots, N\}).$$

There also exist $C = C(y^0, y^1) > 0$ such that $\sum_{i=1}^{N_0} m_i^N \leqslant C$.
We also set $\tau_i^N = \underline{T}_N$ and $m_i^N = 0$ for $i > N_0$.

Convergence results

Theorem

We have $\lim_{N\to\infty} \underline{T}_N = \underline{T}$ *and up to the extraction of a subsequence* $\lim_{N\to\infty} \tau_i^N = \tau_i^\infty \in [0, \underline{T}]$ *and* $\lim_{N\to\infty} m_i^N = m_i^\infty > 0$*, and the sequence* $(\tau_i^\infty, m_i^\infty)$ *satisfies*

$$y_n^1 - e^{-\lambda_n \underline{T}} y_n^0 = \sum_{i=1}^{\infty} e^{-\lambda_n(\underline{T}-\tau_i^\infty)} \gamma_n m_i^\infty \qquad (n \in \mathbb{N}^*).$$

That is to say that the control $\underline{u}^\infty = \sum_{i=1}^\infty m_i^\infty \delta_{\tau_i^\infty}$ *steers* y^0 *to* y^1 *in time* $\underline{T}$*, and* $\underline{u}^\infty$ *is the only nonnegative control in the set of purely impulsive measures doing this job.*

We can also reorganize these sequences such that $(\tau_i^\infty)_{i\in I}$ is increasing and $m_i^\infty > 0$ for some subset I of $\mathbb{N}^*$. And we necessarily have that

1. I is of infinite cardinal;
2. $\lim_{i\to\infty} \tau_i^\infty = \underline{T}$.

Convergence results

Theorem

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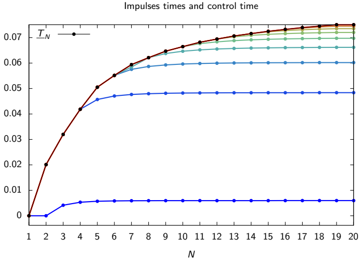
Proof (guide lines):

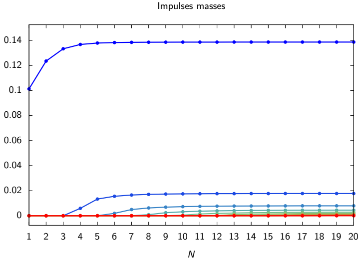
- Use a diagonal extraction to obtain the existence of a limit $(\tau_i^\infty, m_i^\infty)_{i\in\mathbb{N}^*}$;
- Use some estimates on the masses m_i^N to get the vague convergence of $\left(\sum_{i=1}^\infty m_i^N \delta_{\tau_i^N}\right)_N$ to $\sum_{i=1}^\infty m_i^\infty \delta_{\tau_i^\infty}$.

Numerical experiment

Convergence result

We consider $y^0(x) = \cos(\pi x)$ and $y^1(x) = x$.





Legends: τ_n^N and m_n^N for $n = \dots$

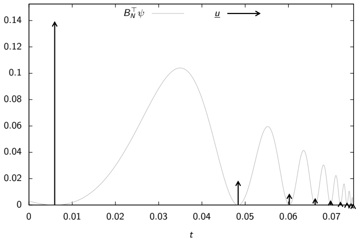
1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Numerical experiment

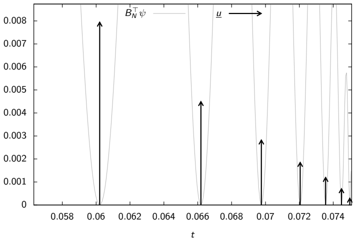
Results with $N = 20$ (control).

For $N = 20$, we obtain $\underline{T}_N \simeq 0.075091$.

Control impulses and adjoint trace

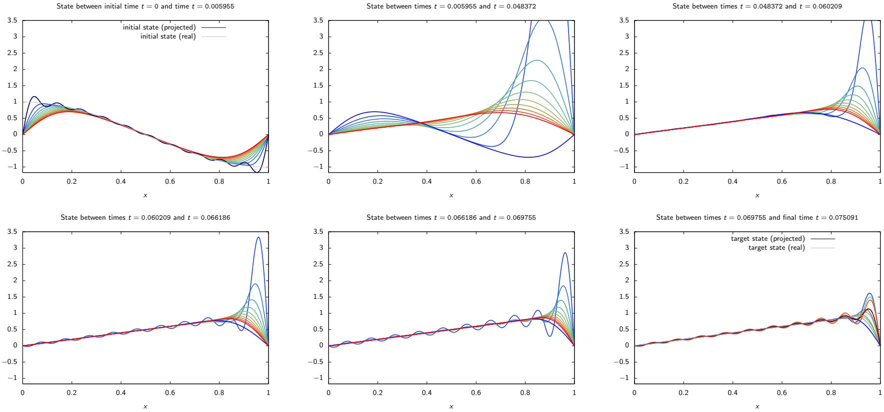


Control impulses and adjoint trace (zoom on final times)



Numerical experiment

Results with $N = 20$ (state).



Other results I

The presented results are taken from

- J. Lohéac, E. Trélat, and E. Zuazua. "Minimal controllability time for finite-dimensional control systems under state constraints". *Automatica J. IFAC* 96 (2018)
- J. Lohéac, E. Trélat, and E. Zuazua. "Nonnegative control of finite-dimensional linear systems". *Ann. Inst. H. Poincaré Anal. Non Linéaire* 38.2 (2021)
- J. Lohéac, E. Trélat, and E. Zuazua. "Minimal controllability time for the heat equation under unilateral state or control constraints". *Math. Models Methods Appl. Sci.* 27.9 (2017)
- J. Lohéac. "Nonnegative boundary control of 1D linear heat equations". *Vietnam Journal of Mathematics* (2021)

Other results are

- For nonlinear finite dimensional systems
  - V. Bezborodov, L. Di Persio, and R. Muradore. "Minimal controllability time for systems with nonlinear drift under a compact convex state constraint". *Automatica J. IFAC* 125 (2021)

Other results II

- For nonlinear parabolic PDEs
  - D. Pighin and E. Zuazua. "Controllability under positivity constraints of semilinear heat equations". *Math. Control Relat. Fields* 8.3-4 (2018)
  - M. R. Nuez-Chvez. *Controllability Under Positive Constraints for Quasilinear Parabolic PDEs*. 2019. arXiv: `1912.01486 [math.AP]`
- For parabolic systems
  - P. Lissy and C. Moreau. "State-constrained controllability of linear reaction-diffusion systems". Nov. 2020
- For fractional heat
  - H. Antil et al. *Controllability properties from the exterior under positivity constraints for a 1-D fractional heat equation*. 2019. arXiv: `1910.14529 [math.OC]`
  - U. Biccari, M. Warma, and E. Zuazua. "Controllability of the one-dimensional fractional heat equation under positivity constraints". *Commun. Pure Appl. Anal.* 19.4 (2020)
- For wave equation
  - D. Pighin and E. Zuazua. "Controllability under positivity constraints of multi-d wave equations". *Trends in control theory and partial differential equations*. Vol. 32. Springer INdAM Ser. 2019

Open problems

General:

- Convergence rates ($T_C^M \to T_C$ and $\underline{T}_N \to \underline{T}$);
- Estimation of the minimal controllability time.

For the heat equation:

- Approximate controllability;
- Heat equation in higher dimension;
- Uniqueness of the minimal time control;
- More adapted numerical methods.

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Merci pour votre attention!