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Input to State Stability (ISS) for PDEs
A Beginner's Guide
Based on (interesting) Examples

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Plan

Definitions (I)

Let X (the state space) and U (the input space) be Banach spaces. Let X_+ (respectively U_+) be Banach spaces.

Definition

A well-posed control system with state space X and input space U is a family $\Sigma = (\Sigma_t)_{t\geqslant 0}$ of continuous maps from $X_+ \times L^p([0,\infty); U_+)$ to X_+ such that, setting $z(t) = \Sigma_t \begin{bmatrix} z_0 \\ u \end{bmatrix}$ we have (in an appropriate sense)

$$\dot{z}(t) = F(z(t), u(t)), \qquad z(0) = z_0.$$

We refer to Mironchenko and Prieur [5] (2020), for a more precise (and slightly different) definition.

Definitions (II)

Definition

The well-posed control system Σ is said input-to-state stable (ISS) if there exist continuous functions $\alpha : [0,\infty) \times [0,\infty) \to [0,\infty)$ and $\gamma : [0,\infty) \to [0,\infty)$ such that:

1. For every $r,\ s,\ t > 0$ we have

$$\alpha(0,s) = \lim_{t\to\infty} \alpha(r,t) = \gamma(0) = 0,$$

$$\alpha(\cdot,t),\ \alpha\left(r,\frac{1}{\cdot}\right) \quad \text{are strictly increasing on } (0,\infty).$$

2. For every $t \geqslant 0$, $z_0 \in X_+$ and $u \in L^\infty([0,\infty); U_+)$ we have

$$\|z(t)\|_X \leqslant \alpha(\|z_0\|_X, t) + \gamma(\sup_{\sigma\in[0,t]} \|u(\sigma)\|_U). \tag{1}$$

Since Sontag [7] (1989), an overwhelming literature is devoted to this subject. For PDEs, see Karafyllis and Krstic [3] (2019), Mironchenko and Prieur [5] (2020).

Some remarks

- A linear system $\dot{z} = Az + Bu$, with $B \in \mathcal{L}(U, X)$ is ISS iff A generates an exponentially stable semigroup.
- The property still holds if B is an admissible control operator for the exponentially stable semigroup generated by A.
- An abstract class of bilinear control systems has been considered in Hosfeld, Jacob and Schwenninger [2] (2020).
- Various PDE examples (mostly parabolic semilinear) have been tackled in the literature.
- Only a few results are available in the nonlinear hyperbolic or quasilinear case.
- For nonlinear system a local concept of ISS is also available.

Plan

The governing equations

$$\begin{cases} \dot{S}(t) = u(t) - v(t)S(t) - \eta S(t)\int_0^\infty \beta(a)i(t,x)\,\mathrm{d}x & (t \geqslant 0), \\ \frac{\partial i}{\partial t}(t,a) = -\frac{\partial i}{\partial a}(t,a) - \nu_I(a)i(t,a) & (t \geqslant 0,\ a \in (0,+\infty)), \\ i(t,0) = \eta S(t)\int_0^\infty \beta(x)i(t,x)\,\mathrm{d}x & (t \geqslant 0), \\ S(0) = S_0, & \\ i(0,a) = i_0(a) & (t \geqslant 0,\ a \in (0,+\infty)), \end{cases}$$

- $S(t)$ denotes the total susceptible population at instant t.
- $i(t,a)$ stands for the number of infected individuals with age of infection a at time t.
- $\eta > 0$ is the rate at which an infectious individual infects the susceptible individuals.
- The nonnegative number $\beta(a)$ designs the probability to be infectious (capable of transmitting the disease) with an age of infection equal to a.
- $t \mapsto v(t)$, the vaccination rate, is L^∞ and positive.
- $u \in L^1_{\mathrm{loc}}[0,\infty)$ is the input (disturbance), designing the flux of susceptible population.

Existence and uniqueness of solutions

Theorem (San Martin, Takahashi, Tucsnak, 2021)

Assume that β is bounded and uniformly continuous from $[0,+\infty)$ to $[0,+\infty)$. Moreover, assume that $\nu_I \in L^\infty[0,\infty)$, that

$$\nu_I(a) \geqslant 0 \qquad (a \in [0,\infty) \text{ a.e.}).$$

Then for every $u \in L^1_{\mathrm{loc}}[0,\infty)$, $v \in L^\infty([0,\infty);\mathbb{R}_+)$, $S_0 > 0$, $i_0 \in L^1[0,\infty)$, with $i_0(a) \geqslant 0$, and $S_0 + \int_0^\infty i_0(a)\,\mathrm{d}a \leqslant 1$ there exists a unique solution with

$$S \in W^{1,\infty}(0,\infty), \qquad i \in C\left([0,\infty), L^1[0,\infty)\right).$$

Some remarks:

- Similar existence and uniqueness results are given in Perthame and Tumuluri [6] (2008) and Magal and Ruan [4].
- The positivity constraints are essential in establishing this result.
- Our methodology seems adaptable to more complicated epidemic models.

Sketch of the proof(I)

Step 1. Let $\tau > 0$ and $C_\tau = S_0 + \|i_0\|_{L^1[0,\infty)} + \|u\|_{L^1[0,\tau]}$. We set

$$K_\tau = \left\{ \varphi \in C([0,\tau]; L^1[0,\infty)) \;\middle|\; \varphi \geqslant 0, \;\int_0^\infty \varphi(t,a)\,\mathrm{d}a \leqslant C_\tau \right\}.$$

For $\varphi \in K_\tau$ we solve

$$\dot{S}_\varphi(t) = u(t) - v(t) S_\varphi(t) - \eta S_\varphi(t) \int_0^\infty \beta(a)\varphi(t,x)\,\mathrm{d}x, \qquad S_\varphi(0) = S_0,$$

$$\begin{cases} \frac{\partial i_\varphi}{\partial t}(t,a) = -\frac{\partial i_\varphi}{\partial a}(t,a) - \nu_I(a) i_\varphi(t,a) & (t \geqslant 0,\; a \in (0,+\infty)), \\ i_\varphi(t,0) = \eta S_\varphi(t) \int_0^\infty \beta(x)\varphi(t,x)\,\mathrm{d}x & (t \geqslant 0), \\ i_\varphi(0,a) = i_0(a) & (t \geqslant 0,\; a \in (0,\infty)), \end{cases} \tag{2}$$

and we define $N_\tau \varphi = i_\varphi$.

Sketch of the proof(II)

Step 2. We check easily that N_τ maps K_τ into K_τ and that, for every $k \in \mathbb{N}$ and every $\varphi_1,\ \varphi_2 \in K_\tau$ we have

$$\|N_\tau^k \varphi_1(t,\cdot) - N_\tau^k \varphi_2(t,\cdot)\|_{L^1[0,\infty)} \leqslant c_\tau^2 \frac{t^k}{k!} \|\varphi_1 - \varphi_2\|_{C([0,\tau];L^1[0,\infty)))}. \qquad (t \in [0,\tau]).$$

N_τ^k is thus, for k large enough, a strict contraction of K_τ, which implies our existence and uniqueness result.

ISS of the epidemic system

Proposition

Assume that v is periodic of period τ with $\int_0^\tau v(t)\,\mathrm{d}t > 0$. Moreover, suppose that $\nu_I(a) \geqslant \nu_0 > 0$ a.e. on $[0,\infty)$. Then the considered system is ISS.

Proof.

It suffices to remark that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(S(t)+\int_0^\infty i(t,x)\,\mathrm{d}x\right) \leqslant u(t) - \min\{v(t),\nu_0\}\left(S(t)+\int_0^\infty i(t,x)\,\mathrm{d}x\right),$$

to obtain that the ISS property holds with

$$\alpha(r,t) = Mr\mathrm{e}^{-\omega t}, \qquad \beta(s) = \gamma s.$$

□

Plan

The viscous Burgers equations with pointwise control

$$
\begin{cases}
\dot{v}(t,y) - v_{yy}(t,y) + v(t,y)v_y(t,y) = u(t)\delta_\xi & t \in (0,\infty),\ y \in (-1,1), \\
v(t,-1) = v(t,1) = 0 & t \in (0,\infty), \\
v(0,y) = v_0(y) & y \in (-1,1).
\end{cases}
\tag{3}
$$

Proposition

The system (3) *is ISS with* $X = L^2(-1,1)$, $U = \mathbb{C}$ *and*

$$\alpha(r,t) = Mr\mathrm{e}^{-\omega t}, \qquad \beta(s) = \gamma s.$$

A simplified fluid-structure system

$$
\begin{cases}
\dot{v}(t,y) - v_{yy}(t,y) + v(t,y)v_y(t,y) = 0 & t \in (0,\infty),\ y \in (-1,1),\ y \neq h(t) \\
v(t,-1) = v(t,1) = 0 & t \in (0,\infty), \\
\dot{h}(t) = v(t,h(t)) & t \in (0,\infty), \\
\ddot{h}(t) = [v_y](t,h(t)) + u(t) & t \in (0,\infty), \\
v(0,y) = v_0(y) & y \in (-1,1), \\
h(0) = h_0, \qquad \dot{h}(0) = g_0. &
\end{cases}
\tag{4}
$$

Conjecture

The system (4) *is ISS with* $X = L^2(-1,1) \times \mathbb{C}$, $U = \mathbb{C}$ *and*

$$
\alpha(r,t) = Mr\mathrm{e}^{-\omega t}, \qquad \beta(s) = \gamma s.
$$

A list of potential ISS systems of interest

- Navier-Stokes in bounded domains with boundary control (what about velocities normal at the boundary?).
- Fluid-structure interactions in several space dimensions.
- Bilinear control.
- More elaborate epidemiological models (with age of infection and age structure).

The SIDHARTE system (I) (Giordano et al. [1] (2020))

$$
\begin{aligned}
\dot{S} &= -S(\alpha I + \beta D + \gamma A + \delta R), && \text{(5a)}\\
\dot{I} &= S(\alpha I + \beta D + \gamma A + \delta R) - (\varepsilon + \zeta + \lambda) I, && \text{(5b)}\\
\dot{D} &= \varepsilon I - (\eta + \rho) D, && \text{(5c)}\\
\dot{A} &= \zeta I - (\theta + \mu + \kappa) A, && \text{(5d)}\\
\dot{R} &= \eta D + \theta A - (\nu + \xi) R, && \text{(5e)}\\
\dot{T} &= \mu A + \nu R - (\sigma(T) + \tau(T)) T, && \text{(5f)}\\
\dot{H} &= \lambda I + \rho D + \kappa A + \xi R + \sigma(T) T, && \text{(5g)}\\
\dot{E} &= \tau(T) T. && \text{(5h)}
\end{aligned}
$$

S - **Susceptible**, **I** - **Infected** (asymptomatic, undetected), **D** - **Diagnosed** (asymptomatic, detected), **A** - **Ailing** (symptomatic, undetected), **R** - **Recognized** (symptomatic, detected), **T** - **Threatened** (symptomatic with life-threatening symptoms, detected), **H** - **Healed** (immune after prior infection, detected or undetected), **E** - **Extinct** (dead, detected).

The SIDHARTE system (II): Various inputs (disturbances)

- α, β, γ describe the infection rates for susceptible individuals, i.e., the rate at which susceptible individuals are infected by the states I, D or R, and A, respectively, and hence join the state I.
- ε, θ describe the testing rate, i.e., at which rate (asymptomatic or symptomatic) infected individuals go from undetected to detected.
- ζ describes the rate of asymptomatic (detected or undetected) infected individuals exhibiting symptoms, i.e., going from states I or D to A or R, respectively.
- μ is the rate at which infected individuals in A or R develop life-threatening symptoms, i.e., join the state T.
- $\lambda, \kappa, \sigma(T)$ are recovery rates for individuals affected by COVID-19. The recovery rate for threatened individuals $\sigma(T)$ depends on T.
- $\tau(T)$ is the mortality rate, i.e., the rate at which individuals with life-threatening symptoms decease, and it depends on T.

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