Nonexistence of holomorphic submersions between complex unit balls equivariant with respect to a lattice and their generalizations

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Abstract. In this article we prove first of all the nonexistence of holomorphic submersions other than covering maps between compact quotients of complex unit balls, with a proof that works equally well in a more general equivariant setting. For a non-equidimensional surjective holomorphic map between compact ball quotients, our method applies to show that the set of critical values must be nonempty and of codimension 1. In the equivariant setting the line of arguments extends to holomorphic mappings of maximal rank into the complex projective space or the complex Euclidean space, yielding in the latter case a lower bound on the dimension of the singular loci of certain holomorphic maps defined by integrating holomorphic 1-forms. In another direction, we extend the nonexistence statement on holomorphic submersions to the case of ball quotients of finite volume, provided that the target complex unit ball is of dimension $m \geq 2$, giving in particular a new proof that a local biholomorphism between noncompact m-ball quotients of finite volume must be a covering map whenever $m \geq 2$. Finally, combining our results with Hermitian metric rigidity, we show that any holomorphic submersion from a bounded symmetric domain into a complex unit ball equivariant with respect to a lattice must factor through a canonical projection to yield an automorphism of the complex unit ball, provided that either the lattice is cocompact or the ball is of dimension at least 2.

In the study of holomorphic mappings and rigidity problems on compact quotients of bounded symmetric domains the case of n-ball quotients has always occupied a special place in terms of formulations of problems and methods of their solutions. Here and in what follows by a quotient of a bounded symmetric domain we will always mean a quotient with respect to a discrete torsion-free subgroup of biholomorphic automorphisms, an n-ball quotient means a quotient of the n-dimensional complex unit ball B^n, and a ball quotient means an n-ball quotient for some $n \geq 1$. The method of harmonic maps of Siu ([Siu1], 1980) makes it possible to obtain holomorphic maps from harmonic maps into m-ball quotients under mild conditions, because the canonical Kähler-Einstein metric on the complex unit ball is of strictly negative curvature in the dual sense of Nakano. In the case where $m = 1$ any harmonic mapping $f : X \to C$ from a compact Kähler manifold X onto a compact Riemann surface C of genus ≥ 2 of maximal rank at some point leads by the study of holomorphic foliations associated to f to a holomorphic mapping $g : X \to C'$ onto some compact Riemann surface C' (Siu [Siu3]). More generally, representations of Kähler groups into automorphism groups $\mathrm{Aut}(B^m) \simeq \mathrm{PU}(m, 1)$ are associated to holomorphic objects. When X is an irreducible compact quotient of a bounded symmetric domain, Margulis superrigidity (Margulis [Ma], 1977) or the method of Hermitian metric rigidity (Mok [Mok1], 1987) implies that any holomorphic mapping $f : X \to Z$ into a compact m-ball quotient Z is necessarily

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constant, unless X is itself an n-ball quotient for some n. From this perspective, holomorphic maps between compact quotients of complex unit balls are special and yet they are essential for completing our understanding of holomorphic maps between compact quotients of bounded symmetric domains, and more generally of linear representations of their fundamental groups.

For the case of a holomorphic mapping $F : B^n \to B^m$, $n > m \geq 1$, between complex unit balls of maximal rank at some point and equivariant with respect to a representation of some cocompact lattice, it is generally believed that any such mapping must be singular at certain points. Especially, conjecturally there ought to be no holomorphic submersions $f : X \to Z$ between compact quotients of complex unit balls $X := B^n/\Gamma$ and $Z := B^m/\Delta$, $n > m \geq 1$. In this article, for convenience we will refer to the confirmation of the latter conjectured statement as the Submersion Problem (for compact quotients of complex unit balls). The Submersion Problem for $n = 2$ and $m = 1$ was settled by Liu ([Liu], 1996), in which the nonexistence of regular holomorphic fibrations on compact 2-ball quotients was proven by means of Chern-number inequalities on surfaces arising from Teichmüller theory. In this article we resolve the Submersion Problem in all dimensions, proving more generally the nonexistence of holomorphic submersions from B^n into B^m, $n > m \geq 1$, equivariant with respect to some representation $\Phi : \Gamma \to \mathrm{Aut}(B^m)$.

Siu ([Siu2], 1984) raised the question whether any holomorphic embedding between compact quotients of complex unit balls must necessarily be totally geodesic provided that the domain manifold is of dimension ≥ 2. The question can be slightly generalized to allow for holomorphic immersions and the generalized problem will be referred to as the Immersion Problem (for compact ball quotients). In Cao-Mok [CM] the Immersion Problem was solved in the affirmative under the additional assumption that the complex dimensions k, resp. ℓ, of the domain manifold, resp. target manifold, satisfy $\ell < 2k$. There, the starting point was an adaptation of an algebraic identity of Feder's ([Fe], 1965) on Chern classes for holomorphic immersions between projective spaces to the case of compact quotients of the complex unit ball, which led to the consideration of the kernel of a closed nonnegative (1,1)-form arising from the second fundamental form of the holomorphic immersion. For the Submersion Problem we adapt the same identity on cohomology classes to the short exact sequence defining the normal bundle of the holomorphic submersion. However, in place of using the second fundamental form, we represent the total Chern class of the normal bundle by differential forms in such a way as to relate it to a nonnegative closed (1,1)-form arising from the Schwarz lemma. More precisely, denoting by ω_k the Kähler form of the complete Kähler manifolds of constant holomorphic sectional curvature -4π on the complex unit ball B^k, for any holomorphic mapping $F : B^n \to B^m$ by the Schwarz lemma we have $F^*\omega_m \leq \omega_n$, and the difference $\omega_n - F^*\omega_m$ is a closed nonnegative (1,1)-form which we will show to have a nontrivial kernel of dimension exactly equal to m at each point. We conclude that $F : B^n \to B^m$ is a Riemannian submersion and derive a contradiction from curvature properties of the complex unit ball.

In relation to non-equidimensional surjective holomorphic maps $f : X \to Z$ between compact ball quotients, it should be noted that Siu also raised in [Siu2] the question whether such maps can exist when $\dim(Z) \geq 2$. While there is at this point no convincing evidence one way or the other regarding the latter question, our results exclude the possibility of holomorphic submersions among the maps $f : X \to Z$ without any restriction on $m = \dim(Z) \geq 1$. Hence, they are also applicable to the case of maps onto compact Riemann surfaces of genus ≥ 2 as studied in [Siu3]. They answer completely a question in some sense dual to the Immersion Problem (whereas the latter problem has only been settled under a certain restriction on dimensions). A ramification from our study is a generalization of the solution of the Submersion Problem to subvarieties of compact ball quotients. As an illustration, we prove that the set of regular values of a non-equidimensional surjective holomorphic map $f : X \to Z$ between compact ball quotients does not contain any compact holomorphic curve C by studying the hypothetical holomorphic submersion of $f^{-1}(C) \subset X$ onto C. Moreover, our methods are also applicable to surjective holomorphic maps from compact ball quotients into complex projective spaces and compact complex tori (although in those cases, the proofs are simpler), and in the latter case a formulation in the equivariant setting yields new information concerning the singular loci of integrals of linearly independent Abelian differentials defined on the covering complex unit ball.

In another direction our methods can be generalized to non-compact complex hyperbolic space forms of finite volume. Here a complex hyperbolic space form means a quotient of the complex unit ball by a torsion-free discrete group of automorphisms. In relation to the Submersion Problem we show that any holomorphic submersion between non-compact complex hyperbolic space forms of finite volume must be a topological covering map (hence equidimensional) provided that the target manifold is of complex dimension ≥ 2. (Note that in the 1-dimensional case there are plenty of unramified holomorphic maps between non-compact complex hyperbolic Riemann surfaces of finite volume which fail to be topological covering maps.) Our methods also lead to a general structure theorem for holomorphic submersions from bounded symmetric domains to the complex unit ball B^m equivariant with respect to a lattice Γ, showing that they must factor through a canonical projection to yield an automorphism of the complex unit ball itself, provided that either Γ is uniform (i.e., cocompact) or $m \geq 2$ (and Γ is a non-uniform lattice).

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§1 Nonexistence of holomorphic submersions equivariant with respect to a cocompact lattice

We start by fixing some conventions and terminology. A complex manifold is understood to be connected. For complex manifolds Y and Z, a mapping $h : Y \to Z$ is said to be a *holomorphic submersion* if and only if h is holomorphic and dh is of rank equal to $\dim(Z)$ at every point. A proper holomorphic submersion $h : Y \to Z$ (which is necessarily surjective by the Proper Mapping Theorem) will be called a *regular holomorphic fibration* if and only if $\dim(Z) \geq 1$ and fibers of h are also positive-dimensional. When fibers of h are connected, $h : Y \to Z$ realizes Y as the total space of a regular family of compact complex manifolds $Y_t := h^{-1}(t)$. When one and hence any fiber of h has $k > 1$ connected components, then there is a k-fold unramified cover $\rho : Z' \to Z$ and a regular holomorphic fibration $h' : Y \to Z'$ with connected fibers such that $h = \rho \circ h'$.

In the study of holomorphic mappings between compact quotients of bounded symmetric domains, in view of Hermitian metric rigidity [Mok1] what remains to be understood is essentially the case where the domain manifold is a compact n-ball quotient. In this direction there was the work of Cao-Mok [CM] on holomorphic immersions between compact quotients of complex unit balls. In our study of the Submersion Problem aiming at proving the nonexistence of regular holomorphic fibrations between compact quotients of complex unit balls it will be clear that the methods are equally applicable to holomorphic submersions which are equivariant with respect to a cocompact lattice on the domain complex unit ball, and the discreteness of the image of the induced homomorphism does not play any essential role. For this reason we will state and prove the first result in this broader context. We have

Theorem 1. *For a positive integer k denote by B^k the k-dimensional complex unit ball, and by $\mathrm{Aut}(B^k)$ its group of biholomorphic automorphisms. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a cocompact lattice of biholomorphic automorphisms. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a homomorphism and $F : B^n \to B^m$ be a holomorphic submersion equivariant with respect to Φ, i.e., $F(\gamma x) = \Phi(\gamma)(F(x))$ for every $x \in B^n$ and for every $\gamma \in \Gamma$. Then, $m = n$ and $F \in \mathrm{Aut}(B^n)$.*

We observe also that torsion-freeness of $\Gamma \subset \mathrm{Aut}(B^n)$ is not needed in the hypothesis as one can always pass to a torsion-free subgroup $\Gamma_0 \subset \Gamma$ of finite index to prove the theorem. As a corollary to Theorem 1 we have

Corollary 1. *In the notations of Theorem 1 suppose furthermore that $\Gamma \subset \mathrm{Aut}(B^n)$ is torsion-free, and write $X = B^n/\Gamma$. Then, there does not exist any regular holomorphic fibration $f : X \to Z$, where Z is a compact m-ball quotient with $1 \leq m < n$.*

Corollary 1 for the case of $n = 2$ was established by Liu ([Liu], 1996) by means of geometric height inequalities in the case of curves over a complex function field of transcendence 1, i.e., for complex surfaces fibered over a compact Riemann surface. Liu's result was used by Kapovich [Ka] to show that the compact 2-ball quotients constructed by Livné [Liv] have incoherent fundamental groups, i.e., they contain finitely generated subgroups which are not finitely presentable. The proof of Liu [Liu] makes use of inequalities arising from Teichmüller theory, and as such does not apply to the case of Γ-equivariant holomorphic maps into the unit disk. It also does not generalize to regular holomorphic fibrations of n-ball quotients over compact Riemann surfaces of genus ≥ 2 for $n \geq 3$, in which case we are dealing with fibers of complex dimension ≥ 2.

In 1965, Feder [Fe] proved that any holomorphic immersion $\tau : \mathbb{P}^k \to \mathbb{P}^\ell$ between complex projective spaces is necessarily a linear embedding whenever $\ell < 2k$. He did this by using Whitney's formula on Chern classes associated to the tangent sequence of the holomorphic sequence, thereby proving that the degree $\tau_* : H_2(\mathbb{P}^k, \mathbb{Z}) \to H_2(\mathbb{P}^\ell, \mathbb{Z})$ must be 1 under the dimension restriction, a condition which forces the vanishing of the k-th Chern class of the holomorphic normal bundle. An adaptation of Feder's identity was used by Cao-Mok [CM] to study the Immersion Problem for the dual situation of holomorphic immersions between compact quotients of complex unit balls. For the Submersion Problem we have an associated short exact sequence, and the dual of that sequence is formally identical to the tangent sequence associated to holomorphic immersions, except that the role of the tangent bundle is played by the cotangent bundle. At the level of Chern classes Feder's identity remains applicable. Representing first Chern classes in terms of the canonical Kähler-Einstein metrics and higher Chern classes by means of the Proportionality Principle of Hirzebruch, we are able to prove the following vanishing result on certain differential forms which serves as a starting point for the proof of Theorem 1.

Here and in what follows we will denote by ω_n the Kähler form of the complete Kähler-Einstein metric of constant holomorphic sectional curvature -4π on the complex unit ball $B^n = \{z \in \mathbb{C}^n : \|z\|^2 < 1\}$. Writing the metric as $ds^2 = 2\mathrm{Re} \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ in local holomorphic coordinates, its curvature tensor is

$$R_{i\bar{j}k\bar{\ell}} = -2\pi (g_{i\bar{j}}\, g_{k\bar{\ell}} + g_{i\bar{\ell}}\, g_{k\bar{j}})$$

(see for example [Mok2, p.84]). The constant is chosen so that for the dual Kähler metric on the complex projective space $\mathbb{P}^n$, of constant holomorphic sectional curvature 4π, the Kähler form $\omega_{\mathbb{P}^n}$ represents the positive generator of $H^2(\mathbb{P}^n, \mathbb{Z})$, as can be seen from the Gauss-Bonnet formula on $\mathbb{P}^1$. For a quotient $X := B^n/\Gamma$ with respect to a torsion-free lattice Γ we will write ω_X for the Kähler form induced by ω_n. When a holomorphic mapping $F : B^n \to B^m$ is equivariant with respect to a representation of Γ, the (1,1)-form $F^*\omega_m$ is invariant under Γ, and it induces a (1,1)-form on X to be denoted by $\overline{\omega_m}$, bearing in mind that there is implicitly the underlying holomorphic map F. We have

Proposition 1. *Let $F : B^n \to B^m$, $X = B^n/\Gamma$ be as in the statement of Theorem 1. Then $\omega_n - F^*\omega_m$ is a nonnegative closed $(1,1)$-form which descends to $\omega_X - \overline{\omega_m}$ on X, and $\big[\omega_X - \overline{\omega_m}\big]^{n-m+1} = 0$ as an $(n-m+1, n-m+1)$–cohomology class. As a consequence $\big(\omega_X - \overline{\omega_m}\big)^{n-m+1} \equiv 0$ on X.*

Proof of Proposition 1. By the choice of normalization of canonical metrics it follows that the total Chern class of $\mathbb{P}^n$ is given by $\big(1+[\omega_{\mathbb{P}^n}]\big)^{n+1}$. By the Hirzebruch Proportionality Principle the total Chern class of the tangent bundle T_X is given by

$$c(T_X) = \big(1 - [\omega_X]\big)^{n+1}. \tag{1}$$

In particular $c_k(T_X)$ is a multiple of $[\omega_X]^k$ for $1 \le k \le n$. From the Φ-equivariant holomorphic submersion $F : B^n \to B^m$, the level sets of F define a Γ-equivariant holomorphic foliation which therefore descends to a holomorphic foliation $\mathcal{F}$ on $X = B^n/\Gamma$. We denote by $T_{\mathcal{F}}$ the associated distribution on X. Consider the short exact sequence $0 \to T_{\mathcal{F}} \to T_X \to N_{\mathcal{F}} \to 0$ on X, which defines the holomorphic normal bundle $N_{\mathcal{F}}$ of the foliation $\mathcal{F}$. Since $N_{\mathcal{F}}$ is obtained by pulling back the tangent bundle of B^m by $F : B^n \to B^m$ and descending to X, an analogue of (1) applies to $N_{\mathcal{F}}$, giving

$$c(N_{\mathcal{F}}) = \big(1 - [\overline{\omega_m}]\big)^{m+1}. \tag{2}$$

On the other hand, since the short exact sequence is defined everywhere on X, by Whitney's formula we have

$$\big(1 - [\omega_X]\big)^{n+1} = c(T_X) = c(T_{\mathcal{F}})c(N_{\mathcal{F}}) = c(T_{\mathcal{F}})\big(1 - [\overline{\omega_m}]\big)^{m+1}. \tag{3}$$

Since (B^n, ω_n) and (B^m, ω_m) are both equipped with complete Kähler-Einstein metrics of constant negative holomorphic sectional curvature of the same negative constant, by the Schwarz lemma we have $F^*\omega_m \le \omega_n$ (see for instance [Ro]). Moreover, the complex vector bundle $T_{\mathcal{F}}$ has rank $n-m$ hence $c_k(T_{\mathcal{F}}) = 0$ for all $k > n-m$. Proposition 1 is then a consequence of the following elementary algebraic identity taken from Feder [Fe] for which we include a proof for the sake of easy reference.

Lemma 1. *Let X be a real manifold and let $\alpha, \beta \in H^2(X, \mathbb{R})$. Let $0 \le m \le n$ be two integers and suppose for $1 \le k \le n-m$ there exists $\gamma_k \in H^{2k}(X, \mathbb{R})$ such that $(1+\alpha)^{n+1} = (1+\gamma_1+\cdots+\gamma_{n-m})(1+\beta)^{m+1}$. Then, $(\alpha-\beta)^{n-m+1} = 0$.*

Proof. Let $\gamma \in \bigoplus_{k=0}^{n} H^{2k}(X, \mathbb{R})$ be the formal quotient $(1+\alpha)^{n+1}(1+\beta)^{-(m+1)}$. Writing γ_k for the component of degree $2k$ in γ, the notation is consistent with those in the statement of Lemma 1, and we have $\gamma_{n-m+1} = 0$. We compute γ_{n-m+1} formally. For an element δ of the graded cohomology groups of even degrees we write δ_k for its element of degree $2k$. We have

$$
\begin{aligned}
0 = \gamma_{n-m+1} &= \big((1+\alpha)^{n+1}(1+\beta)^{-(m+1)}\big)_{n-m+1} \\
&= \sum_{k+\ell=n-m+1} (-1)^\ell \frac{(n+1)!}{k!\,(n-k+1)!}\,\frac{(m+\ell)!}{m!\,\ell!}\alpha^k\beta^\ell \\
&= \sum_{k+\ell=n-m+1} (-1)^\ell \frac{(n+1)!}{k!\,(n-k+1)!}\,\frac{(n-k+1)!}{m!\,\ell!}\alpha^k\beta^\ell \\
&= \sum_{k+\ell=n-m+1} (-1)^\ell \frac{(n+1)!}{(n-m+1)!\,m!}\,\frac{(n-m+1)!}{k!\,\ell!}\alpha^k\beta^\ell \\
&= \frac{(n+1)!}{(n-m+1)!\,m!}(\alpha-\beta)^{n-m+1}
\end{aligned}
$$

as desired. $\square$

We are now ready to give a proof of Theorem 1.

Proof of Theorem 1. First, we are going to show that F is a Riemannian submersion. Consider the closed (1,1)-form $\rho := \omega_X - \overline{\omega_m}$ on X. By Proposition 1, $\rho \geq 0$ and $\rho^{n-m+1} = 0$. Since by definition $F^*\omega_m$ vanishes on the level set $F^{-1}(w)$ for any $w \in B^m$, on B^n the (1,1)-form $\omega_n - F^*\omega_m$ agrees with ω_n on each level set of F, of dimension $n-m$, so that ρ must have at least $n-m$ positive eigenvalues everywhere on X. The identity $\rho^{n-m+1} \equiv 0$ implies that at every point $x \in X$, all other eigenvalues of $\rho(x)$ are zero. In other words, there exists an m-dimensional complex vector subspace $H_x \subset T_{X,x}$ transversal to $T_{\mathcal{F},x}$ such that $\rho(x)|_{H_x} \equiv 0$.

From the short exact sequence $0 \to T_{\mathcal{F}} \to T_X \to N_{\mathcal{F}} \to 0$ there are two different ways to endow the holomorphic vector bundle $N_{\mathcal{F}}$ with a Hermitian metric. First, endowing the holomorphic tangent bundle T_X with the Hermitian metric g_X defined by the Kähler form ω_X, $N_{\mathcal{F}} = T_X/T_{\mathcal{F}}$ inherits a Hermitian metric h as a Hermitian holomorphic quotient vector bundle. On the other hand, Γ acts on the Hermitian holomorphic vector bundle $F^*T_{B^m}$, and the latter descends to a Hermitian holomorphic vector bundle on X which is isomorphic to $N_{\mathcal{F}}$ as a holomorphic vector bundle. From this we obtain another Hermitian metric h' on X. We argue that the two Hermitian metrics h and h' agree with each other. To see this, at a given $x \in X$, write $T_{X,x} = T_{\mathcal{F},x} \oplus H_x$ as a direct sum of complex vector spaces. When measured against the Hermitian inner product $g_X(x)$ on $T_{X,x}$, the direct summands $T_{\mathcal{F},x}$, H_x are eigenspaces of the Hermitian inner product defined by $\rho(x)$ corresponding to the eigenvalues 1, 0 respectively. As a consequence they must be orthogonal to each other. For an element $\eta \in N_{\mathcal{F},x}$, the norm $\|\eta\|_h$ of η with respect to h is given by the minimum norm of $\|\widetilde{\eta}\|_{g_X}$ with respect to g_X, as $\widetilde{\eta}$ ranges over all (1,0)-vectors at x which projects to η modulo $T_{\mathcal{F},x}$. Now since $T_{X,x} = T_{\mathcal{F},x} \oplus H_x$ is an orthogonal decomposition, η lifts to $\eta_0 \in H_x$, and $\|\eta\|_h$ is nothing other than $\|\eta_0\|_{g_X}$. However, as $\rho|_{H_x} \equiv 0$, we have $\overline{\omega_m}|_{H_x} \equiv \omega_X|_{H_x}$, so that $\|\eta_0\|_g = \|\eta_0\|_{h'}$, proving that $h \equiv h'$. In other words, $F : B^n \to B^m$ is an isometric submersion in the sense of Riemannian geometry.

Therefore, if $m = n$, F induces a local isometry of B^n onto itself, so that F sends local, hence global, geodesics of B^n to geodesics of B^n. Since a geodesic in B^n minimizes distance between any two of its points, F must be injective and proper, hence a biholomorphism.

We suppose now that $n > m$ and we wish to get a contradiction. We can compute the curvature tensor of the Chern connection on $N_{\mathcal{F}}$ associated to the metric $h = h'$ in two different ways. Denote by $\sigma \in C^\infty_{1,0}(X, \mathrm{Hom}(T_{\mathcal{F}}, N_{\mathcal{F}}))$ the second fundamental form of the Hermitian holomorphic vector subbundle $T_{\mathcal{F}} \subset T_X$ with respect to the Kähler-Einstein metric g_X and by $\sigma^* \in C^\infty_{0,1}(X, \mathrm{Hom}(N_{\mathcal{F}}, T_{\mathcal{F}}))$ its adjoint. Let $x \in X$ and let $(\xi_1, \ldots, \xi_n)$ be an orthonormal basis of $T_{X,x}$ with respect to $g_X(x)$ such that $\xi_{m+1}, \ldots, \xi_n$ belong to $T_{\mathcal{F},x}$. Writing Θ, resp. Θ', for the curvature tensor of (T_X, g_X), resp. $(N_{\mathcal{F}}, h)$, at x we have by a classical computation of Griffiths

$$\Theta'_{j\bar{k}\lambda\bar{\mu}} = \Theta_{j\bar{k}\lambda\bar{\mu}} + \left\langle \sigma^*_{\bar{\xi}_k}(\xi_\lambda), \overline{\sigma^*_{\bar{\xi}_j}(\xi_\mu)} \right\rangle_{g_X}$$

for any $j, k \in \{1, \ldots, n\}$ and any $\lambda, \mu \in \{1, \ldots, m\}$, where we have identified $N_{\mathcal{F},x}$ with $H_x = T^\perp_{\mathcal{F},x}$ (see for example [Dem, Chapter V, section 14] or [GH, p.78]). But recall that

$$\Theta_{j\bar{k}\lambda\bar{\mu}} = -2\pi(\delta_{jk}\delta_{\lambda\mu} + \delta_{j\mu}\delta_{\lambda k})$$

(cf. last paragraph before Proposition 1). Since $h = h'$, Θ' is the pullback by dF of the curvature tensor of T_{B^m}:

$$\Theta'_{j\bar{k}\lambda\bar{\mu}} = \begin{cases} -2\pi(\delta_{jk}\delta_{\lambda\mu} + \delta_{j\mu}\delta_{\lambda k}), & \forall\, j, k, \lambda, \mu \in \{1, \ldots, m\} \\ 0 \ \text{ in other cases.} \end{cases}$$

Comparing the different relations, we easily deduce that the family

$$\left\{ \frac{\sigma^*_{\bar{\xi}_j}(\xi_\lambda)}{\sqrt{2\pi}} \ : \ (j, \lambda) \in \{m+1, \ldots, n\} \times \{1, \ldots, m\} \right\}$$

of vectors of $T_{\mathcal{F},x}$ is orthonormal. If $m \geq 2$, it is clear (because of the dimensions) that the latter property can never be verified and we get the desired contradiction. If $m = 1$ then $N_{\mathcal{F}}$ is a line bundle. For any $x \in X$, if $\xi, \xi' \in T_{\mathcal{F},x}$ and $\eta \in N_{\mathcal{F},x}$

$$\left\langle \sigma_\xi(\xi'), \overline{\eta} \right\rangle_h = \left\langle \xi', \overline{\sigma^*_{\bar{\xi}}(\eta)} \right\rangle_{g_X}$$

and the above property implies that the restriction of $\sigma : T_{\mathcal{F}} \otimes T_{\mathcal{F}} \longrightarrow N_{\mathcal{F}}$ is everywhere non-degenerate. In other words, if $\Omega_{\mathcal{F}}$ denotes the dual bundle of $T_{\mathcal{F}}$, then $T_{\mathcal{F}}$ and $\Omega_{\mathcal{F}} \otimes N_{\mathcal{F}}$ are isomorphic as smooth complex vector bundles (actually, they are isomorphic as holomorphic vector bundles since σ is in fact holomorphic, but we will not need that in what follows). In particular, their first Chern classes coincide hence $c_1(T_{\mathcal{F}}) = -c_1(T_{\mathcal{F}}) + (n-1)(-2\,[\overline{\omega_1}])$, i.e., $c_1(T_{\mathcal{F}}) = -(n-1)\,[\overline{\omega_1}]$. Now, still as cohomology classes,

$$-(n+1)\,[\omega_X] = c_1(T_X) = c_1(T_{\mathcal{F}}) + c_1(N_{\mathcal{F}}) = -(n+1)\,[\overline{\omega_1}]$$

and this is impossible because $\int_X \omega_X^n > 0$ and $\int_X \overline{\omega_1}^n = 0$ as $n > 1$. The proof of Theorem 1 is complete. $\square$

§2 On the singular loci of holomorphic submersions

From the proof of Theorem 1 we also obtain the following result regarding regular holomorphic fibrations which also applies to the case where the target manifold is a compact complex torus or the complex projective space. The complex unit ball equipped with a complete Kähler-Einstein metric is sometimes referred to as the complex hyperbolic space. In this context a quotient of the complex unit ball by a torsion-free discrete group of biholomorphic automorphisms is sometimes called a complex hyperbolic space form.

Theorem 2. *Let $n > m \geq 1$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free cocompact lattice of biholomorphic automorphisms, $X := B^n/\Gamma$. Let Z be an m-dimensional compact complex hyperbolic space form, compact complex torus or complex projective space. Let $f : X \to Z$ be a surjective holomorphic map and denote by $E \subset Z$ the smallest subvariety such that f is a regular holomorphic fibration over $Z - E$. Then, there is no compact analytic subvariety of positive dimension in $Z - E$. In particular, $E \subset Z$ has at least one irreducible component of complex codimension 1.*

Proof. On $f^{-1}(Z - E)$ we have the short exact sequence of holomorphic vector bundles $0 \to T_f \to T_X \to N \to 0$, where N denotes the holomorphic normal bundle of the holomorphic foliation $\mathcal{F}$ defined by the relative tangent bundle T_f, so that $c(T_X) = c(T_f)c(N)$ holds on it. Let $Q \subset Z - E$ be an irreducible compact complex-analytic curve. Restrict the short exact sequence to the compact complex-analytic subvariety $f^{-1}(Q)$, even if Q may have singularities. Denote by ω_Z the closed (1,1)-form on Z such that $-(m+1)\omega_Z$ is the first Chern form of the canonical Kähler-Einstein metric g_Z on Z. This is consistent with the notation in the proof of Proposition 1 if Z is a compact quotient of the m-ball, and is defined for the case of compact complex tori and the complex projective space in such a way that Proposition 1 remains applicable. We conclude that $[\omega_X - f^*\omega_Z]^{n-m+1} = 0$ as a cohomology class on $f^{-1}(Q)$.

In case Z is a compact complex torus or the complex projective space, this is already a contradiction since ω_Z is nonpositive and hence $\omega_X - f^*\omega_Z$ is strictly positive, and the subtle case is the complex hyperbolic case, where $\omega_X - f^*\omega_Z$ is only known to be nonnegative. In that case, arguing as in the proof of Theorem 1, we find that $f_{|f^{-1}(Q)} : f^{-1}(Q) \to Q$ is a Riemannian submersion above each nonsingular point of Q, if $f^{-1}(Q)$, resp. Q, is endowed with the metric induced by g_X, resp. g_Z. Let $z \in Q$ be a regular point of Q and consider the exact sequence of bundles on the fiber $P = f^{-1}(z)$

$$0 \longrightarrow T_P \longrightarrow T_{f^{-1}(Q)_{|P}} \longrightarrow N_P \longrightarrow 0$$

where T_P is the tangent bundle to the manifold P and $T_{f^{-1}(Q)}$ is the tangent bundle to (the regular part of) $f^{-1}(Q)$, both equipped with the metrics induced by g_X. We also endow the normal bundle $N_P = T_{f^{-1}(Q)_{|P}}/T_P$ with the quotient metric denoted by h. Let $x \in P$ and let $(\xi_1, \ldots, \xi_{n-m+1})$ be an orthonormal basis of $T_{f^{-1}(Q),x}$ such that $\xi_2, \ldots, \xi_{n-m+1}$ are tangent to the fiber P. Writing Φ, resp. Φ', for the curvature tensor of $T_{f^{-1}(Q)_{|P}}$, resp. N_P, we have for any $2 \le j, k \le n-m+1$,

$$0 = \Phi'_{j\bar{k}1\bar{1}} = \Phi_{j\bar{k}1\bar{1}} + \left\langle \tau^*_{\bar{\xi}_k}(\xi_1), \overline{\tau^*_{\bar{\xi}_j}(\xi_1)} \right\rangle_{g_X},$$

where the second fundamental form $\tau \in C^\infty(P, \mathrm{Hom}(S^2 T_P, N_P))$ of the submanifold $P \subset X$ is considered as an element of $C^\infty_{1,0}(P, \mathrm{Hom}(T_P, N_P))$, and $\tau^* \in C^\infty_{0,1}(P, \mathrm{Hom}(N_P, T_P))$ is its adjoint (again, we identify N_P with $T_P^\perp$). In particular, because of the Chern-Weil formula ($T_{f^{-1}(Q)_{|P}}$ being a holomorphic subbundle of $T_{X_{|P}}$), for any $\eta \in N_{P,x}$ and any $\xi \in T_{P,x}$,

$$2\pi \|\xi\|^2 \|\eta\|^2 \le -\Phi_{\xi\bar{\xi}\eta\bar{\eta}} = \left\langle \tau^*_{\bar{\xi}}(\eta), \overline{\tau^*_{\bar{\xi}}(\eta)} \right\rangle_{g_X} = \left\langle \tau_\xi(\tau^*_{\bar{\xi}}(\eta)), \overline{\eta} \right\rangle_h.$$

Since the line bundle N_P is trivial along P, τ can be seen as a symmetric bilinear form on T_P. By the previous inequality, τ is non-degenerate on $T_{P,x}$ and this is true for any $x \in P$. As a consequence, T_P and T_P^* are isomorphic as smooth bundles and therefore $c_1(T_P) = 0$. But $c_1(T_P) < 0$ because X is Kähler-Einstein with negative Einstein constant and T_P is a holomorphic subbundle of $T_{X|P}$ such that the quotient bundle $N_{|P}$ is trivial, thus $c_1(T_P) = c_1(X)_{|P} < 0$, a plain contradiction. Thus there cannot be any irreducible compact complex-analytic curve $Q \subset Z - E$, and the proof of Theorem 2 is complete. $\square$

From the statement of Theorem 2 we deduce readily

Corollary 2. *A compact complex hyperbolic space form does not admit any regular holomorphic fibration over a compact Kähler manifold of constant holomorphic sectional curvature* (*i.e., a compact hyperbolic space form, a compact complex torus, or a complex projective space*)*. As a consequence, a compact complex hyperbolic space form does not admit any regular holomorphic fibration over a compact Riemann surface.*

Remarks. Corollary 2 follows already from an easy extension of the proof of Theorem 1 to cover the case where the target manifold is the complex Euclidean space or the complex projective space (as included in the proof of Theorem 2). The last statement of Corollary 2 was covered by Liu [Liu] in the special case when the domain manifold is of dimension 2. Corollary 2 leaves open the question whether nontrivial regular holomorphic fibrations over higher dimensional base manifolds can exist on compact complex hyperbolic space forms of dimension ≥ 3.

We include some results which follow readily from modifications of the proof of Theorem 1. In the proof of Theorem 1, where we derive a contradiction by assuming that the Φ-equivariant holomorphic mapping $F : B^n \to B^m$ is a holomorphic submersion, the argument actually works to arrive at a contradiction provided that the set of singularities of F, i.e., the subset $\mathrm{Sing}(F) \subset B^n$ over which dF is not of maximal rank, is sufficiently small. More precisely, if $\mathrm{Sing}(F)$ is of dimension $< m-1$, then removing $\mathrm{Sing}(F)$ has no effect on $c_{n-m+1}(N_{\mathcal{F}})$. This is so because a generic linear section of X of dimension $n-m+1$ does not intersect the locus $S \subset X$, where $S := \mathrm{Sing}(F)/\Gamma$. From this we deduce

Theorem 3. *Let $n > m \geq 1$ and let $\Gamma \subset \mathrm{Aut}(B^n)$ be a cocompact lattice of biholomorphic automorphisms. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a homomorphism into the automorphism group $\mathrm{Aut}(B^m)$ of B^m, and $F : B^n \to B^m$ be a Φ-equivariant holomorphic map which is a submersion at some point. Let $\mathrm{Sing}(F) \subset B^n$ be the Γ-invariant subset consisting of points where F fails to be a submersion,* i.e.*, where dF is of rank $< m$, which descends to a complex-analytic subvariety $S \subset X$. Then, S is nonempty and* $\dim(S) \geq m-1$.

Remarks. Note that Theorem 3, applied to the special case of holomorphic submersions $f : X \to Z$ between compact complex hyperbolic space forms, does not imply Theorem 2. In fact, from the statement of Theorem 3, it does not even follow that the image $f(S) = E \subset Z$ is of dimension $m-1$, i.e., that E has codimension 1 in Z, since we do not know that the fiber of $f|_S : S \to E$ over a general point of E is isolated. Beyond saying that $E \subset Z$ is of codimension 1, Theorem 2 actually suggests that the codimension 1 components of E resemble an ample divisor.

For the study of Γ-equivariant holomorphic submersions $F : B^n \to B^m$ as in the proof of Theorem 3, the arguments remain valid (with a simpler proof) when the target manifold is replaced by the Euclidean space or the complex projective space (cf. the proof of Theorem 2). In particular, suppose $\nu_1, \cdots, \nu_m$ are m linearly independent holomorphic 1-forms on a compact quotient $X = B^n/\Gamma$ of the complex unit ball B^n, an analogue of Theorem 3 remains valid for the holomorphic mapping $F : B^n \to \mathbb{C}^m$ obtained by integrating pull-backs of the holomorphic 1-forms $\nu_1, \cdots, \nu_m$ by the universal covering map $\pi : B^n \to X$. In other words, the following statement on singularities of the meromorphic foliation defined by level sets of F holds true.

Theorem 4. *Let $n > m \geq 1$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a cocompact lattice of biholomorphic automorphisms, $X := B^n/\Gamma$. Let $\nu_1, \cdots, \nu_m$ be m holomorphic 1-forms on X which are linearly independent at a general point of X. Let $S \subset X$ be the subvariety where $\nu_1, \cdots, \nu_m$ fail to be linearly independent. Then,* $\dim(S) \geq m-1$.

§3 Generalization to complex hyperbolic space forms of finite volume

In this section, we prove a version of Theorem 1 in the case where $\Gamma \subset \mathrm{Aut}(B^n)$ is a non-uniform lattice. This means that the quotient B^n/Γ is non-compact, but the volume of X with respect to ω_X is finite. Our arguments are quite elementary, in the sense that they do not make use of any compactification of X.

Theorem 1'. *Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a lattice of biholomorphic automorphisms. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a homomorphism and $F : B^n \to B^m$ be a holomorphic* *submersion equivariant with respect to Φ. Suppose either $m \geq 2$ or $\Gamma \subset \mathrm{Aut}(B^n)$ is cocompact. Then, $m = n$ and $F \in \mathrm{Aut}(B^n)$.*

When $\Gamma \subset \mathrm{Aut}(B^n)$ is a non-uniform lattice, the hypothesis $m \geq 2$ is needed in the statement. In fact, for $m = 1$ there are plenty of unramified holomorphic maps between Kobayashi-hyperbolic punctured Riemann surfaces which are not topological coverings (cf. remarks after the proof).

Proof of Theorem 1'. We only have to prove the theorem when $\Gamma \subset \mathrm{Aut}(B^n)$ is torsion-free and $X := B^n/\Gamma$ is non-compact. In this situation, we can argue exactly as in the proof of Theorem 1 if we show that Proposition 1 is still valid. In fact, the proof of the vanishing of the cohomology class $\left[\omega_X - \overline{\omega_m}\right]^{n-m+1}$ goes along the same line of argument, but we need to explain why $\left(\omega_X - \overline{\omega_m}\right)^{n-m+1} \equiv 0$ on X. Let us remark that when $m = 1$, $n - m + 1 = n$, and in this case the vanishing of the class above does not give any information since $H^{2n}(X, \mathbb{R}) = 0$ if X is non-compact.

In order to continue the proof, we need to recall some facts about the geometry of the manifold X (see W. M. Goldman's book [Go] or [KM] for more details). It is the union of a compact part and of a finite number of disjoint cusps. Each cusp C is diffeomorphic to a product $N \times [0, +\infty)$ where N is a compact quotient of a horosphere HS of B^n centered at a point $\infty \in \partial B^n$. The fundamental group Γ_C of C may be identified with the stabilizer in Γ of the horosphere HS.

Let $(z, v, t) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}$ be horospherical coordinates associated to HS. The 1-form $\varsigma = \frac{1}{2\pi} e^{-t}(2\mathrm{Im}\langle\langle z, dz\rangle\rangle - dv)$, as well as dz, t and dt^2 are invariant by Γ_C and the metric g_X takes the form

$$g_X = \frac{1}{2\pi}\left(dt^2 + \varsigma^2 + 4e^{-t}\langle\langle dz, dz\rangle\rangle\right)$$

in the cusp C. The fundamental remark is that on the cusp C, $d\varsigma = \omega_X$ (the invariance of ς by Γ_C allows to descend to C) and, because of the shape of the metric, $|\varsigma|_{g_X}$ is constant.

Let us go back to the proof of the proposition. Let K be a compact subset of X which contains in its interior the compact part of X. Let $\chi \in C^\infty(X, \mathbb{R})$ be a smooth function vanishing on the compact part of X, and equal to 1 on $X \backslash K$. The 1-form $\alpha = \chi\varsigma$ is well-defined on X. (Here the definition of ς depends of course on the cusp.) Moreover, the 2-form $(\omega_X - d\alpha)$ has compact support in X. Therefore,

$$\int_X \left(\omega_X - \overline{\omega_m}\right)^{n-m+1} \wedge \omega_X^{m-2} \wedge (\omega_X - d\alpha) = 0$$

since $\left(\omega_X - \overline{\omega_m}\right)^{n-m+1}$ vanishes in $H^{n-m+1,n-m+1}(M, \mathbb{R})$ and it is integrated against a d-closed form with compact support.

As (X, g_X) is complete, there exists an exhaustive sequence $(K_\nu)_{\nu \in \mathbb{N}}$ of compact subsets of X and smooth cut-off functions $\psi_\nu : X \to [0, 1]$ which are identically equal to one on K_ν, vanish on $X \backslash K_{\nu+1}$, and verify $|d\psi_\nu|_{g_X} \leq 2^{-\nu}$. Now, by the Schwarz lemma, $\left|\left(\omega_X - \overline{\omega_m}\right)^{n-m+1}\right|_{g_X}$ is bounded on X by some constant. Noting that (X, ω_X) is of finite volume,

$$\lim_{\nu \to +\infty} \int_X \left(\omega_X - \overline{\omega_m}\right)^{n-m+1} \wedge \omega_X^{m-2} \wedge d\psi_\nu \wedge \alpha = 0$$

and then

$$\int_X \left(\omega_X - \overline{\omega_m}\right)^{n-m+1} \wedge \omega_X^{m-2} \wedge d\alpha = \lim_{\nu \to +\infty} \int_X \left(\omega_X - \overline{\omega_m}\right)^{n-m+1} \wedge \omega_X^{m-2} \wedge d(\psi_\nu \, \alpha) = 0.$$

We immediately deduce that

$$\int_X \left(\omega_X - \overline{\omega_m}\right)^{n-m+1} \wedge \omega_X^{m-1} = 0$$

and then, that the conclusion of Proposition 1 is true. The proof of Theorem 1' is complete. □

REMARKS.

1) Theorem 1', as it is stated, is trivially false when X is non-compact and $n = 1$. For example, let $\Upsilon \subset \mathrm{Aut}(B^1)$ be any cocompact lattice, $Y = B^1/\Upsilon$. Let X be the same Riemann surface as Y with $p > 0$ punctures. Then, there exists a lattice $\Gamma \subset \mathrm{Aut}(B^1)$ such that X is biholomorphic to B^1/Γ but the embedding $X \hookrightarrow Y$, which is also a submersion, is not isometric.

2) Other than standard examples (i.e. induced by a totally geodesic embedding of B^n into B^m), very few examples of representations of lattices of $\mathrm{Aut}(B^n)$ into $\mathrm{Aut}(B^m)$ are known. Nevertheless, one can find some examples in Deligne-Mostow [DM] — based on a construction of R. Livné [Liv] — of non-trivial holomorphic maps $f : X \longrightarrow Y$ between compact complex hyperbolic manifolds, with $n = 2$ and $m = 1$ (cf. also Kapovich [Ka]). We also refer to the examples of Mostow in the case $n = m = 2$ detailed by Toledo [To]. Notice that all these examples come from "forgetful maps" between Deligne-Mostow ball quotients in the sense that they are induced by a (possibly trivial) contraction of the weights involved in the definition of Deligne-Mostow lattices. In [Der], M. Deraux gives the complete list of the forgetful maps which lead to a holomorphic map between ball quotients. It happens that the method provides one new example with $n = 3$ and $m = 1$, but none for any $n > m$ with $n \geq 4$ or $m \geq 2$. Observe that this result, though interesting, is far from answering Siu's question about the existence of surjective holomorphic maps between compact complex hyperbolic manifolds raised in [Siu2], because the lattices as well as the maps considered in [Der] are very special.

3) When Γ is arithmetic, the Satake-Borel-Baily compactification $\overline{X}$, which is projective-algebraic, is obtained by adding a finite number of cusps, which are isolated singularities of $\overline{X}$. The proof of Theorem 1' can then be obtained by slicing $\overline{X}$ to obtain hyperplane sections which avoid the cusps. When $m \geq 2$ then $n - m + 1 \leq n - 1$. The vanishing of $(\omega_X - \overline{\omega_m})^{n-m+1}$ on all such hyperplane

sections implies its vanishing on X, which gives Theorem 1'. The same argument applies in the case of non-arithmetic quotients. Here it is known that X can be compactified by adding a finite number of points (Siu-Yau [SY]), but the proof of its projective-algebraicity does not seem to be available in the literature. In place of overloading the article with writing down a self-contained proof of the latter, we have chosen to present the more elementary argument here.

4) For $\Gamma \subset \mathrm{Aut}(B^n)$ a non-uniform lattice, the case of Theorem 1' for $n = m \geq 2$ is already non-trivial. It implies in particular that a local biholomorphism from a non-compact complex hyperbolic space form of finite volume into a complex hyperbolic space form is necessarily a covering map, which was established by Mok [Mok2, p.174ff.] in a much more elaborate way by the method of Hermitian metric rigidity applied to certain homogeneous holomorphic vector bundles.

§4 Structure of holomorphic submersions from finite volume quotients of bounded symmetric domains into the complex unit ball

In this section we consider the general structure of holomorphic submersions of quotients of bounded symmetric domains Ω into the complex unit ball. Let $\Omega = \Omega_1 \times \cdots \times \Omega_q$ be the decomposition of Ω into a product of irreducible bounded symmetric domains $\Omega_i, 1 \leq i \leq q$. We assume that each Ω_i is noncompact and denote by $\mathrm{Aut}(\Omega_i)$ the group of biholomorphic automorphisms of Ω_i. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a lattice. After passing to a subgroup of Γ of finite index, one can always assume that Γ is torsion-free and belongs to $\mathrm{Aut}_0(\Omega)$, the identity component of $\mathrm{Aut}(\Omega)$. Then, there exists a partition $I_1, \ldots, I_p$ of $\{1, \ldots, q\}$ and irreducible lattices $\Gamma_{I_k} \subset \Pi_{i \in I_k} \mathrm{Aut}_0(\Omega_i)$ such that $\Gamma = \Pi_{i=1}^p \Gamma_{I_k}$. The fact that the Γ_{I_k} are irreducible means that for any proper subset J of I_k the projection of Γ_{I_k} into $\Pi_{j \in J} \mathrm{Aut}_0(\Omega_j)$ is dense (see [Ra, Cor. 5.21]). We shall denote by X the quotient manifold Ω/Γ. Note that the tangent bundle of X has a natural decomposition $T_X = T_{X,1} \oplus \cdots \oplus T_{X,q}$ coming from the decomposition of Ω.

The following result is a consequence of Theorems 1 and 1', and of Hermitian metric rigidity (see [Mok1] to which we will frequently refer below).

Theorem 5. *Let Ω, Γ and X be as above. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a homomorphism and $F : \Omega \to B^m$ be a holomorphic submersion equivariant with respect to Φ. Suppose that $m \geq 2$ or that X is compact. Then, there exists a Hermitian symmetric space Ω' such that $\Omega = B^m \times \Omega'$ and F is the natural projection onto B^m composed with an element of $\mathrm{Aut}(B^m)$.*

Proof. We show first that there exists $\ell \in \{1, \ldots, q\}$ such that the restriction of dF to $\oplus_{j \neq \ell} T_{X,j}$ vanishes identically. Let $z^{(i)} = (z_1^{(i)}, \ldots, z_{n_i}^{(i)})$ be Euclidean coordinates on Ω_i in terms of its Harish-Chandra embedding and $z = (z^{(1)}, \ldots, z^{(q)})$ be the corresponding coordinates on Ω. We endow Ω_i with the Bergman metric h_i and denote by $\pi_i : \Omega \to \Omega_i$ the natural projection. Consider now the Kähler metric $h = \sum \pi_i^* h_i + F^* g_m$ on Ω. We note once for all that, because of the Schwarz lemma, h is dominated by a constant multiple of $\sum \pi_i^* h_i$. Indeed, the holomorphic sectional curvature of h_i is negative and bounded from below, and the holomorphic sectional curvature on B^m is negative and constant.

The metric h descends to X. From the proof of Theorem 4 in [Mok1] it follows easily that the 2-tensor $F^* g_m$ is given by

$$F^* g_m = 2\,\mathrm{Re}\Big(\sum_{\substack{1\le i\le q\\ 1\le j,k\le n_i}} a^{(i)}_{j\bar{k}}(z^{(i)})\, dz^{(i)}_j \otimes d\bar{z}^{(i)}_k + 2 \sum_{\substack{1\le i<i'\le q\\ 1\le j\le n_i\\ 1\le k\le n_{i'}}} b^{(i,i')}_{j\bar{k}}(z)\, dz^{(i)}_j \otimes d\bar{z}^{(i')}_k \Big)$$

for some functions $a^{(i)}_{j\bar{k}} : \Omega_i \to \mathbb{C}$, $b^{(i,i')}_{j\bar{k}} : \Omega \to \mathbb{C}$ (note that the functions $a^{(i)}_{j\bar{k}}$ depend only on the coordinates of Ω_i). In fact, if Ω_i has rank at least two then the matrix of functions $(a^{(i)}_{j\bar{k}})$ defines the canonical Kähler-Einstein metric on Ω_i (the (ir)reducibility of Γ does not play any role in that case). If Ω_i is of rank one, the functions $a^{(i)}_{j\bar{k}}$ define the canonical Kähler-Einstein metric on Ω_i whenever the cardinality of the multi-index I_k containing i is at least 2. Since the $(1,1)$-form associated to $F^* g_m$ is d-closed, $b^{(i,i')}_{j\bar{k}}$ must be holomorphic in the $z^{(i)}$-coordinates and antiholomorphic in the $z^{(i')}$-coordinates.

Suppose now that there exists $x \in \Omega$, two integers $i \neq i'$ and tangent vectors $\xi = \partial/\partial z^{(i)}_j, \eta = \partial/\partial z^{(i')}_{j'} \in T_{\Omega,x}$ whose images by $d_x F$ do not vanish. We endow the vector bundle $T_{X,i} \oplus T_{X,i'}$ with the metric $\pi^* h_i + \pi^* h_{i'} + (F^* g_m)_{|T_{X,i}\oplus T_{X,i'}}$ and denote by Θ its curvature tensor. In other words, $T_{X,i} \oplus T_{X,i'}$ is identified with a holomorphic vector subbundle of $T_{X,i} \oplus T_{X,i'} \oplus F^* T_{B^m}$ by the embedding $i(\eta_i, \eta_j) = (\eta_i, \eta_j, df(\eta_i + \eta_j))$, from which it inherits a Hermitian metric. Because of the curvature decreasing property of the curvature for holomorphic subbundles, and since bisectional curvatures of B^m are strictly negative, we have $\Theta_{\xi\bar{\xi}\eta\bar{\eta}} \neq 0$ (see the proof of Lemma 2 in [Mok1, p.126]). But in the previous notation,

$$(F^* g_m)_{|T_{X,i}\oplus T_{X,i'}} = 2\,\mathrm{Re}\Big(\sum_{1\le j,k\le n_i} a^{(i)}_{j\bar{k}}(z^{(i)})\, dz^{(i)}_j \otimes d\bar{z}^{(i)}_k + \sum_{1\le j,k\le n_{i'}} a^{(i')}_{j\bar{k}}(z^{(i')})\, dz^{(i')}_j \otimes d\bar{z}^{(i')}_k + 2 \sum_{\substack{1\le j\le n_i\\ 1\le k\le n_{i'}}} b^{(i,i')}_{j\bar{k}}(z)\, dz^{(i)}_j \otimes d\bar{z}^{(i')}_k \Big)$$

and from the properties of the functions $a^{(i)}_{j\bar{k}}$, $a^{(i')}_{j\bar{k}}$ and $b^{(i,i')}_{j\bar{k}}$, we deduce that the value of $\Theta_{\xi\bar{\xi}\eta\bar{\eta}}$ is not affected by the presence of the term $F^* g_m$ and hence that $\Theta_{\xi\bar{\xi}\eta\bar{\eta}} = 0$. This is a contradiction, thus dF must vanish in the direction of all the Ω_i but one at each point of Ω. By holomorphicity of F, dF must vanish identically in the direction of each factor except one, say Ω_1. One can therefore regard F as a map from Ω_1 to B^m.

As a consequence, we can assume that $\Gamma = \Gamma_{I_1}$ with $1 \in I_1$. If $I_1 = \{1\}$ then Theorem 1 of [Mok1] applied to $(\Omega_1/\Gamma_1, h_1)$ and the metric $h_1 + F^* g_m$ implies that Ω_1 must be a complex unit ball, otherwise $X_1 = \Omega_1/\Gamma_1$ is of rank ≥ 2, and $F^* g_m = c\, h_1$ for some constant c by Hermitian metric rigidity, which is impossible as g_m is of strictly negative bisectional curvature. So we can apply Theorem 1 or 1', and Theorem 5 is proved in this case. In other cases, it follows directly from Theorem 4 of [Mok1] that $F^* g_m = c\, h_1$ for some global constant c. Then, necessarily, $\Omega_1 = B^m$ and $F : B^m \to B^m$ must be injective and proper, so $F \in \mathrm{Aut}(B^m)$. $\square$

REMARKS. If, in the previous theorem, the image of Φ is assumed to be discrete (which is the case whenever F is induced by a holomorphic map from X to a complex hyperbolic space form) then Γ must be reducible. More precisely, $\Gamma = \Gamma_0 \times \Gamma'$ where $\Gamma_0 \subset \mathrm{Aut}(B^m)$ and $\Gamma' \subset \mathrm{Aut}_0(\Omega')$ are lattices. Indeed, if this were not the case, the projection $\mathrm{pr}_1(\Gamma)$ into $\mathrm{Aut}(B^m)$ would be dense. This is impossible since $\mathrm{pr}_1(\Gamma)$ is conjugate to $\Phi(\Gamma)$ in $\mathrm{Aut}(B^m)$, by Theorem 5.

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