Weyl formula for the negative dissipative eigenvalues of Maxwell's equations

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Abstract. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup generated by Maxwell's equations in an exterior domain $\Omega \subset \mathbb{R}^3$ with dissipative boundary condition $E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0, \gamma(x) > 0, \forall x \in \Gamma = \partial\Omega$. We study the case when $\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$ and $\gamma \neq 1$ is a constant. We establish a Weyl formula for the counting function of the negative real eigenvalues of G_b.

Keywords. Maxwell system with dissipative boundary conditions, Counting function for negative eigenvalues, Weyl formula.

1. Introduction

Let $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ be an open connected domain and let $\Omega = \mathbb{R}^3 \setminus \bar{K}$ be connected domain with C^∞ smooth boundary Γ. Consider the boundary problem

$$\begin{aligned} &\partial_t E = \operatorname{curl} B, \qquad \partial_t B = -\operatorname{curl} E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega, \\ &E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma, \\ &E(0,x) = E_0(x), \qquad B(0,x) = B_0(x). \end{aligned} \tag{1.1}$$

with initial data $f = (E_0, B_0) \in L^2(\Omega; \mathbb{C}^6) = \mathcal{H}$. Here $\nu(x)$ is the unit outward normal to $\partial\Omega$ at $x \in \Gamma$ pointing into Ω, $\langle\, , \rangle$ denotes the scalar product in $\mathbb{C}^3$, $u_{tan} := u - \langle u, \nu\rangle\nu$, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. Let

$$G = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}$$

and let G_b be the operator G with domain $D(G_b)$ which is the closure in the graph norm

$$|||u||| = (\|u\|_{\mathcal{H}}^2 + \|Gu\|_{\mathcal{H}}^2)^{1/2}$$

of functions $u = (v, w) \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^6)$ satisfying the boundary condition $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$ on Γ. The operator G_b generates a contraction semigroup $V(t)$ in $\mathcal{H}$ (see for instance Theorem 3.1.8 and Section 3.8 in [6]) and the solution of the problem (1.1) is described by

$$(E, B) = V(t)f = e^{tG_b}f, \; t \geq 0.$$

In [1] it was proved that the spectrum of G_b in the open half plan $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is formed by isolated eigenvalues with finite multiplicities. Note that if $G_b f = \lambda f$ with $\operatorname{Re} \lambda < 0$, the solution $u(t,x) = V(t)f = e^{\lambda t} f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are very important for the inverse scattering problems (see [1]). In particular, the eigenvalues λ with $\operatorname{Re} \lambda \to -\infty$ imply a very fast decay of the corresponding solutions. In [2] the existence of eigenvalues of G_b has been studied for the ball $B_3 = \{x \in \mathbb{R}^3 : \; |x| < 1\}$ assuming γ constant. It was proved that for $\gamma = 1$ there are no eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, while for $\gamma = $ const , $\gamma \neq 1$, there is always an infinite number of real eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ and with exception of λ_1 they satisfy the estimate

$$\lambda_n \; \leq \; -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}} = -c_0 \,, \tag{1.2}$$

where $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$.

In this paper we study the distribution of the negative eigenvalues and our purpose is to obtain a Weyl formula for the counting function

$$N(r) = \#\{\lambda \in \sigma_p(G_b) \cap \mathbb{R}^- : \; |\lambda| \leq r\}, \; r > r_0(\gamma),$$

where every eigenvalues λ_n is counted with its algebraic multiplicity given by

$$\operatorname{mult}(\lambda_n) = \operatorname{rank} \frac{1}{2\pi \mathbf{i}} \int_{|\lambda_n - z| = \epsilon} (z - G_b)^{-1} dz,$$

where $0 < \epsilon \ll 1$. Our main result is the following

Theorem 1.1. *Let $\gamma \neq 1$ be a constant and let $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$. Then the counting function $N(r)$ for the ball B_3 has the asymptotic*

$$N(r) = (\gamma_0^2 - 1)r^2 + \mathcal{O}_\gamma(r), \; r \geq r_0(\gamma) > c_0. \tag{1.3}$$

The proof of Theorem 1.1 is based on a precise analysis of the roots of the equation (3.1) involving spherical Hankel functions $h_n^{(1)}(\lambda)$ of first kind. We show in Section 3 that for $\gamma > 1$ this equation has only one real root $\lambda_n < 0$. Moreover, we have $\lambda_{n+1} < \lambda_n$, $\forall n \in \mathbb{N}$, so we have a decreasing sequence of eigenvalues. The geometric multiplicity of λ_n is $2n + 1$. Since C_b is not a self-adjoint operator the geometric multiplicity could be less than the algebraic one. In our case these multiplicities coincide and the proof is based on a representation of $(G_b - z)^{-1}$. To estimate λ_n as $n \to \infty$, we apply an approximation of the exterior semiclassical Dirichlet to Neumann map for the operator $(h^2\Delta + z)$ established in [7] (see also [9]) combined with an application of Rouché theorem.

We conjecture that in the general case of strictly convex obstacles and $\min_{y\in\Gamma}\gamma(y)=\gamma_1>1$ we have the asymptotic

$$N(r)=\frac{1}{4\pi}\Big(\int_\Gamma(\gamma^2(y)-1)dS_y\Big)r^2+\mathcal{O}_\gamma(r),\ r\geq r_0(\gamma).$$

For the ball B_3 this agrees with (1.3).

2. Boundary problem for Maxwell system

Our purpose is to study the eigenvalues of G_b in case the obstacle is the ball $B_3=\{x\in\mathbb{R}^3:|x|\leq 1\}$. Setting $\lambda=\mathbf{i}\mu$, $\operatorname{Im}\mu>0$, an eigenfunction $(E,B)\neq 0$ of G_b satisfies

$$\operatorname{curl}E=-\mathbf{i}\mu B,\qquad \operatorname{curl}B=\mathbf{i}\mu E. \tag{2.1}$$

Replacing B by $H=-B$ yields for $(E,H)\in H^2(\{|x|\geq 1\};\mathbb{C}^6)$ the problem

$$\begin{cases}\operatorname{curl}E=\mathbf{i}\mu H, \qquad \operatorname{curl}H=-\mathbf{i}\mu E, \quad \text{for} \quad x\in B_3,\\ E_{tan}+\gamma(\nu\wedge H_{tan})=0, \quad \text{for} \quad x\in\mathbb{S}^2.\end{cases} \tag{2.2}$$

The functions $E(x),H(x)$ are solutions in $\{x\in\mathbb{R}^3:|x|>1\}$ of the Helmholtz equation

$$\Delta v+\mu^2 v=0$$

and since $(E,H)\in\mathcal{H}$ these solutions are outgoing. By using spherical coordinates ω on $\mathbb{S}^2$, we can expand $E(x),H(x)$ by the spherical functions $Y_n^m(\omega)$, $n=0,1,2,...$, $|m|\leq n$, $\omega\in\mathbb{S}^2$, and the spherical Hankel functions of first kind

$$h_n^{(1)}(z):=\frac{H^{(1)}_{n+1/2}(z)}{\sqrt{z}},n\geq 1.$$

An application of Theorem 2.50 in [3] (in the notation of [3] it is necessary to replace k by $\mu\in\mathbb{C}\setminus\{0\}$) says that the outgoing solution of the system

$$\operatorname{curl}E=\mathbf{i}\mu H,\qquad \operatorname{curl}H=-\mathbf{i}\mu E,\quad \text{for}\quad x\in B_3$$

for $x=|x|\omega, r=|x|>0,\omega=\frac{x}{r}$ has the form

$$E(x)=\sum_{n=1}^{\infty}\sum_{|m|\leq n}\Big[\alpha_n^m\sqrt{n(n+1)}\frac{h_n^{(1)}(\mu r)}{r}Y_n^m(\omega)\omega$$
$$+\frac{\alpha_n^m}{r}(rh_n^{(1)}(\mu r))'U_n^m(\omega)+\beta_n^m h_n^{(1)}(\mu)V_n^m(\omega)\Big], \tag{2.3}$$

$$H(x)=-\frac{1}{\mathbf{i}\mu}\sum_{n=1}^{\infty}\sum_{|m|\leq n}\Big[\beta_n^m\sqrt{n(n+1)}\frac{h_n^{(1)}(\mu r)}{r}Y_n^m(\omega)\omega$$
$$+\frac{\beta_n^m}{r}(rh_n^{(1)}(\mu r))'U_n^m(\omega)+\mu^2\alpha_n^m h_n^{(1)}(\mu)V_n^m(\omega)\Big]. \tag{2.4}$$

Here $U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \mathrm{grad}_{\mathbb{S}^2} Y_n^m(\omega)$ and $V_n^m(\omega) = \nu \wedge U_n^m(\omega)$ for $n \in \mathbb{N}, -n \leq m \leq n$ form a complete orthonormal basis in

$$L_t^2(\mathbb{S}^2) = \{u(\omega) \in (L^2(\mathbb{S}^2; \mathbb{C}^3) : \langle \omega, u(\omega)\rangle = 0 \text{ on } \mathbb{S}^2\}.$$

To find a representation of $\nu \wedge H_{tan}$, observe that $\nu \wedge (\nu \wedge U_n^m) = -U_n^m$, so for $r = 1$ one has

$$(\nu \wedge H_{tan})(\omega) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \Big[\beta_n^m \Big(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r)|_{r=1}\Big) V_n^m(\omega)$$

$$-\mu^2 \alpha_n^m h_n^{(1)}(\mu) U_n^m(\omega)\Big]$$

and the boundary condition in (2.2) is satisfied if

$$\alpha_n^m \Big[h_n^{(1)}(\mu) + \frac{d}{dr}(h_n^{(1)}(\mu r))|_{r=1} - \gamma \mathbf{i}\mu h_n^{(1)}(\mu)\Big] = 0, \ \forall n \in \mathbb{N}, \ |m| \leq n, \quad (2.5)$$

$$-\frac{\beta_n^m \gamma}{\mathbf{i}\mu} \Big[h_n^{(1)}(\mu) + \frac{d}{dr}(h_n^{(1)}(\mu r))|_{r=1} - \frac{\mathbf{i}\mu}{\gamma} h_n^{(1)}(\mu)\Big] = 0, \ \forall n \in \mathbb{N}, \ |m| \leq n. \quad (2.6)$$

3. Roots of the equation $g_n(\lambda) = 0$

To examine the eigenvalues of G_b, it is necessary to find the roots of the equations (2.5) and (2.6). Since $h_n^{(1)}(\mu) \neq 0$ for $\mathrm{Im}\,\mu > 0$, the problem is reduced to study the roots $\lambda \in \mathbb{R}^-$ of the equation

$$1 + \frac{d}{dr} h_n^{(1)}(-\mathbf{i}\lambda r)\Big|_{r=1} (h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma = 0 \quad (3.1)$$

and the same equation with γ replaced by $\frac{1}{\gamma}$. Clearly, if $\mu = -\mathbf{i}\lambda$ is such that the expressions in the brackets [...] in (2.5) and (2.6) are non-vanishing for every $n \geq 1$, we must have $\alpha_n^m = \beta_n^m = 0$ which implies $E_{tan} = B_{tan} = 0$. Hence $(E, B) = 0$ because the boundary problem with $\gamma = 0$ has no eigenvalues in $\{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$. In this section we suppose that $\gamma \neq 1$ and examine the equation

$$g_n(\lambda) := \frac{1}{\lambda} + \frac{d}{d\lambda}\Big(h_n^{(1)}(-\mathbf{i}\lambda)\Big)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma = 0. \quad (3.2)$$

It is well known that (see [5])

$$h_n^{(1)}(-\mathbf{i}\lambda) = (-\mathbf{i})^{n+1} \frac{e^{\lambda}}{-\mathbf{i}\lambda} R_n\Big(\frac{\mathbf{i}}{-2\mathbf{i}\lambda}\Big) = (-\mathbf{i})^n \frac{e^{\lambda}}{\lambda} R_n\Big(-\frac{1}{2\lambda}\Big)$$

with

$$R_n(z) := \sum_{m=0}^{n} a_{m,n} z^m, \ a_{m,n} = \frac{(n+m)!}{m!(n-m)!} > 0.$$

We will prove the following

Proposition 3.1. *For $\lambda < 0$ we have*

$$G_{n,n+1}(\lambda) = \frac{\frac{d}{d\lambda}h^{(1)}_{n+1}(-\mathbf{i}\lambda)}{h^{(1)}_{n+1}(-\mathbf{i}\lambda)} - \frac{\frac{d}{d\lambda}h^{(1)}_{n}(-\mathbf{i}\lambda)}{h^{(1)}_{n}(-\mathbf{i}\lambda)} > 0. \tag{3.3}$$

Proof. The purpose is to show that

$$\Big(h^{(1)}_n(-\mathbf{i}\lambda)\frac{d}{d\lambda}h^{(1)}_{n+1}(-\mathbf{i}\lambda)-h^{(1)}_{n+1}(-\mathbf{i}\lambda)\frac{d}{d\lambda}h^{(1)}_{n}(-\mathbf{i}\lambda)\Big)\Big(h^{(1)}_{n+1}(-\mathbf{i}\lambda)h^{(1)}_{n}(-\mathbf{i}\lambda)\Big)^{-1} > 0.$$

Introduce the functions

$$\xi_n(\lambda) := \frac{e^\lambda}{\lambda}R_n\Big(-\frac{1}{2\lambda}\Big),\ \eta_n(\lambda) := \lambda\xi_n(\lambda).$$

Then $h^{(1)}_n(-\mathbf{i}\lambda) = (-\mathbf{i})^n\xi_n(\lambda)$ and the above inequality is equivalent to

$$\begin{aligned}&\Big(\xi_n(\lambda)\frac{d}{d\lambda}\xi_{n+1}(\lambda) - \xi_{n+1}(\lambda)\frac{d}{d\lambda}\xi_n(\lambda)\Big)\Big(\xi_{n+1}(\lambda)\xi_n(\lambda)\Big)^{-1}\\ &= \Big(\eta_n(\lambda)\frac{d}{d\lambda}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d}{d\lambda}\eta_n(\lambda)\Big)\Big(\eta_{n+1}(\lambda)\eta_n(\lambda)\Big)^{-1} > 0.\end{aligned}$$

Since $\eta_n(\lambda)\eta_{n+1}(\lambda) > 0$ for $\lambda < 0$, it suffices to show that the function

$$F(\lambda) = \eta_n(\lambda)\frac{d}{d\lambda}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d}{d\lambda}\eta_n(\lambda)$$

has positive values for $\lambda \in (-\infty, 0)$. Consider the derivative

$$F'(\lambda) = \eta_n(\lambda)\frac{d^2}{d\lambda^2}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d^2}{d\lambda^2}\eta_n(\lambda).$$

We have

$$\eta_n(\lambda) = \mathbf{i}^{n+1}h^{(1)}_n(-\mathbf{i}\lambda)(-\mathbf{i}\lambda) = \mathbf{i}^{n+1}\Xi_n(-\mathbf{i}\lambda) = -\mathbf{i}^{n-1}\Xi_n(-\mathbf{i}\lambda).$$

The function $\Xi_n(z) = zh^{(1)}_n(z)$ satisfies the equation

$$\Xi''_n(z) + \Big(1 - \frac{n^2+n}{z^2}\Big)\Xi_n(z) = 0$$

and

$$\begin{aligned}\frac{d^2}{d\lambda^2}\eta_n(\lambda) &= \mathbf{i}^{n-1}\Xi''_n(-\mathbf{i}\lambda) = -\mathbf{i}^{n-1}\Big(1 + \frac{n^2+n}{\lambda^2}\Big)\Xi_n(-\mathbf{i}\lambda)\\ &= \Big(1 + \frac{n^2+n}{\lambda^2}\Big)\eta_n(\lambda).\end{aligned}$$

Consequently,

$$\begin{aligned}F'(\lambda) &= \Big[\frac{(n+1)^2+n+1}{\lambda^2} - \frac{n^2+n}{\lambda^2}\Big]\eta_n(\lambda)\eta_{n+1}(\lambda)\\ &= 2(n+2)\frac{\eta_n(\lambda)\eta_{n+1}(\lambda)}{\lambda^2} > 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}F(\lambda) &= e^\lambda R_n\Big(-\frac{1}{2\lambda}\Big)\frac{d}{d\lambda}\Big(e^\lambda R_{n+1}\Big(-\frac{1}{2\lambda}\Big)\Big) - e^\lambda R_{n+1}\Big(-\frac{1}{2\lambda}\Big)\frac{d}{d\lambda}\Big(e^\lambda R_n\Big(-\frac{1}{2\lambda}\Big)\Big)\\ &= \frac{e^{2\lambda}}{2\lambda^2}\Big[R_n\Big(-\frac{1}{2\lambda}\Big)R'_{n+1}\Big(-\frac{1}{2\lambda}\Big) - R_{n+1}\Big(-\frac{1}{2\lambda}\Big)R'_n(-\frac{1}{2\lambda})\Big]\end{aligned}$$

and

$$\lim_{\lambda\to-\infty} F(\lambda) = 0, \ \lim_{\lambda\nearrow 0} F(\lambda) = +\infty$$

since

$$\lim_{w\to+\infty}\Big[R_n(w)R'_{n+1}(w) - R_{n+1}(w)R'_n(w)\Big] = +\infty.$$

Finally, the function $F(\lambda)$ in the interval $(-\infty, 0]$ is increasing from 0 to $+\infty$ and this completes the proof. □

Now if $\lambda_n < 0$ is a solution the equation (3.2) one has

$$g_{n+1}(\lambda_n) = \frac{1}{\lambda_n} + \Big(\frac{d}{d\lambda}h^{(1)}_{n+1}(-i\lambda_n)\Big)(h^{(1)}_{n+1}(-i\lambda_n))^{-1} - \gamma = G_{n,n+1}(\lambda_n) > 0,$$

so λ_n is not a root of the equation

$$g_{n+1}(\lambda) = \frac{1}{\lambda} + \Big(\frac{d}{d\lambda}h^{(1)}_{n+1}(-\mathbf{i}\lambda)\Big)(h^{(1)}_{n+1}(-\mathbf{i}\lambda))^{-1} - \gamma = 0.$$

In the following we assume that $\gamma > 1$. Then for $\lambda \to -\infty$ we have $g_{n+1}(\lambda) \to 1 - \gamma < 0$, and since $g_{n+1}(\lambda_n) > 0$ the equation $g_{n+1}(\lambda) = 0$ has at least one root $-\infty < \lambda_{n+1} < \lambda_n$.

Lemma 3.1. *Let $\gamma > 1$. For every $n \geq 1$ the equation $g_n(\lambda) = 0$ in the interval $(-\infty, 0)$ has exactly one root $\lambda_n < 0$.*

Proof. Setting $w = -\frac{1}{2\lambda}$, we write the equation (3.2) as $\mathcal{R}_n(w) := w^2R'_n(w) + \alpha R_n(w) = 0$, where $\alpha = \frac{1-\gamma}{2} < 0$. We will show that this equation has exactly one positive root. Since

$$w^2R'_n(w) = \sum_{k=1}^{n} ka_{k,n}w^{k+1}, \ R_n(w) = \sum_{k=0}^{n} a_{k,n}w^k,$$

the polynomial $\mathcal{R}_n(w)$ has the representation

$$\mathcal{R}_n(w) = \sum_{k=0}^{n+1} b_{k,n}w^k$$

with

$$\begin{cases} b_{k,n} = (k-1)a_{k-1,n} + \alpha a_{k,n}, \ 0 \leq k \leq n, \ a_{-1,n} = 0, \\ b_{n+1,n} = \frac{(2n)!}{(n-1)!}. \end{cases}$$

Taking into account the form of $a_{k,n}$, we deduce

$$b_{k,n} = \frac{(n+k-1)!}{(n-k+1)!k!}\Big(k(k-1) + \alpha(n+k)(n-k+1)\Big), \ 0 \leq k \leq n+1. \quad (3.4)$$

Thus the sign of $b_{k,n}$ depends on the sign of the function

$$B(k) := (1-\alpha)k^2 + (\alpha - 1)k + \alpha(n^2 + n)$$

which for $k \geq 1$ is increasing since

$$B'(k) = 2(1-\alpha)k + \alpha - 1 \geq 1 - \alpha > 0.$$

Clearly, $b_{0,n} = \alpha < 0$ and $b_{n+1,n} > 0$. There are two cases:

(i) $b_{1,n} \leq 0$. Then there is only one change of sing in the Descartes' sequence $\{b_{n+1,n}, b_{n,n}, ..., b_{1,n}, b_{0,n}\}$.

(ii) $b_{1,n} > 0$. Then $b_{k,n} > 0$ for $1 \leq k \leq n+1$ and in the Descartes' sequence $\{b_{n+1,n}, b_{n,n}, ..., b_{1,n}, b_{0,n}\}$ one has again only one change of sign.

Applying the Descartes' rule of signs, we conclude that the number of the positive roots of $\mathcal{R}_n(w) = 0$ is exactly one.

□

Combining Proposition 3.1 and Lemma 3.1, one obtain the following

Corollary 3.1. *Let $\gamma > 1$. Then the generator G_b has an infinite sequence of real eigenvalues*

$$-\infty < ... < \lambda_n < ... < \lambda_2 < \lambda_1 < 0$$

and λ_n has geometric multiplicity $2n+1$.

The geometric multiplicity is $2n+1$ since the functions $\{Y_{m,n}(\omega)\}_{m=-n}^{m}$ are linearly independent. The algebraic multiplicity of λ_m will be discussed in Section 5.

4. Estimation of the roots

Throughout this section we assume $\gamma > 1$. Set $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$, $0 < h \ll 1$ with $z = -1 + \mathbf{i}\eta$, $0 \leq |\eta| \leq h^{1/2}, \eta \in \mathbb{R}$. Consider the Dirichlet problem

$$\begin{cases} (h^2\Delta + z)w = 0, \ |x| > 1, w \in H^2(|x| > 1), \\ w = f, \ |x| = 1 \end{cases} \tag{4.1}$$

and note that $\Delta + \frac{z}{h^2} = \Delta - \lambda^2$. The solution of (4.1) has the form

$$w(r\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(-\mathbf{i}\lambda r)(h_n^{(1)}(-\mathbf{i}\lambda)^{-1}\alpha_{n,m}Y_{n,m}(\omega),$$

where

$$f(\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n,m}Y_{n,m}(\omega).$$

The semiclassical Dirichlet-to-Neumann operator $\mathcal{N}_{ext}(h,z) = \frac{h}{\mathbf{i}}\frac{d}{dr}w|_{r=1}$ related to (4.1) becomes

$$\mathcal{N}_{ext}(h,z) = -\mathbf{i}\sqrt{z}\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (h_n^{(1)})'(-\mathbf{i}\lambda)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1}\alpha_{n,m}Y_{n,m}$$

$$= \sqrt{z}\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{d}{d\lambda}\Big(h_n^{(1)}(-\mathbf{i}\lambda)\Big)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1}\alpha_{n,m}Y_{n,m}.$$

By using the approximation of $\mathcal{N}_{ext}(h, z)$ established in [9],[7] for $z = -1+\mathbf{i}\eta$, one deduces

$$\|\mathcal{N}_{ext}(h, z)f - Op_h(\rho)f\|_{L^2(\mathbb{S}^2)} \leq C\frac{|\sqrt{z}|}{|\lambda|}\|f\|_{L^2(\mathbb{S}^2)},\ 0 < h \leq h_0$$

with $\rho = \sqrt{z - r_0(x', \xi')}$ and a constant $C > 0$ independent of z, λ and f. Here $r_0(x', \xi')$ is the principal symbol of the semiclasssical Laplace-Beltrami operator $-h^2\Delta_{\mathbb{S}^2} = \frac{z}{\lambda^2}\Delta_{\mathbb{S}^2}$ and $Op_h(\rho)$ is a $h-$pseudodifferential operator with symbol ρ. Moreover, $\sqrt{z} = \mathbf{i}\sqrt{1 - \mathbf{i}\eta} = \mathbf{i}(1 - \frac{\mathbf{i}\eta}{2} + \mathcal{O}(\eta^2))$ and

$$\operatorname{Re}\lambda = -\frac{1}{h} + \mathcal{O}(1),\ \operatorname{Im}\lambda = \mathcal{O}(h^{-1/2}).$$

Hence, for $0 < h \leq h_0$ we get

$$\lambda \in \Lambda_0 = \{z \in \mathbb{C} :\ |\operatorname{Im} z| \leq ch_0^{1/2}|\operatorname{Re} z|,\ \operatorname{Re}\lambda < -\epsilon < 0,\ \ |\lambda| \geq \lambda_0\}.$$

On the other hand, it easy to see that

$$\Big\|Op_h(\rho) - \sqrt{z}\Big(\sqrt{1 - \frac{\Delta_{\mathbb{S}^2}}{\lambda^2}}\Big)\Big\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \leq C_1|\lambda|^{-1},\ \lambda \in \Lambda_0.$$

Applying the spectral calculus for the operator $\Delta_{\mathbb{S}^2}$, one deduces

$$\Big(\sqrt{1 - \frac{\Delta_{S^2}}{\lambda^2}}\Big)f = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big(\sqrt{1 + \frac{n(n+1)}{\lambda^2}}\Big)\alpha_{n,m}Y_{n.m}$$

and

$$\Big\|\Big(\mathcal{N}_{ext}(h, -z) - \sqrt{z}\Big(\sqrt{1 - \frac{\Delta_{S^2}}{\lambda^2}}\Big)f\Big\|^2_{L^2(S^2)} = |z|\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big|\frac{d}{d\lambda}\Big(h_n(-i\lambda)\Big)(h_n(-i\lambda))^{-1}$$

$$-\sqrt{1 + \frac{n(n+1)}{\lambda^2}}\Big|^2|a_{n,m}|^2.$$

This implies

$$\Big|\frac{d}{d\lambda}\Big(h_n^{(1)}(-\mathbf{i}\lambda)\Big)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \sqrt{1 + \frac{n(n+1)}{\lambda^2}}\Big| \leq C_2|\lambda|^{-1},\ \forall n \in \mathbb{N},\ \lambda \in \Lambda_0 \tag{4.2}$$

which we write as

$$\Big|\Big[\frac{1}{\lambda} + \frac{d}{d\lambda}\Big(h_n^{(1)}(-\mathbf{i}\lambda)\Big)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma\Big] - \Big[\sqrt{1 + \frac{n(n+1)}{\lambda^2}} - \gamma\Big]\Big| \leq C_0|\lambda|^{-1}. \tag{4.3}$$

Remark 4.1. *For bounded* $1 \leq n \leq N_0$ *and sufficiently large* $|\lambda|$ *the estimate* (4.2) *follows easily from the fact that* $\frac{R_n'(w)}{R_n(w)} = n(n+1) + \mathcal{O}(|w|)$ *as* $|w| \to 0$.

Remark 4.2. *The estimate* (4.2) *is similar to that in Proposition* 2.1 *in* [8], *where the function* $\frac{J_\nu'(\lambda)}{J_\nu(\lambda)}$ *for* $\nu \geq 0$ *and* $0 < C \leq |\operatorname{Im}\lambda| \leq \delta|\operatorname{Re}\lambda|$, $\operatorname{Re}\lambda > C_1$ *has been approximated. Here* $J_\nu(z)$ *is the Bessel function, while the boundary problem examined in* [8] *is in the bounded domain* $\{x \in \mathbb{R}^3 : |x| < 1\}$.

Put $z = \lambda$ and for $z \in \Lambda_0$ consider the function

$$f_n(z) := \sqrt{1 + \frac{n(n+1)}{z^2}} - \gamma$$

with zeros

$$z_n^{\pm} = \pm\sqrt{\frac{n^2+n}{\gamma^2-1}}.$$

In the following we set $z_n = -\sqrt{\frac{n(n+1)}{\gamma^2-1}}$. Clearly,

$$f_n'(z) = -\frac{1}{z}\frac{\frac{n(n+1)}{z^2}}{\sqrt{1+\frac{n(n+1)}{z^2}}}$$

and $\frac{n(n+1)}{z_n^2} = \gamma^2 - 1$, $f_n'(z_n) = -\frac{\gamma^2-1}{\gamma z_n}$. A calculus yields the second derivative

$$f_n''(z) = \frac{1}{z^2}\Big[\frac{3n(n+1)}{z^2}\Big(\sqrt{1+\frac{n(n+1)}{z^2}}\Big)$$
$$-\frac{n^2(n+1)^2}{z^4}\Big(\sqrt{1+\frac{n(n+1)}{z^2}}\Big)^{-1/2}\Big]\Big(1+\frac{n(n+1)}{z^2}\Big)^{-1}.$$

For n large enough and $a > 0$ to be fixed below introduce the contour

$$C_n(a) := \{z = z_n + ae^{\mathbf{i}\varphi},\ 0 \le \varphi < 2\pi\} \subset \Lambda_0.$$

Our purpose is to choose a so that

$$|f_n(z)| \ge \frac{C_0}{|z|},\ \ \forall z \in C_n(a). \tag{4.4}$$

We have

$$z^2 = z_n^2 + 2z_n a e^{\mathbf{i}\varphi} + a^2 e^{2\mathbf{i}\varphi}$$

and

$$\frac{n(n+1)}{z^2} = (\gamma^2-1)\Big(1 + \mathcal{O}\Big(\frac{1}{n}\Big)a + \mathcal{O}\Big(\frac{1}{n^2}\Big)a^2\Big)^{-1},\ z \in C_n(a). \tag{4.5}$$

On the other hand,

$$\sqrt{\frac{n(n+1)}{z^2}} + 1 = \Big[\frac{\gamma^2 + \mathcal{O}\Big(\frac{1}{n}\Big)a + \mathcal{O}(\frac{1}{n^2})a^2}{1 + \mathcal{O}\Big(\frac{1}{n}\Big)a + \mathcal{O}(\frac{1}{n^2})a^2}\Big]^{1/2}.$$

Clearly, one has the estimate

$$|f_n(z)| \ge \frac{\gamma^2-1}{\gamma|z_n|}a - \frac{a^2}{2}\sup_{z\in C_n(a)}|f_n''(z)|,\ z \in C_n(a). \tag{4.6}$$

Set $C_\gamma = \frac{\gamma^2-1}{\gamma} > 0$ and choose $a > 0$ so that $C_\gamma a > 4C_0$. We fix a and obtain

$$\frac{C_\gamma a}{2|z_n|} > \frac{2C_0}{|z_n|} > \frac{C_0}{|z_n||1+\frac{ae^{\mathbf{i}\varphi}}{z_n}|},\ 0 \le \varphi < 2\pi,$$

taking n large enough to satisfy the inequality

$$\frac{1}{\left|1+\frac{ae^{i\varphi}}{z_n}\right|} < 2.$$

Next we arrange the inequality

$$\frac{C_\gamma a}{2|z_n|} - \frac{a^2}{2} \sup_{z\in C_n(a)} |f_n''(z)| > 0. \tag{4.7}$$

It is clear that

$$f_n''(z) = \frac{1}{z^2} G\Big(\frac{n(n+1)}{z^2}\Big),$$

where

$$G(\zeta) = \Big[3\zeta\sqrt{\zeta+1} - \zeta^2(\zeta+1)^{-1/2}\Big](\zeta+1)^{-1}.$$

Note that for $z \in C_n(a)$ and n large enough according to (4.4), the function $|G(\frac{n(n+1)}{z^2})|$ is bounded by a constant $B_{\gamma,a}$ depending on γ and a . Thus for large n we get

$$\sup_{z\in C_n(a)} |f_n''(z)| \leq B_{\gamma,a} \sup_{z\in C_n(a)} \frac{1}{|z|^2}$$

$$= B_{\gamma,a}\frac{1}{|z_n|^2} \sup_{z\in C_n(a)} \frac{1}{|1+\frac{ae^{i\varphi}}{z_n}|^2} \leq 4B_{\gamma,a}\frac{1}{|z_n|^2}$$

and the proof of (4.7) is reduced to

$$C_\gamma > 4B_{\gamma,a}\frac{a}{|z_n|}$$

which is satisfied taking again n large. Finally, we proved the estimate (4.3) and we can apply Rouché theorem for the functions $g_n(z)$ and $f_n(z)$ and conclude that the function $g_n(z)$ has exactly one simple zero λ_n in $C_n(a)$. Since $g_n(z)$ has only real zeros (see Appendix in [2]), this implies the following

Lemma 4.1. *There exist $n_0(\gamma)$ and $a(\gamma) > 0$ depending on γ such that for $n \geq n_0(\gamma)$ the negative root λ_n of the equation* (3.2) *satisfies the estimate*

$$\Big|\lambda_n + \sqrt{\frac{n(n+1)}{\gamma^2-1}}\Big| \leq a(\gamma). \tag{4.8}$$

Remark 4.3. *According to Proposition* 2.1, *$n_0(\gamma)$ must satisfy the inequality*

$$n_0(\gamma) \geq \frac{\sqrt{\gamma^2-1}}{\max\{\gamma-1, \sqrt{\gamma-1}\}}.$$

5. Weyl asymptotics

We start with the analysis of the algebraic multiplicity of λ_n.

Lemma 5.1. *For* $n \geq n_0(\gamma)$ *we have* $\mathrm{mult}(\lambda_n) = 2n+1$.

Proof. Since the geometric multiplicity of λ_n is $2n+1$, it is sufficient to show that

$$\mathrm{mult}(\lambda_n) \leq 2n+1. \tag{5.1}$$

Let $\lambda \in \Lambda_0$, where Λ_0 is the set introduced in the previous section and let $\lambda \notin \sigma(G_b)$. If $0 \neq (f,g) \in (\mathrm{Image}\, G_b) \cap L^2(\Omega)$, one has $\mathrm{div}\, f = \mathrm{div}\, g = 0$ and for $(u,v) = (G_b - \lambda)^{-1}(f,g)$ we get $\mathrm{div}\, u = \mathrm{div}\, v = 0$. Consider the skew self-adjoint operator

$$A = \begin{pmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix}$$

with boundary condition $\nu \wedge u = 0$ on $\mathbb{S}^2$. Then $\sigma(A) \subset \mathbf{i}\mathbb{R}$ and let

$$(u_0(x;\lambda), v_0(x;\lambda)) = (A-\lambda)^{-1}(f,g),$$

that is

$$\begin{cases} (A-\lambda)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ for } |x| > 1, \\ \nu \wedge u_0 = 0 \text{ on } \mathbb{S}^2. \end{cases} \tag{5.2}$$

Since $\mathrm{div}\, u_0 = \mathrm{div}\, v_0 = 0$, the well known coercive estimates yield $(u_0, v_0) \in H^1(\Omega)$. Moreover the resolvent $(A-\lambda)^{-1}$ is analytic in $\{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$ and $u_0(x;\lambda)$, $v_0(x;\lambda)$ depend analytically on λ. We write $(u,v) = (u_0,v_0) + (u_1,v_1)$, where $(u_1(x;\lambda), v_1(x;\lambda))$ is the solution of the problem

$$\begin{cases} (G-\lambda)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } |x| > 1, \\ (u_1)_{tan} - \gamma(\nu \wedge (v_1)_{tan}) = -\gamma(\nu \wedge (v_0)_{tan}(x;z)) \text{ on } \mathbb{S}^2. \end{cases} \tag{5.3}$$

To solve (5.3), note that $-\gamma(\nu \wedge (v_0)_{tan}(\omega;z)) = F(\omega;\lambda) \in L^2(\mathbb{S}^2)$ with $F(\omega;\lambda)$ analytical in λ for $\lambda \in \Lambda_0$. Thus we may write

$$F(\omega;\lambda) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{\alpha}_n^m(\lambda) U_n^m(\omega) + \tilde{\beta}_n^m(\lambda) V_n^m(\omega)$$

with analytical coefficients $\tilde{\alpha}_n^m(\lambda), \tilde{\beta}_n^m(\lambda)$. Now we can solve (2.5), (2.6) with right hand part $(\tilde{\alpha}_n^m(\lambda),\ \tilde{\beta}_n^m(\lambda))$. Finally, we obtain a representation of the solution of (5.3) with meromorphic coefficients

$$\alpha_n^m(\lambda) = \frac{\tilde{\alpha}_n^m(\lambda)}{h_n^{(1)}(-\mathbf{i}\lambda)\Big[1 + \frac{d}{dr}(h_n^{(1)}(-\mathbf{i}\lambda r))|_{r=1}(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma\Big]},$$

$$\beta_n^m(\lambda) = -\frac{\lambda\tilde{\beta}_n^m(\lambda)}{\gamma h_n^{(1)}(-\mathbf{i}\lambda)\Big[1 + \frac{d}{dr}(h_n^{(1)}(-\mathbf{i}\lambda r))|_{r=1}(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma^{-1}\Big]}.$$

If $\gamma > 1$ the analysis in the previous section shows that for $\lambda \in \Lambda_0$ the meromorphic function $\alpha_n^m(\lambda)$ has a simple pole at $\lambda_n < 0$, while $\beta_n^m(\lambda)$ is analytic in Λ_0. For $0 < \gamma < 1$ the function $\alpha_n^m(\lambda)$ is analytic in Λ_0 and $\beta_n^m(\lambda)$ is meromorphic. Next we integrate $(u(x;\lambda), v(x;\lambda))$ over the circle $|\lambda_n - \lambda| = \epsilon$, where ϵ is sufficiently small. The integral of $(u_0(x;\lambda), v_0(x;\lambda))$ vanishes, while for the integral of $(u_1(x;\lambda), v_1(x;\lambda))$, taking into account the representation of the solution of (5.3), we will obtain a sum

$$S_n = \begin{cases} c_n \sum_{m=-n}^m \tilde{\alpha}_n^m(\lambda_n) U_n^m(\omega), \ c_n \neq 0, \ \gamma > 1, \\ d_n \sum_{m=-n}^m \lambda_n \tilde{\beta}_n^m(\lambda_n)\gamma^{-1} V_n^m(\omega), \ d_n \neq 0, \ 0 < \gamma < 1. \end{cases}$$

This completes the proof of (5.1). □

Passing to the analysis of $N(r)$, consider first the case $\gamma > 1$. The root λ_n has algebraic multiplicity $2n+1$ and to find a lower bound of $N(r)$ we apply the estimate

$$|\lambda_n| \leq \sqrt{\frac{n(n+1)}{\gamma^2 - 1}} + a(\gamma) < \frac{n+1}{\sqrt{\gamma^2-1}} + a(\gamma) \leq r$$

for $r \geq a(\gamma) + \frac{n_0(\gamma)+1}{\sqrt{\gamma^2-1}}$. Then

$$N(r) \geq \sum_{j=n_0(\gamma)}^{[(r-a(\gamma))\sqrt{\gamma^2-1}-1]} (2j+1) = (\gamma^2-1)r^2 + \mathcal{O}_\gamma(r) + A_\gamma.$$

To get a upper bound for $N(r)$, we use the estimate

$$|\lambda_n| \geq \sqrt{\frac{n(n+1)}{\gamma^2 - 1}} - a(\gamma) > \frac{n}{\sqrt{\gamma^2-1}} - a(\gamma) \geq r$$

for

$$n \geq (r + a(\gamma))\sqrt{\gamma^2-1} \geq 2a(\gamma)\sqrt{\gamma^2-1} + n_0(\gamma) + 1,$$

hence

$$N(r) \leq \sum_{j=n_0(\gamma)}^{[(r+a(\gamma))\sqrt{\gamma^2-1}]+1} (2j+1) + D_\gamma = (\gamma^2-1)r^2 + \mathcal{O}_\gamma(r) + A'_\gamma.$$

If $0 < \gamma < 1$, we have $\frac{1}{\gamma} > 1$ and one applies the above argument for the roots of the the equation (2.6). This completes the proof of theorem 1.1

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