WEYL FORMULA FOR THE EIGENVALUES OF THE DISSIPATIVE ACOUSTIC OPERATOR

VESSELIN PETKOV

Abstract. We study the wave equation in the exterior of a bounded domain K with dissipative boundary condition $\partial_\nu u - \gamma(x)\partial_t u = 0$ on the boundary Γ and $\gamma(x) > 0$. The solutions are described by a contraction semigroup $V(t) = e^{tG}$, $t \geq 0$. The eigenvalues λ_k of G with $\operatorname{Re}\lambda_k < 0$ yield asymptotically disappearing solutions $u(t,x) = e^{\lambda_k t} f(x)$ having exponentially decreasing global energy. We establish a Weyl formula for these eigenvalues in the case $\min_{x\in\Gamma}\gamma(x) > 1$. For strictly convex obstacles K this formula concerns all eigenvalues of G.

Keywords: Dissipative boundary conditions, eigenvalues asymptotics

Mathematics Subject Classification 2020: 35P20, 35P25, 47A40, 58J50

1. Introduction

Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain. Let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. and $K \subset \{x \in \mathbb{R}^d : |x| \leq \rho_0\}$. We suppose that the boundary Γ of K is C^∞. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x)\partial_t u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0,x) = f_1,\ u_t(0,x) = f_2 \end{cases} \tag{1.1}$$

with initial data $(f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = \mathcal{H}$. Here $\nu(x)$ is the unit outward normal to Γ pointing into Ω and $\gamma(x) \geq 0$ is a C^∞ function on Γ. The solution of the problem (1.1) is given by $V(t)f = e^{tG}f$, $t \geq 0$, where $V(t)$ is a contraction semi-group in $\mathcal{H}$ whose generator

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

has a domain $D(G)$ which is the closure in the graph norm

$$|||f||| = (\|f\|_{\mathcal{H}}^2 + \|Gf\|_{\mathcal{H}}^2)^{1/2}$$

of functions $f = (f_1, f_2) \in C_{(0)}^\infty(\mathbb{R}^d) \times C_{(0)}^\infty(\mathbb{R}^d)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ. It is well known that the spectrum of G in $\operatorname{Re} z < 0$ is formed by isolated eigenvalues with finite multiplicity (see [7] for d odd and [12] for all $d \geq 2$.) Moreover, G has no eigenvalues on the imaginary axis $\mathbf{i}\mathbb{R}$. Notice that if $Gf = \lambda f$ with $0 \neq f \in D(G)$, $\operatorname{Re}\lambda < 0$ and $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ, we get

$$\begin{cases} (\Delta - \lambda^2) f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda\gamma f_1 = 0 \text{ on } \Gamma \end{cases} \tag{1.2}$$

e-mail:petkov@math.u-bordeaux.fr.

and $u(t,x) = V(t)f = e^{\lambda t}f(x)$ is a solution of (1.1) with exponentially decreasing global energy. Such solutions are called **asymptotically disappearing**. On the other hand, the solutions $u(t,x) = V(t)f$ for which there exists $T > 0$ such that $u(t,x) \equiv 0$ for $t \geq T$ are called **disappearing** (see [8]). For $t_0 > 0$ the closed linear space

$$H(t_0) = \{g \in \mathcal{H} : V(t)g = 0 \, \text{for} \, t \geq t_0\}$$

is invariant under the action of $V(t)$ and if $H(t_0) \neq \{0\}$, then $H(t_0)$ has infinite dimension. If $H(t_0)$ is not trivial, the scattering system is non controllable (see section 4 in [8] for the definition and details). Majda proved in [8] that for obstacles with analytic boundary Γ and analytic $\gamma(x)$ the condition $\gamma(x) \neq 1$, $\forall x \in \Gamma$, implies that there are no disappearing solutions.

In this paper in the case $\min_{x\in\Gamma} \gamma(x) > 1$ we show that there exists a subspace $\mathcal{H}_{sp} \subsetneq \mathcal{H}$ with infinite dimension generated by eigenfunctions of G such that $V(t)g$, $g \in \mathcal{H}_{sp}$ is asymptotically disappearing. The eigenvalues λ_k sufficiently close to $\mathbb{R}^-$ with $\operatorname{Re}\lambda_k \to -\infty$ present a particular interest for applications since they correspond to solutions decreasing sufficiently fast as $t \to +\infty$. It is important to know that such eigenvalues exist and to have their asymptotic. It was proved in [2] that if we have at least one eigenvalue λ of G with $\operatorname{Re}\lambda < 0$, then the wave operators $W_\pm$ are not complete, that is $\mathrm{R}an\, W_- \neq \mathrm{R}an\, W_+$. Hence we cannot define the scattering operator S related to the Cauchy problem for the free wave equations and the boundary problem (1.1) by the product $W_+^{-1} \circ W_-$. When the global energy is conserved in time and the unperturbed and perturbed problems are associated to unitary groups, the corresponding scattering operator $S(z) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ satisfies the identity

$$S^{-1}(z) = S^*(\bar{z}), \; z \in \mathbb{C}, \tag{1.3}$$

providing $S(z)$ invertible at z. Since $S(z)$ and $S^*(z)$ are analytic in the "physical" half plane $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ (see [6]) the above relation implies that $S(z)$ is invertible for $\operatorname{Im} z > 0$. For dissipative boundary problems the relation (1.3) in general is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. For odd dimensions d Lax and Phillips [7] proved that this implies that $\mathbf{i}z_0$ is an eigenvalue of G. Thus the analysis of the eigenvalues of G is important for the location and the existence of points, where the kernel of $S(z)$ is not trivial. A similar connection occurs in the analysis of the interior transmission eigenvalues (see [1] for the definition and more references). More precisely, consider the *far-filed operator*

$$(F(k)f)(\theta) = \int_{\mathbb{S}^{d-1}} a(k,\theta,\omega)f(\omega)d\omega, \; (\theta, \, \omega) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}.$$

Here $a(k,\theta,\omega)$ is the scattering amplitude for the Helmholtz equation $(\Delta + k^2 n(x))u = 0$, $x \in K$ with contrast function $n(x) > 0$ and for d odd the scattering operator has the representation

$$S(k) = Id + \Big(\frac{\mathbf{i}k}{2\pi}\Big)^{(d-1)/2} F(k), \; k \in \mathbb{R}.$$

Therefore if the kernel of $F(k)$ is non trivial, k is an interior transmission eigenvalue [1].

The location in $\mathbb{C}$ of the eigenvalues of G has been studied in [12] improving previous results of Majda [9]. It was proved in [12] that for the case when K is the unit ball $B_3 = \{x \in \mathbb{R}^3 : \; |x| \leq 1\}$ and $\gamma \equiv 1$, the operator G has no eigenvalues. For this reason we study the cases

$$(A) : \max_{x\in\Gamma} \gamma(x) < 1, \; (B) : \min_{x\in\Gamma} \gamma(x) > 1.$$

The results in [12] say that in the case (B) for every $0 < \epsilon \ll 1$ and every $M \in \mathbb{N}$, $M \geq 1$ the eigenvalues lie in $\Lambda_\epsilon \cup \mathcal{R}_N$, where

$$\Lambda_\epsilon = \{z \in \mathbb{C} : \; |\operatorname{Re} z| \leq C_\epsilon(1 + |\operatorname{Im} z|^{1/2+\epsilon}), \; \operatorname{Re} z < 0\},$$

$$\mathcal{R}_M = \{z \in \mathbb{C} : \; |\operatorname{Im} z| \leq A_M(1 + |\operatorname{Re} z|)^{-M}, \; \operatorname{Re} z < 0\}.$$

Moreover, for strictly convex obstacles K there exists $R_0 > 0$ such that the eigenvalues lie in $\mathcal{R}_M \cup \{|z| \leq R_0\}$. In the case (A) the eigenvalues lie in Λ_ϵ. By using the results in [18], it is possible to improve the eigenvalue free regions replacing Λ_ϵ by $\{z \in \mathbb{C} : \; -A_0 \leq \operatorname{Re} z < 0\}$ with sufficiently large $A_0 > 0$.

The existence of eigenvalues has been proved (see Appendix in [12]) only for the ball B_3 and $\gamma \equiv const > 1$ and in this particular case we have

$$\sigma_p(G) \subset (-\infty, -\frac{1}{\gamma - 1}]. \tag{1.4}$$

Moreover, we have infinite number of real eigenvalues and as $\gamma \searrow 1$ one gets a large strip $\{z \in \mathbb{C} : \; -\frac{1}{\gamma-1} < \operatorname{Re} z < 0\}$ without eigenvalues.

The purpose of this paper is to establish a Weyl formula for the eigenvalues in $\mathcal{R}_M \cap \{z \in \mathbb{C} : \; \operatorname{Re} z < -C_0 \leq -1\}$ in the case (B). Introduce the set

$$\Lambda = \{\lambda \in \mathbb{C} : \; |\operatorname{Im} \lambda| \leq C_1(1 + |\operatorname{Re} \lambda|)^{-2}, \; \operatorname{Re} \lambda \leq -C_0 \leq -1\}$$

containing $\mathcal{R}_M$, $\forall M \geq 2$, modulo a compact set and denote by $\sigma_p(G)$ the point spectrum of G. Increasing the constant $C_0 > 0$ in the definition of Λ, we subtract a compact set and this is not important for the asymptotic (1.5) below. In the following we assume that $C_0 \geq 2C_1$. Given $\lambda \in \sigma_p(G)$, we define the algebraic multiplicity of λ by

$$\text{mult}\,(\lambda) = \operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{|z-\lambda|=\epsilon} (z - G)^{-1} dz$$

with $0 < \epsilon \ll 1$ sufficiently small. Our main result is the following

Theorem 1. *Assume $\gamma(x) > 1$ for all $x \in \Gamma$. Then the counting function of the eigenvalues in Λ taken with their multiplicities has the asymptotic*

$$\begin{aligned} &\sharp\{\lambda_j \in \sigma_p(G) \cap \Lambda : \; |\lambda_j| \leq r, \; r \geq C_\gamma\} \\ &= \frac{\omega_{d-1}}{(2\pi)^{d-1}} \Big(\int_\Gamma (\gamma^2(x) - 1)^{(d-1)/2} dS_x \Big) r^{d-1} + \mathcal{O}_\gamma(r^{d-2}), \; r \to \infty, \end{aligned} \tag{1.5}$$

ω_{d-1} being the volume of the unit ball $\{x \in \mathbb{R}^{d-1} : \; |x| \leq 1\}$.

The example concerning the ball B_3 and (1.4) show that the condition $r \geq C_\gamma$ is natural since the coefficient before r^{d-1} in (1.5) goes to 0 as $\max_{x \in \Gamma} \gamma(x) \searrow 1$. Notice that for strictly convex obstacles K in the case (B) we obtain a Weyl formula for *all eigenvalues* of G. For Maxwell's equations with dissipative boundary conditions in the particular case $K = B_3, \gamma \equiv const \neq 1$, the formula (1.5) has been obtained in [4]. Weyl formula for the transmission eigenvalues have been obtained by several authors. We refer to [13] and [11] for more references. It is important to note that in [13] the Weyl formula is established with remainder which depends on the eigenvalue free region. In [11] the relation with the eigenvalues free regions is not exploited and the argument is based on a Tauberian theorem which yields a weak remainder. In the present paper we apply the eigenvalue free results in [12] and the remainder in (1.5) is optimal.

To prove Theorem 1, we apply the approach of [15] and the construction of a semi-classical parametrix $T(h,z)$, $0 < h \leq h_0$, $z = -\frac{1}{(1+\mathbf{i}\eta)^2}$, $|\eta| \leq h^2$ for the semi-classical exterior Dirichlet-to-Neumann map $N(h,z)$ given in [17], [12]. For $z = -1$ the operator $P(h) := T(h,-1) - \gamma(x)$ is self-adjoint and we denote by $\mu_1(h) \leq \mu_2(h) \leq ...$ its eigenvalues counted with their multiplicities. The points $0 < h_k \leq h_0$ for which $\mu_k(h_k) = 0$ correspond to points h for which $P(h)$ is not invertible. For large fixed k_0, depending on h_0, the eigenvalues $\mu_k(h_0)$ are positive, whenever $k > k_0$. Thus if $\mu_k(r^{-1}) < 0$, $k > k_0$, we have $\mu_k(h_k) = 0$ for some $r^{-1} < h_k < h_0$ and by a more fine analysis we prove that such a h_k is unique. The operator $P(h)$ can be extended as holomorphic one for complex $\tilde{h} = h(1+\mathbf{i}\eta) \in L$ with $|\eta| \leq h^2$ and L defined in (2.12). For the resolvent $(\lambda - G)^{-1}$ a trace formula has been established in [12] (see Proposition 1). Similarly, a trace formula involving $P^{-1}(\tilde{h})$ and the derivative $\dot{P}(\tilde{h})$ can be proved. These two trace formulas differs by negligible terms and this leads to a map between the points $h_k \in L$, where $P(h_k)$ is not invertible and the eigenvalues of G. To obtain (1.5), one counts the number of the negative eigenvalues of $P(r^{-1})$, $r \geq C_\gamma$ which is given by well known formula.

The analysis of the counting function of the eigenvalues of G lying in a strip $\{z \in \mathbb{C} : -A_0 \leq \operatorname{Re} z \leq 0\}$, $A_0 > 0$, as well as the study of the case (A) are open problems. There is a conjecture that there exists a sequence of eigenvalues λ_k, $|\operatorname{Im} \lambda_k| \to \infty$. For the investigation of these problems it seems convenient to use the semi-classical parametrix $T(h,z)$ for the exterior Dirichlet-to-Neumann problem constructed in [16] for strictly convex obstacles in the hyperbolic region $\{z \in \mathbb{C} :\ z = 1 + \mathbf{i}hw\}$, $|w| \leq B_0$.

The paper is organised as follows. In Section 2 we collect some facts concerning the operator $\mathcal{C}(\lambda) = \mathcal{N}(\lambda) - \lambda\gamma$ for $\operatorname{Re}\lambda < 0$, where $\mathcal{N}(\lambda)$ is exterior Dirichlet-to-Neumann map defined in the beginning of Section 2. We recall a the trace formula involving the resolvent $(G-\lambda)^{-1}$ established in [12]. In Section 3 one presents some information for the semi-classical parametrix for $N(h,z)$ and $z \in Z_e = \{z \in \mathbb{C} :\ z = -\frac{1}{(1+\mathbf{i}\eta)^2}\}$, $|\eta| \leq h^2$ based on the construction in [17], [19]. The properties of the operator $P(h)$ for h real are treated in Section 4. In Section 5 we compare the trace formulas for $\mathcal{C}(\lambda)$ and for $P(\tilde{h})$ and we prove Theorem 1. Finally, in Section 6 we discuss some generalisations and a dissipative boundary problem for Maxwell's equations.

2. Preliminaries

We start with some facts which are necessary for our exposition (see [12]). For $\operatorname{Re}\lambda < 0$ introduce the exterior Dirichlet-to-Neumann map

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_\nu u|_\Gamma \in H^{s-1}(\Gamma),$$

where u is the solution of the problem

$$\begin{cases} (-\Delta + \lambda^2)u = 0 \text{ in } \Omega,\ u \in H^2(\Omega), \\ u = f \text{ on } \Gamma, \\ u : (\mathbf{i}\lambda) - \text{outgoing}. \end{cases} \tag{2.1}$$

A function $u(x)$ is $(\mathbf{i}\lambda)$-outgoing if there exists $R > \rho_0$ and $g \in L^2_{comp}(\mathbb{R}^d)$ such that

$$u(x) = (-\Delta_0 + \lambda^2)^{-1} g,\ |x| \geq R,$$

where $R_0(\lambda) = (-\Delta_0 + \lambda^2)^{-1}$ is the outgoing resolvent of the free Laplacian $-\Delta_0$ in $\mathbb{R}^d$ which is analytic in $\mathbb{C}$ for d odd and on the logarithmic covering of $\mathbb{C}$ for d even. The resolvent

$R_0(\lambda)$ has kernel

$$R_0(\lambda, x-y) = -\frac{\mathbf{i}}{4}\Big(\frac{-\mathbf{i}\lambda}{2\pi|x-y|}\Big)^{(n-2)/2}\Big(H^{(1)}_{\frac{n-2}{2}}(u)\Big)\Big|_{u=-\mathbf{i}\lambda|x-y|}, \tag{2.2}$$

$H^{(1)}_\nu(z)$ being the Hankel function of first kind and we have the asymptotic

$$H^{(1)}_\nu(z) = \Big(\frac{2}{\pi r}\Big)^{1/2} e^{\mathbf{i}(z-\frac{\nu\pi}{2}-\frac{\pi}{4})} + \mathcal{O}(r^{-3/2}),\ -\pi < \arg z < 2\pi, |z| = r \to +\infty. \tag{2.3}$$

The solution of the problem (2.1) with $f \in H^{3/2}(\Gamma)$ has the representation

$$u = e(f) + (-\Delta_D + \lambda^2)^{-1}((\Delta - \lambda^2)(e(f)),$$

where $e(f) : H^{3/2}(\Gamma) \ni f \to e(f) \in H^2_{comp}(\Omega)$ is an extension operator and $R_D(\lambda) = (-\Delta_D + \lambda^2)^{-1}$ is the outgoing resolvent of the Dirichlet Laplacian Δ_D in Ω. The cut-off resolvent $R_\chi(\lambda) = \chi(x)R_D(\lambda)\chi(x)$ with $\chi(x) \in C_0^\infty(\mathbb{R}^d)$ equal to 1 in a neighbourhood of $K \cup \operatorname{supp} e(f)$ is analytic for $\operatorname{Re}\lambda < 0$ and meromorphic in $\mathbb{C}$ for d odd and on the logarithmic covering of $\mathbb{C}$ for d even. Consequently, $\mathcal{N}(\lambda) : H^{3/2}(\Gamma) \to H^{1/2}(\Gamma)$ is a meromorphic operator-valued function with the same poles as $R_\chi(\lambda)$. The same result holds for the action of $\mathcal{N}(\lambda)$ on other Sobolev spaces. Consider the set $\Lambda \subset \{z \in \mathbb{C} : \operatorname{Re} z < -C_0 \leq -1\}$ introduced in Section 1. By using the estimates for $R_\chi(\lambda)$ for $\operatorname{Re}\lambda < -C_0$, we obtain

$$\|\mathcal{N}(\lambda)\|_{H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)} \leq A_0|\lambda|^2,\ \lambda \in \Lambda. \tag{2.4}$$

Applying Green's representation for the solution $u(y)$ of (2.1) and taking the limit

$$\Omega \ni y_n \to x \in \Gamma,$$

we have

$$(C_{00}(\lambda)f)(x) - (C_{01}(\lambda)\mathcal{N}(\lambda)f)(x) = \frac{f(x)}{2},\ x \in \Gamma$$

where

$$(C_{00}(\lambda)f)(x) = \int_\Gamma f(y)\frac{\partial}{\partial\nu(y)}R_0(\lambda, x-y)dS_y,$$

$$(C_{01}(\lambda)g)(x) = \int_\Gamma g(y)R_0(\lambda, x-y)dS_y$$

are the Calderón operators or double and single layer potentials which have the same analytic properties as $R_0(\lambda, x-y)$. Melrose showed ([10], Section 3) that there exists an entire family $P_D(\lambda)$ of compact pseudo-differential operators of order -1 on Γ such that

$$-2(-\Delta_\Gamma + 1)^{1/2}C_{01}(\lambda) = Id + P_D(\lambda),$$

Δ_Γ being the Laplace Beltrami operator on Γ equipped with the Riemannian metric induced by the Euclidean one in $\mathbb{R}^d$. In fact, $-C_{01}(\lambda)$ is a pseudo-differential operator of order -1 with principal symbol $\frac{1}{2}(-\Delta_\Gamma)^{-1/2}$ (see [10]) and one takes the composition of the operators $\sqrt{-\Delta_\Gamma + 1}$ and $(-\Delta_\Gamma)^{-1/2}$. Consequently, $(Id + P_D(\lambda))^{-1}$ is a meromorphic operator-valued function and for $\operatorname{Re}\lambda < 0$ one deduces

$$\mathcal{N}(\lambda) = (Id + P_D(\lambda))^{-1}(-\Delta_\Gamma + 1)^{1/2}(Id - 2C_{00}(\lambda)). \tag{2.5}$$

Since $\mathcal{N}(\lambda)$ is analytic for $\operatorname{Re}\lambda < 0$,- 1 is not an eigenvalue of $P_D(\lambda)$ for $\operatorname{Re}\lambda < 0$. On the other hand, $C_{00}(\lambda)$ is a pseudo-differential operator of order -1, hence it is compact one. The Neumann problem

$$\begin{cases} (-\Delta + \lambda^2)u = 0 \text{ in } \Omega, \; u \in H^2(\Omega), \\ \partial_\nu u = 0 \text{ on } \Gamma, \\ u : (\mathbf{i}\lambda) - \text{outgoing}. \end{cases} \tag{2.6}$$

has a non-trivial solution if the operator $2C_{00}(\lambda)$ has eigenvalue 1 and this occurs only if λ coincides with a resonance $\nu_j, \operatorname{Re}\nu_j > 0$, of the Neumann problem (see [6]). By Fredholm theorem one deduces that

$$\mathcal{N}(\lambda)^{-1} = (Id - 2C_{00}(\lambda))^{-1}(-\Delta_\Gamma + 1)^{-1/2}(Id + P_D(\lambda)) : H^s(\Gamma) \to H^{s+1}(\Gamma)$$

is meromorphic with poles ν_j.

Going back to the problem (1.2), for $\operatorname{Re}\lambda < 0$ we write the boundary condition as follows

$$\mathcal{C}(\lambda)v := (\mathcal{N}(\lambda) - \lambda\gamma)v = \mathcal{N}(\lambda)\Big(Id - \lambda\mathcal{N}(\lambda)^{-1}\gamma\Big)v = 0, \; v = f_1 \in H^{1/2}(\Gamma).$$

Clearly, for $\operatorname{Re}\lambda < 0$ the operator $\mathcal{C}(\lambda)$ has the same singularities as $\mathcal{N}(\lambda)$, hence $\mathcal{C}(\lambda) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is analytic and satisfies the estimate (2.4) with another constant A_0. The operator $\mathcal{N}(\lambda)^{-1}$ is compact and by the results in [12] there are points λ_0, $\operatorname{Re}\lambda_0 < 0$, for which $Id - \lambda_0\mathcal{N}(\lambda_0)^{-1}\gamma$ is invertible. Applying the analytic Fredholm theorem for the operator $\Big(Id - \lambda\mathcal{N}(\lambda)^{-1}\gamma\Big)$ in the half plane$\operatorname{Re}\lambda < 0$, one concludes that

$$\mathcal{C}(\lambda)^{-1} = \Big(Id - \lambda\mathcal{N}(\lambda)^{-1}\gamma\Big)^{-1}\mathcal{N}(\lambda)^{-1} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{2.7}$$

is a meromorphic operator-valued function. Notice that for $\lambda \in \mathbb{R}^-$ the operators $\mathcal{N}(\lambda), \mathcal{C}(\lambda)$ are self-adjoint. This follows from the Green formula for $(-\Delta + \lambda^2)$.

Remark 1. *It is important to note that the analyticity of the resolvent* $(-\Delta_D + \lambda^2)^{-1}$ *for* $\operatorname{Re}\lambda < 0$ *and the absence of resonances of the Neumann problem in the half plan* $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ *imply that* $\mathcal{C}(\lambda)^{-1}$ *is meromorphic for* $\operatorname{Re}\lambda < 0$ *and* (2.5) *is not necessary for the proof of this statement.*

For the resolvent $(\lambda - G)^{-1}$ in [12] the following trace formula has been proved.

Proposition 1. *Let* $\delta \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ *be a closed positively oriented curve without self intersections. Assume that* $\mathcal{C}(\lambda)^{-1}$ *has no poles on* δ *. Then*

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi i}\int_\delta (\lambda - G)^{-1}d\lambda = \operatorname{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi i}\int_\delta \mathcal{C}(\lambda)^{-1}\frac{\partial\mathcal{C}}{\partial\lambda}(\lambda)d\lambda. \tag{2.8}$$

Since G has only point spectrum in $\operatorname{Re}\lambda < 0$, the left hand term in (2.8) is equal to the number of the eigenvalues of G in the domain ω bounded by δ counted with their algebraic multiplicities. Setting $\tilde{\mathcal{C}}(\lambda) = \frac{\mathcal{N}(\lambda)}{\lambda} - \gamma$, we write the right hand side of (2.8) as

$$\operatorname{tr}\frac{1}{2\pi i}\int_\delta \tilde{\mathcal{C}}(\lambda)^{-1}\frac{\partial\tilde{\mathcal{C}}}{\partial\lambda}(\lambda)d\lambda. \tag{2.9}$$

Set $\lambda = -\frac{1}{\tilde{h}}$, $0 < \operatorname{Re} \tilde{h} \ll 1$ and consider the problem

$$\begin{cases} (-\tilde{h}^2 \Delta + 1)u = 0 \text{ in } \Omega, \\ -\tilde{h}\partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{2.10}$$

We introduce the operator $C(\tilde{h}) := -\tilde{h}\mathcal{N}(-\tilde{h}^{-1}) - \gamma$ and using (2.9), the trace formula (2.8) becomes

$$\operatorname{tr} \frac{1}{2\pi i} \int_\delta (\lambda - G)^{-1} d\lambda = \operatorname{tr} \frac{1}{2\pi i} \int_{\tilde{\delta}} C(\tilde{h})^{-1} \dot{C}(\tilde{h}) d\tilde{h}, \tag{2.11}$$

where $\dot{C}$ denote the derivative with respect to $\tilde{h}$ and $\tilde{\delta}$ is the curve $\tilde{\delta} = \{z \in \mathbb{C} : z = -\frac{1}{w},\, w \in \delta\}$.

Obviously, for $\lambda \in \Lambda$ one has $|\operatorname{Im} \lambda| \leq 1$ and this implies $\tilde{h} \in L$, where

$$L := \{\tilde{h} \in \mathbb{C} : \ |\operatorname{Im} \tilde{h}| \leq C_1 |\tilde{h}|^4,\ |\tilde{h}| \leq C_0^{-1},\ \operatorname{Re} \tilde{h} > 0\}. \tag{2.12}$$

We write the points in L as $\tilde{h} = h(1 + \mathbf{i}\eta)$ with $0 < h \leq h_0 \leq C_0^{-1}$, $\eta \in \mathbb{R}$. Recall that $\frac{2C_1}{C_0} \leq 1$. Then $\frac{C_1}{C_0^3} \leq 1/2$ and for $\tilde{h} \in L$ we get

$$|\eta| \leq \frac{1}{2}\sqrt{1 + \eta^2},$$

hence $\eta^2 \leq 1/3$. This implies

$$|\eta| \leq C_1 h (1 + \eta^2)^2 h^2 \leq h^2,\ h(1 + \mathbf{i}\eta) \in L,$$

since $\frac{16 C_1 h}{9} \leq 1$. Therefore the problem (2.10) becomes

$$\begin{cases} (-h^2 \Delta - z)u = 0 \text{ in } \Omega, \\ -(1 + \mathbf{i}\eta)h\partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{2.13}$$

with $z = -\frac{1}{(1+\mathbf{i}\eta)^2} = -1 + s(\eta)$, $|s(\eta)| \leq (2 + h^2)h^2 \leq 3h^2$. On the other hand,

$$C(\tilde{h}) = -(1 + \mathbf{i}\eta)h\mathcal{N}(-\tilde{h}^{-1}) - \gamma(x).$$

3. Parametrix for $N(h, z)$ in the elliptic region

In our exposition we will use h-pseudo-differential operators and we refer to [5] for more details. Let X be a C^∞ smooth compact manifold without boundary with dimension $d - 1 \geq 1$. Let (x, ξ) be the coordinates in $T^*(X)$ and let $a(x, \xi; h) \in C^\infty(T^*(X) \times (0, h_0])$. Given $\ell, m \in \mathbb{R}$, one denotes by $S^{\ell, m}$ the set of symbols so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha,\beta} h^{-\ell} (1 + |\xi|)^{m - |\beta|},\ \forall \alpha, \forall \beta, \quad (x, \xi) \in T^*(X).$$

If $\ell = 0$, we denote $S^{\ell, m}$ by S^m. The $h-$pseudo-differential operator with symbol $a(x, \xi; h)$ is defined by

$$(Op_h(a)f)(x) := \ (2\pi h)^{-d+1} \int_{T^*X} e^{\mathbf{i}\langle x - y, \xi\rangle / h} a(x, \xi; h) f(y) dy d\xi.$$

We define the space of symbols $S_{cl}^{\ell,m}$ which have an asymptotic expansion

$$a(x,\eta;h) \sim \sum_{j=0}^{\infty} h^{j-\ell} a_j(x,\eta),\ a_j \in S^{m-j}$$

and the corresponding classical pseudo-differential operator is given by

$$(Op(a)f)(x) := \ (2\pi)^{-d+1} \int_{T^*X} e^{\mathbf{i}\langle x-y,\eta\rangle} a(x,\eta;h) f(y) dy d\eta.$$

It is clear that by a change of variable $\xi = h\eta$ we may write a $h-$ pseudo-differential operator as a classical one with parameter h. We will use this fact in Section 4. The operators with symbols in $S^{\ell,m}, S_{cl}^{\ell,m}$ are denoted by $L^{\ell,m}, L_{cl}^{\ell,m}$, respectively. The wave front $\widetilde{WF}(A) \subset \widetilde{T^*(\Gamma)}$ of an operator $A \in L^{\ell,m}$ is defined as in [15], where $\widetilde{T^*(\Gamma)}$ is the compactification of $T^*(\Gamma)$.

We will recall some results for the *exterior* semi-classical Dirichlet-to-Neumann map (see [16], [17], [12]). Consider the operator

$$\mathcal{P}(h,z)u = (-h^2\Delta_x - z)u,\ z = -1 + s(\eta).$$

In local normal geodesic coordinates $(y_1, y'), y_1 = \mathrm{dist}\,(y,\Gamma)$ in a neighbourhood $\mathcal{U}$ of $x_0 \in \Gamma$ the operator $\mathcal{P}$ has the form (see [14])

$$\mathcal{P}(h,z) = h^2 D_{y_1}^2 + r(y, hD_{y'}) + h^2 q(x) D_{y_1} - z,\ D_j = -\mathbf{i}\partial_{y_j}$$

with $r(y,\eta') = \langle R(y)\eta', \eta'\rangle,\ q(y) \in C^\infty$. Here

$$R(y) = \Big\{ \sum_{k=1}^{d} \frac{\partial y_m}{\partial x_k} \frac{\partial y_j}{\partial x_k} \Big\}_{m,j=2}^{d} = \Big\{ \Big\langle \frac{\partial y_m}{\partial x}, \frac{\partial y_j}{\partial x} \Big\rangle \Big\}_{m,j=2}^{d}$$

is a symmetric $((d-1)\times(d-1))$ matrix and $r(0,y',\eta') = r_0(y',\eta')$, where $r_0(y',\eta')$ is the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$ on Γ equipped with the Riemannian metric induced by the Euclidean one in $\mathbb{R}^d$. For $z = -1 + s(\eta)$ introduce $\rho(y',\eta',z) = \sqrt{z - r_0(y',\eta')} \in C^\infty(T^*\Gamma)$ as the root of the equation

$$\rho^2 + r_0(y',\eta') - z = 0$$

with $\mathrm{Im}\,\rho(y',\eta',z) > 0$. We have $\rho \in S^1$ and

$$\sqrt{-1 + s(\eta) - r_0} = \mathbf{i}\sqrt{1 + r_0} - \frac{s(\eta)}{\sqrt{1 - s(\eta) + r_0} + \mathbf{i}\sqrt{1 + r_0}}$$

which implies $\rho - \mathbf{i}\sqrt{1+r_0} \in S^{-1}$.

Let u be the solution of the Dirichlet problem

$$\begin{cases} (-h^2\Delta - z)u = 0 \text{ in } \Omega, \\ u = f \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{3.1}$$

Consider the semi-classical Sobolev spaces $H_h^k(\Gamma)$ with norm $\|(1 - h^2\Delta)^{s/2} u\|_{L^2(\Gamma)}$ and introduce the exterior semi-classical Dirichlet-to-Neumann map

$$N(h,z) : H_h^s(\Gamma) \ni f \longrightarrow -h\partial_\nu u|_\Gamma \in H_h^{s-1}(\Gamma).$$

G. Vodev [17] established for bounded domains $K \subset \mathbb{R}^d$, $d \geq 2$, with C^∞ boundary and solutions u of the Helmoltz equation $(-h^2\Delta - z)u = 0$, $x \in K$, an approximation of the interior Dirichlet-to-Neumann map. With some modifications his results can be applied for the exterior Dirichlet-to-Neumann map $N(h, z)$ (see [12]). We need some information for the parametrix build in [17], [19] in the elliptic region $Z_e := \{z \in \mathbb{C} : \ z = -1 + s(\eta)\}$.

For the reader convenience we recall some points of the construction in [17], [19] for $z \in Z_e$. Let $\psi \in C_0^\infty(U_0)$, $\psi = 1$ in a neighbourhood U_0 of $x_0 \in \Gamma$. Denote the local normal geodesic coordinates by (x_1, x') and the dual variables by (ξ_1, ξ'). We search a parametrix u_ψ of the problem (3.1) with boundary data ψf in the form

$$\tilde{u}_\psi(x) = (2\pi h)^{-d+1} \iint e^{\frac{i}{h}(\varphi(x,\xi',z)+\langle y',\xi'\rangle)} \phi^2(\frac{x_1}{\delta}) a(x, \xi', h, z) f(y') d\xi' dy'.$$

Here $0 < \delta \ll 1$ and $\phi(t) \in C_0^\infty(\mathbb{R})$ is equal to 1 for $|t| \leq 1$ and to 0 for $|t| \geq 2$. We write

$$R(x) = \sum_{k=0}^{N-1} x_1^k R_k(x') + x_1^N \mathcal{R}_N(x), \ q(x) = \sum_{k=0}^{N-1} x_1 q_k(x') + x_1^N \mathcal{Q}_N(x).$$

For φ the eikonal equation modulo x_1^N becomes $(\partial_{x_1}\varphi)^2 + \langle R(x)\partial_{x'}\varphi, \partial_{x'}\varphi\rangle - z = x_1^N \Phi_N$ and one obtains a smooth solution having the form

$$\varphi = \sum_{k=0}^{N} x_1^k \varphi_k(x', \xi', z), \ \varphi_0 = -\langle x', \xi'\rangle, \ \partial_{x_1}\varphi|_{x_1=0} = \varphi_1 = \rho.$$

The functions φ_k satisfy for $0 \leq K \leq N-2$ the equalities

$$\sum_{k+j=K} (k+1)(j+1)\varphi_{k+1}\varphi_{j+1} + \sum_{k+j+\ell=K} \langle R_\ell \nabla_{x'}\varphi_k, \nabla_{x'}\varphi_j\rangle - z = 0. \tag{3.2}$$

Clearly, we can determine φ_{K+1} from the above equality since $\rho \neq 0$. For $z = -1$ we have $\rho = \mathbf{i}\sqrt{1 + r_0}$ and by recurrence one deduces $\varphi_k = \mathbf{i}\tilde{\varphi}_k$ with real-valued function $\tilde{\varphi}_k$. Thus for $z = -1$ we have $\varphi = -\langle x', \xi'\rangle + \mathbf{i}\tilde{\varphi}$ with real-valued function $\tilde{\varphi}$. The amplitude of the parametrix has the form

$$a = \sum_{j=0}^{N-1} h^j a_j(x, \xi', z), \ a_0|_{x_1=0} = \psi, \ a_j|_{x_1=0} = 0, \ j \geq 1$$

with $a_j = \sum_{k=0}^N x_1^k a_{k,j}(x', \xi', z)$, $a_{0,0} = \psi$, $a_{0,j} = 0$, $j \geq 1$. The functions a_j satisfy the transport equations

$$2\mathbf{i}\frac{\partial\varphi}{\partial x_1}\frac{\partial a_j}{\partial x_1} + 2\mathbf{i}\langle R(x)\nabla_{x'}\varphi, \nabla_{x'}a_j\rangle + \mathbf{i}(\Delta\varphi)a_j + \Delta a_{j-1}$$
$$= x_1^N A_N^{(j)}, \ 0 \leq j \leq N-1, \ a_{-1} = 0.$$

We write (see Section 3 in [19])

$$\Delta\varphi = \sum_{k=0}^{N-1} x_1^k \varphi_k^\Delta + x_1^N E_N(x), \ \Delta a_{j-1} = \sum_{k=0}^{N-1} x_1^k a_{k,j-1}^\Delta + x_1^N F_N^{(j-1)}(x)$$

with

$$\varphi_k^\Delta = (k+1)(k+2)\varphi_{k+2} + \sum_{\ell+\nu=k} \Big(\langle R_\ell \nabla_{x'}, \nabla_{x'}\varphi_\nu\rangle + q_\ell(\nu+1)\varphi_{\nu+1}\Big),$$

$$a^{\Delta}_{k,j-1} = (k+1)(k+2)a_{k+2,j-1} + \sum_{\ell+\nu=k}\Big(\langle R_\ell \nabla_{x'}, \nabla_{x'} a_{\nu,j-1}\rangle + q_\ell(\nu+1)a_{\nu+1,j-1}\Big).$$

This leads to the equality (see (3.18) in [19])

$$2\mathbf{i}\sum_{k_1+k_2=k}(k_1+1)(k_2+1)\varphi_{k_1+1}a_{k_2+1,j} + 2\mathbf{i}\sum_{k_1+k_2+k_3=k}\langle R_{k_1}\nabla_{x'}\varphi_{k_3}, \nabla_{x'}a_{k_3,j}\rangle$$
$$+\sum_{k_1+k_2=k}\mathbf{i}\varphi^{\Delta}_{k_1}a_{k_2,j} = -a^{\Delta}_{k,j-1} \text{ for } 0 \le k \le N-1,\ 0 \le j \le N-1. \tag{3.3}$$

We can determine $a_{k,j}$ by recurrence from the above equality so that $a_{0,0} = \psi,\ a_{0,j} = 0,\ j \geq 1,\ a_{k,-1} = 0,\ k \geq 0$. Next introduce the operator

$$T_\psi(h,z)f = -h\frac{\partial \tilde{u}_\psi}{\partial x_1}|_{x_1=0} = Op_h(\tau_\psi)f$$

with

$$\tau_\psi = -\mathbf{i}\rho\psi - \sum_{j=0}^{N-1} h^{j+1}a_{1,j},\ a_{1,j} \in S^{-j}.$$

By using the outgoing resolvent $(h^2\Delta_D - z)^{-1}$ for the Dirichlet Laplacian in Ω, we obtain a parametrix u_ψ in Ω and for $z \in Z_e$ we have (see Prop. 2.2 in [12] and [17])

$$\|\mathcal{N}(h,z)(\psi f) - T_\psi f\|_{H^N_h(\Gamma)} \leq C_N h^{-s_d+N}\|f\|_{L^2(\Gamma)},\ \forall N \in \mathbb{N} \tag{3.4}$$

with $C_N > 0, s_d > 0$ independent of f, h and z and s_d independent of N. Taking a partition of unity $\sum_{j=1}^M \psi_j(x') \equiv 1$ on Γ, we construct a parametrix and define the operator $T(h,z) = \sum_{j=1}^M T_{\psi_j}(h,z)$. For $z = -1$ the symbol $\sqrt{1+r_0} + \sum_{j=0}^{N-1} h^{j+1}a_{1,j}$ of $T(h,-1)$ is real valued and we have the estimate (3.4) with $T_\psi(h,z)$ replaced by $T(h,z)$. Clearly, we may extend the symbol of $T(h,z)$ holomorphically for $\tilde{h} \in L$.

4. Properties of the operator $P(h)$

In this section we assume that $\gamma(x) > 1,\ \forall x \in \Gamma$ and we study the operator $P(h) = T(h,-1) - \gamma(x)$ when h is real. Set

$$\min_{x\in\Gamma}\gamma(x) = c_0 > 1,\ \max_{x\in\Gamma}\gamma(x) = c_1 \geq c_0$$

and choose a constant $C = \frac{2}{c_1^2}$. As we mentioned in Section 3, we can consider the operator $P(h)$ as a classical pseudo-differential operator $Op(P)$ with parameter h with classical symbol $P = \sqrt{1+h^2r_0} - \gamma + hP_0(x,h\xi),\ P_0(x,\xi) \in S^0$. We denote by $(.,.)$ the scalar product in $L^2(\Gamma)$ and for two self adjoint operators L_1, L_2 the inequality $L_1 \geq L_2$ means $(L_1u,u) \geq (L_2u,u),\ \forall u \in L^2(\Gamma)$.

Proposition 2. *Let $\langle h\Delta\rangle = (1-h^2\Delta_\Gamma)^{1/2}$ and let $\epsilon = C(c_0-1)^2 < 2$. Then for h sufficiently small we have*

$$h\frac{\partial P(h)}{\partial h} + CP(h)\langle h\Delta\rangle^{-1/2}P(h) \geq \epsilon(1-C_2h)\langle h\Delta\rangle \tag{4.1}$$

with a constant $C_2 > 0$ independent of h.

Proof. The principal symbol of the operator on the left hand side in (4.1) has the form

$$\begin{aligned} q_1 &= 2h^2r_0(1+h^2r_0)^{-1/2} + C\sqrt{1+h^2r_0} - 2C\gamma(x) + C\gamma^2(x)(1+h^2r_0)^{-1/2} \\ &= (2+C-\epsilon)\sqrt{1+h^2r_0} - 2C\gamma(x) + (C\gamma^2(x)-2)(1+h^2r_0)^{-1/2} \\ &\qquad + \epsilon\sqrt{1+h^2r_0}. \end{aligned} \tag{4.2}$$

Clearly,

$$((2+C-\epsilon)\langle hD\rangle u, u) \geq ((2+C-\epsilon)u, u)$$

and

$$C\gamma^2(x) - 2 \leq Cc_1^2 - 2 = 0.$$

Therefore,

$$\begin{aligned} &((C\gamma^2(x)-2)\langle hD\rangle^{-1/2}u, u) = ((C\gamma^2(x)-2)(\langle hD\rangle^{-1/2}-1)u, u) \\ &+((C\gamma^2(x)-2)u, u) \geq ((C\gamma^2(x)-2)u, u) - hC_1\|u\|^2, \; 0 < h \leq h_0. \end{aligned}$$

Here the operator $(C\gamma^2(x)-2)(\langle hD\rangle^{-1/2}-1)$ has non-negative (classical) principal symbol

$$\frac{(2-C\gamma^2(x))h^2r_0}{1+h^2r_0+\sqrt{1+h^2r_0}}$$

and applied the semi-classical sharp Gärding inequality (see for instance, [5], Theorem 7.12). Taking into account (4.2) and the inequality $C(\gamma(x)-1)^2 - \epsilon \geq C(c_0-1)^2 - \epsilon = 0$, one deduces

$$\begin{aligned} (Op(q_1)u, u) &\geq ((C(\gamma(x)-1)^2) - \epsilon)u, u) + \epsilon(\langle hD\rangle u, u) - hC_1\|u\|^2 \\ &\geq \epsilon(\langle hD\rangle u, u) - hC_1h\|u\|^2. \end{aligned}$$

The full symbol of the operator on the right hand side of (4.1) has the form $q_1 + hq_0$. The term $h(Op(q_0)u, u) - hC_1\|u\|^2$ can be absorbed by $\epsilon C_2 h(\langle hD\rangle u, u)$ taking $\epsilon C_2 \geq C_1 + \|Op(q_0)\|_{L^2\to L^2}$ and this complets the proof. □

Remark 2. *The values of ϵ depends on $(c_0-1)^2$ and $\epsilon \searrow 0$ as $c_0 \searrow 1$. In the case when $\gamma \equiv const$ and K is the ball $\{x : \|x\| \leq 1\}$ the operator G has no eigenvalues if $\gamma \equiv 1$ (see* [12]*). Moreover, in this case for $\gamma > 1$ the eigenvalues of G lie in the interval $(-\infty, -\frac{1}{\gamma-1})$. Thus as $\gamma \searrow 1$, in the domain $\operatorname{Re}\lambda > -\frac{1}{\gamma-1}$ there are no eigenvalues.*

Next we follow the argument of Section 4, [15] with some modifications. Consider the semi-classical Sobolev space $H^s(\Gamma)$ with norm $\|u\|_s = \|\langle hD\rangle^s u\|_{L^2}$. The operator $P(h) : H^1 \to L^2$ has derivative $\dot{P}(h) = \mathcal{O}(h^{-1}) : H^1 \to L^2$. Denote by

$$\mu_1(h) \leq \mu_2(h) \leq ... \leq \mu_k(h) \leq ...$$

the eigenvalues of $P(h)$ repeated with their multiplicities.

Let h_1 be small and let $\mu_k(h_1)$ have multiplicity m. For h close to h_1 one has exactly m eigenvalues and we denote by $F(h)$ the space spanned by them. We can find a small interval (α, β) around $\mu_k(h_1)$, independent on h, containing the eigenvalues spanning $F(h)$. Given $h_2 > h_1$ close to h_1, consider a normalised eigenfunction $e(h_2)$ with eigenvalue $\mu_k(h_2)$. Let $\pi(h) = E_{(\alpha,\beta)}$ be the spectral projection of $P(h)$, hence $F(h) = \pi(h)L^2(\Gamma)$. Then $(\pi(h) - I)\pi(h) = 0$ yields $\pi(h)\dot{\pi}(h)\pi(h) = 0$ and we deduce $\dot{\pi}(h)|_{F(h)} = 0$. We construct a smooth extension $e(h) \in F(h)$, $h \in [h_1, h_2]$ of $e(h_2)$ with $\|e(h)\| = 1$, $\dot{e}(h) \in F(h)^\perp$. Obviously, $e(h_1)$ will be normalised eigenfunction with eigenvalue $\mu_k(h_1)$.

Considering the eigenvalues $\mu_k(h)$ of $P(h)$ in a small interval $[-\delta, \delta]$, $\delta > 0$, one gets $\|P(h)e(h)\| \leq \delta$. On the other hand,

$$h\dot{P}(h) = h^2\Delta\langle hD\rangle^{-1} + hL_0 = P(h) - \langle hD\rangle^{-1} + hL_1$$

with zero order operators L_0, L_1 and this implies $|(\dot{P}(h)e(h), e(h))| \leq C_0 h^{-1}, \ h \in [h_1, h_2]$. Therefore

$$|\mu_k(h_2) - \mu_k(h_1)| = \Big|\int_{h_1}^{h_2} \frac{d}{dh}(P(h)e(h), e(h))dh\Big| \leq C_0 \int_{h_1}^{h_2} h^{-1}dh \leq \frac{C_0}{h_1}(h_2 - h_1).$$

Assuming $\mu_k(h) \in [-\delta, \delta]$, we deduce that $\mu_k(h)$ is locally Lipschitz function in h and its almost defined derivative satisfies $|\frac{\partial \mu_k(h)}{\partial h}| \leq C_0 h^{-1}$.

To estimate $h\frac{\partial \mu_k(h)}{\partial h}$ from below, we exploit Proposition 2 and apply (4.1). For $h \leq h_0 \leq \frac{1}{8C_2}$ and $\mu_k(h) \in [-\delta, \delta]$ we have

$$h\frac{\partial \mu_k(h)}{\partial h} = (h\dot{P}(h)e(h), e(h)) \geq \epsilon(1 - C_2 h)(\langle hD\rangle e(h), e(h)) - C(\langle hD\rangle^{-1}P(h)e(h), P(h)e(h))$$
$$\geq \epsilon(1 - C_2 h) - C\delta^2 \geq \frac{3\epsilon}{4},$$

choosing

$$\delta = (c_0 - 1)\sqrt{\frac{1}{4} - C_2 h_0} \geq \frac{(c_0 - 1)}{2\sqrt{2}}.$$

Consequently, for $h \in [h_1, h_2]$ one has

$$\mu_k(h_2) - \mu_k(h_1) \geq \frac{3\epsilon}{4}\int_{h_1}^{h_2} h^{-1}dh \geq \frac{3\epsilon}{4h_2}(h_2 - h_1)$$

and we obtain

$$\frac{3\epsilon}{4} \leq h\frac{d\mu_k(h)}{dh} \leq C_0.$$

Fixing $h_0 >$ small, we conclude that the eigenvalue $\mu_k(h)$ increases when h increases and $\mu_k(h) \in [-\delta, \delta]$. It is well known (see for instance, [5]) that

$$\sharp\{k : \mu_k(h_0) \leq 0\} = \kappa_0 = \frac{1}{(2\pi h_0)^{d-1}}\int_{p_1(x,\xi)\leq 0} dxd\xi + \mathcal{O}(h_0^{-d+2}),$$

$p_1(x, \xi)$ being the principal symbol of $P(\operatorname{Re} h)$. Then for $k > \kappa_0$ we have $\mu_k(h_0) > 0$ and if for $h < h_0$ one has $\mu_k(h) < 0$, then there exists a point $h < h_k < h_0$ with the properties $\mu_k(h_k) = 0$, $\mu_k(h) < 0$ for $0 < h < h_k$. This implies that there exists a sequence $h_{k_0} \geq h_{k_0+1} \geq ...$ of values $0 < h \leq h_0$ such that $\mu_k(h_k) = 0$, $k_0 > \kappa_0$. These values h_k are precisely those for which $P(h)$ is not invertible. Next we choose $p > d$ and construct the intervals $I_{k,p}$ containing h_k with length $|I_{k,p}| \sim h^{p+1}$ and $|\mu_k(h)| \geq h^p$ for $h \in (0, h_0] \setminus I_{k,p}$. As in [15], one constructs the disjoint intervals $J_{k,p}$, and we obtain the following

Proposition 3 (Prop. 4.1, [15]). *Let $p > d$ be fixed. The inverse operator $P(h)^{-1} : L^2 \to L^2$ exists and has norm $\mathcal{O}(h^{-p})$ for $h \in (0, h_0] \setminus \Omega_p$, where Ω_p is a union of disjoint closed intervals $J_{1,p}, J_{2,p}, ...$ with $|J_{k,p}| = \mathcal{O}(h^{p+2-d})$ for $h \in J_{k,p}$. Moreover, the number of such intervals that intersect $[h/2, h]$ for $0 < h \leq h_0$ is at most $\mathcal{O}(h^{1-p})$.*

5. Relations between the trace integrals for $C(h)$ and $P(h)$

In this section we study the operators $C(h)$ and $P(h)$ for complex $h \in L$. We use the notation h instead of $\tilde{h}$ used in Sections 2, 3. For $z = -1$ the operator $T(\operatorname{Re} h, -1)$ constructed in Section 3 has principal semi-classical symbol $\sqrt{1 + r_0}$, so it is elliptic. The ellipticity holds also for the operator $T(h, z)$, $h \in L, z = -1 + s(\eta)$, holomorphic with respect to h, provided $|h|$ small enough. On the other hand, $P(h) = (1 + \mathbf{i}\eta)T(h, z) - \gamma(x)$ is not elliptic and for $h \in \mathbb{R}$, $\eta = 0$, $z = -1$ its semi-classical principal symbol vanishes on the set

$$\Sigma = \{(x, \xi) \in T^*(\Gamma) :\ r_0(x, \xi) = \gamma^2 - 1\}.$$

For the symbol $r_0(x, \xi)$ of the Laplace-Beltrami operator on Γ there exists a constant $C_3 > 0$ such that $r_0(x, \xi) \geq C_3 \|\xi\|^2$, $(x, \xi) \in T^*(\Gamma)$. Choose a constant $B_0 > 0$ so that $\sqrt{C_3} B_0 \geq 2c_1$ and consider a symbol $\chi(x, \xi) \in C_0^\infty(T^*(\Gamma))$, $0 \leq \chi(x, \xi) \leq 2$ such that

$$\chi(x, \xi) = \begin{cases} 2, & x \in \Gamma,\ \|\xi\| \leq B_0, \\ 0, & x \in \Gamma,\ \|\xi\| \geq B_0 + 1. \end{cases}$$

Introduce the operator

$$\tilde{M}(h) = P(\operatorname{Re} h) + \gamma(x)\chi(x, hD_x) = T(\operatorname{Re} h, -1) + \gamma(x)(\chi(x, hD_x) - 1).$$

The principal symbol of $\tilde{M}(h)$ has the form

$$\tilde{m}(x, \xi) = \sqrt{1 + r_0} + \gamma(x)(\chi(x, \xi) - 1).$$

Clearly, $\tilde{M}(h)$ is elliptic since for $\|\xi\| \leq B_0$ one gets $\operatorname{Re} \tilde{m}(x, \xi) \geq c_0$, while for $\|\xi\| > B_0$ we have

$$|\tilde{m}(x, \xi)| \geq \sqrt{C_3}\|\xi\| - c_1 \geq \frac{\sqrt{C_3}}{2}\|\xi\| + \frac{\sqrt{C_3}}{2} B_0 - c_1 \geq \frac{\sqrt{C_3}}{2}\|\xi\|.$$

Consequently, $\tilde{m}(x, \xi) \in S_0^1$, the operator $\tilde{M}(h)^{-1} : H^s - H^{s+1}$ is bounded by $\mathcal{O}_s(1)$ and $\widetilde{WF}(P(\operatorname{Re} h) - \tilde{M}(h)) \cap \{\|\xi\| \gg B_0 + 1\} = \emptyset$. Since $\chi(x, \xi)$ vanishes for $\|\xi\| \geq B_0 + 1$, by applying Proposition A.1 in [15], we can extend holomorphically $\chi(x, hD_x)$ to $\eta(x, \tilde{h}D_x)$ in the domain L. As we mentioned in Section 3, the operator $P(h)$ also has a holomorphic extension for $\tilde{h} \in L$.Thus $\tilde{M}(h)$ has a holomorphic extension

$$M(h) = P(h) + \gamma(x)(\eta(x, \tilde{h}D_x) - 1)$$

for $\tilde{h} \in L$ and $\widetilde{WF}(P(h) - M(h)) \cap \{\|\xi\| \gg B_0 + 1\} = \emptyset$. The last relation implies $P(h) - M(h) : \mathcal{O}(1) : H^{-s} \to H^s$, $\forall s$.

Now we can repeat without any change the proof of Lemma 5.1 in [15], exploiting Proposition 2. First we obtain

$$\|P(h)^{-1}\|_{\mathcal{L}(H^{-1/2}, H^{1/2})} \leq C \frac{\operatorname{Re} h}{|\operatorname{Im} h|},\ \operatorname{Re} h > 0,\ \operatorname{Im} h \neq 0. \tag{5.1}$$

Next, one deduces the estimate

$$\|P(h)^{-1}\|_{\mathcal{L}(H^s, H^{s+1})} \leq C_s \frac{\operatorname{Re} h}{|\operatorname{Im} h|},\ \operatorname{Re} h > 0,\ \operatorname{Im} h \neq 0 \tag{5.2}$$

applying (5.1) and the representation

$$P^{-1} = M^{-1} - M^{-1}(P - M)M^{-1} + M^{-1}(P - M)P^{-1}(P - M)M^{-1},$$

combined with the property of $P(h) - M(h)$ mentioned above. Following [15], introduce a piecewise smooth simply positively oriented curve $\gamma_{k.p}$ as a union of four segments: $\operatorname{Re} h \in J_{k,p}$, $\operatorname{Im} h = \pm(\operatorname{Re} h)^{p+1}$ and $\operatorname{Re} h \in \partial J_{k,p}$, $|\operatorname{Im} h| \leq (\operatorname{Re} h)^{p+1}$, where $J_{k,p}$ is one of the intervals in Ω_p defined in Proposition 3. Then we have

Proposition 4 (Prop. 5.2, [15]). *For every $h \in \gamma_{k,p}$ the inverse operator $P(h)^{-1}$ exists and*

$$\|P(h)^{-1}\|_{\mathcal{L}(H^s, H^{s+1})} \leq C_s(\operatorname{Re} h)^{-p},\ h \in \gamma_{k,p}.$$

To estimate $C(h)^{-1}$, we write

$$C(h) = -(1 + \mathbf{i}\eta)hN(\operatorname{Re} h, z) - \gamma(x) = (1 + \mathbf{i}\eta)T(\operatorname{Re} h, z) - \gamma(x) + \mathcal{R}_m(\operatorname{Re} h, z)$$

$$= P(h) + \mathcal{R}_m(h, z),\ m \gg 2p$$

with $\mathcal{R}_m(h, z) : \mathcal{O}((\operatorname{Re} h)^m) : H^s \to H^{s+m-1}$. Therefore

$$C(h)P(h)^{-1} = Id + \mathcal{R}_m(\operatorname{Re} h, z)P(h)^{-1} \tag{5.3}$$

and Proposition 4 imply

$$\Big\|\mathcal{R}_m(\operatorname{Re} h, z)P(h)^{-1}\Big\|_{\mathcal{L}(H^s, H^{s+m})} \leq C_s(\operatorname{Re} h)^{-p+m}.$$

For small $\operatorname{Re} h$ the operator on the right hand side of (5.3) is invertible and

$$C(h)P(h)^{-1}\Big(Id + \mathcal{R}_m(\operatorname{Re} h, z)P(h)^{-1}\Big)^{-1} = Id.$$

On the other hand, the operator $C(h)$ is elliptic for $|\xi| \gg 1$ and this implies that $C(h) : H^{1/2} \to H^{-1/2}$ is a Fredholm operator. The index of $C(h)$ is constant for $h \in L$ and according to the results in [12], this index is 0. Hence the right inverse to $C(h)$ is also a left inverse, so it is two side inverse. Thus we obtain

$$\|C(h)^{-1}\|_{\mathcal{L}(H^s, H^{s+1})} \leq C_s(\operatorname{Re} h)^{-p},\ h \in \gamma_{k,p}. \tag{5.4}$$

Moreover,

$$C(h)^{-1} - P(h)^{-1} = P(h)^{-1}\Big(\Big(Id + \mathcal{R}_m(\operatorname{Re} h, z)P(h)^{-1}\Big)^{-1} - Id\Big) = K(h) \tag{5.5}$$

with $K(h) = \mathcal{O}_s(|h|^{m-2p}) : H^s \to H^{s+m+1}$, $\forall s,\ h \in \gamma_{k,p}$. To estimate $\dot{C}(h) - \dot{P}(h)$, notice that $C(h) - P(h)$ is holomorphic with respect to h in L_0 and by Cauchy formula

$$\dot{C}(h) - \dot{P}(h) = \frac{1}{2\pi\mathbf{i}}\int_{\tilde{\gamma}_{k,p}} \frac{C(\zeta) - P(\zeta)}{\zeta - h}d\zeta = \frac{1}{2\pi\mathbf{i}}\int_{\tilde{\gamma}_{k,p}} \frac{\mathcal{R}_m(\operatorname{Re} h, z)}{\zeta - h}d\zeta = K'(h),$$

where $\tilde{\gamma}_{k,p}$ is the boundary of a domain containing $\gamma_{k,p}$ with the property $\operatorname{dist}(\tilde{\gamma}_{k,p}, \gamma_{k.p}) \geq (\operatorname{Re} h)^p$. Thus yileds

$$K'(h) = \mathcal{O}_s(|h|^{m-p}) : H^s \to H^{s+m+1},\ \forall s,\ h \in \gamma_{k,p}.$$

Concerning the operator $P(h)$, we obtain a trace formula repeating without any change the argument in [15]. Let $\mu_k(h_k) = 0$, $k \geq k_0$. It is easy to see that $\mu_k(h)$ has no other zeros for $0 < h \leq h_0$, exploiting the fact that $\mu_k(h)$ in increasing for $\mu_k(h) \in [-\delta, \delta]$. One defines the multiplicity of h_k as the multiplicity of the eigenvalue $\mu_k(h_k)$. Then we have

Proposition 5 (Prop. 5.3, [15]). *Let $\beta \subset L$ be a closed positively oriented C^1 curve without self intersections which avoids the points h_k with $\mu_k(h_k) = 0$. Then*

$$\mathrm{tr}\frac{1}{2\pi \mathbf{i}}\int_\beta P(h)^{-1}\dot{P}(h)dh$$

is equal to the number of h_k in the domain bounded by β.

Now we may compare the trace formula for $C(h)$ and $P(h)$. First we compare the integrals over $\gamma_{k,p}$. We have

$$\mathrm{tr}\,\frac{1}{2\pi \mathbf{i}}\int_{\gamma_{k,p}} C(h)^{-1}\dot{C}(h)dh = \mathrm{tr}\,\frac{1}{2\pi \mathbf{i}}\int_{\gamma_{k,p}} (C(h)^{-1} - P(h)^{-1})\dot{C}(h)dh$$

$$+\mathrm{tr}\,\frac{1}{2\pi \mathbf{i}}\int_{\gamma_{k,p}} P(h)^{-1}\dot{C}(h)dh = \mathrm{tr}\,\frac{1}{2\pi \mathbf{i}}\int_{\gamma_{k,p}} P(h)^{-1}\dot{C}(h)dh + \mathcal{O}_p((\mathrm{Re}\, h)^{m-2p}).$$

Here we have used (5.5) and the estimate

$$\|\dot{C}(h)\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \leq C|h|^{-2},\ h \in L.$$

which follows from (2.4). Next the property of $K'(h)$ yields

$$\mathrm{tr}\,\frac{1}{2\pi \mathbf{i}}\int_{\gamma_{k,p}} P(h)^{-1}\dot{C}(h)dh = \mathrm{tr}\,\frac{1}{2\pi \mathbf{i}}\int_{\gamma_{k,p}} P(h)^{-1}\dot{P}(h)dh + \mathcal{O}_p((\mathrm{Re}\, h)^{m-2p}).$$

For small h and $m \gg 2p$ the terms $\mathcal{O}_p((\mathrm{Re}\, h)^{m-2p})$ are negligible and we obtain, as in [15], a map ℓ_p between the set of points $h_k \in (0, h(p)]$ counted with their multiplicities and the eigenvalues $\ell_p(h_k) \in \Lambda$ counted with their multiplicities. The number of points $h_k \in J_{k,p}$ counted with their multiplicities is equal to the number of eigenvalues $\lambda_j = \ell(h_k)$ of G counted with their multiplicities lying in $\Lambda_{k,p} = \{z \in \mathbb{C} : z = -\frac{1}{\zeta},\ \zeta \in \omega_{k,p}\}$, $\omega_{k,p} \subset L$ being the domain bounded by $\gamma_{k,p}$. Notice that for a point h_k we could have many $\lambda_j \in \ell_p(h_k) \subset \Lambda_{k,p}$. On the other hand, for every $\lambda_j \in \ell_p(h_k)$ one has

$$|\lambda_j + \frac{1}{h_k}| \leq C_p h_k^{p+2-d}.$$

The integral over β in Proposition 5 can be presented as a sum of integrals over $\gamma_{k,p}$ plus integrals over curves $\alpha_{j,p}$ which are the boundary of domains $\beta_{j,p}$ such that $\beta_{j,p} \cap \Omega_p = \emptyset$, $\forall j$. By Proposition 3 for $h \in (0, h_0] \setminus \Omega_p$ the operator $P(h)$ is invertible. Applying an argument similar to that used above, one concludes that $P(h)$ is invertible for $h \in \beta_{j,p}$. Consequently, there are no contributions from the integrals over $\alpha_{j,p}$ and we must sum the contributions over the integrals over $\gamma_{k,p}$, that is the sum of the number of the corresponding points h_k.

Consider the counting function

$$\mathbf{N}(r) = \sharp\{\lambda \in \sigma_p(G) \cap \Lambda :\ |\lambda| \leq r,\ \mathrm{Re}\,\lambda \leq -C_0\},\ r > C_0$$

with $h_0^{-1} = C_0 > 0$ large enough. Then for $|\lambda| \leq r$ we must consider $r^{-1} < |\tilde{h}|$, $\tilde{h} \in L$. Modulo a finite number eigenvalues (see Section 4 and the number κ_0), we are going to count the points $r^{-1} < h_k \leq h_0$ and the number of the eigenvalues $\mu_k(h)$ for which we have $\mu_k(h_k) = 0$. Hence we have $\mu_k(r^{-1}) < 0$, since otherwise we obtain a contradiction. The problem is reduced to find the number of the negative eigenvalues of $P(r^{-1})$ which is given by well known formula

$$\frac{r^{d-1}}{(2\pi)^{d-1}}\int_{p_1(x,\xi)\leq 0} dx d\xi + \mathcal{O}_\gamma(r^{d-2}).$$

Clearly,

$$\int_{p_1(x,\xi)\leq 0} dx d\xi = \int_{r_0(x,\xi)\leq \gamma^2(x)-1} dx d\xi = \int_\Gamma (\gamma^2(x)-1)^{(d-1)/2}\Big(\int_{r_0(x,\eta)\leq 1} d\eta\Big) dx.$$

For the induced Riemannian metric on Γ the integral over the dual variable η yields the volume ω_{d-1} of the unit ball $\{x \in \mathbb{R}^{d-1} : |x| \leq 1\}$ and we obtain the asymptotic (1.5). This completes the proof of Theorem 1.

6. Generalisations

We may study with some modifications a more general dissipative boundary problem

$$\begin{cases} u_{tt} - \Delta_x u + c(x) u_t = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x)\partial_t u - \sigma(x) u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0,x) = f_1,\ u_t(0,x) = f_2, \end{cases} \tag{6.1}$$

where $c(x) \geq 0$, $\sigma(x) \geq 0$ are smooth functions defined respectively in $\mathbb{R}^d$ and Γ and $c(x) = 0$ for $|x| \geq R_0 > 0$ (see [8]). The solution is given by a semi-group $V(t) = e^{tG}$, $t \geq 0$ with $f = (f_1, f_2)$ in the energy space $\mathcal{H}_E$ with norm

$$\|f\|^2_{\mathcal{H}_E} = \int_\Omega (|\nabla_x f_1|^2 + |f_2|^2) dx + \int_\Gamma \sigma |f_1|^2 dy.$$

The generator of $V(t)$ has the form

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & c \end{pmatrix}$$

with a domain $D(G)$ being the closure in the graph norm

$$|||f|||_E = (\|f\|^2_{\mathcal{H}_E} + \|Gf\|^2_{\mathcal{H}_E})^{1/2}$$

of functions $f = (f_1, f_2) \in C^\infty_{(0)}(\mathbb{R}^d) \times C^\infty_{(0)}(\mathbb{R}^d)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 - \sigma f_1 = 0$ on Γ. If we have an eigenfunction $f = (f_1, f_2)$ with $Gf = \lambda f$, and $\lambda = -\frac{1}{\tilde{h}}$ (for simplicity we keep the notation of Section 2), then $u = f_1$ is a solution of the problem

$$\begin{cases} (-\tilde{h}^2\Delta + 1 - \tilde{h}c)u = 0 \text{ in } \Omega, \\ -\tilde{h}\partial_\nu u - \gamma u + \tilde{h}\sigma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{6.2}$$

Therefore with $\tilde{h} = h(1 + \mathbf{i}\eta)$, $\eta \in \mathbb{R}, z = -\frac{1}{(1+\mathbf{i}\eta)^2}$ we obtain the problem

$$\begin{cases} (-h^2\Delta - z - \frac{h}{1+\mathbf{i}\eta} c)u = 0 \text{ in } \Omega, \\ -(1+\mathbf{i}\eta)h\partial_\nu u - \gamma u + h(1+\mathbf{i}\eta)\sigma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{6.3}$$

We need to consider the semi-classical exterior Dirichlet-to-Neumann operator $N(h,z)$ related to the operator $-h^2\Delta - z - \frac{h}{1+\mathbf{i}\eta} c$. The construction of the semi-classical paramterix for

$N(h,z)$ is the same as in [17], [12] . The term $\frac{h}{1+\mathbf{i}\eta}c$ is lower order operator and the principal symbol of $N(h,z)$ is $\sqrt{1+r_0}$. Next we deal with the operator

$$P(\tilde{h}) = (1+\mathbf{i}\eta)N(h,z) - \gamma - h(1+\mathbf{i}\eta)\sigma$$

and the self-adjoint operator $P(h) = N(h,z) - \gamma - h\sigma$. Hear $h(1+\mathbf{i}\eta)\sigma$ is a lower order operator and we may repeat the arguments of Sections 4, 5. Under the assumptions of Theorem 1 one obtains a Weyl formula (1.5) with the same leading term. We leave the details to the reader.

We hope that our arguments combined with the construction of a semi-classical parametrix in [20] can be applied for the analysis of the eigenvalues of Maxwell's equations with dissipative boundary conditions

$$\begin{cases} \partial_t E - \operatorname{curl} H = 0,\ \partial_t H + \operatorname{curl} E = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \nu \wedge E - \gamma(x)(\nu \wedge \nu \wedge H) = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ E(0,x) = E_0(x),\ H(0,x) = H_0(x), \end{cases} \tag{6.4}$$

where $d = 3$, $(E_0, H_0) \in L^2(\mathbb{R}_t^+ \times \Omega : \mathbb{C}^6)$, $\gamma(x) > 0$, $\forall x \in \Gamma$. The solution of (6.4) is given by a contraction semi-group $V_b(t) = e^{tG_b}$, $t \geq 0$ (see [4] for the definition of G_b) and the spectrum of G_b in the half-plan $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is formed by isolated eigenvalues with finite multiplicities [2].

We sketch briefly below the similitudes with the analysis in Section 2. If $(E,H) \neq 0$ is an eigenfunction of G_b with eigenvalue λ, then

$$\begin{cases} \operatorname{curl} E = -\lambda H,\ \operatorname{curl} H = \lambda E \text{ in } \Omega, \\ \frac{1}{\gamma(x)}\Big(\nu \wedge \nu \wedge E\Big) + \nu \wedge H = 0 \text{ on } \Gamma, \\ (E,H) : (\mathbf{i}\lambda) - \textit{outgoing}. \end{cases} \tag{6.5}$$

Consider the problem

$$\begin{cases} \operatorname{curl} E = -\lambda H,\ \operatorname{curl} H = \lambda E \text{ in } \Omega, \\ \nu \wedge E = f \text{ on } \Gamma, \\ (E,H) : (\mathbf{i}\lambda) - \textit{outgoing}. \end{cases} \tag{6.6}$$

In the space $\mathcal{H}_s^t(\Gamma) := \{u \in H^s(\Gamma) : \langle \nu, u\rangle = 0\}$ introduce the operator $\mathcal{N}_b(\lambda) : H_{s+1}^t(\Gamma) \to H_s^t(\Gamma)$ defined by

$$\mathcal{N}_b(\lambda) f = \nu \wedge H|_\Gamma,$$

(E,H) being the solution of the problem (6.6). The operator $\mathcal{N}_b(\lambda)$ is the analog of the exterior Dirichlet-to-Neumann operator in Section 2 (see [20]) and the boundary condition on (6.5) can be written as

$$\mathcal{C}_b(\lambda) f = \frac{1}{\gamma(x)}(\nu \wedge f) + \mathcal{N}_b(\lambda) f = 0,\ f = \nu \wedge E|_\Gamma.$$

The outgoing resolvent of the problem

$$\begin{cases} \operatorname{curl} E = -\lambda H + F_1,\ \operatorname{curl} H = \lambda E + F_2 \text{ in } \Omega, \\ \nu \wedge E = 0 \text{ on } \Gamma, \\ (E,H) : (\mathbf{i}\lambda) - \textit{outgoing}, \end{cases}$$

is analytic for $\operatorname{Re}\lambda < 0$ since the above problem corresponds to a self-adjoint operator. Therefor we can prove that $\mathcal{C}_b(\lambda)$ is analytic for $\operatorname{Re}\lambda < 0$. In the same way from the fact that for $\operatorname{Re}\lambda < 0$ there are no non trivial solutions of the problem

$$\begin{cases} \operatorname{curl} E = -\lambda H, \ \operatorname{curl} H = \lambda E \text{ in } \Omega, \\ \nu \wedge H = 0 \text{ on } \Gamma, \\ (E, H) : (\mathbf{i}\lambda) - outgoing, \end{cases}$$

one concludes that $\mathcal{N}_b(\lambda)^{-1}$ is analytic for $\operatorname{Re}\lambda < 0$. As in Section 2, one deduces that $\mathcal{C}_b(\lambda)^{-1}$ is a meromorphic operator valued function for $\operatorname{Re}\lambda < 0$ (see (2.7) and Remark 1). Assuming $\gamma(x) \neq 1$, $\forall x \in \Gamma$, according to the results in [3], for every $\epsilon > 0$ and every $M \in \mathbb{N}$, $M \geq 1$ the eigenvalues of G_b lie in $\Lambda_\epsilon \cup \mathcal{R}_M$. Next one can establish a trace formula involving $(\lambda - G_b)^{-1}$ and for the analysis of the counting function of the eigenvalue of G_b in Λ it is possible to apply the strategy of Sections 4, 5 combined with the semi-classical parametrix constructed in [20].

Acknowledgments

Thanks are due to the referees for their critical comments and useful suggestions leading to an improvement of the previous version of the paper.

References

[1] F. Cakoni and H. Haddar, *Transmission eigenvalues in Inverse Scattering Theory.* Inside and Out II, MSRI Publications 60, 2012.
[2] F. Colombini, V. Petkov and J. Rauch, *Spectral problems for non elliptic symmetric systems with dissipative boundary conditions*, J. Funct. Anal. **267** (2014), 1637-1661.
[3] F. Colombini, V. Petkov and J. Rauch, *Eigenvalues of the Maxwell's equations with dissipative boundary conditions*, Asymptotic Analysis, **99** (2016), 105-124.
[4] F. Colombini and V. Petkov, *Weyl formula for the negative dissipative eigenvalues of Maxwell's equations*, Archiv der Mathematik, **110** (2018), 183-195.
[5] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in semi-classical limits*, London Mathematical Society, Lecture Notes Series, **268**, Cambridge University Press, 1999.
[6] P. Lax and R. Phillips, *Scattering Theory*, Second edition, Academic Press, New York, 1989.
[7] P. Lax and R. Phillips, *Scattering theory for dissipative systems*, J. Funct. Anal. **14** (1973), 172-235.
[8] A. Majda, *Disappearing solutions for dissipative wave equation*, Indiana Univ. Math. J. **24** (12) (1974), 1119-1133.
[9] A. Majda, *The location of the spectrum of the dissipative acoustic operator*, Indiana Univ. Math. J. **25** (1976), 973-987.
[10] R. Melrose, *Polynomial bound on the distribution of poles in scattering by an obstacle*, Journées Equations aux Dérivées Partielles, (1984), 1-8.
[11] Hoai-Minh Nguyen and Quoc-Hung Hguyen, The Weyl law of the transmission eigenvalues and the completeness of generalised transmission eigenfunctions, J. Funct. Anal, **281** (2001), 1091.
[12] V. Petkov, *Location of eigenvalues for the wave equation with dissipative boundary conditions*, Inverse Probl. Imagining, **10** (2016), 1111-1139.
[13] V. Petkov and G. Vodev, *Asymptotics of the number of the interior transmission eigenvalues*, J. Spectral Theory, **7** (2017), 1-31.
[14] L. Robiano, *Spectral analysis of the interior transmission eigenvalue problem*, Inverse Problems, **29**, (2013), 104001.
[15] J. Sjöstrand and G. Vodev, *Asymptotics of the number of Rayleigh resonances*, Math. Ann. **309** (1997), 287-306.

[16] J. Sjöstrand, *Weyl law for semi-classical resonances with random perturbed potentials*, Mémoires de SMF, **136** (2014).
[17] G. Vodev, *Transmission eigenvalue-free regions*. Commun. Math. Phys. **336** (2015), 1141-1166.
[18] G. Vodev, *High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues*, Anal. & PDF, **11** (1) (2018), 213-236.
[19] G. Vodev, *Parabolic transmission eigenvalues in the degenerate isotropic case*, Asymptotic Analysis, **106** (2018), 147-168.
[20] G. Vodev, *Semiclassical parametrix for the Maxwell equation and applications to the electromagnetic transmission eigenvalues*, Res. Math. Sci. **8** (3) (2021), Paper 35.

Université de Bordeaux, Institut de Mathématiques de Bordeaux, 351, Cours de la Libération, 33405 Talence, France
E-mail address: petkov@math.u-bordeaux.fr