ABSENCE OF EMBEDDED EIGENVALUES FOR HAMILTONIAN WITH CROSSED MAGNETIC AND ELECTRIC FIELDS

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Abstract. In the presence of the homogeneous electric field $\mathbf{E}$ and the homogeneous perpendicular magnetic field $\mathbf{B}$, the classical trajectory of a quantum particle on $\mathbb{R}^2$ moves with drift velocity α which is perpendicular to the electric and magnetic fields. For such Hamiltonians the absence of the embedded eigenvalues of perturbed Hamiltonian has been conjectured. In this paper one proves this conjecture for the perturbations $V(x, y)$ which have sufficiently small support in direction of drift velocity.

1. Introduction

We consider the quantum dynamics on the plane $\mathbb{R}^2$ in the presence of a homogeneous constant electric field which lies on this plane and a constant magnetic field which is perpendicular to this plane. Therefore the quantum system can be described by the following magnetic Stark Hamiltonian acting on $L^2(\mathbb{R}^2)$

$$H_{LS} := \frac{1}{2m}\left(D_X + \frac{B}{2}Y\right)^2 + \frac{1}{2m}\left(D_Y - \frac{B}{2}X\right)^2 - qE\cdot\mathbf{X} + V,$$

where $D_X = -i\partial_X$, $D_Y = -i\partial_Y$, $\mathbf{X} = (X, Y) \in \mathbb{R}^2$, $m > 0$, $q \neq 0$ are the position, the mass and the charge of a quantum particle and $\mathbf{E} = E = (E_1, E_2) \neq (0, 0)$, $\mathbf{B} = (0, 0, B)$, $B \neq 0$ stand for the electric field and the magnetic field, respectively. Next $V \colon \mathbb{R}^2 \to \mathbb{R}$ is the multiplication operator by $V(\mathbf{X})$. We assume that $V(\mathbf{X})$ is bounded and decays as $|\mathbf{X}| \to \infty$. Under some decaying conditions for the potential V, in [5] and [1] it was established that

$$\sigma(H_{LS} - V) = \sigma_{ac}(H_{LS} - V) = \mathbb{R},\ \sigma_{ess}(H_{LS}) = \mathbb{R}.$$

Here $\sigma(L)$, $\sigma_{ac}(L)$, $\sigma_{ess}(L)$, $\sigma_{pp}(L)$ denote the spectrum, the absolutely continuous spectrum, the essential spectrum and the point spectrum, respectively, of the operator L. In the physical literature it was conjectured that $\sigma_{pp}(H_{LS}) = \emptyset$. This property has been proved in the following cases:

(I) $|qE|^2 - qE\cdot\nabla V > 0$ for all $\mathbf{X} \in \mathbb{R}^2$ (see [4]),

(II) $|qE|$ is sufficiently large [1] or sufficiently small [4].

Moreover, it was shown in [5] that

(III) There exists $R_0 > 0$ such that $\sigma_{pp}(H_{LS}) \cap \Big((-\infty, -R_0] \cup [R_0, \infty)\Big) = \emptyset$ and, moreover, there exist at most a finite number of eigenvalues with finite multiplicities.

In particular, (II) implies that if eigenvalues exist, then $|qE|$ is not small as well as not large. This condition seems very strange and it is natural to show that for any $|qE|$, H_{LS} has no eigenvalues.

Key words and phrases. Crossed magnetic and electric fields, Embedded eigenvalues.

The absence of point spectrum of H_{LS} is an open and challenging problem. There are two major difficulties in the investigation of this problem. If we consider the operator $H_0 = H_{LS} - V$, first H_0 has double characteristics and second the electric and magnetic fields are not decreasing as $|\mathbf{X}| \to \infty$. Consequently, even for H_0 it is quite difficult to obtain weighted estimates for the resolvent

$$\| \langle \mathbf{X} \rangle^{-s} (H_0 - \lambda - \pm i\epsilon)^{-1} \langle \mathbf{X} \rangle^{-s} \|, \; s > 1/2, \; |\lambda| \gg 1$$

uniformly with respect to $\epsilon \neq 0$. In the literature there are a lot of works treating weighted resolvent estimates for the perturbations of the Laplacian. Recently the proof of suitable Carleman estimates led to several important results. We cite only some recent works [16], [17], [18], [8], where the reader may find other references. However, in these papers some decay of the potentials is assumed and this plays a crucial role in the analysis. Studying H_0, we cannot treat H_0 as a perturbation of $-\Delta$ since electromagnetic potentials do not decrease in $\mathbf{X}$ but increase quadratically. Usually, a Hamiltonian with quadratic potential may have only bound states. Nevertheless, the presence of electric potential $qE \cdot \mathbf{X}$ implies that H_0 has only a continuous spectrum. In this direction the Hamiltonian H_0 is an exceptional model in quantum physics. In the case with a potential V, it is natural to consider H_0 as an unperturbed operator and to obtain resolvent estimates for H_0.

We examine the situation when the support of $V(X, Y)$ in direction of *drift velocity*

$$\alpha = (E_2/B, -E_1/B)$$

is sufficiently small. Passing to new coordinates (x, y), this means that the support of $V(x, y)$ has small support with respect to y (see Assumption 1.1 below). We do not impose conditions on $|qE|$.

Concerning the velocity α, notice that according to Proposition 4.4 in Adachi-Kawamoto [1], we have the estimate

$$\mathrm{s-}\lim_{t \to \infty} \chi(q^2 B^2 \alpha \cdot \mathbf{X} \leq c_1 t) \varphi(H_{LS}) e^{-itH_{LS}} = 0,$$

where χ is the characteristic function such that $\chi(s \leq a) = 1$ if $s \leq a$ and $\chi(s \leq a) = 0$ if $s > a$, $\varphi \in C_0^\infty(\mathbb{R})$ and $c_1 > 0$ is a suitable constant. This proposition shows that the quantum particle described by this system undergo a uniform linear motion in direction α. By using this proposition, Kawamoto [12] characterized the space $L^2_{pp}(H_{LS})$ of all eigenstates of H_{LS}, as follows:

$$\psi \in L^2_{pp}(H_{LS}) \Leftrightarrow \lim_{R \to \infty} \sup_{t \in \mathbb{R}} \left\| \chi(R \leq |\alpha \cdot \mathbf{X}|) e^{-itH_{LS}} \psi \right\|_{L^2(\mathbb{R}^2)} = 0.$$

Hence the norms of the eigenfunctions over the region $|\alpha \cdot \mathbf{X}| \geq R$ goes to 0 as $R \to \infty$, it is expected that the behavior of the potential in direction perpendicular to α must be negligible for the existence of eigenvalues. It is easy to see that

$$|\alpha|^{-2} (\alpha \cdot \mathbf{X})^2 + |qE|^{-2} (qE \cdot \mathbf{X})^2 = |\mathbf{X}|^2, \quad \alpha \cdot qE = 0,$$

(see §1 of [12]) which implies that the direction α is perpendicular to qE.

In the following up to the end of the paper for simplicity we assume $m = 1/2, B = 1, q = 1$. Introduce the change of variables

$$|E| x = -E \cdot \mathbf{X}, \qquad |\alpha| y = -\alpha \cdot \mathbf{X}, \tag{1.1}$$

hence

$$x = -\frac{E_1 X + E_2 Y}{|E|}, \quad y = \frac{-E_2 X + E_1 Y}{|E|}, \quad X = -\frac{E_1 x + E_2 y}{|E|}, \quad Y = -\frac{E_2 x - E_1 y}{|E|}$$

and

$$\partial_X = -\frac{E_1}{|E|} \partial_x - \frac{E_2}{|E|} \partial_y, \quad \partial_Y = -\frac{E_2}{|E|} \partial_x + \frac{E_1}{|E|} \partial_y.$$

By using these variables, the Hamiltonian H_{LS} is reduced to

$$H_{LS} = \left(D_x + \frac{1}{2}y\right)^2 + \left(D_y - \frac{1}{2}x\right)^2 + |E|x + V(x,y)$$

and with the unitary transform $e^{ixy/2}$, we have

$$e^{-ixy/2} H_{LS} e^{ixy/2} = (D_x + y)^2 + D_y^2 + |E|x + V(x,y).$$

The potential V changes but we will denote again the new potential by $V(x,y)$. Throughout this paper we assume $|E| = 1$ and consider the reduced Hamiltonians

$$H := H_0 + V,$$
$$H_0 := (D_x + y)^2 + D_y^2 + x,$$

acting on $\mathscr{D}(H) = \mathscr{D}(H_0) \subset L^2(\mathbb{R}^2)$. In the exposition we will use the notation $\langle r \rangle = (1+r^2)^{1/2}$, $r = (x^2+y^2)^{1/2}$ and similar notation for $\langle x \rangle, \langle y \rangle$.

The purpose of this paper is to study two problems:

(A) Estimates for the resolvent $\langle r \rangle^{-\delta} (H_0 - \lambda - i\nu)^{-1} \langle r \rangle^{-\delta}$ for $|\lambda| \gg 1$, $\nu > 0$ and $\delta > 0$.

(B) Absence of eigenvalues of the operator H.

The problem **(A)** is examined in Section 3 and we prove the estimate

$$\| \langle r \rangle^{-\delta} (H_0 - \lambda - i\nu)^{-1} \langle r \rangle^{-\delta} \|_{L^2 \to L^2} \leq C_\delta \nu^{-3} |\lambda|^{-\delta/4}, \tag{1.2}$$

where $0 < \delta \leq 2$, $0 < \nu \leq 1$ and $|\lambda| \gg 1$. For $\delta = 2$ we obtain the optimal decay $\mathcal{O}(|\lambda|^{-1/2})$. The proof of Proposition 3.1 is based on a representation of operator e^{itH_0} established in [1]. It seems that this is the first result where we have an estimate of the resolvent of H_0 as $|\lambda| \to \infty$. For operators with magnetic and electric potentials having some decay a similar result with bound $\mathcal{O}(|\lambda|^{-1/2})$ and constant $C > 0$ uniform with respect to $\nu > 0$ has been obtain by Vodev [16]. For Stark Hamiltonian without magnetic field estimates of the resolvent are given in [2]. Concerning H_{LS}, it is an open problem to improve (1.2) with a constant independent of $\nu > 0$.

The problem **(B)** is studied under the assumption

Assumption 1.1. *We have $V(x,y), \partial_x V(x,y) \in C(\mathbb{R}^2)$ and there exists $\eta_0 > 0$ such that*

$$\operatorname{supp}(V) \subset \left\{(x,y) \in \mathbb{R}^2 \,:\, -\eta_0 \leq y \leq \eta_0\right\}. \tag{1.3}$$

Moreover, the potential $V(x,y)$ satisfies the estimates

$$\sup_{x,y\in\mathbb{R}} \langle x \rangle^{2s} |V(x,y)| \leq A_0, \ \sup_{x,y\in\mathbb{R}} \langle x \rangle |V_x(x,y)| \leq A_1, \tag{1.4}$$

with constants $s > 1/2$, $A_k > 0, k = 0, 1$.

Remark 1.2. *Under Assumption* 1.1*, it is easy to prove that the operators*

$$V(H+i)^{-1}, \quad (H-i)^{-1} (\partial_x V) (H+i)^{-1}$$

are compact ones.

Our goal is to prove the absence of embedded eigenvalues of H, when V satisfies (1.3) with small η_0. To examine the non-existence of eigenvalues of H, first we prove in Proposition 4.2 that without *any assumption* on the support of V there exist $R_1 > 0, R_2 > 0$ *independent of the support* of V such that

$$\sigma_{pp}(H) \cap \Big((-\infty, -R_1) \cup (R_2, \infty) \Big) = \emptyset.$$

This statement is more precise than the result in [5], where the dependence of the support of V was implicit. Moreover, in contrast to [5], the constants R_1, R_2 are explicitly given and we have

$$R_1 = C_1 + \|V\|_{L^\infty},\ R_2 = C_2 \| \langle x \rangle \langle y \rangle V_x \|_{L^\infty}^8,$$

where $C_1 > 0, C_2 > 0$ are independent of η_0 and V. We see that the **eigenvalues-free region** depends only on the *amplitudes* A_0, A_1. The argument in Section 4 is based on Lemmas 2.2, 2.3 and 2.7. and we show that with a suitable weight $\varphi(x) > 0$ one has the estimates

$$\| \sqrt{\varphi(x)} (D_x + y)(H - \lambda - i)^{-1} \|_{L^2 \to L^2} \leq C \langle \lambda \rangle^{1/2},$$

$$\| \sqrt{\varphi(x)} D_y (H - \lambda - i)^{-1} \|_{L^2 \to L^2} \leq C \langle \lambda \rangle^{1/2},$$

with constant $C > 0$ independent of λ. In the literature such type of estimates with $\lambda = 0$ have been used without a weight $\sqrt{\varphi(x)}$. However we show in Appendix A that the operators $(D_x + y)(H - i)^{-1}$, $D_y(H - i)^{-1}$ are *unbounded* (see Remark 2.4). We expect that the properties of these operators as well as Appendix A will be useful for further analysis.

Obviously, if V satisfies Assumption 1.1 with $s \geq 3/4$, then V satisfies (1.4) with $s < 3/4$. Next in the exposition without loss of generality we assume that $1/2 < s < 3/4$. Fixing $R = \max\{R_1, R_2\} > 0$, we establish a Mourre type estimate for the operator H_0. More precisely, setting $\gamma = 2s - 1 < 1/2$, there exists a constant $C_{R,\gamma}$ independent of η_0 such that

$$\sup_{\lambda \in [-R,R], \nu > 0} \left\| |y|^{-\gamma} F\Big(\frac{y}{\eta_0}\Big) \langle x \rangle^{-1/2-\gamma/2} (H_0 - \lambda \mp i\nu)^{-1} \langle x \rangle^{-1/2-\gamma/2} F\Big(\frac{y}{\eta_0}\Big) |y|^{-\gamma} \right\|_{L^2 \to L^2} \leq C_{R,\gamma}, \tag{1.5}$$

where $F(t) \in C_0^\infty(\mathbb{R} : [0,1])$ is a cut-off function such that $F(t) = 1$ for $|t| \leq 1$, $F(t) = 0$ for $|t| \geq 2$ (see Proposition 5.2). It is well known that H_0 has no eigenvalues in $\mathbb{R}$, however the above estimate for the resolvent of H_0 is non-trivial. Since H_0 has only continuous spectrum, the starting point is the Mourre estimate (5.1). Since $\gamma < 1/2$, the weight $|y|^{-\gamma}$ is integrable around 0 and this plays an essential role.

Our main result is the following

Theorem 1.3. *Let V satisfy the Assumption* 1.1, $\gamma = 2s - 1$ *and let $C_{R,\gamma} > 0$ be the constant in* (1.5). *Assume that*

$$\eta_0^{2\gamma} C_{R,\gamma} A_0 = c_{R,\gamma,\eta_0} < 1. \tag{1.6}$$

Then the operator H has no embedded eigenvalues.

The condition (1.6) does not imply the smallness of the potential, it is not related to $q|E|$ as well as to the cases (I) and (II). In fact, given a potential satisfying Assumption 1.1, one can choose the constant η_0 small enough in function of A_0 and $C_{R,\gamma}$, to obtain that there are no embedded eigenvalues of H.

Our result may be generalized to cover the case when the support of V is included in a strip $\{(x,y) \in \mathbb{R}^2 : |y - \beta| \leq \eta_0\}$ with fixed $\beta > 0$. Also, we can consider potentials having some singularities for $|x| + |y| \le K$. These generalizations need some technical modifications, but the idea of the proof is the same. For simplicity of the exposition we are not going to treat them.

Considering the case $\eta_0 \to 0$, one can choose A_0 large enough and such a case is closely related to the one where the potential is a delta function. When the potential V is a delta function, there are interesting results due to Hauge-van Leeuwen [10], Gyger-Martin [9] concerning the non-existence of embedded eigenvalues. Finally, notice that the absence of embedded eigenvalues is important for the analysis of the resonances widths (see [6], [7]) .

Setting $s = 1/2 + \gamma/2$, the proof of Theorem 1.3 is based on the equality

$$|y|^{-\gamma} F\Big(\frac{y}{\eta_0}\Big) \langle x \rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1} = |y|^{-\gamma} F\Big(\frac{y}{\eta_0}\Big) \langle x \rangle^{-s} (H_0 - \lambda - i\nu)^{-1} |D_x + i|^{-1}$$
$$-|y|^{-\gamma} F\Big(\frac{y}{\eta_0}\Big) \langle x \rangle^{-s} (H_0 - \lambda - i\nu)^{-1} V (H - \lambda - i\nu)^{-1} |D_x + i|^{-1}, \nu > 0,\ 0 < \gamma < 1/2,$$

where $\lambda \in \mathbb{R}$ and $F\Big(\frac{y}{\eta_0}\Big)$ is a cut-off function equal to 1 for $|y| \le \eta_0$.

By the condition of V, we have $V = |y|^{-2\gamma} F(y/\eta_0) \cdot |y|^{2\gamma} V$ and for $\eta_0^{2\gamma} C_{R,\gamma} \| \langle x \rangle^{1+\gamma} V \|_{L^\infty} = c_{R,\gamma,\eta_0} < 1$ we may estimate

$$\sup_{\lambda \in [-R,R], \nu > 0} \Big\| |y|^{-\gamma} F\Big(\frac{y}{\eta_0}\Big) \langle x \rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1} \Big\|_{L^2 \to L^2} \le \frac{B_{R,\gamma}}{1 - c_{R,\gamma,\eta_0}}.$$

If $\psi \in L^2(\mathbb{R}^2)$ is an eigenfunction of H with eigenvalues $\lambda \in [-R, R]$, we show in the Appendix B that $D_x \psi \in L^2$ and one obtains easily a contradiction with the above estimate. Following this approach, one needs to establish uniform estimate (1.5). To cover more general cases of potentials, it is necessary to obtain estimate similar to (1.5) with more general weights and this is an interesting open problem.

The plan of the paper is as follows. In Section 2 we prove some preliminary results including Lemma 2.3, Lemma 2.7 and Proposition 2.9 which are used in the next sections. In Section 3 we examine the estimates of the resolvent of H_0. The absence of large eigenvalues of H is studied in Section 4. Mourre type estimates are proved in Section 5 and Theorem 1.3 is established in Section 6. In the Appendix A and B we prove some technical results. Finally, notice that we use the Assumption 1.1 for the support of $V(x, y)$ with respect to y only in Section 6. The results in other sections concerning H hold without any restriction on the support of V.

2. Preliminaries

In this section we prove some lemmas which are necessary for the exposition. Throughout this section to the end of this paper, $\|\cdot\|_{L^2(\mathbb{R}^2)}$ and $\|\cdot\|_{\mathscr{B}(L^2(\mathbb{R}^2))}$ are denoted as $\|\cdot\|$ and $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbb{R}^2)$. Also we denote by $\mathscr{D}(A)$ the domain of the operator A. We write $r := \sqrt{x^2 + y^2}$ and $\langle \cdot \rangle = (1 + \cdot^2)^{1/2}$.

Lemma 2.1 (Interpolation Theorem)**.** *Let A and B be positive selfadjoint operators on $L^2(\mathbb{R}^2)$ and let T be a bounded operator on $L^2(\mathbb{R}^2)$. Assume that with constants $\alpha_0, \beta_0, \alpha_1, \beta_1 \ge 0$ and $C_0, C_1 > 0$ we have*

$$\Big\| A^{\alpha_0} T B^{\beta_0} \Big\| \le C_0,$$
$$\Big\| A^{\alpha_1} T B^{\beta_1} \Big\| \le C_1.$$

Then for all $0 < \theta' < 1$*, setting* $\alpha_{\theta'} = \alpha_0(1-\theta') + \alpha_1\theta'$ *and* $\beta_{\theta'} = \beta_0(1-\theta') + \beta_1\theta'$*, one has*

$$\left\| A^{\alpha_{\theta'}} T B^{\beta_{\theta'}} \right\| \leq C_0^{1-\theta'} C_1^{\theta'}.$$

Proof. We can find the proof of this lemma for example in §6 in Isozaki [11]. We will give a sketch of the proof based on the Hadamard's three line theorem. Recall this theorem.
Let $f(z)$ be an analytic function on $\Omega_0 := \{z = x+iy : \ 0 < x < 1, -\infty < y < \infty\}$ which is bounded on $\overline{\Omega_0} := \{z = x+iy : \ 0 \leq x \leq 1, -\infty < y < \infty\}$. Then if one has

$$\sup_{-\infty<y<\infty} |f(iy)| \leq M_0,$$
$$\sup_{-\infty<y<\infty} |f(1+iy)| \leq M_1,$$

then for all $0 < x < 1$ we have the estimate

$$\sup_{-\infty<y<\infty} |f(x+iy)| \leq M_0^{1-x} M_1^x.$$

Let $u, v \in L^2(\mathbb{R}^2)$, and let $E_A(\cdot)$, $E_B(\cdot)$ be the spectral decompositions of A and B, respectively. Let $I \subset \mathbb{R}$ be some bounded interval. Define

$$f(z) := \Big(E_A(I) A^{\alpha_z} T B^{\beta_z} E_B(I) u, v\Big),$$

where $\alpha_z := (\alpha_1 - \alpha_0)z + \alpha_0$ and $\beta_z := (\beta_1 - \beta_0)z + \beta_0$. Then $f(z)$ is an analytic function on $0 < \mathrm{Re} z < 1$ which is continuous and bounded on $0 \leq \mathrm{Re} z \leq 1$ and for all $y \in \mathbb{R}$ one gives

$$|f(iy)| \leq C_0 \|u\|\|v\|, \quad |f(1+iy)| \leq C_1 \|u\|\|v\|.$$

By the theorem above, for all $0 < \theta' < 1$, we get

$$|f(\theta')| \leq C_0^{1-\theta'} C_1^{\theta'} \|u\|\|v\|.$$

Since C_0, C_1 are independent of I, by taking $I \to \mathbb{R}$, one completes the proof of Lemma 2.1. □

Introduce a positive function $\rho(x) \in C^\infty(\mathbb{R})$ such that for some fixed $a > 1$ we have $\rho(x) = -\frac{1}{x}$, $x \leq -a$, $\rho(x) = 2x, 2x > a$.

Lemma 2.2. *For* $f \in \mathscr{D}(H)$ *we have*

$$\iint_{\mathbb{R}^2} \rho(x)\Big(|(D_x+y)f|^2 + |D_y f|^2\Big) dxdy \leq C(\|Hf\|^2 + \|f\|^2). \tag{2.1}$$

Proof. Consider for $f \in C_0^\infty(\mathbb{R}^2)$ the integral

$$0 \leq \iint_{\mathbb{R}^2} |(H_0 - \rho)f|^2 dxdy = \iint_{\mathbb{R}^2} \Big(|H_0 f|^2 + (\rho^2 - 2x\rho + \rho'')|f|^2\Big) dxdy$$
$$-2\iint_{\mathbb{R}^2} \rho\Big(|(D_x+y)f|^2 + |D_y f|^2\Big) dxdy.$$

Here we have used that by integration by parts one obtains

$$-\mathrm{Re} \iint_{\mathbb{R}^2} \Big[((D_x+y)^2 f)\rho\bar{f} + \rho f\overline{(D_x+y)^2 f}\Big] dxdy = \iint_{\mathbb{R}^2} \Big(-2\rho|(D_x+y)f|^2 + \rho''|f|^2\Big) dxdy$$

and similarly one transforms the integral with $D_y^2 f$. Clearly with a constant $C_0 > 0$, one has

$$\rho^2 - 2x\rho + \rho'' < C_0,$$

hence

$$2\iint_{\mathbb{R}^2} \rho(x)\Big(|(D_x+y)f|^2 + |D_y f|^2\Big) dxdy \leq C_1(\|H_0 f\|^2 + \|f\|^2) \leq C_2(\|Hf\|^2 + \|f\|^2).$$

Since H is a closed operator, for every $f \in \mathscr{D}(H)$ there exists a sequence of functions $f_n \in C_0^\infty(\mathbb{R}^2)$ such that $f_n \to f$, $Hf_n \to Hf$ in L^2. Taking the limit $n \to \infty$, we obtain the result. □

The above Lemma is analogous to Lemma 1 in [15] for Stark Hamiltonian.

Consider a function $0 < \varphi(x) \le A$ defined by

$$\varphi(x) = \begin{cases} \tan^{-1}(x - a + \pi/4), & x \ge a, \\ \varphi_1(x), & -a < x < a \\ -\frac{1}{x}, & x \le -a, \end{cases}$$

where $\frac{1}{a} \le \varphi_1(x) \le 1$, $|x| \le a$ is a smooth function so that $\varphi(x) \in C^3(\mathbb{R})$ and $\varphi'(x) > 0$ for all $x \in \mathbb{R}$.

Lemma 2.3. *We have the estimates*

$$\left\| \varphi^{1/2}(x)(D_x + y)(H - \lambda - i)^{-1} \right\|_{L^2 \to L^2} \le C \langle \lambda \rangle^{1/2}, \tag{2.2}$$

$$\left\| \varphi^{1/2}(x) D_y (H - \lambda - i)^{-1} \right\|_{L^2 \to L^2} \le C \langle \lambda \rangle^{1/2} \tag{2.3}$$

with a constant $C = C_a > 0$ independent of λ.

Proof. By using the resolvent equality

$$(H - \lambda - i)^{-1} = (H_0 - \lambda - i)^{-1} - (H_0 - \lambda - i)^{-1} V (H - \lambda - i)^{-1},$$

it is sufficient to prove the estimates with H replaced by H_0. We apply the unitary operator $e^{i\lambda D_x}$ giving a shift $x \to x + \lambda$, and obtain

$$e^{i\lambda D_x} \varphi^{1/2}(x)(D_x + y) e^{-i\lambda D_x} e^{i\lambda D_x} (H_0 - \lambda - i)^{-1} e^{-i\lambda D_x}$$
$$= \varphi^{1/2}(x + \lambda) \rho^{-1/2}(x) \Big(\rho^{1/2}(x)(D_x + y)(H_0 - i)^{-1} \Big).$$

On the other hand,

$$\left\| \rho^{1/2}(x)(D_x + y)(H_0 - i)^{-1} \right\|_{L^2 \to L^2} \le C.$$

In fact, we apply (2.1) replacing H by H_0 and choose $f = (H_0 - i)^{-1} g$. This yields

$$\|\rho^{1/2}(x)(D_x + y)(H_0 - i)^{-1} g\|^2 \le C\Big(\|H_0 (H_0 - i)^{-1} g\|^2 + \|(H_0 - i)^{-1} g\|^2 \Big) \le C_1 \|g\|^2.$$

It remains to prove the estimate

$$\varphi(x + \lambda) \rho^{-1}(x) \le C_2 (1 + |\lambda|) \tag{2.4}$$

with $C_2 = C_2(a) > 0$ independent of λ. For $x \ge -2a$ the function $\rho^{-1}(x)$ is bounded by a constant B_a depending on a and $\varphi(x+\lambda) \le A$, hence we have (2.4) . We are going to study the case $x < -2a$. We have three subcases: (i) $|\lambda| \le a$, $x < -2a$, (ii) $|\lambda| > a$, $-2|\lambda| \le x < -2a$, (iii) $|\lambda| > a$, $x < -2|\lambda|$. Clearly, in the subcase (i) one has $x + \lambda \le -2a + |\lambda| \le -a$ and $\varphi(x + \lambda)\rho^{-1}(x) = \frac{2|x|}{|x+\lambda|} \le 2 + \frac{2|\lambda|}{a}$. In the subcase (ii) we have $\rho^{-1}(x) = |x| \le 2|\lambda|$. In the subcase (iii) we have $x + \lambda < -|\lambda| \le -a$ and $\varphi(x + \lambda)\rho^{-1}(x) = \frac{|x|}{|x+\lambda|} \le 1 + \frac{|\lambda|}{a}$. Thus we obtain (2.4). For D_y we apply the same argument. □

Remark 2.4. *The presence of the factor $\varphi^{1/2}(x)$ in the estimates* (2.2), (2.3) *is important for the boundedness of these operators. In fact, the domain $\mathscr{D}(H)$ is not included in the domains $\mathscr{D}(D_x + y)$, $\mathscr{D}(D_y)$ and both operators $(D_x + y)(H - i)^{-1}$, $D_y(H - i)^{-1}$ are unbounded. We prove this property in Appendix A.*

Remark 2.5. *It is clear that the estimates* (2.2), (2.3) *hold with $\varphi(x)$ replaced by $\langle x \rangle^{-1}$.*

Corollary 2.6. *For every $0 < \gamma \leq 1$ we have the estimate*

$$\left\| \langle x \rangle^{-\gamma/2} \langle D_y \rangle^{\gamma} (H - \lambda - i)^{-1} \right\|_{L^2 \to L^2} \leq C_\gamma \langle \lambda \rangle^{\gamma/2} \tag{2.5}$$

Proof. Writing $\langle D_y \rangle = \frac{\langle D_y \rangle}{D_y + i}(D_y + i)$, we deduce that (2.5) holds with $\gamma = 1$. Next we apply the interpolation Lemma 2.1 between

$$\| \langle x \rangle^0 \langle D_y \rangle^0 \langle x \rangle^0 (H - \lambda - i)^{-1} \| \leq 1$$

and

$$\left\| \langle x \rangle^{-1/4} \langle D_y \rangle \langle x \rangle^{-1/4} (H - \lambda - i)^{-1} \right\| \leq C \langle \lambda \rangle^{1/2}.$$

□

Notice that we have

$$\varphi'(x) = \begin{cases} \left(1 + (x - a + \pi/4)^2\right)^{-1}, \; x \geq a, \\ \varphi_1'(x), \; -a \leq x < a, \\ x^{-2}, \; x < -a, \end{cases}$$

which implies

$$(\varphi'(x))^{1/2} \leq C_2 \sqrt{\varphi(x)},$$

hence the estimates (2.2), (2.3) hold with $\varphi(x)$ replaced by $\varphi'(x)$.

For the eigenfunctions of H we need a more precise result.

Lemma 2.7. *Let ψ and λ be an eigenfunction and eigenvalue of H. Moreover, suppose that $\|\psi\| = 1$. Then we have the estimates*

$$\left\| \sqrt{\varphi'(x)} (D_x + y) \psi \right\| \leq C \langle \lambda \rangle^{1/4}, \tag{2.6}$$

$$\left\| \sqrt{\varphi(x)} \sqrt{\varphi'(y)} D_y \psi \right\| \leq C \langle \lambda \rangle^{3/8} \tag{2.7}$$

with $C = C_a > 0$ independent of ψ and λ and the support of V.

Proof. By a direct calculus we obtain a representation for the commutator

$$\begin{aligned} 0 &= \Big(i \left[H, \varphi(x)(D_x + y) + (D_x + y)\varphi(x) \right] \psi, \psi \Big) \\ &= \Big(\big(4(D_x + y)\varphi'(x)(D_x + y) + 4\varphi(x) D_y - \varphi(x)(1 + V_x) + 2\varphi'''(x) \big) \psi, \psi \Big). \end{aligned}$$

Hence we have

$$\left\| \sqrt{\varphi'(x)} (D_x + y) \psi \right\|^2 \leq C_1 + \| \varphi(x) D_y \psi \|,$$

where the constant $C_1 > 0$ depends only on $\varphi(x)$ and $\|V_x\|_{L^\infty}$.

Applying Lemma 2.3, one deduces

$$\| \varphi(x) D_y \psi \| \leq C \sqrt{A} \| \varphi^{1/2}(x) D_y (H - \lambda - i)^{-1} \psi \| \leq C_2 \langle \lambda \rangle^{1/2}$$

and

$$\left\| \sqrt{\varphi'(x)} (D_x + y) \psi \right\|^2 \leq C_3 \langle \lambda \rangle^{1/2}.$$

Now we pass to the analysis of the estimate containing D_y. In a similar way one has

$$\begin{aligned}
0 &= \Big(i\,[H, D_y\varphi(x)\varphi(y) + \varphi(y)\varphi(x)D_y]\,\psi,\psi\Big)\\
&= \Big((4D_y\varphi(x)\varphi'(y)D_y - 2\varphi(x)\varphi'''(y))\psi,\psi\Big)\\
&\quad + 4\mathrm{Re}\Big((D_x+y)\varphi'(x)\varphi(y)D_y\psi,\psi\Big) - 2\mathrm{Im}\,(D_y\varphi(y)\varphi(x)\psi, V\psi)\\
&\quad - 2\mathrm{Im}\,(\varphi(x)\varphi(y)D_y\psi, V\psi)\\
&\geq 4\left\|\sqrt{\varphi(x)\varphi'(y)}D_y\psi\right\|^2\\
&\quad - 4A\left\|\sqrt{\varphi'(x)}(D_x+y)\psi\right\|\left\|\sqrt{\varphi'(x)}D_y\psi\right\|\\
&\quad - C\Big(1+\left\|\sqrt{\varphi(x)}D_y\psi\right\|\Big).
\end{aligned}$$

Applying the estimates

$$\left\|\sqrt{\varphi'(x)}(D_x+y)\psi\right\| \leq C_a\,\langle\lambda\rangle^{1/4},$$

$$\left\|\sqrt{\varphi'(x)}D_y\psi\right\| \leq C_a\|\,\langle x\rangle^{-1}\,D_y\psi\| \leq C_a\|\sqrt{\varphi(x)}D_y\psi\| \leq C_a\,\langle\lambda\rangle^{1/2},$$

we obtain the result. □

It is obvious that the estimates (2.6) and (2.7) hold with $\sqrt{\varphi'(x)}$ and $\sqrt{\varphi'(y)}$ replaced by $\langle x\rangle^{-1}$ and $\langle y\rangle^{-1}$, respectively.

Remark 2.8. *Notice that by Lemma* 2.2*, we obtain*

$$\langle r\rangle^{-1/2}(D_x+y)\varphi \in L^2,\ \langle r\rangle^{-1/2}D_y\varphi\in L^2,\ \varphi\in\mathscr{D}(H).$$

Let H_{ev} be the space generated by the eigenfunctions of H. By the closed graph theorem the operators $\langle r\rangle^{-1/2}(D_x+y)$ and $\langle r\rangle^{-1/2}D_y$ are bounded as operators from H_{ev} to $L^2(\mathbb{R}^2)$. Therefore for every eigenfunction ψ of H we have the estimate

$$\|\,\langle r\rangle^{-1/2}(D_x+y)\psi\| + \|\,\langle r\rangle^{-1/2}D_y\psi\| \leq B\|\psi\|, \tag{2.8}$$

where $B>0$ is independent of ψ. However, the constant B in general could depend on the support of V.

Let $F(t)\in C_0^\infty(\mathbb{R})$ be a function such that $0\leq F(t)\leq 1$, $F(t)=1$ for $|t|\leq 1$, $F(t)=0$ for $|t|\geq 2$. For $c>0$ define $F_c(t)=F(\frac{t}{c})$.

Proposition 2.9. *Let $0<\gamma<1/2$, $\beta\in\mathbb{R}$ and $0<\eta_0<1$. Then the operator*

$$|y-\beta|^{-\gamma}F_{\eta_0}(y-\beta)\langle x\rangle^{-\gamma/2}\,\langle H\rangle^{-\gamma}$$

is bounded and its bound is independent of η_0 and β.

Proof. First consider the case $\beta=0$. Let $f(t)\in C_0^\infty(\mathbb{R})$ be a function such that $f(t)=1$ for $|t|\leq 2$. Then $f(t)F_{\eta_0}(t)=F_{\eta_0}(t)$ and for any $\phi\in C_0^\infty(\mathbb{R}^2)$, it is enough to prove that

$$\mathcal{A} := \left\||y|^{-\gamma}f(y)\langle x\rangle^{-\gamma/2}\,\langle H\rangle^{-\gamma}\,\phi\right\| \leq C_\gamma\|\phi\|$$

with a constant C_γ dependent of γ but independent of η_0. By simple calculation one has

$$\begin{aligned}
\mathcal{A} &\leq \left\||y|^{-\gamma}f(y)\langle x\rangle^{-\gamma/2}\,\langle H\rangle^{-\gamma}\,\phi\right\|\\
&\leq \left\||y|^{-\gamma}\,\langle D_y\rangle^{-\gamma}\right\|_{L^2(\mathbb{R}_y)\to L^2(\mathbb{R}_y)}\left\|\langle D_y\rangle^{\gamma}\,f(y)\langle x\rangle^{-\gamma/2}\,\langle H\rangle^{-\gamma}\,\phi\right\|.
\end{aligned}$$

Here we apply the fractional Sobolev inequality (see, e.g., Stein-Weiss [14] and Yafaev [19])

$$\left\| |y|^{-\gamma} u \right\|_{L^2(\mathbb{R}_y)} \le C_\gamma \left\| (D_y^2)^{\gamma/2} u \right\|_{L^2(\mathbb{R}_y)}$$

for $u \in \mathscr{D}\left((D_y^2)^{\gamma/2}\right) \subset L^2(\mathbb{R}^2)$. Then $\mathcal{A}$ can be estimated by

$$\mathcal{A} \le C_\gamma \| \langle D_y \rangle^\gamma f(y) \langle x \rangle^{-\gamma/2} \langle H \rangle^{-\gamma} \phi \|.$$

On the other hand, the norms of the operators

$$\left\| \langle D_y \rangle^0 \langle x \rangle^0 \cdot f(y) \cdot \langle H \rangle^{-0} \right\|_{L^2 \to L^2},$$
$$\left\| \langle D_y \rangle^1 \langle x \rangle^{-1/2} \cdot f(y) \cdot \langle H \rangle^{-1} \right\|_{L^2 \to L^2} \le \left\| \langle x \rangle^{-1/2} \langle D_y \rangle f(y) (H-i)^{-1} \cdot (H-i) \langle H \rangle^{-1} \right\|_{L^2 \to L^2}$$

are bounded. In fact, we write $(H-i)^{-1} = (H_0 - i)^{-1} - (H_0 - i)^{-1} V (H-i)^{-1}$ and one applies Corollary 2.6 to estimate $\langle x \rangle^{-1/2} D_y (H_0 - i)^{-1}$. Since $\langle x \rangle^{-1/4} \langle D_y \rangle \langle x \rangle^{-1/4}$ is selfadjoint, by using the interpolation Lemma 2.1, we conclude that

$$\left\| \langle D_y \rangle^\gamma \langle x \rangle^{-\gamma/2} \cdot f(y) \cdot \langle H \rangle^{-\gamma} \right\|_{L^2 \to L^2}$$

is bounded. Therefore $\mathcal{A} \le C_\gamma \|\phi\|$ and we obtain the estimate.

Now consider the case when $\beta \ne 0$. We have

$$\begin{aligned} &\left\| |y-\beta|^{-\gamma} f(y-\beta) \langle x \rangle^{-\gamma/2} \langle H \rangle^{-\gamma} \right\|_{L^2 \to L^2} \\ &\le \left\| |y-\beta|^{-\gamma} \langle D_y \rangle^{-\gamma} \right\|_{\mathscr{B}(L^2(\mathbb{R}_y))} \left\| \langle D_y \rangle^\gamma f(y-\beta) \langle x \rangle^{-\gamma/2} \langle H \rangle^{-\gamma} \right\|_{L^2 \to L^2} \\ &\le \left\| e^{-i\beta D_y} |y|^{-\gamma} \langle D_y \rangle^{-\gamma} e^{i\beta D_y} \right\|_{\mathscr{B}(L^2(\mathbb{R}_y))} \left\| \langle D_y \rangle^\gamma f(y-\beta) \langle x \rangle^{-\gamma/2} \langle H \rangle^{-\gamma} \right\|_{L^2 \to L^2} \le C, \end{aligned}$$

noting that $\sup_y |f'(y-\beta)|$ is independent of β. □

3. Estimates of the resolvent of H_0

In the section we establish a decay estimate for

$$\|f(H_0 - \lambda - i\nu)^{-1} g\|_{L^2 \to L^2}$$

with $\nu > 0$ and $|\lambda| \to \infty$. In [7] the case when $f, g \in C_0^\infty(\mathbb{R}^2)$ has been studied, while in [1] the situation with $f, g \in L^p(\mathbb{R}^2)$, $p > 2$ was examined. We prove the following more precise result which has an independent interest.

Proposition 3.1. *Consider the operator*

$$M_\delta(\lambda, \nu) := \langle r \rangle^{-\delta} (H_0 - \lambda - i\nu)^{-1} \langle r \rangle^{-\delta}, \ 0 < \delta \le 2,$$

where $\lambda \in \mathbb{R}$, $0 < \nu \le 1$. *Then for* $0 < \theta \le 1/2$ *and* $|\lambda| \ge 1$, *there exists a constant* $C = C(\theta) > 0$ *such that*

$$\|M_\delta(\lambda,\nu)\|_{L^2 \to L^2} \le C\nu^{-1} \left(|\lambda|^{-\theta} + (1+\nu)|\lambda|^{-1} + (1+\nu^{-2})|\lambda|^{\theta-1} \right)^{\delta/2}. \tag{3.1}$$

Proof. We consider only the case $\lambda > 0$, since for $\lambda < 0$ the proof is similar. Set

$$\omega_n := \left\{ t \in [0, \infty) \ : \ |t - n\pi| \le \lambda^{-\theta} \right\}, \quad n \in \mathbb{N} \cup \{0\},$$

and $\Omega = \bigcup_{n=0}^{\infty} \omega_n$. By the integral formula for the resolvent, we have

$$M_2(\lambda,\nu) = i\,\langle r\rangle^{-2}\int_0^{\infty} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\,dt = K_1 + K_2$$

with

$$K_1 := i\,\langle r\rangle^{-2}\int_{\Omega} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\,dt$$

and

$$\begin{aligned} K_2 &:= i\,\langle r\rangle^{-2}\int_{[0,\infty)\setminus\Omega} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\,dt \\ &= i\,\langle r\rangle^{-2}\int_{[0,\infty)\setminus\Omega} \frac{1}{i\lambda}\left(\frac{d}{dt}e^{it\lambda}\right) e^{-it(H_0-i\nu)}\,\langle r\rangle^{-2}\,dt \\ &= \frac{1}{\lambda}\sum_{n=0}^{\infty}\left[\langle r\rangle^{-2} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\right]\Big|_{t=n\pi+\lambda^{-\theta}}^{t=(n+1)\pi-\lambda^{-\theta}} \\ &\quad + \frac{i}{\lambda}\int_{[0,\infty)\setminus\Omega}\langle r\rangle^{-2} e^{-it(H_0-\lambda-i\nu)}(H_0-i\nu)\,\langle r\rangle^{-2}\,dt. \end{aligned}$$

For $\phi\in L^2$ one gets

$$\begin{aligned} \|K_1\phi\| &\le \sum_{n=0}^{\infty}\left\|\int_{\omega_n}\langle r\rangle^{-2} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\,\phi dt\right\|_{L^2(\mathbb{R}^2)} \\ &\le \sum_{n=0}^{\infty}\int_{\omega_n}\left\|\langle r\rangle^{-2} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\,\phi\right\|_{L^2(\mathbb{R}^2)}dt \\ &\le \sum_{n=0}^{\infty}\int_{n\pi-\lambda^{-\theta}}^{n\pi+\lambda^{-\theta}}\left\|\langle r\rangle^{-2}\right\|_{L^\infty}^2\|\phi\|_{L^2(\mathbb{R}^2)}e^{-t\nu}dt \\ &= \frac{\|\langle r\rangle^{-2}\|_{L^\infty}^2\|\phi\|_{L^2(\mathbb{R}^2)}}{\nu}\left(e^{\nu\lambda^{-\theta}}-e^{-\nu\lambda^{-\theta}}\right)\sum_{n=0}^{\infty}e^{-n\nu\pi} \le 2C\frac{e^{\nu\lambda^{-\theta}}}{1-e^{-\nu\pi}}\lambda^{-\theta}\|\phi\|_{L^2(\mathbb{R}^2)} \\ &\le C_1\nu^{-1}\lambda^{-\theta}\|\phi\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.2}$$

Here for $0<\nu\le 1$ we have used the elementary inequality $1-e^{-\nu\pi}\ge\frac{\pi\nu}{2e^{\pi}}$. Next

$$\begin{aligned} &\frac{1}{\lambda}\left\|\sum_{n=0}^{\infty}\left[\langle r\rangle^{-2} e^{-it(H_0-\lambda-i\nu)}\,\langle r\rangle^{-2}\right]\Big|_{t=n\pi+\lambda^{-\theta}}^{t=(n+1)\pi-\lambda^{-\theta}}\phi\right\| \\ &\le \lambda^{-1}\|\langle r\rangle^{-2}\|_{L^\infty}^2\|\phi\|_{L^2(\mathbb{R}^2)}\sum_{n=0}^{\infty}\left(e^{-\nu((n+1)\pi-\lambda^{-\theta})}+e^{-\nu(n\pi+\lambda^{-\theta})}\right) \\ &\le 2C\lambda^{-1}e^{\nu\lambda^{-\theta}}\|\phi\|_{L^2}\sum_{n=0}^{\infty}e^{-\nu n\pi} \le C\frac{2e^{\nu\lambda^{-\theta}}}{1-e^{-\nu\pi}}\lambda^{-1}\|\phi\|_{L^2} \le C\nu^{-1}\lambda^{-1}\|\phi\|_{L^2} \end{aligned}$$

and

$$\frac{1}{\lambda}\left\|\int_{[0,\infty)\setminus\Omega}\langle r\rangle^{-2} e^{-it(H_0-\lambda-i\nu)}\nu\,\langle r\rangle^{-2}\,dt\right\|_{L^2} \le \frac{\nu}{\lambda}\|\langle r\rangle^{-2}\|^2\int_0^{\infty}e^{-\nu t}dt = \lambda^{-1}\|\langle r\rangle^{-2}\|^2.$$

Therefore,

$$\|K_2\|_{L^2\to L^2} \le C\nu^{-1}\lambda^{-1}\| \langle r\rangle^{-2} \|^2 + \lambda^{-1}\|K_3\|_{L^2\to L^2} \tag{3.3}$$

with

$$K_3 = \int_{[0,\infty)\setminus\Omega} \langle r\rangle^{-2} H_0 e^{-it(H_0-\lambda-i\nu)} \langle r\rangle^{-2} dt.$$

Now we estimate $\|K_3\|_{L^2\to L^2}$. Let $\mathbf{x} = (x, y)$. By an application of the formula (4.6) in [1], we have the following representation of the operator e^{-itH_0} (For the operator H_{LS} in [1] one chooses the constants $q = B = 1$, $m = 1/2$, $\omega = 2$, $E_1 = -1$, $E_2 = 0$, $\nu = 0$, $\tilde{\nu} = \omega = 2$, $\theta = \pi$ and $E_0 = 1$)

$$\begin{aligned}
&\left(e^{-itH_0}\phi\right)(\mathbf{x}) = (e^{-ixy/2}e^{-itH_{LS}}e^{ixy/2}\phi)(\mathbf{x})\\
&= \frac{1}{4\pi i\sin(t)}\int_{\mathbb{R}^2} e^{-ia(t)}e^{-ixy/2}e^{ib(t)\cdot\mathbf{x}}e^{-ic(t)\cdot\mathbf{A}(\mathbf{x})}e^{-i\mathbf{w}\cdot\mathbf{A}(\mathbf{x}-c(t))}e^{i(\cot t)(\mathbf{x}-c(t)-\mathbf{w})^2/4}e^{iw_1w_2/2}\phi(\mathbf{w})d\mathbf{w}\\
&=: \frac{1}{4\pi i\sin(t)}\int_{\mathbb{R}^2} K(t,\mathbf{x},\mathbf{w})\phi(\mathbf{w})d\mathbf{w}
\end{aligned}$$

with $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$,

$$\mathbf{A}(\mathbf{x}) = (-y/2, x/2), \quad a(t) = \int_0^t \left(b(s)^2 + 2b(s)\cdot\mathbf{A}(c(s))\right) ds$$

and $b(t) = (b_1(t), b_2(t))$, $c(t) = (c_1(t), c_2(t))$ with

$$b_1(t) = -(\sin(2t))/2, \quad b_2(t) = (1-\cos(2t))/2,$$

$$c_1(t) = \cos(2t), \quad c_2(t) = t - \sin(2t).$$

Simple calculation shows that

$$y\partial_x K(t,\mathbf{x},\mathbf{w}) = y\left(-i\frac{y}{2} + ib_1(t) - i\frac{c_2(t)}{2} - i\frac{w_2}{2} + i\frac{\cot t}{2}(x - c_1(t) - w_1)\right)K(t,\mathbf{x},\mathbf{w}),$$

$$\partial_x^2 K(t,\mathbf{x},\mathbf{w}) = \left(\left(-i\frac{y}{2} + ib_1(t) - i\frac{c_2(t)}{2} - i\frac{w_2}{2} + i\frac{\cot t}{2}(x - c_1(t) - w_1)\right)^2 + i\frac{\cot t}{2}\right)K(t,\mathbf{x},\mathbf{w})$$

and

$$\begin{aligned}
&\partial_y^2 K(t,\mathbf{x},\mathbf{w})\\
&= \left(\left(-i\frac{x}{2} + ib_2(t) + i\frac{c_1(t)}{2} + i\frac{w_1}{2} + i\frac{\cot t}{2}(y - c_2(t) - w_2)\right)^2 + i\frac{\cot t}{2}\right)K(t,\mathbf{x},\mathbf{w}).
\end{aligned}$$

Thus we deduce

$$\begin{aligned}
&\langle r\rangle^{-2} H_0 K(t,\mathbf{x},\mathbf{w})\langle \mathbf{w}\rangle^{-2}\\
&= \langle r\rangle^{-2}\left(D_x^2 + 2yD_x + y^2 + D_y^2 + x\right)K(t,\mathbf{x},\mathbf{w})\langle \mathbf{w}\rangle^{-2}\\
&= \sum_{k=1}^{7} \mathcal{Q}_{1,k}(t,\mathbf{x})K(t,\mathbf{x},\mathbf{w})\mathcal{Q}_{2,k}(t,\mathbf{w}),
\end{aligned}$$

where for $k = 1, ..., 7$ we have

$$Q_{1,k}(t,\mathbf{x}) = q_{1,k}(t)m_{1,k}(\mathbf{x}), Q_{2,k}(t,\mathbf{w}) = q_{2,k}(t)m_{2,k}(\mathbf{w})$$

and

$$|q_{1,k}(t)q_{2,k}(t)| \le C\Big(1 + |\cot t|(1+t) + |\cot t|^2(1+t+t^2)\Big),$$

$$\|m_{1,k}(\mathbf{x})\|_{L^\infty_{\mathbf{x}}} \le C,\ \|m_{2,k}(\mathbf{w})\|_{L^\infty_{\mathbf{w}}} \le C.$$

Hence we have a smoothing effect

$$\begin{aligned}
\|K_3\phi\|_{L^2(\mathbb{R}^2)} &= C\left\|\int_{[0,\infty)\setminus\Omega}(\sin t)^{-1}e^{-it(-\lambda-i\nu)}\int_{\mathbb{R}^2}\sum_{k=1}^{7}\mathcal{Q}_{1,k}(t,\mathbf{x})K(t,\mathbf{x},\mathbf{w})\mathcal{Q}_{2,k}(t,\mathbf{w})\phi(\mathbf{w})d\mathbf{w}dt\right\|_{L^2(\mathbb{R}^2)}\\
&\le C\int_{[0,\infty)\setminus\Omega}\left\|\sum_{k=1}^{7}\mathcal{Q}_{1,k}(t,\mathbf{x})\left(e^{-it(H_0-\lambda-i\nu)}\mathcal{Q}_{2,k}(t,\cdot)\phi(\cdot)\right)(\mathbf{x})\right\|_{L^2(\mathbb{R}^2_{\mathbf{x}})}dt\\
&\le C\int_{[0,\infty)\setminus\Omega}\sum_{k=1}^{7}\|\mathcal{Q}_{1,k}(t,\mathbf{x})\|_{L^\infty_{\mathbf{x}}}\left\|\left(e^{-it(H_0-\lambda-i\nu)}\mathcal{Q}_{2,k}(t,\cdot)\phi(\cdot)\right)(\mathbf{x})\right\|_{L^2(\mathbb{R}^2_{\mathbf{x}})}dt\\
&\le C\int_{[0,\infty)\setminus\Omega}\sum_{k=1}^{7}\|\mathcal{Q}_{1,k}(t,\cdot)\|_{L^\infty}\|\mathcal{Q}_{2,k}(t,\cdot)\|_{L^\infty}\|\phi\|_{L^2(\mathbb{R}^2)}e^{-\nu t}dt\\
&\le C\sum_{n\in\mathbb{N}}\int_{n\pi+\lambda^{-\theta}}^{(n+1)\pi-\lambda^{-\theta}}e^{-\nu t}\left(1+\lambda^\theta(1+t)+(\sin t)^{-2}(1+t+t^2)\right)\|\phi\|_{L^2(\mathbb{R}^2)}dt\\
&\le C\nu^{-1}\lambda^\theta(1+\nu^{-1}+\nu^{-2})\|\phi\|_{L^2(\mathbb{R}^2)}. \qquad (3.4)
\end{aligned}$$

Here we used an integration by parts for the term involving $(\sin t)^{-2}$ combined with the fact that for $t \in (n\pi + \lambda^{-\theta}, (n+1)\pi - \lambda^{-\theta})$ and $\lambda^{-\theta} \le \frac{\pi}{6}$ one has a lower bound

$$|\sin t| = |\sin(t-n\pi)| \ge |\sin(\lambda^{-\theta})| \ge \frac{\lambda^{-\theta}}{2}.$$

Taking together (3.2), (3.3) and (3.4), we get

$$\|M_2(\lambda,\nu)\|_{L^2\to L^2} \le C(\theta)\nu^{-1}\left(\lambda^{-\theta}+(1+\nu)\lambda^{-1}+(1+\nu^{-2})\lambda^{\theta-1}\right).$$

Clearly

$$\|M_0(\lambda,\nu)\|_{L^2\to L^2} \le \nu^{-1}$$

and by Lemma 2.1 with $A = B = \langle r\rangle^{-1}$, $T = (H_0-\lambda-i\nu)^{-1}$, $\alpha_0 = \beta_0 = 2$, $\alpha_1 = \beta_1 = 0$ and $\theta' = 1-\delta/2$, one deduces

$$\|M_\delta(\lambda,\nu)\|_{L^2\to L^2} \le (C(\theta))^{\delta/2}\nu^{-1}\left(\lambda^{-\theta}+(1+\nu)\lambda^{-1}+(1+\nu^{-2})\lambda^{\theta-1}\right)^{\delta/2}.$$

□

4. Absence of large embedded eigenvalues

In this section we study the relation

$$\sigma_{pp}(H)\cap\Big((-\infty,-R_1)\cup(R_2,\infty)\Big) = \emptyset.$$

and we work without any assumption on the support of V. The absence of large eigenvalues has been established by Dimassi-Petkov [5]. However, the fact that $R_1, R_2 > 0$ do not depend on the support of V has not been proven in [5]. Here we establish this result and, moreover, we obtain bounds for R_1, R_2.

In Appendix B we prove the following

Proposition 4.1. *Assume that we have*

$$\|\langle x\rangle^{1/2}\langle y\rangle V\|_{L^\infty} \leq A_2, \langle x\rangle^{1/2}V \to 0,\ \langle x\rangle^{1/2}V_x \to 0,\ \langle y\rangle V_x \to 0 \text{ as } (x^2+y^2) \to \infty. \tag{4.1}$$

Let ψ be an eigenfunction of H with eigenvalues λ. Then $D_x\psi \in L^2(\mathbb{R}^2)$.

Proposition 4.2. *Assume that V satisfies the conditions* (4.1) *and*

$$\sup_{(x,y)\in\mathbb{R}^2} |\langle x\rangle\langle y\rangle V_x(x,y)| \leq A_1.$$

Then there exist constants $R_1 > 0, R_2 > 0$ independent of η_0 such that

$$\sigma_{pp}(H) \cap \Big((-\infty, -R_1) \cup (R_2, \infty)\Big) = \emptyset.$$

Moreover, we have

$$R_2 \leq (C_a A_1)^8,\ R_1 \leq C_a + A_0, \tag{4.2}$$

where $C_a > 0$ is a constant depending on the choice of the function $\varphi(x)$ in Section 2 *and $a > 0$.*

Notice that in the case when V satisfies Assumption 1.1 the conditions of Proposition 4.2 are fulfilled.

Proof. Let ψ and λ be an eigenfunction and an eigenvalue of H, respectively. Let $\|\psi\| = 1$ and let $|\lambda| \geq 1$. The operator D_x is a conjugated operator for H in the sense of [13] and D_x satisfies the conditions (a)-(e) in [13] (see for more details Section 3 in [4]). In particular, the condition (c) in [13] means that for $\Psi \in \mathscr{D}(H) \cap \mathscr{D}(D_x)$ the symmetric form

$$(\Psi, i[H, D_x]\Psi) = i(H\Phi, D_x\Psi) - i(D_x\Psi, H\Psi)$$

is bounded from below and closable and we can define the self-adjoint operator $[H, D_x]^o$ associated to its closure ([13]). According to Proposition 4.1, we have $\psi \in \mathscr{D}(H) \cap \mathscr{D}(D_x)$. Thus $i[H, D_x]^o\psi$ is well defined and $0 = (\psi, i[H, D_x]^o\psi) = ((1 + V_x)\psi, \psi)$. Consequently,

$$\begin{aligned} 0 = \Big|(i[H, D_x]^o\psi, \psi)\Big| &\geq 1 - \Big|((\partial_x V)\psi, \psi)\Big| \\ &\geq 1 - \|\langle x\rangle\langle y\rangle \partial_x V\|_{L^\infty} \Big\|\langle x\rangle^{-1}\langle y\rangle^{-1}\psi\Big\|. \end{aligned} \tag{4.3}$$

Let $\varphi(x)$ be the function introduced in Section 2. Obviously, with a constant $c_a > 0$ one has

$$|\varphi''(x)(\varphi'(x))^{-1}| \leq c_a,\ \forall x \in \mathbb{R}. \tag{4.4}$$

Recall that from Lemma 2.7 we have

$$\|\sqrt{\varphi'(x)}(D_x + y)\psi\| \leq C_1|\lambda|^{1/4},\ \|\sqrt{\varphi'(x)\varphi'(y)}D_y\psi\| \leq C_1|\lambda|^{3/8}. \tag{4.5}$$

We need the following

Lemma 4.3. *We have the equality*

$$\begin{aligned} \Gamma(\psi) :=& \|\sqrt{\varphi'(x)\varphi'(y)}(D_x + y)\psi\|^2 + \|\sqrt{\varphi'(x)\varphi'(y)}D_y\psi\|^2 \\ &- \operatorname{Im}\Big(\varphi''(x)(\varphi'(x))^{-1}\sqrt{\varphi'(x)\varphi'(y)}(D_x + y)\psi,\ \sqrt{\varphi'(x)\varphi'(y)}\psi\Big) \\ &- \operatorname{Im}\Big((\varphi''(y)(\varphi'(y))^{-1}\sqrt{\varphi'(x)\varphi'(y)}D_y\psi,\ \sqrt{\varphi'(x)\varphi'(y)}\psi\Big) \\ &+ \Big(\sqrt{\varphi'(x)\varphi'(y)}(x + V)\psi,\ \sqrt{\varphi'(x)\varphi'(y)}\psi\Big) \\ =& \lambda(\psi, \varphi'(x)\varphi'(y)\psi). \end{aligned} \tag{4.6}$$

Notice that by (4.5) all scalar products in (4.6) are well defined.

Proof. Choose a sequence of functions $f_n \in C_0^\infty(\mathbb{R}^2)$ such that $f_n \to \psi, Hf_n \to H\psi$ in L^2. Clearly,

$$((H-\lambda)f_n, \varphi'(x)\varphi'(y)f_n) \to ((H-\lambda)\psi, \varphi'(x)\varphi(y)\psi) = 0.$$

By integration by parts, we will show that

$$\Gamma(f_n) = (Hf_n, \varphi'(x)\varphi'(y)f_n) \tag{4.7}$$

which yields

$$((H-\lambda)f_n, \varphi'(x)\varphi'(y)f_n) = \Gamma(f_n) - \lambda(f_n, \varphi'(x)\varphi'(y)f_n).$$

To do this, we transform the term

$$((D_x+y)^2 f_n + D_y^2 f_n, \varphi'(x)\varphi'(y)f_n).$$

First consider

$$\begin{aligned}(D_y^2 f_n, \varphi'(x)\varphi'(y)f_n) &= (\varphi'(y)D_y\sqrt{\varphi'(x)}D_y f_n, \sqrt{\varphi'(x)}f_n)\\ &= (\sqrt{\varphi'(x)\varphi'(y)}D_y f_n, \sqrt{\varphi'(x)\varphi'(y)}D_y f_n)\\ &+i(\varphi''(y)(\varphi'(y))^{-1}\sqrt{\varphi'(x)\varphi'(y)}D_y f_n, \sqrt{\varphi'(x)\varphi'(y)}f_n).\end{aligned}$$

Second, by the same argument we get

$$\begin{aligned}((D_x+y)^2 f_n, \varphi'(x)\varphi'(y)f_n) &= \Big(\varphi'(x)(D_x+y)\sqrt{\varphi'(y)}(D_x+y)f_n, \sqrt{\varphi'(y)}f_n\Big)\\ &= (\sqrt{\varphi'(x)\varphi'(y)}(D_x+y)f_n, \sqrt{\varphi'(x)\varphi'(y)}(D_x+y)f_n)\\ &+i(\varphi''(x)(\varphi'(x))^{-1}\sqrt{\varphi'(x)\varphi'(y)}(D_x+y)f_n, \sqrt{\varphi'(x)\varphi'(y)}f_n).\end{aligned}$$

Thus we obtain (4.7). We take the limit $n \to \infty$ and deduce $\Gamma(f_n) \to \Gamma(\psi)$. Indeed, by Lemma 2.2 we have in L^2 the convergence

$$\sqrt{\varphi'(x)}(D_x+y)f_n \to \sqrt{\varphi'(x)}(D_x+y)\psi, \ \sqrt{\varphi'(x)}D_y f_n \to \sqrt{\varphi'(x)}D_y\psi$$

and the function $\varphi'(x)x$ is bounded for all $x \in \mathbb{R}$. □

Applying (4.4) and (4.5), one has

$$\Big|\Big(\varphi''(y)(\varphi'(y))^{-1}\sqrt{\varphi'(x)\varphi'(y)}D_y\psi, \sqrt{\varphi'(x)\varphi'(y)}\psi\Big)\Big|$$

$$\le C\|\sqrt{\varphi(x)\varphi'(y)}D_y\psi\| \le C|\lambda|^{3/8},$$

$$\Big|(\varphi''(x)(\varphi'(x))^{-1}\sqrt{\varphi'(x)\varphi'(y)}(D_x+y)\psi, \sqrt{\varphi'(x)\varphi'(y)}\psi)\Big|$$

$$\le C\|\sqrt{\varphi(x)\varphi'(y)}(D_x+y)\psi\| \le C|\lambda|^{1/4}.$$

Consequently, from (4.6) one deduces

$$\|\sqrt{\varphi'(x)\varphi'(y)}\psi\|^2 \le C(|\lambda|^{3/4}+1)|\lambda|^{-1} \le C|\lambda|^{-1/4}, \ |\lambda| \ge 1,$$

hence

$$\|\langle x\rangle^{-1}\langle y\rangle^{-1}\psi\| \le C_0\|\sqrt{\varphi'(x)\varphi'(y)}\psi\| \le C_1|\lambda|^{-1/8}.$$

Going back to (4.3), we deduce that for $|\lambda| \ge (2C_1A_1)^8$ we have no eigenvalues of H.

For $\lambda \le 0$ we have better result. For simplicity of notations denote

$$\|\sqrt{\varphi'(x)\varphi'(y)}(D_x+y)\psi\| = B_1, \ \|\sqrt{\varphi'(x)\varphi'(y)}D_y\psi\| = B_2, \|\sqrt{\varphi'(x)\varphi'(y)}\psi\| = D.$$

Since $-\lambda\|\sqrt{\varphi'(x)\varphi'(y)}\psi\| \geq 0$, the equality (4.6) implies

$$B_1^2 + B_2^2 - C_2 B_1 D - C_3 B_2 D + \Big(\sqrt{\varphi'(x)\varphi'(y)}(x+V)\psi, \sqrt{\varphi'(x)\varphi'(y)}\psi\Big) \leq -|\lambda| D^2$$

with constants $C_2 > 0, C_3 > 0$ independent of λ. Therefore,

$$\Big(B_1 - \frac{C_2}{2}D\Big)^2 + \Big(B_2 - \frac{C_3}{2}D\Big)^2 - \Big(\frac{C_2^2}{4} + \frac{C_3}{4}\Big)D^2 \leq (C_4 + A_0 - |\lambda|)D^2$$

with a constant $C_4 > 0$ depending on $\varphi(x)$ and independent of λ. Consequently, one deduces

$$|\lambda| D^2 \leq \Big(\frac{C_2^2}{4} + \frac{C_3}{4}\Big)D^2 + (C_4 + A_0)D^2 = (C_5 + A_0)D^2$$

If $|\lambda| > C_5 + A_0$, we have

$$\|\varphi'(x)\varphi'(y)\psi\| = 0,$$

hence $\psi = 0$. □

5. Mourre type estimate for the operator H_0

In this section we fix $R \geq \max\{R_1, R_2\}$, where $R_k,\ k = 1, 2$, are given by Proposition 4.2. The following result follows from [13].

Proposition 5.1. *There exists a constant $C_R > 0$ such that*

$$\sup_{\lambda \in [-R,R], \nu > 0} \big\| |D_x + \beta + i|^{-1}(H_0 - \lambda \mp i\nu)^{-1}|D_x + \beta + i|^{-1}\big\|_{L^2 \to L^2} \leq C_R. \tag{5.1}$$

We have

$$i[D_x + \beta, H_0] = 1.$$

As it was mentioned in the previous section, the conjugate operator $D_x + \beta$ satisfies the conditions (a)-(e) in [13] and the principal theorem in [13] implies the estimate (5.1).

Proposition 5.2. *Let $0 < \gamma < 1/2$, $s = 1/2 + \gamma/2$, $\beta \in \mathbb{R}$ and $\lambda \in [-R, R]$. Then we have the estimate*

$$\sup_{\lambda \in [-R,R], \nu > 0} \big\| |y - \beta|^{-\gamma} F_{\eta_0}(y - \beta) \langle x \rangle^{-s} (H_0 - \lambda \mp i\nu)^{-s} \langle x \rangle^{-s} F_{\eta_0}(y - \beta)|y - \beta|^{-\gamma}\big\|_{L^2 \to L^2} \leq C_{R,\gamma} \tag{5.2}$$

with a constant $C_{R,\gamma} > 0$ independent of η_0 and β.

Proof. For simplicity we treat the case $\beta = 0$. Define

$$\mathcal{J}_{H_0} := |D_x + i|^{-1}(H_0 - \lambda \mp i\nu)^{-1}|D_x + i|^{-1}.$$

We write

$$\begin{aligned}
&|y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} (H_0 - \lambda \mp i\nu)^{-1} \langle x \rangle^{-s} F_{\eta_0}(y)|y|^{-\gamma} \\
&= |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} F_{2R}(H_0)(H_0 - \lambda \mp i\nu)^{-1} F_{2R}(H_0) \langle x \rangle^{-s} F_{\eta_0}(y)|y|^{-\gamma} \\
&\qquad + |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} (1 - F_{2R}^2(H_0))(H_0 - \lambda \mp i\nu)^{-1} \langle x \rangle^{-s} F_{\eta_0}(y)|y|^{-\gamma} \\
&= I_1 \mathcal{J}_{H_0} I_1^* + I_2
\end{aligned}$$

with

$$I_1 := |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} F_{2R}(H_0)|D_x + i|$$

and

$$I_2 := |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} (1 - F_{2R}^2(H_0))(H_0 - \lambda \mp i\nu)^{-1} \langle x \rangle^{-s} F_{\eta_0}(y)|y|^{-\gamma}.$$

First, we show that $\|I_1\|_{L^2\to L^2} \le C_{1,R,\gamma}$.

To do this, one considers the product

$$I_1 = I_{1,1} I_{1,2}$$

with

$$\begin{aligned} I_{1,1} &:= |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} F_{2R}(H_0)(D_x + i), \\ I_{1,2} &:= (D_x + i)^{-1} |D_x + i|. \end{aligned}$$

Clearly, $I_{1,2}$ is a bounded operator.

Next we write

$$\begin{aligned} I_{1,1} &= |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} (D_x + i) F_{2R}(H_0) \\ &\quad + |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} \langle H_0 \rangle^{-\gamma} \langle H_0 \rangle^{\gamma} [F_{2R}(H_0), D_x] = J_1 + J_2. \end{aligned}$$

The term J_1 can be estimated by

$$\begin{aligned} &\left\| |y|^{-\gamma} F_1(y) \langle x \rangle^{-s} (D_x + y + i) F_{2R}(H_0) \right\| + \left\| |y|^{1-\gamma} F_1(y) \langle x \rangle^{-s} F_{2R}(H_0) \right\| \\ &\le C \left\| \langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-s} (D_x + y + i) F_{2R}(H_0) \right\| + C_1 \\ &\le C \left\| \langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-s} (D_x + y + i)(H_0 + i)^{-2} \right\| + C_1. \end{aligned}$$

To handle the operator on the right hand side, write

$$\begin{aligned} &\langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-s} (D_x + y + i)(H_0 + i)^{-2} \\ &= \langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-\gamma/2} (H_0 + i)^{-1} \langle x \rangle^{-1/2} (D_x + y + i)(H_0 + i)^{-1} \\ &\quad + \langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-\gamma/2} [\langle x \rangle^{-1/2} (D_x + y + i), (H_0 + i)^{-1}](H_0 + i)^{-1}. \end{aligned} \tag{5.3}$$

According to (2.2), (2.3) and Corollary 2.6, the first term in right hans side of (5.3) is bounded. For the second term one has

$$\begin{aligned} &\langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-\gamma/2} [\langle x \rangle^{-1/2} (D_x + y + i), (H_0 + i)^{-1}](H_0 + i)^{-1} \\ &= \langle D_y \rangle^{\gamma} F_1(y) \langle x \rangle^{-\gamma/2} (H_0 + i)^{-1} [H_0, \langle x \rangle^{-1/2} (D_x + y + i)](H_0 + i)^{-2}. \end{aligned}$$

Clearly,

$$\begin{aligned} &[H_0, \langle x \rangle^{-1/2} (D_x + y + i)](H_0 + i)^{-2} \\ &= [(D_x + y)^2 + D_y^2 + x, \langle x \rangle^{-1/2} (D_x + y) + i \langle x \rangle^{-1/2}](H_0 + i)^{-2} \\ &= ix \langle x \rangle^{-5/2} (D_x + y)^2 (H_0 + i)^{-2}/2 + \mathcal{B}_0, \end{aligned}$$

where $\mathcal{B}_0$ is a bounded operator. It remains to show that the operator

$$\mathcal{B}_1 = \langle x \rangle^{-1} (D_x + y)^2 (H_0 - i)^{-2}$$

is bounded. Set $Q = (D_x + y)^2 + D_y^2$. Then

$$(D_x + y)^2 (H_0 - i)^{-1} = (D_x + y)^2 (Q - i)^{-1} + (D_x + y)^2 (Q - i)^{-1} x (H_0 - i)^{-1}.$$

The pseudo-ffiferential operator $(D_x + y)^2 (Q - i)^{-1}$ has symbol in $S^0(\mathbb{R}^4_{(x,y,\xi,\eta)})$, hence it is bounded (see [4]). Consequently, the operator

$$\langle x \rangle^{-1} (D_x + y)^2 (Q - i)^{-1} x$$

is also bounded since by composition of pseudo-differential operators its principal symbol is in $S^0(\mathbb{R}^4_{(x,y,\xi,\eta)})$. This implies that $\mathcal{B}_1$ is bounded.

To prove the boundedness of J_2, let $\tilde{g}(z) \in C_0^\infty(\mathbb{C})$ be an almost analytic continuation of $g(s) = F_{2R_0}(s)$ such that

$$\bar{\partial}_z \tilde{g}(z) = \mathcal{O}(|\mathrm{Im}\, z|^N),\ \forall N \in \mathbb{N}.$$

Consider the representation

$$F_{2R}(H_0) = \frac{1}{\pi} \int \bar{\partial}_z \tilde{g}(z)(H_0 - z)^{-1} L(dz),$$

where $L(dz)$ is the Lebesgue measure on $\mathbb{C}$. Therefore

$$\begin{aligned} i[F_{2R}(H_0), D_x] &= \frac{i}{\pi} \int \bar{\partial}_z \tilde{g}(z)[(H_0 - z)^{-1}, D_x] L(dz) \\ &= -\frac{i}{\pi} \int \bar{\partial}_z \tilde{g}(z)(H_0 - z)^{-1}[H_0, D_x](H_0 - z)^{-1} L(dz) \\ &= \frac{1}{\pi} \int \bar{\partial}_z \tilde{g}(z)(H_0 - z)^{-2} L(dz). \end{aligned}$$

On the other hand, the operator

$$|y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} \langle H_0 \rangle^{-\gamma} = \langle x \rangle^{-1/2} |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-\gamma/2} \langle H_0 \rangle^{-\gamma}$$

is bounded applying Proposition 2.9 with H replaced by H_0, while

$$\frac{1}{\pi} \int \bar{\partial}_z \tilde{g}(z) \langle H_0 \rangle^{\gamma} (H_0 - z)^{-2} L(dz)$$

is trivially bounded. Combining the above estimates, one concludes that

$$\|I_1\|_{L^2 \to L^2} \le C_{1,R,\gamma}.$$

Concerning I_2, notice that for $|\lambda| \le R$ by the spectral Theorem the operator

$$\langle H_0 \rangle^{\gamma} (1 - (F_{2R}(H_0))^2)(H_0 - \lambda \mp i\nu)^{-1} \langle H_0 \rangle^{\gamma}$$

is bounded. Next one obtains the estimate

$$\|I_2\|_{L^2 \to L^2} \le C \Big\| |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-\gamma/2} \langle H_0 \rangle^{-\gamma} \Big\|^2_{L^2 \to L^2} \le C_{2,R,\gamma}$$

by applying once more Proposition 2.9. The case $\beta \neq 0$ can be treated by a similar argument. □

In the next section we need a modification of Proposition 5.2 when we have a product with a right factor $|D_x + i|^{-1}$.

Proposition 5.3. *Let $0 < \gamma < 1/2$, $s = 1/2 + \gamma/2$, $\beta \in \mathbb{R}$. Then we have*

$$\sup_{\lambda \in [-R,R], \nu > 0} \big\| |y - \beta|^{-\gamma} F_{\eta_0}(y - \beta) \langle x \rangle^{-s} (H_0 - \lambda \mp i\nu)^{-1} |D_x + i|^{-1} \big\|_{L^2 \to L^2} \le B_{R,\gamma} \tag{5.4}$$

with constant $B_{R,\gamma} > 0$ independent of η_0 and β.

Proof. We use the notations of the proof of Proposition 5.2. For $\beta = 0$ one has

$$|y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} (H_0 - \lambda \mp i\nu)^{-1} |D_x + i|^{-1} = I_1 \mathcal{J}_{H_0} + J_1,$$

where I_1 and $\mathcal{J}_{H_0}$ are the same as in the proof of Proposition 5.2 and

$$J_1 = |y|^{-\gamma} F_{\eta_0}(y) \langle x \rangle^{-s} (1 - F_{2R}(H_0))(H_0 - \lambda \mp i\nu)^{-1} |D_x + i|^{-1}.$$

Notice that the operator J_1 can be bounded by $C_{3,R,\gamma}$ by a calculation similar to that used for I_2 in the proof of Proposition 5.2 and we leave the details to the reader. The case $\beta \neq 0$ is treated by a similar argument. □

6. Absence of embedded eigenvalues for potentials with small support

In this section we prove Theorem 1.3.

Proof. Concerning H_0 and $0 < \gamma < 1/2$, $s = 1/2 + \gamma/2$, we have the estimates (5.2) with $\beta = 0$ and (5.4). For the operator $H = H_0 + V$ with $\operatorname{supp} V \subset \{(x, y) :\ |y| \leq \eta_0\}$ write

$$(H - \lambda - i\nu)^{-1} = (H_0 - \lambda - i\nu)^{-1}\Big[1 - V(H - \lambda - i\nu)^{-1}\Big] \tag{6.1}$$

which yields

$$\begin{aligned} &|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1} \\ &= |y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H_0 - \lambda - i\nu)^{-1} |D_x + i|^{-1} \\ &- \Big[|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H_0 - \lambda - i\nu)^{-1} \langle x\rangle^{-s} |y|^{-\gamma} F_{\eta_0}(y)\Big]\Big(\langle x\rangle^{2s} |y|^{2\gamma} V\Big) \\ &\quad \times \Big[|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1}\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big(I + \Big[|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H_0 - \lambda - i\nu)^{-1} \langle x\rangle^{-s} |y|^{-\gamma} F_{\eta_0}(y)\Big]\Big(\langle x\rangle^{1+\gamma} |y|^{2\gamma} V\Big)\Big) \\ &\qquad \times |y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1} \\ &= |y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H_0 - \lambda - i\nu)^{-1} |D_x + i|^{-1}. \end{aligned}$$

Clearly,

$$\Big\|\langle x\rangle^{1+\gamma} |y|^{2\gamma} V\Big\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \leq \eta_0^{2\gamma} \| \langle x\rangle^{1+\gamma} V\|_{L^\infty(\mathbb{R}^2)}.$$

Consequently, assuming $\eta_0^{2\gamma} C_{R,\gamma} \| \langle x\rangle^{1+\gamma} V\|_{L^\infty(\mathbb{R}^2)} = c_{R,\gamma,\eta_0} < 1$, we deduce that the operator in the brackets $\Big(...\Big)$ is invertible and

$$\sup_{|\lambda| \leq R, \nu > 0} \Big\| |y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1}\Big\|_{L^2 \to L^2} \leq \frac{B_{R,\gamma}}{1 - c_{R,\gamma,\eta_0}}.$$

This estimate implies that H has no eigenvalues in $[-R, R]$. In fact, let ψ be an eigenfunction of H with eigenvalue $\lambda \in [-R, R]$. By Proposition 4.1 we know that $D_x\psi \in L^2(\mathbb{R}^2)$, hence $|D_x + i|\psi = |D_x + i|(D_x + i)^{-1}(D_x + i)\psi \in L^2(\mathbb{R}^2)$. Then we conclude that

$$|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} (H - \lambda - i\nu)^{-1} |D_x + i|^{-1} |D_x + i|\psi = |y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s}\, i\nu^{-1}\psi.$$

If $F_{\eta_0}(y)\psi(x, y) = 0$, then $V(x, y)\psi(x, y) = 0$ and ψ will be an eigenfunction of H_0 which is impossible. Thus $|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} \psi \neq 0$ and as $\nu \searrow 0$ the $L^2(\mathbb{R}^2)$ norm of the function $|y|^{-\gamma} F_{\eta_0}(y) \langle x\rangle^{-s} \nu^{-1}\psi$ is not bounded. We obtain a contradiction and the proof is complete. □

Appendix A

We prove in this Appendix the following

Lemma A.1. *The operators*

$$x(H_0 - i)^{-1}, \quad (D_x + y)^k (H_0 - i)^{-1}, \quad (D_y)^k (H_0 - i)^{-1},\ k = 1, 2$$

are unbounded from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$.

Proof. Set $U_1 = e^{iD_xD_y}$. We have

$$U_1^{-1}(D_x + y)U_1 = y,\ \ U_1^{-1}xU_1 = x - D_y.$$

Combining this with the fact that U_1 commutes with D_y, we get

$$U_1^{-1}(D_x + y)^k U_1 U_1^{-1}\left((D_x + y)^2 + D_y^2 + x - i\right)^{-1} U_1 = y^k\left(y^2 + D_y^2 + x - D_y - i\right)$$
$$= y^k\left(y^2 + (D_y - \frac{1}{2})^2 + x - \frac{1}{4} - i\right)^{-1}.$$

Hence, applying the unitary transformation $e^{iy/2}$, one deduces that $(D_x + y)^k(H_0 + i)^{-1}$ is unitarily equivalent to

$$L_k = y^k\left(y^2 + D_y^2 + x - \frac{1}{4} - i\right)^{-1} =: y^k(B - i)^{-1}.$$

Next, we prove that L_1 is unbounded from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$. Let $\varphi \in C_0^\infty(]1, 2[; \mathbb{R})$ be a function such that $\int \varphi(x)^2 dx = 1$, and let $\psi_n(y)$ be the normalized eigenfunction of the harmonic oscillator corresponding to $\lambda_n = 2n + 1$, that is

$$(D_y^2 + y^2)\psi_n(y) = (2n + 1)\psi_n(y), \|\psi_n\| = 1. \tag{A.1}$$

Set $\Psi_n(x, y) = \psi_n(y)\varphi(x + 2n + 1)$. Clearly,

$$L_1\Psi_n(x, y) = y\left(2n + 1 + x - \frac{1}{4} - i\right)^{-1}\Psi_n(x, y),\ \ \|\Psi_n\| = 1.$$

Therefore,

$$\|L_1\Psi_n\|^2 = \int_{\mathbb{R}} y^2\psi_n^2(y)dy \int_{\mathbb{R}} \frac{\varphi^2(x + 2n + 1)}{(x + 2n + \frac{3}{4})^2 + 1}dx.$$

On the support of $\varphi(x + 2n + 1)$ we have $\frac{3}{4} \leq x + 2n + \frac{3}{4} \leq 1 + \frac{3}{4}$, hence

$$\frac{1}{(x + 2n + \frac{3}{4})^2 + 1} \geq \frac{16}{65}.$$

This yields

$$\|L_1\Psi_n\|^2 \geq \frac{16}{65}\int_{\mathbb{R}} y^2\psi_n^2(y)dy \int_{\mathbb{R}} \varphi^2(x + 2n + 1)dx = \frac{16}{65}\int_{\mathbb{R}} y^2\psi_n^2(y)dy. \tag{A.2}$$

By using the Fourier transform $\mathcal{F}_{y\to\eta}$ with respect to y, one obtains $\mathcal{F}(D_y^2 + y^2)\mathcal{F}^{-1} = D_\eta^2 + \eta^2$ and

$$\|\psi_n(y)\| = \|\hat{\psi}_n(\eta)\|.$$

Thus we deduce that $\hat{\psi}_n(\eta)$ is also a solution of (A.1) and $\hat{\psi}_n(\eta) = \psi(\eta)$. Therefore

$$\|\psi_n'(y)\| = \|\eta\psi_n(\eta)\| = \|y\psi_n(y)\|.$$

Combining this with the obvious equality

$$2n + 1 = \langle(D_y^2 + y^2)\psi_n, \psi_n\rangle = \|\psi_n'\|^2 + \|y\psi_n\|^2,$$

we deduce that $\|y\psi_n\|^2 = \frac{2n+1}{2}$. Consequently, (A.2) yields

$$\|L_1\Psi_n\|^2 \geq \frac{16(2n + 1)}{130}.$$

Letting $n \to \infty$, we conclude that L_1 is unbounded from L^2 into L^2. On the other hand, from

$$\|L_1u\|^2 = |\langle L_2u, (B - i)^{-1}u\rangle| \leq \|L_2u\|\|u\|,$$

we deduce that L_2 is also unbounded. This shows that $(D_x+y)^k(H_0-i)^{-1}$, $k=1,2$ are unbounded. Similar arguments show that $x(H_0-i)^{-1}$ and $(D_y)^k(H_0-i)^{-1}$ are unbounded. □

Appendix B

In this Appendix we establish Proposition 4.1.

Proof of Proposition 4.1 Let ψ be a normalized by $\|\psi\|=1$. Suppose that

$$D_x\psi \notin L^2(\mathbb{R}^2) \tag{B.1}$$

and for $\epsilon>0$, introduce the function $f_\epsilon(x)=\ln\left(\frac{\langle x\rangle}{1+\epsilon\langle x\rangle}\right)$. The operators $F_\epsilon = e^{f_\epsilon(D_x)} = \frac{\langle D_x\rangle}{1+\epsilon\langle D_x\rangle}$ and its inverse $F_\epsilon^{-1}=e^{-f_\epsilon(D_x)}=\frac{1+\epsilon\langle D_x\rangle}{\langle D_x\rangle}$ are bounded. Therefore, $F_\epsilon\psi\in L^2(\mathbb{R}^2)$. The condition (B.1) implies $\lim_{\epsilon\searrow 0}\|F_\epsilon\psi\|=\infty$. Let $\mathcal{F}_x=\mathcal{F}_{x\to\xi}$ denotes the Fourier transform with respect to x. The dominated convergence theorem yields

$$\begin{aligned}\lim_{\epsilon\searrow 0}\iint_{\mathbb{R}^2} e^{f_\epsilon(D_x)}\psi(x,y)\overline{\mathcal{F}_x^{-1}g(\xi,y)}dxdy &= \lim_{\epsilon\searrow 0}\iint_{\mathbb{R}^2} e^{f_\epsilon(\xi)}(\mathcal{F}_x\psi)(\xi,y)\overline{g(\xi,y)}d\xi dy\\ &= \iint_{\mathbb{R}^2}\langle\xi\rangle(\mathcal{F}_x\psi)(\xi,y)\overline{g(\xi,y)}d\xi dy,\end{aligned}$$

for all $g(\xi,y)\in\mathcal{F}_x(C_0^\infty(\mathbb{R}^2))$. This implies

$$\lim_{\epsilon\searrow 0}\iint_{\mathbb{R}^2} e^{f_\epsilon(D_x)}\psi(x,y)\overline{h(x,y)}\frac{dxdy}{\|F_\epsilon\psi\|}=0,\quad \forall h\in C_0^\infty(\mathbb{R}^2).$$

Consequently, the normalized function $\varphi_\epsilon := \frac{F_\epsilon\psi}{\|F_\epsilon\psi\|}$ converges weakly to zero.

By using $F_\epsilon^{-1}xF_\epsilon = x+i(\partial_x f_\epsilon)(D_x)$, and taking into account that F_ϵ commutes with the operator $H-x-V$, we get

$$H\varphi_\epsilon = F_\epsilon\left(H+i(\partial_x f_\epsilon)(D_x)-V+F_\epsilon^{-1}VF_\epsilon\right)\frac{\psi}{\|F_\epsilon\psi\|}=(\lambda+i(\partial_x f_\epsilon)(D_x)+V-F_\epsilon VF_\epsilon^{-1})\varphi_\epsilon. \tag{B.2}$$

Notice that the operators $\frac{1}{1+\epsilon\langle D_x\rangle}$ and $\frac{\epsilon\langle D_x\rangle}{1+\epsilon\langle D_x\rangle}$ are bounded from L^2 into L^2 uniformly with respect to $\epsilon\in[0,1]$. Since $V,\partial_x V\in L^\infty(\mathbb{R}^2)$, the operator $\langle D_x\rangle V\langle D_x\rangle^{-1}$ is bounded. Hence,

$$F_\epsilon VF_\epsilon^{-1} = \frac{\langle D_x\rangle}{1+\epsilon\langle D_x\rangle}V\frac{1+\epsilon\langle D_x\rangle}{\langle D_x\rangle} = \frac{1}{1+\epsilon\langle D_x\rangle}\left(\langle D_x\rangle V\langle D_x\rangle^{-1}\right)+\frac{\epsilon\langle D_x\rangle}{1+\epsilon\langle D_x\rangle}V, \tag{B.3}$$

is uniformly bounded for $\epsilon\in[0,1]$.

From now on we denote

$$K_\epsilon := i(\partial_x f_\epsilon)(D_x)+V-F_\epsilon VF_\epsilon^{-1}.$$

Let $G(x,y)$ be a continuous function going to zero as $(x^2+y^2)\to\infty$. It is well known that

$$\phi\langle D_x\rangle^{-s}(H+i)^{-1},$$

is a compact operator for every $\phi\in C_0^\infty(\mathbb{R}^2)$ and all $s\geq 0$ (see for instance, [4]). Thus, by an approximation argument, $G\langle D_x\rangle^{-s}(H+i)^{-1}$ is also compact. We claim that

$$G\langle D_x\rangle^{-s}\varphi_\epsilon \text{ converges strongly to zero as } \epsilon\searrow 0. \tag{B.4}$$

To prove this, we use (B.2). Write

$$G\langle D_x\rangle^{-s}\varphi_\epsilon = G\langle D_x\rangle^{-s}(H+i)^{-1}(H+i)\varphi_\epsilon = G\langle D_x\rangle^{-s}(H+i)^{-1}\left(\lambda+i+K_\epsilon\right)\varphi_\epsilon.$$

Since $(\lambda + i + K_\epsilon)$ is bounded uniformly for $\epsilon \in [0,1]$, and φ_ϵ converges weakly to zero, it follows from the compactness of $G\langle D_x\rangle^{-s}(H+i)^{-1}$ that the right hand side of the above equality converges strongly to zero.

For $t > 1$, let $\chi_t(x)$ be an odd smooth function satisfying

$$\chi_t(x) = \begin{cases} x, \ 0 \le x \le t, \\ 2t, \ x \ge 2t, \end{cases} \tag{B.5}$$

$\chi_t^{(k)}(x) = \mathcal{O}(t^{-k+1}), k \ge 1$, and $\chi_t'(x) \ge 0$. Clearly, $i[x, -\chi_t(D_x)] = \chi_t'(D_x)$ and

$$i \lim_{t\to\infty} i([x, -\chi_t(D_x)]\varphi_\epsilon, \varphi_\epsilon) = (\varphi_\epsilon, \varphi_\epsilon). \tag{B.6}$$

Next, we claim that for every fixed $\epsilon > 0$ we have

$$\lim_{t\to\infty} i([V, -\chi_t(D_x)]\varphi_\epsilon, \varphi_\epsilon) = -2 \lim_{t\to\infty} \mathrm{Im}(\varphi_\epsilon, \chi_t(D_x)V\varphi_\epsilon) = (V_x\varphi_\epsilon, \varphi_\epsilon). \tag{B.7}$$

First, it follows from (B.2) that $h_\epsilon := (H_0 - i)\varphi_\epsilon$ is uniformly bounded in L^2 with respect to $\epsilon \in [0,1]$. On the other hand, Lemma 2.3 and the conditions (4.1) show that

$$D_xV(H_0-i)^{-1}h_\epsilon = V_x(H_0-i)^{-1}h_\epsilon + V(D_x+y)(H_0-i)^{-1}h_\epsilon - yV(H_0-i)^{-1}h_\epsilon \in L^2.$$

Combining this with the fact that $|\chi_t(\xi) - \xi| \le C|\xi|$ (uniformly for $t \ge 1$), we deduce

$$|(\chi_t(\xi) - \xi)H_\epsilon(\xi)| \le C|\xi| \ |H_\epsilon(\xi)| \in L^2, \quad \text{where} \quad H_\epsilon(\xi) = \mathcal{F}_{x\to\xi}\Big(V(H_0-i)^{-1}h_\epsilon\Big)(\xi).$$

Hence, the dominated convergence theorem yields

$$\lim_{t\to+\infty} \chi_t(D_x)V(H_0-i)^{-1}h_\epsilon = V_x(H_0-i)^{-1}h_\epsilon = V_x\varphi_\epsilon, \quad \text{in} \ \ L^2,$$

and the proof of the claim is complete. Taking together (B.6), (B.7) and the equality $[H, \chi_t(D_x)] = [x + V, \chi_t(D_x)]$, we obtain

$$\lim_{t\to+\infty} i\left([H, -\chi_t(D_x)]\,\varphi_\epsilon, \varphi_\epsilon\right) = ((1+V_x)\varphi_\epsilon, \varphi_\epsilon).$$

Now applying (B.4) with $G = \partial_x V$ and $s = 0$, we deduce that

$$\left((1 + \partial_x V)\,\varphi_\epsilon, \varphi_\epsilon\right) \ge \frac{1}{2} \tag{B.8}$$

for ϵ small enough. To complete the proof, we will show that the left hand side of (B.8) is less than $\frac{1}{4}$ for ϵ small enough. This leads to a contradiction.

Equation (B.2) implies

$$\begin{aligned} i\left([H, -\chi_t(D_x)]\,\varphi_\epsilon, \varphi_\epsilon\right) &= i\left(\chi_t(D_x)H\varphi_\epsilon, \varphi_\epsilon\right) - i\left(\chi_t(D_x)\varphi_\epsilon, H\varphi_\epsilon\right) \\ &= i\left(\chi_t(D_x)(\lambda + K_\epsilon)\varphi_\epsilon, \varphi_\epsilon\right) - i\left(\chi_t(D_x)\varphi_\epsilon, (\lambda + K_\epsilon)\varphi_\epsilon\right) \\ &= -2\mathrm{Im}\left(\chi_t(D_x)K_\epsilon\varphi_\epsilon, \varphi_\epsilon\right). \end{aligned} \tag{B.9}$$

On the other hand, the inequality

$$\chi_t(x)\left(\partial_x f_\epsilon\right)(x) = \frac{x\chi_t(x)}{\langle x\rangle^2\left(1 + \epsilon\langle x\rangle\right)} \ge 0,$$

yields

$$\chi_t(D_x)\left(\partial_x f_\epsilon\right)(D_x) \ge 0,$$

in the sense of self-adjoint operators. Consequently,

$$\begin{aligned} &-2\mathrm{Im}\,(\chi_t(D_x)K_\epsilon\varphi_\epsilon,\varphi_\epsilon) \\ &= -2\mathrm{Im}\left(i\chi_t(D_x)(\partial_x f_\epsilon)(D_x) + V - F_\epsilon V F_\epsilon^{-1}\varphi_\epsilon,\varphi_\epsilon\right) \\ &\le 2\,\mathrm{Im}\left(\chi_t(D_x)\left(F_\epsilon V F_\epsilon^{-1} - V\right)\varphi_\epsilon,\varphi_\epsilon\right). \end{aligned} \tag{B.10}$$

From (B.3), we have

$$F_\epsilon V F_\epsilon^{-1} - V = \frac{1}{1+\epsilon\langle D_x\rangle}\left(\langle D_x\rangle V\langle D_x\rangle^{-1}\right) - \frac{1}{1+\epsilon\langle D_x\rangle}V = \frac{1}{1+\epsilon\langle D_x\rangle}\left[\langle D_x\rangle, V\right]\langle D_x\rangle^{-1}.$$

Therefore

$$\begin{aligned} \lim_{t\to\infty}\Big(\chi_t(D_x)\left(F_\epsilon V F_\epsilon^{-1} - V\right)\varphi_\epsilon,\varphi_\epsilon\Big) &= \Big(D_x\left(F_\epsilon V F_\epsilon^{-1} - V\right)\varphi_\epsilon,\varphi_\epsilon\Big) \\ &= \Big(\frac{D_x}{1+\epsilon\langle D_x\rangle}\left[\langle D_x\rangle, V\right]\langle D_x\rangle^{-1}\varphi_\epsilon,\varphi_\epsilon\Big) \end{aligned} \tag{B.11}$$

is bounded uniformly for $\epsilon\in[0,1]$. Letting $t\to\infty$, we deduce from (B.7), (B.10) and (B.11)

$$\left((1+\partial_x V)\,\varphi_\epsilon,\varphi_\epsilon\right) \le 2\mathrm{Im}\Big(\frac{D_x}{1+\epsilon\langle D_x\rangle}\left[\langle D_x\rangle, V\right]\langle D_x\rangle^{-1}\varphi_\epsilon,\varphi_\epsilon\Big). \tag{B.12}$$

To complete the proof of Proposition 4.1, we apply the following

Lemma B.1. *We have*

$$\lim_{\varepsilon\to 0}\Big(\frac{1}{1+\varepsilon\langle D_x\rangle}D_x[\langle D_x\rangle, V]\langle D_x\rangle^{-1}\varphi_\varepsilon,\varphi_\varepsilon\Big) = 0.$$

Proof. Write

$$\begin{aligned} D_x[\langle D_x\rangle, V]\langle D_x\rangle^{-1} &= \Big(\langle D_x\rangle D_x V - D_x V\langle D_x\rangle\Big)\langle D_x\rangle^{-1} \\ &= \Big[\langle D_x\rangle(V_x + VD_x) - (V_x+VD_x)\langle D_x\rangle\Big]\langle D_x\rangle^{-1} \\ &= [\langle D_x\rangle, V_x]\langle D_x\rangle^{-1} + \Big(\langle D_x\rangle V - V\langle D_x\rangle\Big)D_x\langle D_x\rangle^{-1} \\ &= [\langle D_x\rangle, V_x]\langle D_x\rangle^{-1} + [\langle D_x\rangle, V]D_x\langle D_x\rangle^{-1} \end{aligned}$$

and set

$$L_1 := \frac{1}{1+\varepsilon\langle D_x\rangle}[\langle D_x\rangle, V_x]\langle D_x\rangle^{-1},\ L_2 := \frac{1}{1+\varepsilon\langle D_x\rangle}\Big([\langle D_x\rangle, V]D_x\langle D_x\rangle^{-1}\Big).$$

Therefore,

$$\begin{aligned} D_x\langle D_x\rangle^{-1}\varphi_\varepsilon &= D_x\langle D_x\rangle^{-1}(H_0-i)^{-1}(H_0-i)\varphi_\varepsilon \\ &= (H_0-i)^{-1}D_x\langle D_x\rangle^{-1}h_\varepsilon + (H_0-i)^{-1}[D_x\langle D_x\rangle^{-1}, x](H_0-i)^{-1}h_\varepsilon. \end{aligned}$$

Clearly, the operator $[D_x\langle D_x\rangle^{-1}, x] = \langle D_x\rangle^{-1}(1-\frac{D_x^2}{\langle D_x\rangle^2})$ is bounded and this implies that

$$D_x\langle D_x\rangle^{-1}\varphi_\varepsilon = (H_0-i)^{-1}\tilde h_\varepsilon \tag{B.13}$$

with $\tilde h_\varepsilon$ bounded in L^2 uniformly with respect to ε. Recall that the operator

$$\langle x\rangle^{-1/2}(D_x+y)(H_0-i)^{-1}$$

is bounded by Lemma 2.3. By using this, one deduces that the operator

$$(H_0+i)^{-1}VD_x(H_0-i)^{-1} = (H_0+i)^{-1}V\langle x\rangle^{1/2}\langle x\rangle^{-1/2}(D_x+y)(H_0-i)^{-1} - (H_0+i)^{-1}Vy(H_0-i)^{-1}$$

is compact since $V\langle x\rangle^{1/2} \to 0, Vy \to 0$ as $(x^2+y^2) \to \infty$ by conditions (4.1). To handle the operator $\langle D_x\rangle$, we exploit the following representation

$$\begin{aligned} V\langle D_x\rangle(H_0-i)^{-1} &= V(D_x+i)\langle D_x\rangle(D_x+i)^{-1}(H_0-i)^{-1} \\ &= V(D_x+i)(H_0-i)^{-1}\langle D_x\rangle(D_x+i)^{-1} \\ &\quad + V(D_x+i)(H_0-i)^{-1}[\langle D_x\rangle(D_x+i)^{-1}, x](H_0-i)^{-1}. \end{aligned}$$

Obviously, the commutator $[\langle D_x\rangle(D_x+i)^{-1}, x]$ is a bounded operator and $(D_x+i) = (D_x+y)-(y-i)$. So as above we obtain that $(H_0+i)^{-1}V\langle D_x\rangle(H_0-i)^{-1}$ is compact. In the same way we show that the operators

$$(H_0+i)^{-1}V_xD_x(H_0-i)^{-1},\ (H_0+i)^{-1}V_x\langle D_x\rangle(H_0-i)^{-1}$$

are compact because $V_x\langle x\rangle^{1/2} \to 0, V_xy \to 0$ as $(x^2+y^2) \to \infty$ according to conditions (4.1).

To deal with the operator L_1, write

$$\begin{aligned} &\Big(\frac{1}{1+\varepsilon\langle D_x\rangle}[\langle D_x\rangle, V_x]\langle D_x\rangle^{-1}\varphi_\varepsilon, \varphi_\varepsilon\Big) \\ &= \Big(\frac{1}{1+\varepsilon\langle D_x\rangle}[\langle D_x\rangle, V_x]\langle D_x\rangle^{-1}\varphi_\varepsilon, (H_0-i)^{-1}h_\varepsilon\Big) \\ &= \Big(\frac{1}{1+\varepsilon\langle D_x\rangle}(H_0+i)^{-1}[\langle D_x\rangle, V_x]\langle D_x\rangle^{-1}\varphi_\varepsilon, h_\varepsilon\Big) \\ &\quad - \Big((H_0+i)^{-1}\frac{\varepsilon D_x}{\langle D_x\rangle(1+\varepsilon\langle D_x\rangle)^2}(H_0+i)^{-1}[\langle D_x\rangle, V_x]\langle D_x\rangle^{-1}\varphi_\varepsilon, h_\varepsilon\Big). \end{aligned}$$

We have $(H_0+i)^{-1}[\langle D_x\rangle, V_x]\langle D_x\rangle^{-1} = (H_0+i)^{-1}(\langle D_x\rangle V_x\langle D_x\rangle^{-1} - V_x)$. The analysis of the term with V_x is easy since $(H_0+i)^{-1}V_x$ is compact. For the other term we get

$$\begin{aligned} \langle D_x\rangle^{-1}\varphi_\varepsilon &= \langle D_x\rangle^{-1}(H_0-i)^{-1}h_\varepsilon \\ &= (H_0-i)^{-1}\langle D_x\rangle^{-1}h_\varepsilon - (H_0-i)^{-1}\frac{\varepsilon D_x}{\langle D_x\rangle(1+\varepsilon\langle D_x\rangle)^2}(H_0-i)^{-1}h_\varepsilon \end{aligned}$$

and notice that $(H_0+i)^{-1}\langle D_x\rangle V_x(H_0-i)^{-1}$ is compact.

Passing to the analysis of the operator L_2, we have $[\langle D_x\rangle, V] = \langle D_x\rangle V - V\langle D_x\rangle$. For $\langle D_x\rangle VD_x\langle D_x\rangle^{-1}$ we repeat the above argument by using (B.13) and the fact that $(H_0+i)^{-1}\langle D_x\rangle V(H_0-i)^{-1}$ is compact since its adjoint $(H_0+i)^{-1}V\langle D_x\rangle(H_0-i)^{-1}$ is compact. On the other hand, applying (B.13) once more, we have

$$V\langle D_x\rangle D_x\langle D_x\rangle^{-1}\varphi_\varepsilon = V\langle D_x\rangle(H_0-i)^{-1}\tilde{h}_\varepsilon.$$

The operator $V\langle D_x\rangle(H_0-i)^{-1}$ has been treated above and the proof is complete. □

Acknowledgment. Thanks are due to the referees for their critical comments and useful suggestions leading to an improvement of the previous version of the paper.

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