Local energy decay and Strichartz estimates for the wave equation with time-periodic perturbations

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Abstract. We examine the memorphic continuaiton of the cut-off resolvent $R_\chi(z) = \chi(U(T,0) - z)^{-1}\chi,\ \chi(x) \in C_0^\infty(\mathbb{R}^n)$, where $U(t,s)$ is the propagator related to the wave equation with non-trapping time-periodic perturbations (potential $V(t,x)$ or a periodically moving obstacle) and $T > 0$ is the period. Assuming that $R_\chi(z)$ has no poles z with $|z| \geq 1$, we establish a local energy decay and we obtain global Strichartz estimates. We discuss the case of trapping moving obstacles and we present some results and conjectures concerning the behavior of $R_\chi(z)$ for $|z| > 1$.

Keywords: monodromy operator, local energy decay, Strichartz estimates

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1 Introduction

In this talk we present a survey of some recent results concerning two problems for the wave equation with time-periodic perturbations. The first one is the Cauchy problem with time-periodic potential

$$\begin{cases} \partial_t^2 u - \Delta u + V(t,x)u = F(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(\tau, x) = f_0(x),\ u_t(\tau, x) = f_1(x),\ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where the potential $V(t,x) \in C^\infty(\mathbb{R}^{n+1}), n \geq 2$, satisfies the conditions:

(H_1) there exists $R_0 > 0$ such that $V(t,x) = 0$ for $|x| \geq R_0$, $\forall t \in \mathbb{R}$,

(H_2) $V(t+T, x) = V(t,x)$, $\forall (t,x) \in \mathbb{R}^{n+1}$ with $T > 0$.

Consider the homogeneous Sobolev spaces $\dot{H}^\gamma(\mathbb{R}^n) = \Lambda^{-\gamma}L^2(\mathbb{R}^n)$, where $\Lambda = \sqrt{-\Delta}$ and $-\Delta$ is the Laplacian in $\mathbb{R}^n$ and set $\dot{\mathcal{H}}_\gamma(\mathbb{R}^n) = \dot{H}^\gamma(\mathbb{R}^n) \oplus \dot{H}^{\gamma-1}(\mathbb{R}^n)$ The solution of (1) with $F = 0$ is given by the propagator

$$U(t,\tau) : \dot{\mathcal{H}}_\gamma(\mathbb{R}^n) \ni (f_0,\ f_1) \longrightarrow U(t,\tau)(f_0, f_1)$$
$$= (u(t,x),\ u_t(t,x)) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n).$$

Let $U_0(t) = e^{itG_0}$ be the unitary group in $\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$ related to the Cauchy problem (1) with $V = 0,\ F = 0,\ \tau = 0$ and let $U(T) = U(T, 0)$. Let $\chi,\ \psi_1$ be functions in $C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = \psi_1(x) = 1$ for $|x| \leq R_0 + T$. We suppose also that

$$(1 - \psi_1)U(0, s)Q(s) = 0,\ 0 \leq s \leq T, \tag{2}$$

where

$$Q(s) = \begin{pmatrix} 0 & 0 \\ V(s,x) & 0 \end{pmatrix}.$$

Consider the cut-off resolvent

$$R_\chi(\theta) = \chi(U(T) - e^{-i\theta}I)^{-1}\psi_1 :\ \dot{\mathcal{H}}_1(\mathbb{R}^n) \mapsto \dot{\mathcal{H}}_1(\mathbb{R}^n),$$

where $\operatorname{Im}\theta \geq A > 0,\ -\pi < \operatorname{Re}\theta \leq \pi$ and ψ_1 is fixed. We show that $R_\chi(\theta)$ admits a meromorphic extension in $\mathbb{C}$ for $n \geq 3$, n odd, and to

$$\mathbb{C}^{'} = \{\theta \in \mathbb{C} :\ \theta \neq 2\pi k - i\mu,\ \mu > 0,\ k \in \mathbb{Z}\}$$

for $n \geq 2$, n even. The poles of $R_\chi(\theta)$ play an essential role in the problems of local energy decay, global Strichartz estimates, trace formulae and blow up of the local energy (see [7], [1], [2], [15], [21]).

The second problem we deal with is the Dirichlet problem for the wave equation outside a time-periodic moving obstacle. Let $Q \subset \mathbb{R}^{n+1}$, $n \geq 3$, be an open domain with C^∞ smooth boundary ∂Q. Set

$$\Omega(t) = \{x \in \mathbb{R}^n : (t,x) \in Q\},$$
$$\emptyset \not\equiv K(t) = \{x \in \mathbb{R}^n : (t,x) \notin Q\} \subset \{x : |x| \leq R_0\}.$$

We suppose that the obstacle is periodically moving

$$K(t + T) = K(t),\ \forall t \in \mathbb{R},\ T > 0$$

and for each $(t,x) \in \partial Q$ the exterior unit normal (ν_t, ν_x) to ∂Q at (t,x) satisfies $|\nu_t| < |\nu_x|$. We study the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } Q, \\ u = 0 \text{ on } \partial Q, \\ u(\tau,x) = f_0(x),\ u_t(\tau,x) = f_1(x). \end{cases} \tag{3}$$

The solution is given by a propagator $U(t,\tau) : H(\tau) \longrightarrow H(t)$, where $H(t)$ is the energy space related to $\Omega(t)$ (see [7], [14] for a precise definition). As above we introduce the monodromy operator $U(T) = U(T,0)$ and the cut-off resolvent $R_\chi(\theta) = \chi(U(T) - e^{-i\theta}I)^{-1}\chi$ with $\chi = 1$ on $\{x : \ |x| \leq R_0 + T\}$.

We examine the problem of the meromorphic continuation of the cut-off resolvents $R_\chi(\theta)$ for time-periodic potentials and non-trapping moving obstacles. In contrast to stationary perturbations, the absence of trapping rays is not sufficient to guarantee an uniform local energy decay. To obtain the last property, we must exclude the existence of poles of $R_\chi(\theta)$ with $\operatorname{Im}\theta \geq 0$ and for this purpose we introduce the condition $(\mathcal{R})$ in Section 2. In Section 3 we show that the local energy decay of solutions with initial data having compact support leads to a L^2-integrability of the local energy of solutions with data in the energy space. This is the crucial point in the proof of global Strichartz estimates for time-periodic non-trapping perturbations.

The investigation of trapping moving obstacles is more complicated and many problems are still open. In some recent works (see [3], [4]) it was proved that for stationary trapping obstacles the cut-off resolvent $\chi(U(t)-z)^{-1}\chi$ has a singularity as $z \to z_0$, $|z| > 1$, for every $z_0 \in \mathbb{S}$ and almost all $t \in \mathbb{R}^+$ (see Theorem 3). Thus we have not a meromorphic extension across the unit circle $\mathbb{S}$ as in the case of non-trapping perturbations. Moreover, it is not known if for trapping moving obstacles $\chi(U(T) - z)^{-1}\chi$ has a meromorphic continuation from $\{z \in \mathbb{C} : |z| \geq A \gg 1\}$ to $\{z \in \mathbb{C} : e^{\epsilon T} \leq |z| \leq A\}$, $\epsilon > 0$. We conjecture that for obstacles having at least one δ-trapping bicharacteristic the cut-off resolvent $\chi(U(T)-z)^{-1}\chi$ is not meromorphic in $\{z \in \mathbb{C} : e^{\epsilon T} \leq |z|\}$, $0 < \epsilon < \delta$ (see Section 5 for the notations).

2 Resonances for time-periodic potentials

In this section we study the problem (1) and $U(t,s)$ denotes the corresponding propagator. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a fixed cut-off such that $\psi(x) = 1$ for $|x| \leq R_0 + T$. By a finite speed of propagation argument we get

$$(1-\psi)U(T,s)Q(s) = 0, \ Q(s)U_0(s)(1-\psi) = 0, \ 0 \leq s \leq T. \tag{4}$$

For $A > 0$ large enough and $\operatorname{Im}\theta \geq A$ the resolvents $(U_0(T) - e^{-i\theta}I)^{-1}$, $(U(T) - e^{-i\theta}I)^{-1}$ exist, we have the equality

$$U(T) - zI = \Big[I - \psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\Big](U_0(T)-zI), z = e^{-i\theta}$$

and

$$(U_0(T)-zI)^{-1} = (U(T)-zI)^{-1}\Big[I - \psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\Big].$$

Assume that $\psi_1 \in C_0^\infty(\mathbb{R}^n)$ satisfies (2) and let $\psi_1(x) = 1$ on supp ψ. We take an arbitrary cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi = 1$ on supp ψ and multiply the above equality by χ and ψ_1 to get

$$\chi(U_0(T)-zI)^{-1}\psi_1 = \chi(U(T)-zI)^{-1}\psi_1\Big[I-\psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\psi_1\Big].$$

Introduce the operator

$$K(z) = \psi\int_0^T U(T,s)Q(s)U_0(s)ds\psi(U_0(T)-zI)^{-1}\psi_1.$$

For $n \geq 3$, n odd, the operator $\psi(U_0(T) - e^{-i\theta}I)^{-1}\psi_1$ admits an analytic continuation with respect to θ in $\mathbb{C}$ and this follows immediately from the Huygens principle and the expansion

$$\psi(U_0(T) - e^{-i\theta}I)^{-1}\psi_1 = -\sum_{k=0}^{N(\psi,\psi_1)} \psi U_0(kT)\psi_1 e^{i(k+1)\theta}$$

which holds for $\operatorname{Im}\theta \geq A > 0$. On the other hand, the operator $K(z)$ is compact in $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ and an application of the analytic Fredholm theorem leads to a meromorphic continuation of $R_\chi(\theta)$ in $\mathbb{C}$. For n even a similar argument leads to a meromorphic continuation of $R_\chi(\theta)$ in

$$\mathbb{C}^{'} = \{z \in \mathbb{C} :\ z \neq 2\pi k - i\mu,\ \mu \geq 0,\ k \in \mathbb{Z}\},$$

but the analysis of the analytic extension of $\psi(U_0(T) - e^{-i\theta}I)^{-1}\psi_1$ in $\mathbb{C}^{'}$ is more complicated (see [20], [21], [15]). Thus we have the following

Proposition 1 *The cut-off resolvent $R_\chi(\theta)$ admits a meromorphic continuation in $\mathbb{C}$ for n odd and in $\mathbb{C}^{'}$ for n even.*

The time-periodic potentials are non-trapping perturbations. Nevertheless, some exponentially growing modes could exist. To establish a local energy decay, we introduce the following condition

($\mathcal{R}$) The operator $R_\chi(\theta)$ admits a holomorphic extension from $\{\theta \in \mathbb{C} :\ \operatorname{Im}\theta \geq A > 0\}$ to $\{\theta \in \mathbb{C} :\ \operatorname{Im}\theta \geq 0\}$, for $n \geq 3$, odd, and to $\{\theta \in C : \operatorname{Im}\theta \geq 0, \theta \neq 2\pi k,\ k \in \mathbb{Z}\}$ for $n \geq 2$, even . Moreover, for n even we have

$$\lim_{\lambda \to 0,\ \lambda > 0} \|R_\chi(i\lambda)\|_{\dot{\mathcal{H}}_1 \to \dot{\mathcal{H}}_1} < \infty.$$

This condition is independent of the choice of $\chi,\ \psi_1$. Let $\varphi \in C_0^\infty(\mathbb{R}^n),\ f \in \dot{\mathcal{H}}_1,\ f = 0$ for $|x| \leq R$. We denote the norm in $\dot{\mathcal{H}}_1(\mathbb{R}^n)$ by $\|.\|$ and we use the same notation for the norm of bounded operators in $\dot{\mathcal{H}}_1(\mathbb{R}^n)$.

Theorem 1 ([*15*]) *Assume the condition* ($\mathcal{R}$) *fulfilled. Then for* $0 \leq s \leq t,\ t - s \geq t_0 > 1$ *we have*

$$\|\varphi U(t,s)f\| \leq C(n,\varphi,R)p(t-s)\|f\|,$$

where

$$p(t) = \begin{cases} e^{-\delta t},\ \delta > 0,\ n \geq 3,\ \text{odd}, \\ t^{-1}(\ln t)^{-2},\ n \geq 2,\ \text{even}. \end{cases}$$

The local energy decay has been established for n odd, by Bachelot and Petkov [1] assuming that the Lax-Phillips operator $Z^b(T) = P_+^b U(T) P_-^b$, $b > R_0 + T$ has no eigenvalues $z \in \mathbb{C}$, $|z| \geq 1$, (see Section 4 for the definition of the projectors $P_\pm^b$) and by Vainberg [21] for $n \geq 2$ assuming a similar condition for an operator $R(\theta)$ having complicated form. The novelty of our approach is the role of the cut-off resolvent $R_\chi(\theta)$. It is worth remarking that the resolvent of the monodromy operator plays an essential role in the analysis of time-periodic perturbations of the Schrödinger operator (see for example, [8]). On the other hand, the link between the poles of $R_\chi(\theta)$ and the spectrum of $Z^b(T)$ has been established in [2].

Sketch of the proof. We have the representation

$$U(t,0)f = U_0(t)f - \int_0^t U(t,s)Q(s)U_0(s)f ds,$$

and we will deal with

$$I(\varphi,f) = \int_{-\infty}^t \varphi U(t,s)Q(s)U_0(s)f ds$$

extending $U_0(s)f$ as 0 for $s < 0$. Introduce the Fourier-Block-Gelfand transform

$$g(\theta,s) = F(U_0(s)f)(\theta,s) = \sum_{k=-\infty}^{\infty} U_0(kT+s)e^{ik\theta}f$$

which is well defined for $\operatorname{Im}\theta \geq \alpha > 0$.

Applying the inverse transform of F, we are going to examine

$$J(t) = \frac{1}{2\pi}\int_{-\infty}^t \varphi U(t,s)Q(s)\int_{d_\alpha} g(\theta,s)d\theta ds,$$

where $d_\alpha = [i\alpha - \pi, i\alpha + \pi]$ and $\alpha > 0$ will be chosen large enough below. Choose an integer $m \in \mathbb{Z}$ so that $t' = t - mT \in [0,T[$. Then $J(t)$ has the form

$$\frac{1}{2\pi}\int_0^{t'} \varphi U(t',s')Q(s')U_0(s')\int_{d_\alpha} e^{-im\theta}g(\theta,0)d\theta ds'$$

$$+\frac{1}{2\pi}\sum_{k=0}^{\infty}\int_{-kT-T}^{-kT}\varphi U(t',s')Q(s')\int_{d_\alpha}e^{-im\theta}g(\theta,s')d\theta ds'$$

$$= I_1(t) + I_2(t).$$

We write $I_2(t)$ as follows

$$\int_{d_\alpha}\int_0^T \varphi U(t'+T,0)\chi(e^{-i\theta}I - U(T))^{-1}\psi_1$$

$$\times U(0,\xi)Q(\xi)U_0(\xi)e^{-im\theta}\psi g(\theta,0)d\xi d\theta,$$

where $\chi = 1$ on supp ψ and $\varphi U(t'+T,0)(1-\chi) = 0$.

Assume $n \geq 3$, n odd. Then $(\mathcal{R})$ implies that $R_\chi(\theta)$ has no poles θ with $\operatorname{Im}\theta \geq 0$ and we can choose $\delta > 0$ so that $R_\chi(\theta)$ has no poles θ with $\operatorname{Im}\theta \geq -\delta T, -\pi < \operatorname{Re}\theta \leq \pi$. Let $d_{-\delta T} = [-i\delta T - \pi, -i\delta T + \pi]$. Recall that $t = mT + t'$, so $e^{-m\delta T} \leq Ce^{-\delta t}$ with $C > 0$ independent of m and t. On the other hand,

$$\psi g(\theta,0) = e^{-i\theta}\psi(e^{-i\theta} - U_0(T))^{-1}f,\ \operatorname{Im}\theta > 0$$

and we conclude that $\psi g(\theta, 0)$ admits an analytic continuation in $\mathbb{C}$. We shift the contour of the integration from d_α to $d_{-\delta T}$ (see Figure 1) and we obtain

$$\|I_2(t)\| \leq C_1 e^{-\delta t}\|f\|,\ t \geq 0.$$

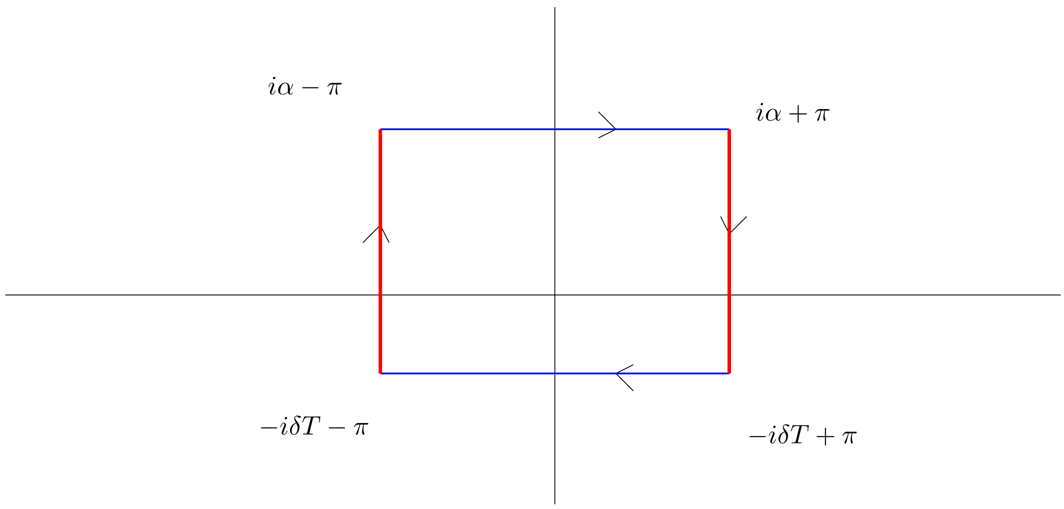

Figure 1

By the same argument we get an estimate for $I_1(t)$ and we conclude that

$$\|\varphi U(t,s)f\| \leq C(n,\varphi,f)e^{-\delta(t-s)}\|f\|,\ t - s \geq 1.$$

For n even we apply a similar argument by shifting the contour of integration to a curve γ going around 0 (see [15]). For the analysis of the integral in a neighborhood of 0 we use the hypothesis on the behavior of $R_\chi(\theta)$ and a result of Vainberg [20], to obtain

$$\|I_k(t)\| \leq C_2 t^{-1}(\ln t)^{-2}\|f\|,\ t \geq t_0 > 1,\ k = 1, 2.$$

We refer to [15] for more details.

3 Strichartz estimates

We say that the real numbers

$$1 \leq \tilde{p}, \tilde{q} \leq 2 \leq p, q \leq +\infty,\ 0 \leq \gamma \leq 1$$

are **admissible** for the free wave equation if the following estimate holds: For data $(f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, $F \in L^r_t(\mathbb{R}; L^s_x(\mathbb{R}^n))$ and $u(t,x)$ solution of (1) with $\tau = 0$, $V = 0$ we have

$$\begin{aligned}\|u\|_{L^p_t(\mathbb{R};\, L^q_x(\mathbb{R}^n))} &+ \|u(t,x)\|_{\dot{H}^\gamma_x} + \|\partial_t u(t,x)\|_{\dot{H}^{\gamma-1}_x} \\ &\leq C\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R};\, L^{\tilde{q}}_x(\mathbb{R}^n))}\Big)\end{aligned} \tag{5}$$

with a constant $C = C(n, p, q, \tilde{p}, \tilde{q}, \gamma) > 0$ independent of $t \in \mathbb{R}$. We refer to Lindblad-Sogge [11] and Keel-Tao [12] and to the references given there for global Strichartz estimates for the free wave equation and to [18] for some results for perturbations depending only of t.

Notice that if q, $\tilde{q}' < \frac{2(n-1)}{n-3}$, then $p, q, \tilde{p}, \tilde{q}, \gamma$ are admissible if the following conditions hold:

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{\tilde{p}} + \frac{n}{\tilde{q}} - 2,$$

$$\frac{1}{p} \leq \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{q}\Big),\ \frac{1}{\tilde{p}'} \leq \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{\tilde{q}'}\Big).$$

Theorem 2 ([*15*]) *Let the condition* $(\mathcal{R})$ *be fulfilled and let* $1 \leq \tilde{p}, \tilde{q} \leq 2 \leq p, q \leq +\infty$, $0 \leq \gamma \leq \min\{1, (n-1)/2\}$, $p > 2$ *be admissible for the free wave equation. Moreover, if* n *is even assume that* $\tilde{p} < 2$. *Then for data* $(f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$, $F \in L^{\tilde{p}}_t(\mathbb{R};\, L^{\tilde{q}}_x(\mathbb{R}^n))$ *and* $u(t,x)$ *solution of* (1) *with* $\tau = 0$ *we have the estimate*

$$\begin{aligned}\|u\|_{L^p_t(\mathbb{R};\, L^q_x(\mathbb{R}^n))} &+ \|u(t,x)\|_{\dot{H}^\gamma_x} + \|\partial_t u(t,x)\|_{\dot{H}^{\gamma-1}_x} \\ &\leq C\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R};\, L^{\tilde{q}}_x(\mathbb{R}^n))}\Big)\end{aligned} \tag{6}$$

with a constant $C = C(n, p, q, \tilde{p}, \tilde{q}, \gamma) > 0$ *independent of* $t \in \mathbb{R}$.

Sketch of the proof. The proof is based on the following propositions.

Proposition 2 ([*15*]) *Assume* $(\mathcal{R})$ *fulfilled and* $0 \le \gamma \le \min\{1, (n-1)/2\}$. *Let* $(f_0, f_1) \in \dot{\mathcal{H}}_\gamma(\mathbb{R}^n)$ *and let* $F \in L^2_t(\mathbb{R};\ \dot{H}^\gamma_x(\mathbb{R}^n))$ *be supported in* $\{x : |x| \le R\}$. *Then for every fixed* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *the solution* $u(t,x)$ *of* (1) *with* $\tau = 0$ *satisfies the estimate*

$$\int_{-\infty}^{\infty} \|(\varphi u(t,x), \varphi\partial_t u(t,x))\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} dt$$

$$\le C(n, \varphi, R)\Big(\|f_0\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|f_1\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L^2_t(\mathbb{R};\dot{H}^\gamma_x(\mathbb{R}^n))}\Big)^2.$$

Proposition 3 ([*19*], [*15*]) *Let* $(p, q, \tilde{p}, \tilde{q}, \gamma)$, f_0, f_1, F *be as in Theorem* 2. *Let* $u_0(t,x)$ *be the solution of* (1) *with* $\tau = 0$, $V = 0$. *Then for every* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *we have*

$$\int_{-\infty}^{\infty} \|(\varphi u_0(t,x), \varphi\partial_t u_0(t,x))\|^2_{\dot{\mathcal{H}}_\gamma(\mathbb{R}^n)} dt$$

$$\le C(n, \varphi)\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R};L^{\tilde{q}}_x(\mathbb{R}^n))}\Big)^2.$$

For n odd and $1 \le \tilde{p} \le 2$ Proposition 3 has been established in [19]. To obtain the L^2-integrability of the local energy in Proposition 2, we use the local energy decay given by Theorem 1 and for this purpose we need the condition $(\mathcal{R})$. To prove the estimate (6), we write the solution of (1) as a sum $u = u_0 + v$, where u_0 is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta)u_0 = F, \\ u_0|_{t=0} = f_0,\ \partial_t u_0|_{t=0} = f_1, \end{cases}$$

while v is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta + V)v = -Vu_0, \\ v|_{t=0} = \partial_t v|_{t=0} = 0. \end{cases}$$

Applying Proposition 3 for Vu_0, we obtain the estimate

$$\|Vu_0\|_{L^2_t(\mathbb{R};\ \dot{H}^\gamma_x(\mathbb{R}^n))} \le C_0\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{p}}_t(\mathbb{R};\ L^{\tilde{q}}_x(\mathbb{R}^n))}\Big). \tag{7}$$

In fact, choosing a function $\beta \in C_0^\infty(\mathbb{R}^n)$ such that $\beta = 1$ on $\mathrm{supp}_x\, V(t,x)$, we have

$$\|V(t,x)u_0\|_{\dot{H}^\gamma_x(\mathbb{R}^n)} \le C_{\gamma,V}\|\beta u_0\|_{\dot{H}^\gamma_x(\mathbb{R}^n)}.$$

The estimate of $\|u_0\|_{L^p_t(\mathbb{R};\ L^q_x(\mathbb{R}^n))}$ follows form (5). Next we have

$$v(t,x) = -\int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}(Vu_0 + Vv)(s,x)ds.$$

The function Vu_0 satisfies the estimate (7) and by Proposition 2 applied to the equation $(\partial_t^2 - \Delta + V)v = -Vu_0$ we deduce

$$\|Vu_0 + Vv\|_{L_t^2(\mathbb{R};\, \dot{H}_x^\gamma(\mathbb{R}^n))} \le C_1\Big(\|f_0\|_{\dot{H}^\gamma} + \|f_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L_t^{\tilde{p}}(\mathbb{R};\, L_x^{\tilde{q}}(\mathbb{R}^n))}\Big). \quad (8)$$

We wish to show that

$$\Big\|\int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}(Vu_0+Vv)(s,x)ds\Big\|_{L_t^p(\mathbb{R}^+;\, L_x^q(\mathbb{R}^n))} \le C_2\|Vu_0+Vv\|_{L_t^2(\mathbb{R}^+;\, \dot{H}_x^\gamma(\mathbb{R}^n))}. \quad (9)$$

Following the argument of [19], we conclude that the operator

$$T : \dot{H}^{-\gamma}(\mathbb{R}^n) \ni g \mapsto \beta e^{\pm it\Lambda} g \in L_t^2(\mathbb{R}^+; \dot{H}_x^{-\gamma}(\mathbb{R}^n))$$

is bounded. The adjoint operator

$$(T^*G)(x) = \int_0^\infty e^{\mp is\Lambda}\beta G(s,x))ds$$

is bounded as an operator from $L_t^2(\mathbb{R}^+; \dot{H}_x^\gamma(\mathbb{R}^n))$ to $\dot{H}_x^\gamma(\mathbb{R}^n)$ and this yields

$$\Big\|\int_0^\infty e^{\pm is\Lambda}\beta h(s,x)(s,x)ds\Big\|_{\dot{H}^\gamma(\mathbb{R}^n)} \le C_2\|h\|_{L_t^2(\mathbb{R}^+;\, \dot{H}_x^\gamma(\mathbb{R}^n))}. \quad (10)$$

Consider the integral operators

$$J : L_t^2(\mathbb{R}^+; \dot{H}_x^\gamma(\mathbb{R}^n)) \ni h(t,x) \longrightarrow \int_0^t K(s,t)h(s,x)ds \in L_t^p(\mathbb{R}^+;\, L_x^q(\mathbb{R}^n)),$$

where $K(s,t) = \Lambda^{-1}\sin((t-s)\Lambda)\beta$. To apply Christ-Kiselev lemma [6], it is sufficient to have an estimate for

$$\Big\|\int_0^\infty \frac{\sin((t-s)\Lambda)}{\Lambda}\beta h(s,x)ds\Big\|_{L_t^p(\mathbb{R}^+;L_x^q(\mathbb{R}^n))}.$$

By (5) and (10), we get

$$\Big\|e^{\pm it\Lambda}\Lambda^{-1}\int_0^\infty e^{\pm is\Lambda}\beta h(s,x)ds\Big\|_{L_t^p(\mathbb{R}^+;\, L_x^q(\mathbb{R}^n))}$$

$$\le C_3\Big\|\int_0^\infty e^{\pm is\Lambda}\beta h(s,x)ds\Big\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \le C_2C_3\|h\|_{L_t^2(\mathbb{R}^+;\, \dot{H}_x^\gamma(\mathbb{R}^n))}.$$

We take $h = Vu_0 + Vv$ and we use the addition formula for $\sin((t-s)\Lambda)$ to conclude that

$$\Big\|\int_0^\infty \frac{\sin((t-s)\Lambda)}{\Lambda}(Vu_0+Vv)ds\Big\|_{L_t^p(\mathbb{R}^+;L_x^q(\mathbb{R}^n))} \le C_4\|Vu_0+Vv\|_{L_t^2(\mathbb{R}^+;\, \dot{H}_x^\gamma(\mathbb{R}^n))}. \quad (11)$$

By hypothesis $p > 2$, so an application of Christ-Kiselev lemma [6] yields immediately (9). Consequently, (8) implies an estimate for $\|v\|_{L_t^p(\mathbb{R}^+;L_x^q(\mathbb{R}^n))}$ and, similarly, we deal with the norm $\|v\|_{L_t^p(\mathbb{R}^-;L_x^q(\mathbb{R}^n))}$. To estimate $\|v(t_0,x)\|_{\dot{H}^\gamma(\mathbb{R}_x^n)}$ uniformly with respect to t_0, notice that

$$\Big\| e^{\pm it\Lambda} \Lambda^{-1} \int_0^{t_0} e^{\pm is\Lambda} (Vu_0 + Vv)(s,x) ds \Big\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}$$

$$\leq C_5 \Big\| \int_0^{t_0} e^{\pm is\Lambda} (Vu_0 + Vv)(s,x) ds \Big\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}$$

with a constant $C_5 > 0$ independent of t_0. As above, we can estimate the right hand part by $\|Vu_0 + Vv\|_{L^2_t(\mathbb{R};\, \dot{H}^{\gamma}_x(\mathbb{R}^n))}$ uniformly with respect to t_0 and apply (8). A similar argument works for $\|\partial_t v(t_0, x)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n_x)}$ and the proof of Theorem 2 is complete.

4 Non-trapping moving obstacles

Throughout this and the following sections we assume that n is odd. To make a precise definition of non-trapping obstacles we must consider the generalized bicharacteristics of the wave operator $\Box = \partial_t^2 - \Delta_x$ determined as the trajectories of the generalized Hamiltonian flow $\mathcal{F}_\sigma$ in $\overline{Q}$ related to the symbol $\sum_{i=1}^n \xi_i^2 - \tau^2$ of $\Box$ (see [13] for a precise definition). In general, $\mathcal{F}_\sigma$ is not smooth and in some cases there may exist two different integral curves issued from the same point in the phase space. To avoid this situation, we assume that for every $(t, x, \tau, \xi) \in T^*(\overline{Q}) \setminus \{0\}$ the flow $\mathcal{F}_\sigma$ is uniquely determined. To deal with a continuous flow, following [13] we consider the compressed cotangent bundle $\tilde{T}^*(\overline{Q})$ which for $(t, x) \in \partial Q$ can be identified with

$$T^*_{t,x}(\overline{Q})/N_{t,x}(\partial Q),$$

$N_{t,x}(\partial Q)$ being the fiber of the formers vanishing on $T_{t,x}(\partial Q)$.

Thus given $\rho = (t, x, \tau, \xi) \in \tilde{T}^*(\overline{Q}) \setminus \{0\} = \dot{T}^*(Q)$, there exists a unique generalized (compressed) bicharacteristic $\gamma(\sigma) = (t(\sigma), x(\sigma), \tau(\sigma), \xi(\sigma)) \in \dot{T}^*(Q)$ such that $\gamma(0) = \rho$ and we define $\mathcal{F}_\sigma(\rho) = \gamma(\sigma)$ for all $\sigma \in \mathbb{R}$ (see [13]). We obtain a flow $\mathcal{F}_\sigma : \dot{T}^*(Q) \longrightarrow \dot{T}^*(Q)$ which is called also *generalized geodesic flow* on $\dot{T}^*(Q)$. The projections of the compressed generalized bicharacteristics on $\overline{Q}$ are called generalized geodesics.

Definition. The obstacle Q is called non-trapping if for each $R > R_0$ there exists $T(R) > 0$ such that there are no generalized geodesics of $\Box$ with length T_R lying entirely in $\overline{Q} \cap \{(t, x) :\ |x| \leq R\}$.

Let $P^b_\pm$ be the orthogonal projections on the orthogonal complements of the Lax-Phillips spaces

$$D^b_\pm = \{f \in \dot{\mathcal{H}}_1 : U_0(t)f = 0,\ |x| < \pm t + b, \pm t > 0\},$$

where $U_0(t)$ is the unitary group introduced in Section 1. Set

$$Z^b(T) = P_+^b U(T,0) P_-^b.$$

Following the general results of propagation of singularities (see [13]), it is not difficult to show that if Q is non-trapping, given a function $\varphi \in C_0^\infty(\mathbb{R}_x^n)$ with $\operatorname{supp} \varphi \subset \{x : |x| \le a\}$, $a \ge R_0$, the operator $\varphi U(t,0) P_-^a : H(0) \longrightarrow H(t)$ for $t > 4a + T_{4a}$ is compact (see [7], [14]). In fact, set $M(t,s) = U(t,s) - U_0(t-s)$ and let $\Phi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off such that $\Phi = 1$ for $|x| \le 3a$, $\Phi = 0$ for $|x| \ge 4a$. Then for $t > 4a + T_{4a}$ we have

$$\varphi U(t,0) P_-^a = \varphi M(t, t-2a) \Phi U(t-2a, 2a) \Phi M(2a, 0) P_-^a$$

and the operator at the right hand side is compact. Next we take $a = R_0$ and by a similar argument choosing $kT > 4a + T_{4a}$, we deduce that the operator $(Z^a(T))^k$ is compact. This implies that the spectrum of the operator $Z^a(T)$ is discrete with finite multiplicity. For $b \ge a$ we can use the same argument and show that $(Z^b(T))^{m(b)}$ is compact with some integer $m(b) \in \mathbb{N}$ depending of b. Consequently, the spectrum of $Z^b(T)$ is also discrete and with finite multiplicity. According to [7], the eigenvalues of $Z^b(T)$ and their multiplicities are independent of b. Next given a cut-off $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ for $|x| \le R_0$, $\operatorname{supp} \chi \subset \{x : |x| \le b\}$, $b > a$, we deduce $P_\pm^b \chi = \chi = \chi P_\pm^b$. It is clear that for $|z| \ge A \gg 1$ we have

$$\chi (Z^b(T) - z)^{-1} \chi = \chi (U(T) - z)^{-1} \chi.$$

The left hand side admits a meromorphic continuation for $|z| \le A$ and the same is true for the cut-off resolvent $\chi(U(T) - z)^{-1}\chi$, hence the poles of $\chi(U(T) - z)^{-1}\chi$ are between the poles of $(Z^b(T) - z)^{-1}$ which are independent of b.

To prove that the poles of $\chi(U(T) - z)^{-1}\chi$ coincide with those of $(Z^b(T) - z)^{-1}$, we apply with some modification an argument used in [3] for stationary obstacles. Choose a function $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $\psi = 1$ for $|x| \le R_0 + 1$, $\psi = 0$ for $|x| \ge R_0 + 2$ and consider the operator

$$L_\psi(g, h) = \Big(0, \langle \nabla_x \psi, \nabla_x g \rangle + (\Delta \psi) g\Big).$$

In particular, we define $L_\psi(U(t,s)f)$ and $L_\psi(U_0(t)f)$ and will write simply $L_\psi U(t,s)$ and $L_\psi U_0(t)$. It is easy to see that we have

$$(1 - \psi) U(t,0) = U_0(t)(1 - \psi) + \int_0^t U_0(t) L_\psi U(t,s) ds, \tag{12}$$

$$U(t,0)(1 - \psi) = (1 - \psi) U_0(t) + \int_0^t U(t,s) L_\psi U_0(s) ds. \tag{13}$$

An application of these equalities yields

$$\begin{aligned}U(t,0) =& U(t,0)\psi + (1-\psi)U_0(t) + \int_0^t \psi U(t,s)L_\psi U_0(s)ds \\ &+ \int_0^t U_0(t-s)(1-\psi)L_\psi U_0(s)ds + \int_0^t\int_0^{t-s} U_0(\tau)L_\psi U(t-s,\tau)L_\psi U_0(s)d\tau ds \\ &= \psi U(t,0)\psi + U_0(t)\psi(1-\psi) + (1-\psi)U_0(t) + \int_0^t \psi U(t,s)L_\psi U_0(s)ds \\ &+ \int_0^t U_0(s)L_\psi U(t,s)\psi ds + \int_0^t U_0(t-s)(1-\psi)L_\psi U_0(s)ds \\ &+ \int_0^t\int_0^{t-s} U_0(\tau)L_\psi U(t-s,\tau)L_\psi U_0(s)d\tau ds.\end{aligned}$$

Let $g \in C_0^\infty(B_{R_0+3})$ be a cut-off function equal to 1 on B_{R_0+2}. We choose the projectors $P_\pm^b$ so that

$$P_\pm^b\psi = \psi = \psi P_\pm^b,\ P_\pm^b g = g = gP_\pm^b.$$

Next we fix $b > 0$ and the projectors $P_\pm^b$ with these properties and will write $P_\pm,\ Z(T)$ instead of $P_\pm^b,\ Z^b(T)$. Note that $gL_\psi = L_\psi = L_\psi g$ and let $T_0 > 0$ be chosen so that $P_+U_0(t)P_- = 0$ for $t \geq T_0$. For A large enough and $z \in \mathbb{C},\ |z| \geq A$, we have

$$(Z(T)-z)^{-1} = -\sum_{j=0}^\infty z^{-j-1}P_+U(jT,0)P_-.$$

Now we apply the above representation of $U(jT,0)$ for $P_+U(jT,0)P_-$, $j \in \mathbb{N}$, and write

$$\begin{aligned}(Z(T)-z)^{-1} &= \psi(U(T)-z)^{-1}\psi \\ &- \sum_{jt\leq T_0} z^{-j-1}P_+U_0(jT)\psi(1-\psi)P_- \\ &- \sum_{jT\leq T_0} z^{-j-1}P_+(1-\psi)U_0(jT)P_- \\ &+ \int_0^{T_0} P_+U_0(s)L_\psi(U(T)-z)^{-1}\Phi U(0,s)\psi P_- ds \\ &+ \int_0^{T_0} P_+\psi(U(T)-z)^{-1}\Phi U(0,s)L_\psi U_0(s)P_- ds \\ &- \sum_{jT\leq T_1}\int_0^{\min(jT,T_0)} z^{-j-1}P_+U_0(jT-s)(1-\psi)L_\psi U_0(s)P_- ds \\ &+ \int_0^{T_0}\int_0^{T_0} P_+U_0(\tau)L_\psi U(-s,0)\Phi(U(T)-z)^{-1}\Phi \\ &\times U(0,\tau)L_\psi U_0(s)P_- d\tau ds + G(z)\end{aligned}$$

with an operator $G(z)$ holomorphic for $z \neq 0$. Here Φ is a cut-off function with compact support determined by the finite speed of propagation so that

$$(1-\Phi)U_0(t)g = 0 \text{ and } (1-\Phi)U(t,\tau)g = gU(t,\tau)(1-\Phi) = 0$$

for $|t| \leq 2T_0$, $0 \leq \tau \leq T_0$. The terms given by finite sums are holomorphic operators with respect to $z \neq 0$. Choose a function $\Psi \in C_0^\infty(|x| \leq c+1)$ equal to 1 for $|x| \leq c$ and fix $c > b$ large enough. Thus we conclude that if $\Psi(U(T)-z)^{-1}\Psi$ is analytic in a neighborhood of z_0, $0 < |z_0| < A$, the same is true for $(Z(T)-z)^{-1}$, hence $\Psi(U(T)-z)^{-1}\Psi$ and $(Z(T)-z)^{-1}$ have the same poles. The analysis of the multiplicities of the corresponding poles is more difficult and we refer to [2] for the results in this direction.

To study the local energy decay for non-trapping obstacles, we can follow the approach in [7] (see also Chapter 6 in [14]). In fact, assume that $\Psi(U(T)-z)^{-1}\Psi$ has no poles $z \in \mathbb{C}$, $|z| \geq 1$, for a cut-off function Ψ given above. Then choosing $b > R_0$ large enough, we get

$$\sigma(Z^b(T)) \cap \{z \in \mathbb{C} : |z| \geq 1\} = \emptyset,$$

where $\sigma(L)$ denotes the spectrum of the operator L. The same property of $\sigma(Z^a(T))$ holds for all $a \geq R_0$ and we deduce

$$\|Z^a(t,s)\| \leq C_a e^{-\delta_a(t-s)},\ t \geq s \tag{14}$$

with $C_a > 0$, $\delta_a > 0$ independent of t and s. Thus given a function $f \in H(s)$ with supp $f \in \{|x| \leq R\}$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ for $|x| \leq R_0$, we conclude that

$$\|\varphi U(t,s)f\|_{H(t)} \leq C(\varphi, R)e^{-\gamma(t-s)}\|f\|_{H(s)},\ t \geq s$$

with $\gamma > 0$ independent of t and s. For this purpose we choose suitably b and apply (14) with $a = b$.

Finally, to establish the L^2-integrability of the local energy, we exploit (13) and using the notations in (13), we write

$$U(t,0)f = U(t,0)\psi f + (1-\psi)U_0(t)f + \int_0^t U(t,s)L_\psi u_0(s)ds.$$

The estimate of $\int_0^\infty \|\varphi U(t,0)\psi f\|^2_{H(t)}dt$ is trivial, while for

$$\int_0^\infty \Big\| \int_0^t \varphi U(t,s)L_\psi u_0(s)ds \Big\|^2_{H(t)} dt$$

we apply Young's inequality. Thus we obtain

$$\int_0^\infty \|\varphi U(t,0)f\|^2_{H(t)}dt \leq C(\varphi)\|f\|^2_{H(0)}.$$

Under the condition that we have no poles $z \in \mathbb{C}$ with $|z| \geq 1$ of the cut-off resolvent, we can obtain Strichartz estimates modifying the arguments of Section 3.

5 Trapping moving obstacles

First let us consider a stationary obstacle $K(t) = K$, $\forall t \in \mathbb{R}$ and set $\Omega = \mathbb{R}^n \setminus \overline{K}$. Let $U(t) = e^{itG}$ be the unitary group related to the Dirichlet problem (2) in $\mathbb{R} \times \Omega$ and let $\mathcal{H} = H_D(\Omega) \oplus L^2(\Omega)$ be the energy space (see [10]). Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function equal to 1 on $\overline{K}$ and let $R_\chi(\lambda) = \chi(-\Delta_D - \lambda^2)^{-1}\chi$ be the cut-off resolvent of the Dirichlet Laplacian Δ_D in Ω which is bounded in $L^2(\Omega)$ for $\operatorname{Im} \lambda > 0$. For non-trapping obstacles K we have the estimate (see for instance, [20])

$$\|\lambda R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \leq C, \ \forall \lambda \in \mathbb{R}. \tag{15}$$

On the other hand, the existence of at least one trapped ray leads to the following

Proposition 4 ([*4*]) *If the generalized compressed Hamiltonian flow $\mathcal{F}_\sigma$ in $\mathbb{R} \times \overline{\Omega}$ is continuous and if we have at least one (generalized) trapping ray in $\overline{\Omega}$, then*

$$\sup_{\lambda\in\mathbb{R}} \|\lambda R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} = +\infty. \tag{16}$$

Proof. Our hypothesis imply the existence of a sequence of ordinary reflecting rays γ_n with sojourn times $T_{\gamma_n} \to \infty$ (see for instance, [13]) and we may apply the result of Ralston [17] which says that we have not an uniform decay of local energy. On the other hand, according to the results in [22], the uniform decay of the local energy is equivalent to (15) and we deduce that the estimate (15) fails. Consequently, we get (16).

The existence of one trapping ray γ leads to several results (see [3], [4]) which hold without having any knowledge of the geometry of K outside a small neighborhood of γ. In particular, we are interested on the analytic properties of the cut-off resolvent of the monodromy operator $U(T)$ introduced in Section 1. Since a stationary obstacle K is periodic with period every $t > 0$, it is natural to study the analytic properties of the cut-off resolvent $\Psi(U(t) - z)^{-1}\Psi$ with $\Psi \in C_0^\infty(|x| \leq c + 1)$, $\Psi = 1$ for $|x| \leq c$, where $c > R_0$ is large and fixed. For trapping obstacles we cannot obtain a meromorphic continuation across the unit circle $\mathbb{S}^1$ and we have the following

Theorem 3 ([*3*]) *Assume the obstacle K stationary and the condition* (16) *fulfilled. Then for almost all $t \in \mathbb{R}^+$ and all $z_0 \in \mathbb{S}^1$ we have*

$$\lim_{z\to z_0,\, |z|>1} \|\Psi(U(t) - z)^{-1}\Psi\|_{\mathcal{H}\to\mathcal{H}} = +\infty.$$

The proof is based on the following idea. Taking $b \geq c+1$, we have $P_\pm^b \Psi = \Psi = \Psi P_\pm^b$, where $P_\pm^b$ have been introduced in the previous section. Consider the Lax-Phillips semigroup $Z^b(t) = P_+^b U(t) P_-^b$. We fix b with the above property and for simplicity of the notations we write $Z(t)$ instead of $Z^b(t)$. Let B be the generator of $Z(t)$, that is $Z(t) = e^{tB}$. Therefore, it is easy to see that the condition (16) implies

$$\sup_{\lambda \in \mathbb{R}} \|(iB - \lambda)^{-1}\|_{\mathcal{H} \to \mathcal{H}} = +\infty.$$

By applying a result of I. Herbst [9], we deduce that for almost all $t \in \mathbb{R}^+$ we have the inclusion

$$\mathbb{S}^1 \subset \sigma(Z(t)). \tag{17}$$

Next we obtain a representation of $(Z(t) - z)^{-1}$, $|z| > 1$, as a sum of terms involving the cut-off resolvent

$$\Psi \sum_{j=0}^{\infty} z^{-j-1} U(jt) \Psi = -\Psi (U(t) - z)^{-1} \Psi$$

as we have done this for the operator $Z(T)$ and the propagator $U(T, 0)$ in the previous section. Consequently, if the norm of $\Psi(U(t) - z)^{-1}\Psi$ has a limit as $z \to z_0 \in \mathbb{S}^1$, $|z| > 1$, we obtain a contradiction with (17).

Passing to trapping moving obstacles, introduce the normal speed of ∂Q by

$$\mathbf{v}(z) = \frac{\nu_t(z)}{|\nu_x(z)|} \frac{\nu_x(z)}{|\nu_x(z)|}.$$

Given a point $z = (t, x) \in \partial Q$, and a bicharacteristic

$$\gamma = (t(\sigma), x(\sigma), \tau(\sigma), \xi(\sigma)) \in T^*(\overline{Q})$$

reflecting at z, denote the incident direction of γ by $\frac{-\xi_i}{\tau_i}$ and the reflecting direction by $\frac{-\xi_r}{\tau_r}$ with $|\xi_i|^2 = \tau_i^2$, $|\xi_r|^2 = \tau_r^2$. Then $\tau_r = \mu(z)\tau_i$ and

$$\mu(z) = \frac{(1 - 2|\mathbf{v}(z)| \cos\varphi + |\mathbf{v}(z)|^2)}{(1 - |\mathbf{v}(z)|^2)^{-1}} > 0,$$

where $0 \leq \varphi \leq \pi$ is the angle between $\frac{-\xi_i}{\tau_i}$ and $\mathbf{v}(z)$. We say that a bicharacteristic (ray) γ issued from $(s, y, \tau, \eta) \in \dot{T}^*(Q)$ with infinite number of reflection points $z_j \in \partial Q$, $j \in \mathbb{N}$, at times $t_j \to \infty$ is **δ-trapping** if

$$\prod_{0 \leq t_j \leq t} \mu(z_j) \geq C e^{\delta t},\ t \in [0, \infty], \delta > 0. \tag{18}$$

It turns out that for stationary obstacles we have always $\mu(z) = 1$ and the existence of δ-trapping rays is possible only for trapping moving obstacles. Next we consider a example examined by Popov and Rangelov.

Example. (see [16]) Let $K(t) = \mathcal{O}_1 \cup \mathcal{O}_2(t)$,

$$\mathcal{O}_1 \cap \mathcal{O}_2(t) = \emptyset,\ \mathcal{O}_2(t+T) = \mathcal{O}_2(t),\ \forall t \in \mathbb{R}.$$

Suppose that for all t the obstacles $\mathcal{O}_1$ and $\mathcal{O}_2(t)$ are strictly convex and set

$$d(t) = \text{dist}\Big(\mathcal{O}_1, \mathcal{O}_2(t)\Big),\ d_1 = \min d(t),\ d_2 = \max d(t).$$

Assume that the obstacle $K(t)$ and its exterior normal satisfy the hypothesis in Section 1 and the conditions:

$$(i)\ \ d_1 < T/2 < d_2,$$

(ii) there exists $y_1 \in \partial\mathcal{O}_1$ and $y_2(t) \in \partial\mathcal{O}_2(t)$ so that

$$d(t) = |y_1 - y_2(t)|,\ \forall t \in \mathbb{R},$$

(iii) the normal speed $\mathbf{v}(t, y_2(t))$ of $\mathcal{O}_2(t)$ vanishes only if $d(t) = d_i, i = 1, 2$.

We have $|d'(t)| < 1$ and by our assumptions there exists $s_0 > 0$ so that $d(s_0) = T/2, d'(s_0) < 0$. We choose $s < s_0$ and set $y = y_2(s_0) + (s - s_0)\omega$, $\omega = \frac{y_2(t)-y_1}{|y_2(t)-y_1|}$. The bichracteristic $\gamma(\sigma) = (t(\sigma), x(\sigma), \tau(\sigma), \xi(\sigma))$ issued from $(s, y, 1-\omega)$ has an infinite number of reflections at $z_k = (t_k, x_k),\ k \in \mathbb{N}$, with

$$t_k = s_0 + (k-1)T/2,\ x_{2k-1} = y_2(s_0),\ x_{2k} = y_1$$

and

$$\mu(z_{2k}) = 1,\ \mu(z_{2k+1}) = \frac{1 + |d'(s_0)|}{1 - |d'(s_0)|} > 1.$$

Moreover, $\gamma(\sigma)$ is δ-trapping with

$$\delta = \frac{1}{T}\Big(\ln(1 + |d'(s_0)|) - \ln(1 - |d'(s_0)|)\Big) > 0.$$

The following general result of Popov and Rangelov leading to solutions with exponentially growing local energy can be considered as a generalization of that of Ralston [17] for stationary obstacles.

Theorem 4 ([*16*]) *Assume that there exists a* δ*-trapping bicharacteristic* $\gamma(\sigma)$ *issued from* $(s, y, \tau, \eta) \in \dot{T}^*(Q)$*. Then for every neighborhood* W *of* y *in* $\Omega(s)$ *and every* $0 < \epsilon < \delta$ *there exists* $f = (f_0, f_1) \in H(s)$ *with supp* $f \subset W$ *so that for* $R \geq R_0 + T$ *we have*

$$\|U(t+s, s)f\|_{H_{\Omega(t+s)\cap\{|x|\leq R\}}} \geq C(\epsilon, s, f)e^{\epsilon t},\ t \in [s, \infty[, \tag{19}$$

$\|.\|_{H_{\Omega(t+s)\cap\{|x|\leq R\}}}$ *being the energy norm over* $\Omega(t+s) \cap \{|x| \leq R\}$.

In particular, the above result shows that if we have a δ-trapping bicharacteristic $\gamma(\sigma)$, then the spectral radius of $Z^b(T) = P_+^b U(T, 0)P_-^b$ for $b > R_0 + T$ is greater or equal to $e^{\delta T}$.

Following the argument of the previous section, we may compare the analytic singularities of $(Z^b(T) - z)^{-1}$ and those of the cut-off resolvent $\Psi(U(T) - z)^{-1}\Psi$, where $\Psi \in C_0^\infty(|x| \leq c+1)$ and $c > R_0$ is large enough and fixed.

Theorem 5 *Under the hypothesis of Theorem* 4 *for every* $0 < \epsilon < \delta$ *the cut-off resolvent of the monodromy operator*

$$\Psi(U(T) - z)^{-1}\Psi$$

has not an analytic continuation from $\{z \in \mathbb{C} : |z| \geq A \gg 1\}$ *to*

$$\{z \in \mathbb{C} :\ e^{\epsilon T} \leq |z| \leq A\}.$$

The analysis of the spectrum of $Z(T) = Z^b(T)$ for $|z| > 1$ is an open problem. We conjecture that the existence of a δ-trapping bicharacteristic implies that $(Z(T) - z)^{-1}$ has not a meromorphic continuation in

$$\{z \in \mathbb{C} :\ e^{\epsilon T} \leq |z| \leq A\},\ 0 < \epsilon < \delta.$$

More precisely, we expect that the continuous spectrum of the operator $Z(T)$ is not empty. In this direction it is interesting to note that for two strictly convex disjoint stationary obstacles K_i, $i = 1, 2$, for almost all $t \in \mathbb{R}^+$ we have the inclusion (17). In fact, a more stronger result holds.

Theorem 6 ([*4*]) *Let* $K = K_1 \cup K_2$, *where* $K_i, i = 1, 2$, *are strictly convex and disjoint and let* $\Omega = \mathbb{R}^n \setminus \overline{K}$. *Consider the semigroup* $Z^b(t) = P^b_+ U(t) P^b_-$, $b > R_0$, *where* $U(t)$ *is the unitary group related to the Dirichlet problem* (3) *in* $\mathbb{R} \times \Omega$. *Then for almost all* $t \in \mathbb{R}^+$ *we have*

$$\{z \in \mathbb{C} :\ |z| \leq 1\} = \sigma(Z^b(t)). \tag{20}$$

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