SPECTRAL PROBLEMS FOR OPERATORS WITH CROSSED MAGNETIC AND ELECTRIC FIELDS

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In memory of Pierre Duclos

Abstract. We obtain a representation formula for the derivative of the spectral shift function $\xi(\lambda; B, \epsilon)$ related to the operators $H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x$ and $H(B, \epsilon) = H_0(B, \epsilon) + V(x, y)$, $B > 0, \epsilon > 0$. We prove that the operator $H(B, \epsilon)$ has at most a finite number of embedded eigenvalues on $\mathbb{R}$ which is a step to the proof of the conjecture of absence of embedded eigenvalues of H in $\mathbb{R}$. Applying the formula for $\xi'(\lambda, B, \epsilon)$, we obtain a semiclassical asymptotics of the spectral shift function related to the operators $H_0(h) = (hD_x - By)^2 + h^2 D_y^2 + \epsilon x$ and $H(h) = H_0(h) + V(x, y)$.

1. Introduction

Consider the two-dimensional Schrödinger operator with homogeneous magnetic and electric fields

$$H = H(B, \epsilon) = H_0(B, \epsilon) + V(x, y),\ D_x = -\mathbf{i}\partial_x,\ D_y = -\mathbf{i}\partial_y,$$

where

$$H_0 = H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x.$$

Here $B > 0$ and $\epsilon > 0$ are proportional to the strength of the homogeneous magnetic and electric fields and $V(x, y)$ is a $L^\infty(\mathbb{R}^2)$ real valued function satisfying the estimates

$$|V(x, y)| \leq C(1 + |x|)^{-2-\delta}(1 + |y|)^{-1-\delta}, \delta > 0. \tag{1.1}$$

For $\epsilon \neq 0$ we have $\sigma_{\text{ess}}(H_0(B, \epsilon)) = \sigma_{\text{ess}}(H(B, \epsilon)) = \mathbb{R}$. On the other hand, for decreasing potentials V it is possible to have embedded eigenvalues $\lambda \in \mathbb{R}$ and this situation is quite different from that with $\epsilon = 0$ when the spectrum of $H(B, 0)$ is formed by eigenvalues with finite multiplicities which may accumulate only to Landau levels $\lambda_n = (2n + 1)B$, $n \in \mathbb{N}$ (see [7], [11], [13] and the references cited there). The analysis of the spectral properties of H and the existence of resonances have been studied in [5], [6], [3] under the assumption that $V(x, y)$ admits a holomorphic extension in the x-variable into a domain

$$\Gamma_{\delta_0} = \{z \in \mathbb{C} :\ 0 \leq |\operatorname{Im} z| \leq \delta_0\}.$$

On the other hand, without any assumption on the analyticity of $V(x, y)$, it was proved in [3] that the operator $(H - z)^{-1} - (H_0 - z)^{-1}$ for $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, is trace class. Thus, following the general setup [9], [19], we may define the spectral shift function $\xi(\lambda) = \xi(\lambda; B, \epsilon)$ related to $H_0(B, \epsilon)$ and $H(B, \epsilon)$ by

$$\langle \xi', f \rangle = \operatorname{tr}\Big(f(H) - f(H_0)\Big),\ f \in C_0^\infty(\mathbb{R}).$$

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By this formula $\xi(\lambda)$ is defined modulo a constant but for the analysis of the derivative $\xi'(\lambda)$ this is not important. For the analysis of the behavior of $\xi(\lambda; B, \epsilon)$ it is important to have a representation of the derivative $\xi'(\lambda; B, \epsilon)$. Such representation has been obtained in [3] for strong magnetic fields $B \to +\infty$ under the assumption that $V(x, y)$ admits an analytic continuation in x-direction.

In this paper we consider the operator H without *any assumption* on the analytic continuation of $V(x, y)$ and without the *restriction* $B \to +\infty$. For such potentials we cannot use the techniques in [5], [6] and [3] related to the resonances of the perturbed problem. Our purpose is to study $\xi'(\lambda; B, \epsilon)$ and the existence of embedded eigenvalues of H. The key point in this direction is the following

Theorem 1. *Let $V, \partial_x V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ and assume that* (1.1) *holds for V and $\partial_x V$. Then for every $f \in C_0^\infty(\mathbb{R})$ and $\epsilon \neq 0$ we have*

$$\mathrm{tr}\, \Big(f(H) - f(H_0)\Big) = -\frac{1}{\epsilon} \mathrm{tr}\, \Big((\partial_x V) f(H)\Big). \tag{1.2}$$

Notice that in (1.2) by $\partial_x V$ we mean the operator of multiplication by $\partial_x V$. The formula (1.2) has been proved by D. Robert and X. P. Wang [17] for Stark Hamiltonians in absence of magnetic field ($B = 0$). In fact, the result in [17] says that

$$\xi'(\lambda; 0, \epsilon) = -\frac{1}{\epsilon} \int_{\mathbb{R}^2} \partial_x V(x, y) \frac{\partial e}{\partial \lambda}(x, y, x, y; \lambda, 0, \epsilon) dx dy, \tag{1.3}$$

where $e(., .; \lambda, 0, \epsilon)$ is the spectral function of $H(0, \epsilon)$. On the other hand, the spectral shift function in [17] is related to the trace of the *time delay* operator $T(\lambda)$ defined via the corresponding scattering matrix $S(\lambda)$ (see [16]). The presence of magnetic filed $B \neq 0$ and Stark potential lead to some serious difficulties to follow this way. Recently, Theorem 1 has been established by the authors in [4] but the proof in [4] is technical, long and based on the trace class properties of the operators

$$\psi(H \pm \mathbf{i})^{-N},\ \partial_x \circ \psi(H \pm \mathbf{i})^{-N},\ (H \pm \mathbf{i}) \partial_x \circ \psi(H \pm \mathbf{i})^{-N-2} \tag{1.4}$$

with $\psi \in C_0^\infty(\mathbb{R})$ and $N \geq 2$. The idea is to use the commutators with the operators $\chi_R \partial_x$, where $\chi_R(x, y) = \chi\Big(\frac{x}{R}, \frac{y}{R}\Big)$ and $\chi \in C_0^\infty(\mathbb{R}^2)$ is a cut-off such that $\chi = 1$ for $|(x, y)| \leq 1$. One shows that

$$\mathrm{tr}\, \Big([\chi_R \partial_x, H] f(H) - [\chi_R \partial_x, H_0] f(H_0)\Big) = 0 \tag{1.5}$$

and next we are going to examine the limit $R \to \infty$ of the trace of the operators in (1.5). The commutators with ∂_x and the presence of magnetic field lead to operators involving $D_x - By$ and this is one of the main difference with the case $B = 0$. To overcome this difficulty we used in [4] the trace class operators (1.4) which led to technical problems. On the other hand, the operator ∂_x is often used for operators with Stark potential ϵx and this influenced our approach in [4]. One of the goal of this work is to present a new shorter and elegant proof of Theorem 1. The new idea is to apply the shift operator $U_\tau : f(x, y) \longrightarrow f(x + \tau, y)$ instead of ∂_x. In Proposition 1 we show that

$$\mathrm{tr}\, \Big([U_\tau, H] f(H) - [U_\tau, H] f(H_0)\Big) = 0.$$

The proof of the later equality is much easier than that of (1.5) and we don't need the trace class properties of the operators (1.4). Moreover, applying the operator U_τ, we may generalize the result of Theorem 1 for Schrödinger operators $(D_x - C(y))^2 + D_y^2 + \epsilon x + V(x, y)$ with variable magnetic filed as well as for operators with magnetic potentials in $\mathbb{R}^n, n \geq 3$.

The second question examined in this work is the existence of embedded real eigenvalues of H. In the physical literature one conjectures that for $\epsilon \neq 0$ there are no embedded eigenvalues. We established in [4] a weaker result saying that in every interval $[a, b]$ we may have at most a finite number of embedded eigenvalues with finite multiplicities. Under the assumption for analytic continuation of V it was proved in [5] that in some finite interval $[\alpha(B, \epsilon), \beta(B, \epsilon)]$ there are no resonances z of $H(B, \epsilon)$ with $\operatorname{Re} z \notin [\alpha(B, \epsilon), \beta(B, \epsilon)]$. Since the real resonances z coincide with the eigenvalues of $H(B, \epsilon)$, we obtain some information for the embedded eigenvalues. We prove in Section 3 without the condition of analytic continuation of $V(x, y)$ that H has no embedded eigenvalues outside an interval $[\alpha(B, \epsilon), \beta(B, \epsilon)]$. Combining this with the result in [4], we conclude that H has at most a finite number of embedded eigenvalues. Finally, applying the representation formula for the derivative of the spectral shift function $\xi_h(\lambda) = \xi_h(\lambda, B, \epsilon)$ related to the operators $H_0(h) = (hD_x - By)^2 + h^2 D_y^2 + \epsilon x$ and $H(h) = H_0(h) + V(x, y)$, we obtain a semiclassical asymptotics of $\xi_h(\lambda)$ as $h \searrow 0$ uniformly with respect to $\lambda \in [E_0, E_1]$ under some assumptions on the critical values of the symbol of $H(h)$.

2. Representation of the spectral shift function

We suppose without loss of generality that $B = \epsilon = 1$. Set $\langle z \rangle = (1 + |z|^2)^{1/2}$. For reader convenience we recall the following lemma proved in [4]

Lemma 1. *Let $\delta > 0$ and let $k_j(x, y) = \langle x \rangle^{-j(1+\delta)} \langle y \rangle^{-j(\frac{1}{2}+\delta)}, j = 1, 2$. The operators $G_2 := k_2(H_0 + \mathbf{i})^{-2},\ G_2^*$, (resp. $G_1 := k_1(H_0 + \mathbf{i})^{-1},\ G_1^*$), are trace class (resp. Hilbert-Schmidt).*

As an application of Lemma 1 recall that Proposition 1 in [4] says that for $g \in C_0^\infty(\mathbb{R})$ the operators $Vg(H)$ and $Vg(H_0)$ are trace class. Obviously, the same is true for $V(x + \tau, y)g(H)$ and we will use this fact below. Consider the shift operator

$$U_\tau : f(x, y) \longrightarrow f(x + \tau, y).$$

Let $H_0 = (D_x - y)^2 + D_y + x, H = H_0 + V(x, y)$. It is clear that

$$[U_\tau, H_0]u = U_\tau H_0 u - H_0 U_\tau u = U_\tau(xu) - xU_\tau u = \tau U_\tau u,$$

hence $[U_\tau, H_0] = \tau U_\tau$. Next

$$[U_\tau, V] = U_\tau(Vu) - VU_\tau u = V(x + \tau)U_\tau u - VU_\tau u = \Big(V(x + \tau, y) - V(x, y)\Big)U_\tau u.$$

Thus given a function $f \in C_0^\infty(\mathbb{R})$, we get

$$\begin{aligned}[U_\tau, H]f(H) - [U_\tau, H_0]f(H_0) &= \Big[\tau + (V(x + \tau, y) - V(x, y))\Big]U_\tau f(H) - \tau U_\tau f(H_0) \\ &= \tau U_\tau\Big(f(H) - f(H_0)\Big) + \Big(V(x + \tau, y) - V(x, y)\Big)U_\tau f(H).\end{aligned}$$

Proposition 1. *We have the equality*

$$\operatorname{tr}\Big([U_\tau, H]f(H) - [U_\tau, H_0]f(H_0)\Big) = 0. \tag{2.1}$$

Proof. We write

$$\begin{aligned}&\operatorname{tr}\Big[U_\tau Hf(H) - U_\tau H_0 f(H_0) + H_0 U_\tau f(H_0) - HU_\tau f(H)\Big] \\ &= \operatorname{tr} U_\tau\Big(Hf(H) - H_0 f(H_0)\Big) + \operatorname{tr}\Big(H_0 U_\tau f(H_0) - HU_\tau f(H)\Big) = (I) + (II).\end{aligned}$$

For the term (I), by using the cyclicity of the trace, we have

$$(I) = \operatorname{tr}\Big((Hf(H) - H_0 f(H_0))U_\tau\Big) = \operatorname{tr}\Big(f(H)H - f(H_0)H_0\Big)U_\tau. \tag{2.2}$$

On the other hand,

$$(II) = \operatorname{tr}\Big((H_0 - H)U_\tau f(H_0)\Big) + \operatorname{tr}\Big[HU_\tau\Big(f(H_0) - f(H)\Big)\Big] = (II_1) + (II_2).$$

and we justify below the trace class properties of the operators (II_1) and (II_2). For (II_1) we write

$$-(II_1) = VU_\tau f(H_0) = U_\tau[U_\tau^{-1}VU_\tau]f(H_0) = U_\tau V(x - \tau, y)f(H_0)$$

and the operator on the right hand side is trace class.

It is easy to see that the operator $\Big(f(H_0) - f(H)\Big)(H + \mathbf{i})$ is trace class since

$$\Big(f(H_0) - f(H)\Big)(H + \mathbf{i}) = \Big[f(H_0)(H_0 + \mathbf{i}) - f(H)(H + \mathbf{i})\Big] + f(H_0)V,$$

where on the right hand side we have a sum of two trace class operators. The same argument shows that the operator $H(f(H_0) - f(H))$ is trace class. Next we show that the operator $H(f(H_0) - f(H))(H + \mathbf{i})$ is trace class. To do this, we write

$$H(f(H_0) - f(H))(H + \mathbf{i}) = \Big(H_0 f(H_0)(H_0 + \mathbf{i}) - Hf(H)(H + \mathbf{i})\Big) + Vf(H_0)(H_0 + \mathbf{i})$$
$$+Vf(H_0)V + H_0 f(H_0)V$$

and the four operators on the right hand side are trace class. This implies that $HU_\tau(f(H_0) - f(H))(H + \mathbf{i})$ is trace class since the commutator $[H, U_\tau]$ is a bounded operator. After these preparations we write

$$(II_2) = HU_\tau(f(H_0) - f(H)) = U_\tau H(f(H_0) - f(H)) + [H, U_\tau](f(H_0) - f(H))$$

which obviously is trace class. Exploiting the trace class properties, we can write

$$(II_2) = \operatorname{tr}\Big[HU_\tau(f(H_0) - f(H)(H + \mathbf{i})(H + \mathbf{i})^{-1}\Big]$$
$$= \operatorname{tr}\Big[(H + \mathbf{i})^{-1}HU_\tau(f(H_0) - f(H))(H + \mathbf{i})\Big]$$
$$= \operatorname{tr}\Big((f(H_0) - f(H))(H + \mathbf{i})(H + \mathbf{i})^{-1}HU_\tau\Big) = \operatorname{tr}\Big((f(H_0) - f(H))HU_\tau\Big).$$

Combining the above expressions, we get

$$(I) + (II_1) + (II_2) = \operatorname{tr}\Big((H_0 - H)U_\tau f(H_0)\Big) + \operatorname{tr}\Big(f(H_0)(H - H_0)U_\tau\Big)$$
$$= \operatorname{tr}\Big(-VU_\tau f(H_0)\Big) + \operatorname{tr}\Big(U_\tau f(H_0)V\Big).$$

It remains to show that $\operatorname{tr}\Big(VU_\tau f(H_0)\Big) = \operatorname{tr}\Big(U_\tau f(H_0)V\Big)$. To do this, choose a function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ for $|(x, y)| \leq 1$. For $R > 0$ set

$$\chi_R(x, y) = \chi\Big(\frac{x}{R}, \frac{y}{R}\Big)$$

and consider

$$\operatorname{tr}\Big(VU_\tau f(H_0)\chi_R\Big) = \operatorname{tr}\Big(U_\tau f(H_0)V\chi_R\Big).$$

The operator χ_R converges strongly to identity as $R \to \infty$ and applying the well known property of trace class operators (see for instance, Proposition 1 in [4]), we conclude that

$$\operatorname{tr}\Big(VU_\tau f(H_0)\Big) = \operatorname{tr}\Big(U_\tau f(H_0)V\Big)$$

and the proof is complete. $\square$

Proof of Theorem 1. According to Proposition 1, we have

$$\operatorname{tr}\Big(U_\tau(f(H) - f(H_0))\Big) = -\operatorname{tr}\Big(\frac{V(x+\tau, y) - V(x,y)}{\tau} U_\tau f(H)\Big). \tag{2.3}$$

We take the limit $\tau \to 0$ and observe that

$$U_\tau \longrightarrow I,\ \frac{V(x+\tau, y) - V(x,y)}{\tau} U_\tau \longrightarrow \partial_x V$$

strongly. Since $(f(H) - f(H_0))$ is a trace class operator, applying once more the property of trace class operators, we get

$$\lim_{\tau \to 0} \operatorname{tr}\Big(U_\tau(f(H) - f(H_0))\Big) = \operatorname{tr}\,(f(H) - f(H_0)).$$

To treat the limit $\tau \to 0$ in the right hand term of (2.3), consider the function,

$$g_\delta(x,y) = \langle x \rangle^{-2-\delta} \langle y \rangle^{-1-\delta}$$

$\delta > 0$ being the constant of (1.1). Following Lemma 1, the operator $g_\delta(H_0 + \mathbf{i})^{-2}$ is trace class. Hence

$$g_\delta f(H) = g_\delta(f(H) - f(H_0)) + g_\delta(H_0 + \mathbf{i})^{-2}(H_0 + \mathbf{i})^2 f(H_0)$$

is also a trace class operator.

To treat the limit $\tau \to 0$, we use the representation

$$\Big(\frac{V(x+\tau, y) - V(x,y)}{\tau} g_\delta^{-1}\Big)\Big[g_\delta U_\tau g_\delta^{-1}]g_\delta f(H).$$

The operators in the brackets $\Big(...\Big)$, $[...]$ converge strongly as $\tau \to 0$ to $(\partial_x V)g_\delta^{-1}$ and I, respectively. Letting $\tau \to 0$, we obtain

$$\lim_{\tau \to 0} \operatorname{tr}\Big(\frac{V(x+\tau, y) - V(x,y)}{\tau}\Big) U_\tau f(H) = \operatorname{tr}\Big((\partial_x V) f(H)\Big)$$

and the proof is complete.

Remark 1. *The proof of Theorem 1 works for operators $M = (D_x - C(y))^2 + D_y^2 + \epsilon x + V(x,y)$ with non-linear $C(y)$ assuming that we have an analog of Lemma 1 for H and H_0 replaced by M and $M_0 = (D_x - C(y))^2 + D_y^2 + \epsilon x$, respectively. Also we may examine the operators in $\mathbb{R}^3$ having the form*

$$\Big(D_x + \frac{B}{2}y\Big)^2 + \Big(D_y - \frac{B}{2}x\Big)^2 + D_z^2 + \epsilon z + V(x,y,z)$$

applying the shift operator $U_\tau : f(x,y,z) \longrightarrow f(x,y,z+\tau)$. Some operators with magnetic potentials and Stark potential in $\mathbb{R}^n$, $n \geq 3$, can be investigated by the same approach.

Now consider the operators $H_0(h) = (hD_x - By)^2 + h^2 D_y^2 + \epsilon x$, $H(h) = H_0(h) + V(x, y)$, $h > 0$. Under the assumption (1.1) for $V(x, y)$ we have the statement of Lemma 1 for H_0 replaced by $H_0(h)$. Moreover, the operators $Vg(H(h))$ and $Vg(H_0(h))$ are trace class for every $g \in C_0^\infty(\mathbb{R})$. Thus for every $f \in C_0^\infty(\mathbb{R})$ the operator $f(H(h)) - f(H_0(h))$ is trace class and we can define the spectral shift function $\xi_h = \xi_h(\lambda, B, \epsilon)$ modulo constant by the formula

$$\langle \xi_h', f \rangle = \mathrm{tr}\,\Big(f(H(h) - f(H_0)\Big),\ f \in C_0^\infty(\mathbb{R}).$$

Under the assumption of Theorem 1, we obtain repeating the proof of (1.2) the representation

$$\mathrm{tr}\Big(f(H(h)) - f(H_0(h))\Big) = -\frac{1}{\epsilon}\mathrm{tr}\Big((\partial_x V) f(H(h))\Big). \tag{2.4}$$

3. Embedded eigenvalues of H

In this section we use the notation

$$L = H(0) = (D_x - By)^2 + D_y^2 + \epsilon x.$$

Our purpose is to prove the following

Theorem 2. *There exists $C > 0$ such that H has no eigenvalues λ, $|\lambda| \geq C$.*

Proof. First notice that for every function $f \in C_0^\infty(\mathbb{R})$ we have

$$f(H)[\partial_x, H] f(H) = \epsilon f^2(H) + f(H)\partial_x V f(H). \tag{3.1}$$

We will show the absence of embedded eigenvalues $\lambda > C > 0$. The case $\lambda < -C$ can be treated by the same argument. Assume that there exists a sequence of eigenvalues $\lambda_n \longrightarrow +\infty$, $\lambda_{n+1} > \lambda_n + 1$, $\forall n$ and let $H\varphi_n = \lambda_n \varphi_n$, $n \in \mathbb{N}$ with $(\varphi_i, \varphi_j) = \delta_{i,j}$. Choose cut-off functions $f_n(t) \in C_0^\infty(\mathbb{R})$ so that $f_n(\lambda_n) = 1$, $0 \leq f_n(t) \leq 1$ and $f_n(t) = 0$ for $|t - \lambda_n| \geq 1/2$. It is clear that $f_n(H)\varphi_n = \varphi_n$ and

$$(\varphi_n, f_n(H)[\partial_x, H] f_n(H)\varphi_n) = 0,\ \forall n \in \mathbb{N}.$$

We wish to prove that for n large enough we have

$$\Big|(\varphi_n, f_n(H)\partial_x V f_n(H)\varphi_n)\Big| = \Big|(\varphi_n, \partial_x V f_n(H)\varphi_n)\Big| \leq \epsilon/2 \tag{3.2}$$

which leads to a contradiction with (3.1) since $(\varphi_n, f_n^2(H)\varphi_n) = 1$. Consider the operator

$$f_n(H) = -\frac{1}{\pi}\int_{W_n} \bar{\partial}\tilde{f}_n(z)(z - H)^{-1} L(dz),$$

where $\tilde{f}_n(z)$ is an almost analytic continuation of f_n with $\mathrm{supp}\, \tilde{f}_n(z) \subset W_n$, $W_n = \{z \in \mathbb{C} : |z - \lambda_n| \leq 2/3\}$ is a complex neighborhood of λ_n and

$$\bar{\partial}\tilde{f}_n(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$$

uniformly with respect to n. Here $L(dz)$ is the Lebesgue measure in $\mathbb{C}$. We write

$$(\varphi_n, \partial_x V f_n(H)\varphi_n) = -\frac{1}{\pi}\int_{W_n \cap \{|\operatorname{Im} z| \leq \eta\}} \bar{\partial}\tilde{f}_n(z)(\varphi_n, (\partial_x V)(z - H)^{-1}\varphi_n) L(dz)$$

$$-\frac{1}{\pi}\int_{W_n \cap \{|\operatorname{Im} z| > \eta\}} \bar{\partial}\tilde{f}_n(z)(\varphi_n, (\partial_x V - V_0)(z - H)^{-1}\varphi_n) L(dz)$$

$$-\frac{1}{\pi}\int_{W_n\cap\{|\operatorname{Im} z|>\eta\}} \bar\partial \tilde f_n(z)(\varphi_n, V_0(z-H)^{-1}\varphi_n)L(dz) = R_n + Q_n + S_n,$$

where $V_0(x,y)\in C_0^\infty(\mathbb{R}^2)$. We choose $\eta>0$ small enough to arrange $|R_n|\le \epsilon/6$ for all $n\in\mathbb{N}$. Next we fix $0<\eta<1$ and we will estimate Q_n and S_n. For the resolvent $(z-L)^{-1}$ we will exploit the following

Proposition 2. ([6]) *Let f, g be bounded functions with compact support in $\mathbb{R}^2$. Then for every compact $\mathcal{K}\subset\mathbb{R}\setminus\{0\}$ we have*

$$\lim_{\lambda\to\pm\infty}\|f(\lambda+\mathbf{i}\gamma-L)^{-1}g\| = 0$$

uniformly for $\gamma\in\mathcal{K}$.

We choose V_0 so that $\|\partial_x V - V_0\|$ is sufficiently small in order to arrange $|Q_n|\le\epsilon/6$, $\forall n\in\mathbb{N}$. Now we pass to the estimation of S_n. We have

$$V_0(z-H)^{-1} = V_0(z-L)^{-1} + V_0(z-L)^{-1}(V-V_1)(z-H)^{-1} + V_0(z-L)^{-1}V_1(z-H)^{-1}. \quad (3.3)$$

We replace $V_0(z-H)^{-1}$ in S_n by the right hand side (3.3) choosing $V_1\in C_0^\infty(\mathbb{R}^2)$. For the term involving $(V-V_1)$ in (3.3) we take V_1 so that $\|V-V_1\|$ is small enough, to obtain a term bounded by $\epsilon/18$. Next we fix the potentials V_0, V_1 with compact support. By Proposition 2 setting $z=\lambda+\mathbf{i}\gamma$, $\eta\le|\gamma|\le 1$, we get

$$\|\bar\partial\tilde f_n(z)V_0(\lambda+\mathbf{i}\gamma-L)^{-1}V_1(H-z)^{-1}\|\le C_2\eta^{-1}\|V_0(\lambda+\mathbf{i}\gamma-L)^{-1}V_1\|\le\frac{9}{4\pi^2}\frac{\epsilon}{18}$$

for $\operatorname{Re} z=\lambda\ge C_{\epsilon,\eta}$. We choose $n\ge n_0=n_0(\epsilon,\eta)$, so that $\operatorname{Re} z\ge C_{\epsilon,\eta}$ for $z\in W_n$ and $n\ge n_0$. Thus we can estimate the term involving $V_0(z-L)^{-1}V_1$ in (3.3) by $\epsilon/18$. It remains to deal with the term containing $V_0(z-L)^{-1}$. Let $\psi(x,y)\in C_0^\infty(\mathbb{R}^2)$ be a cut-off function such that $\psi=1$ on the support of V_0. We write

$$\begin{aligned}\psi V_0(z-L)^{-1} &= V_0(z-L)^{-1}\psi - V_0(z-L)^{-1}[(D_x-By)^2+D_y^2,\psi](z-L)^{-1}\\ &= V_0(z-L)^{-1}\psi - V_0(z-L)^{-1}\psi_1[(D_x-By)^2+D_y^2,\psi](z-L)^{-1},\end{aligned}$$

where ψ_1 is a cut-off function equal to 1 on the support of ψ. For n large enough we will have $\operatorname{Re} z=\lambda\ge C'_{\epsilon,\eta}$ for $z\in\operatorname{supp} W_n$ and can treat $V_0(z-L)^{-1}\psi$ and $V_0(z-L)^{-1}\psi_1$ as above. On the other hand,

$$[(D_x-By)^2+D_y^2,\psi] = -2\mathbf{i}\partial_x\psi(D_x-By)-2\mathbf{i}\partial_y\psi D_y-\Delta_{x,y}\psi \quad (3.4)$$

and the operators $\partial_x\psi(D_x-By)(z-L)^{-1}$ and $\partial_y\psi D_y(z-L)^{-1}$ are bounded by $C\eta^{-1}$ for $|\operatorname{Im} z|\ge\eta$. Indeed, we have

$$(z-L) = (\mathbf{i}-L)^{-1}[I+(\mathbf{i}-z)(z-L)^{-1}]$$

and it suffices to show that $\partial_x\psi(D_x-By)(\mathbf{i}-L)^{-1}$ and $\partial_y\psi D_y(\mathbf{i}-L)^{-1}$ are bounded. Next, $(\mathbf{i}-L)^{-1}$ is a pseudodifferential operator and the symbol of the pseudodifferential operator $(D_x-By)(\mathbf{i}-L)^{-1}$ becomes

$$\frac{\xi-By}{\mathbf{i}-(\xi-By)^2-\eta^2-\epsilon x}-\frac{\mathbf{i}B\eta}{(\mathbf{i}-(\xi-By)^2-\eta^2-\epsilon x)^2}.$$

From the well known results for the L^2 boundedness of pseudodifferential operators (see [1]) we deduce that (3.4) is bounded by $C|\operatorname{Im} z|^{-1}$. Consequently, applying Proposition 2 once more, we can arrange the norm of the operator

$$V_0(z-L)^{-1}\psi_1[(D_x-By)^2+D_y^2,\psi](z-L)^{-1}$$

to be sufficiently small for $z \in W_n$, $|\operatorname{Im} z| \geq \eta$ and $n \geq n_1 > n_0$. Combining this with the previous estimates, we get $|S_n| \leq \epsilon/6$, hence $|R_n + Q_n + S_n| \leq \epsilon/2$ for n large enough. This implies (3.2) and the proof is complete. $\square$

Corollary 1. *Assume in addition to* (1.1) *that* $\partial_x^2 V \in C_0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Then* H *has at most finite number of embedded eigenvalues in* $\mathbb{R}$.

This result follows from Theorem 2 and Corollary 1 in [4] which guarantees that H has at most finite number of embedded eigenvalues in every interval $[a, b] \subset \mathbb{R}$. The conjecture is that H has no embedded eigenvalues on $\mathbb{R}$.

4. Asymptotics of the spectral shift function

Our purpose in this section is to apply Theorem 1 and (2.4) to give a Weyl type asymptotics with optimal remainder estimates for the spectral shift function $\xi_h(\lambda) := \xi(\lambda; H(h), H_0(h))$ corresponding to the operators

$$H(h) = (hD_x - y)^2 + h^2 D_y^2 + x,\ H_0(h) = H(h) + V(x, y),\ h > 0.$$

For simplicity of the exposition in this section we assume that $B = \epsilon = 1$. Let $p_2(x, y, \zeta, \eta) = (\zeta - y)^2 + \eta^2 + x + V(x, y)$. For the analysis of $\xi_h(\lambda)$ we need the following theorems.

Theorem 3. *Let* $\psi \in C_0^\infty(\mathbb{R}^2)$ *and let* $f \in C_0^\infty(]0, +\infty[; \mathbb{R})$. *Then we have*

$$\operatorname{tr}\Big[\psi f(H(h))\Big] \sim \sum_{j=0}^{\infty} a_j h^{j-2},\ h \searrow 0, \tag{4.1}$$

with

$$a_0 = \frac{1}{(2\pi)^2} \iint \psi(x, y) f(p_2(x, y, \zeta, \eta)) dx dy d\zeta d\eta. \tag{4.2}$$

Theorem 4. *Assume that* $\psi \in C_0^\infty(\mathbb{R}^2)$. *Let* $f \in C_0^\infty([E_0, E_1[)$ *and* $\theta \in C_0^\infty(] - \frac{1}{C_0}, \frac{1}{C_0}[; \mathbb{R})$, $\theta = 1$ *in a neighborhood of* 0. *Assume that if* $p_2(x, y, \zeta, \eta) = \tau$, $\tau \in [E_0, E_1]$, *then* $dp_2 \neq 0$. *Then there exists* $C_0 > 0$ *such that for all* $N, m \in \mathbb{N}$ *there exists* $h_0 > 0$ *such that*

$$\operatorname{tr}\Big(\psi \breve{\theta}_h(\tau - H(h)) f(H(h))\Big) = (2\pi h)^{-2}\Big(f(\tau)\sum_{j=0}^{N-1} \gamma_j(\tau) h^j + \mathcal{O}(h^N \langle \tau \rangle^{-m})\Big), \tag{4.3}$$

uniformly with respect to $\tau \in \mathbb{R}$ *and* $h \in]0, h_0]$, *where*

$$\gamma_0(\tau) = -(2\pi \mathbf{i})^{-1} \int\int_{\mathbb{R}^4} \psi(x, y)\Big((\tau + \mathbf{i}0 - p_2(x, y, \zeta, \eta))^{-1} - (\tau - \mathbf{i}0 - p_2(x, y, \zeta, \eta))^{-1}\Big) dx dy d\zeta d\eta.$$

Here

$$\breve{\theta}_h(\tau) = (2\pi h)^{-1} \int e^{i\tau t/h} \theta(t) dt.$$

Proof of Theorem 3 *and Theorem* 4. Here and below $\psi \prec \varphi$ means that $\varphi(x) = 1$ on the support of ψ. Let $G \in C_0^\infty(\mathbb{R}^2)$ with $\psi \prec G$. Introduce the operator

$$\tilde{H}(h) = (hD_x - G(x, y)y)^2 + h^2 D_y^2 + G(x, y)x + V(x, y),$$

and set

$$I = \operatorname{tr}\Big[\psi\Big(f(H(h)) - f(\tilde{H}(h))\Big)\Big].$$

Let $\tilde f(z) \in C_0^\infty(\mathbb{C})$ be an almost analytic continuation of f with $\bar\partial_z \tilde f(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$. From Helffer-Sjöstrand formula it follows that

$$I = \frac{1}{\pi}\int \bar\partial_z \tilde f(z) \mathrm{tr}\left[\psi\Big((z-\tilde H(h))^{-1} - (z-H(h))^{-1}\Big)\right] L(dz),$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}$.

Let $\psi_1 \in C^\infty(\mathbb{R}^2)$ be a function with $\psi_1 = 1$ near supp $(1-G)$ and $\psi_1 = 0$ near supp ψ, and let $\tilde\psi \in C_0^\infty(\mathbb{R}^2)$ be equal to one near supp$(\nabla\psi_1)$ and $\tilde\psi = 0$ near supp ψ. A simple calculus shows that $\tilde H(h) - H(h) = \psi_1(\tilde H(h) - H(h))$ and $[\tilde H(h), \psi_1] = \tilde\psi[\tilde H(h), \psi_1]\tilde H$. Then

$$\begin{aligned}\psi\Big((z-\tilde H(h))^{-1} - (z-H(h))^{-1}\Big) &= \psi(z-\tilde H(h))^{-1}\psi_1(\tilde H(h) - H(h))(z-H(h))^{-1} \\ &= \psi(z-\tilde H(h))^{-1}\tilde\psi[\tilde H(h),\psi_1](z-\tilde H(h))^{-1}(\tilde H(h)-H(h))(z-H(h))^{-1}.\end{aligned} \tag{4.4}$$

Let $\chi_1, ..., \chi_N \in C_0^\infty(\mathbb{R}^2; [0,1])$ with $\psi_1 \prec \chi_1 \prec ... \prec \chi_N$ and $\chi_i\tilde\psi = 0$, $i = 1, ..., N$. By using the equalities $\chi_1\psi_1 = ... = \chi_N\psi_1 = \psi_1$, $\chi_k\,\tilde\psi = 0$, $\chi_{k-1}[\chi_k, \tilde H(h)] = 0$ and the fact that

$$[\chi_k, (z-\tilde H(h))^{-1}] = (z-\tilde H(h))^{-1}[\chi_k, \tilde H(h)](z-\tilde H(h))^{-1},$$

we get

$$\begin{aligned}&\psi(z-\tilde H(h))^{-1}\tilde\psi[\tilde H(h),\psi_1] \\ &= \psi(z-\tilde H(h))^{-1}[\chi_1,\tilde H(h)](z-\tilde H(h))^{-1}...[\chi_N,\tilde H(h)](z-\tilde H(h))^{-1}\tilde\psi[\tilde H(h),\psi_1] =: L_N(h).\end{aligned}$$

Here

$$L_N(h) = \mathcal{O}_N(1)\Big(\frac{h^N}{|\operatorname{Im} z|^N}\Big) : H^s(\mathbb{R}^2) \to H^{s+N}(\mathbb{R}^2),$$

where we equip $H^N(\mathbb{R}^2)$ with the h-dependent norm $\|\langle hD\rangle^N u\|_{L^2}$. Choose $N > 2$ and let $s = -N$. According to Theorem 9.4 of [1], we have

$$\Big\|\Big(-h^2\Delta + 1\Big)^{-N/2}\tilde\psi\Big\|_{\mathrm{tr}} = \mathcal{O}(h^{-2}).$$

Then

$$\begin{aligned}\|\psi(z-\tilde H(h))^{-1}\tilde\psi[\tilde H(h),\psi_1]\tilde\psi\|_{\mathrm{tr}} &= \Big\|L_N(h)\Big(-h^2\Delta+1\Big)^{N/2}\Big(-h^2\Delta+1\Big)^{-N/2}\tilde\psi\Big\|_{\mathrm{tr}} \\ &\le C\Big\|\Big(-h^2\Delta+1\Big)^{-N/2}\tilde\psi\Big\|_{\mathrm{tr}}\Big(\frac{h^N}{|\operatorname{Im} z|^N}\Big) \le C_1\Big(\frac{h^{N-2}}{|\operatorname{Im} z|^N}\Big).\end{aligned} \tag{4.5}$$

Combining this with (4.4) and using the fact that

$$\|(z-\tilde H(h))^{-1}(\tilde H(h)-H(h))(z-H(h))^{-1}\| = \|(z-\tilde H(h))^{-1} - (z-H(h))^{-1}\| = \mathcal{O}\Big(|\operatorname{Im} z|^{-1}\Big),$$

we obtain

$$\Big\|\psi\Big((z-\tilde H(h))^{-1} - (z-H(h))^{-1}\Big)\Big\|_{\mathrm{tr}} = \mathcal{O}\Big(\frac{h^{N-2}}{|\operatorname{Im} z|^{N+1}}\Big).$$

Since $\bar\partial_z\tilde f(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, we have

$$I = \mathcal{O}(h^\infty).$$

Summing up, we have proved that

$$\mathrm{tr}\,\Big(\psi f(H(h))\Big) = \mathrm{tr}\,\Big(\psi f(\tilde H(h))\Big) + \mathcal{O}(h^\infty). \tag{4.6}$$

In the same way, we obtain

$$\operatorname{tr}\left(\psi\breve{\theta}_h(\tau - H(h))f(H(h))\right) = \operatorname{tr}\left(\psi\breve{\theta}_h(\tau - \tilde{H}(h))f(\tilde{H}(h))\right) + \mathcal{O}(h^\infty). \tag{4.7}$$

The operator $\tilde{H}(h)$ is elliptic semi-bounded h-pseudodifferential operator, so Theorem 3 and Theorem 4 follow from the h-pseudodifferential calculus and the analysis of elliptic operators in Chapters 8, 9, 12 in [1] (see also [15]). The calculus of the leading terms is given by trivial modification of the argument of Section 7 in [2] and we omit the details. □

Remark 2. *Notice that $dp_2 \neq 0$ on $p_2 = \tau$ is equivalent to*

$$\nabla_{x,y}(x + V(x,y)) \neq 0, \ \text{ on } \ \{(x,y);\ \ x + V(x,y) = \tau\}. \tag{4.8}$$

Now we will apply Theorem 3 and Theorem 4 to obtain a Weyl-type asymptotics for $\xi_h(\lambda)$ when $h \searrow 0$.

Theorem 5. *Assume that $V \in C_0^\infty(\mathbb{R}^2)$ and suppose that (4.8) holds for $\tau = \lambda_1, \lambda_2$. Then there exists $h_0 > 0$ such that for $h \in]0, h_0]$ we have*

$$\xi_h(\lambda_2) - \xi_h(\lambda_1) = (2\pi h)^{-2}(c_0(\lambda_2) - c_0(\lambda_1)) + \mathcal{O}(h^{-1}), \tag{4.9}$$

where

$$c_0(\lambda) = -\pi \int_{\mathbb{R}^2} \partial_x V(x,y)(\lambda - x - V(x,y))_+ dxdy. \tag{4.10}$$

Proof. Choose a large constant M such that

$$M \geq \|\partial_x V\|_\infty + 1.$$

Let $\psi \in C_0^\infty(\mathbb{R}^2; [0,1])$ with $\partial_x V \prec \psi^2$. According to (2.4), by using the cyclicity of the trace, we get

$$\begin{aligned} \langle \xi_h', f\rangle = \operatorname{tr}\left(f(H(h)) - f(H_0(h)\right) &= -\operatorname{tr}\Big((\partial_x V) f(H(h))\Big) \\ = \operatorname{tr}\Big((M - \partial_x V)^{1/2}\psi f(H(h)\psi(M - \partial_x V)^{1/2}\Big) &- M\operatorname{tr}\Big(\psi f(H(h))\psi\Big) \\ &=: \langle \xi_1', f\rangle - \langle \xi_2', f\rangle. \end{aligned}$$

Since

$$f \to \operatorname{tr}\Big((M - \partial_{x_1} V)^{1/2}\psi f(H(h)\psi(M - \partial_{x_1} V)^{1/2}\Big)$$

and

$$f \to M\operatorname{tr}\Big(\psi f(H(h))\psi\Big)$$

are positive functions for $f \geq 0$, we deduce that the functions $\lambda \to \xi_i(\lambda)$, $i = 1, 2$ are monotonic.

Consequently, we may apply Tauberian arguments for the analysis of the asymptotics of $\xi_i(\lambda)$, $i = 1, 2$. We treat below $\xi_2(\lambda)$. Let $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geq 0$, and suppose that (4.8) holds for all $\tau \in \operatorname{supp}\varphi$. Consider the function

$$F_\varphi(\lambda) = \int_{-\infty}^{\lambda} \xi_2'(\mu)\varphi(\mu)d\mu.$$

Applying (4.3) with $N = 1$ and $m = 2$, we obtain

$$\frac{d}{d\lambda}(\breve{\theta}_h * F_\varphi)(\lambda) = \int \breve{\theta}_h(\lambda - \mu)\xi_2'(\mu)\varphi(\mu)d\mu = (2\pi h)^{-2}\Big(\varphi(\lambda)\gamma_0(\lambda) + \mathcal{O}\Big(\frac{h}{\langle\lambda\rangle^2}\Big)\Big). \tag{4.11}$$

We integrate from $-\infty$ to λ and we get

$$\int \Big(\int_{-\infty}^{\lambda} \breve{\theta}_h(\lambda' - \mu) d\lambda' \Big) \xi_2'(\mu) \varphi(\mu) d\mu \tag{4.12}$$
$$= \frac{1}{(2\pi h)^2} \Big(\int \int_{p_2 \leq \lambda} M \psi^2(x,y) \varphi(p_2) dx dy d\eta d\zeta + \mathcal{O}(h) \Big).$$

In the following we choose $\theta \in C_0^\infty(\mathbb{R})$ with $\breve{\theta}_h \geq 0$. Let $h\breve{\theta}_h(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \theta(u) du \geq 2C_1 > 0$. Therefore, it follows that there exist $C_2 > 0$ such that

$$|t| < \frac{h}{C_2} \Longrightarrow h\breve{\theta}_h(t) \geq C_1.$$

Combining this with the fact that $\breve{\theta}_h \geq 0$, and using $\langle \xi_2', f \rangle \geq 0$ for $f \geq 0$, we obtain

$$C_1 \int_{\{|\lambda - \mu| < \frac{h}{C_0}\}} \xi_2'(\mu) \varphi(\mu) d\mu \leq h \int_{\{|\lambda - \mu| < \frac{h}{C_0}\}} \breve{\theta}_h(\lambda - \mu) \xi_2'(\mu) \varphi(\mu) d\mu$$
$$\leq h \int_{\mathbb{R}} \breve{\theta}_h(\lambda - \mu) \xi_2'(\mu) \varphi(\mu) d\mu = h \frac{d}{d\lambda} (\breve{\theta}_h * F_\varphi)(\lambda) = \mathcal{O}(h^{-1}), \tag{4.13}$$

uniformly with respect to $\lambda \in \mathbb{R}$. On the other hand, a simple calculus shows that

$$\int_{-\infty}^{\lambda} \breve{\theta}_h(\lambda' - \mu) d\lambda' = \int_{-\infty}^{\frac{\lambda - \mu}{h}} \breve{\theta}_1(t) dt = \mathbf{1}_{]-\infty, \lambda[}(\mu) + \mathcal{O}\Big(\langle \frac{\lambda - \mu}{h} \rangle^{-\infty} \Big). \tag{4.14}$$

Indeed, for $\mu < \lambda$ and all $k \in \mathbb{N}$ we have

$$\int_{-\infty}^{\frac{\lambda - \mu}{h}} \breve{\theta}_1(t) dt - 1 = - \int_{\frac{\lambda - \mu}{h}}^{\infty} t^k \breve{\theta}_1(t) \frac{1}{t^k} dt$$

and

$$\int_{\frac{\lambda - \mu}{h}}^{\infty} t^k \breve{\theta}_1(t) \frac{1}{t^k} dt \leq \Big(\frac{\lambda - \mu}{h} \Big)^{-k} \int_{\mathbb{R}} t^k \breve{\theta}_1(t) dt.$$

A similar argument works for $\mu > \lambda$. From (4.13) we have for $k \geq 2$ the estimate

$$\int_{\mathbb{R}} \langle \frac{\lambda - \mu}{h} \rangle^{-k} \xi_2'(\mu) \varphi(\mu) d\mu = \sum_{m=-\infty}^{\infty} \int_{\frac{m}{C_0} \leq \frac{\mu - \lambda}{h} < \frac{m+1}{C_0}} \langle \frac{\lambda - \mu}{h} \rangle^{-k} \xi_2'(\mu) \varphi(\mu) d\mu \tag{4.15}$$
$$\leq \sum_{m=0}^{\infty} \Big(1 + \Big(\frac{m}{C_0} \Big)^2 \Big)^{-k/2} \int_{\lambda + \frac{mh}{C_0}}^{\lambda + \frac{(m+1)h}{C_0}} \xi_2'(\mu) \varphi(\mu) d\mu$$
$$+ \sum_{m=-\infty}^{-1} \Big(1 + \Big(\frac{|m+1|}{C_0} \Big)^2 \Big)^{-k/2} \int_{\lambda + \frac{mh}{C_0}}^{\lambda + \frac{(m+1)h}{C_0}} \xi_2'(\mu) \varphi(\mu) d\mu \leq \sum_{m=-\infty}^{\infty} \frac{1}{(C_0 + |m|)^k} \mathcal{O}(h^{-1}),$$

where in the last inequality at the right hand side we used the fact that (4.13) holds uniformly with respect to $\lambda \in \mathbb{R}$ and we can estimate the integrals involving $\xi_2'(\mu)\varphi(\mu)$ by $\mathcal{O}(h^{-1})$ uniformly with respect to m.

Inserting the right hand side of (4.14) in the left hand side of (4.12) and using (4.15), we get

$$F_\varphi(\lambda) = (2\pi h)^{-2} \Big(\int \int_{p_2 \leq \lambda} M \psi^2(x,y) \varphi(p_2) dx dy d\eta d\zeta + \mathcal{O}(h) \Big).$$

We apply the same argument for $\xi_1(h)$ and introduce the function

$$G_\varphi(\lambda) = \int_{-\infty}^{\lambda} \xi_1'(\mu)\varphi(\mu)d\mu.$$

Replacing the function ψ by $(M - \partial_x V)^{1/2}\psi$, we get

$$G_\varphi(\lambda) = \frac{1}{(2\pi h)^2}\Big(\int\int_{p_2 \leq \lambda} (M - \partial_x V)\psi^2(x,y)\varphi(p_2)dxdyd\eta d\zeta + \mathcal{O}(h)\Big).$$

Since $\xi_h = \xi_1 - \xi_2$, the above results yield

$$M_\varphi(\lambda) = \int_{-\infty}^{\lambda} \xi_h'(\mu)\varphi(\mu)d\mu = \frac{1}{(2\pi h)^2}\Big(\int\int_{p_2\leq\lambda} -\partial_x V(x,y)\varphi(p_2)dxdyd\eta d\zeta + \mathcal{O}(h)\Big). \tag{4.16}$$

Now, we are ready to prove Theorem 5. Assume that $\lambda_1 < \lambda_2$, and let $\epsilon > 0$ be small enough. Let $\varphi_1, \varphi_2, \varphi_3 \in C_0^\infty(]\lambda_1 - \epsilon, \lambda_2 + \epsilon[)$ with $\varphi_1 + \varphi_2 + \varphi_3 = 1$ on $[\lambda_1, \lambda_2]$, $\operatorname{supp}\varphi_1 \subset]\lambda_1 - \epsilon, \lambda_1 + \epsilon[$, $\operatorname{supp}\varphi_2 \subset]\lambda_2 - \epsilon, \lambda_2 + \epsilon[$ and $\operatorname{supp}\varphi_3 \subset]\lambda_1, \lambda_2[$. We choose ϵ small enough so that (4.8) holds for all $\tau \in]\lambda_1 - \epsilon, \lambda_1 + \epsilon[\cup]\lambda_2 - \epsilon, \lambda_2 + \epsilon[$. We write

$$\xi_h(\lambda_2) - \xi_h(\lambda_1) = \int_{\lambda_1}^{\lambda_2} (\varphi_1 + \varphi_2 + \varphi_3)(\lambda)\xi_h'(\lambda)d\lambda$$

$$= M_{\varphi_2}(\lambda_2) + M_{\varphi_1}(\lambda_2) - M_{\varphi_2}(\lambda_1) - M_{\varphi_1}(\lambda_1) - \operatorname{tr}(\partial_x V\varphi_3(H)),$$

where for the function φ_3 we have exploited (2.4). Next for the term involving φ_3 we apply Theorem 3 and obtain

$$\operatorname{tr}(\partial_x V\varphi_3(H)) = \frac{1}{(2\pi h)^2}\int\int \partial_x V\varphi_3(p_2)dxdyd\zeta d\eta + \mathcal{O}(h^{-1}).$$

For $M_{\varphi_1}(\lambda_i)$ and $M_{\varphi_2}(\lambda_i)$, $i = 1, 2$, we exploit the above argument and we deduce the asymptotics taking into account (4.16). Summing the terms involving φ_j, $j = 1, 2, 3$, we conclude that

$$\xi_h(\lambda_2) - \xi_h(\lambda_1) = (2\pi h)^{-2}d(\lambda_2, \lambda_1) + \mathcal{O}(h^{-1}).$$

For the leading term we have

$$d(\lambda_2, \lambda_1) = \int\int_{\lambda_1 \leq p_2 \leq \lambda_2} -\partial_x V(x,y)\Big(\varphi_1(p_2) + \varphi_2(p_2) + \varphi_3(p_2)\Big)dxdyd\zeta d\eta$$

$$= -\int\int_{p_2\leq\lambda_2} \partial_x V(x,y)dxdyd\zeta d\eta + \int\int_{p_2\leq\lambda_1} \partial_x V(x,y)dxdyd\zeta d\eta.$$

Finally, notice that

$$c_0(\lambda) = -\int\int_{p_2\leq\lambda} \partial_x V(x,y)dxdyd\zeta d\eta = -\int_{\mathbb{R}^2} \partial_x V(x,y)\Big(\int_{(\zeta-y)^2+\eta^2\leq(\lambda-x-V(x,y))_+} d\zeta d\eta\Big)dxdy$$

$$= -\pi\int_{\mathbb{R}^2} \partial_x V(x,y)(\lambda - x - V(x,y))_+ dxdy$$

and the proof of Theorem 5 is complete. $\square$

Remark 3. *If $\lambda \gg 1$ is large enough (resp. $\lambda \ll -1$) then on* supp $(\partial_x V)$, *we have*

$$(\lambda - x - V)_+ = \lambda - x - V, \ \text{(resp. } (\lambda - x - V)_+ = 0).$$

Consequently,

$$c_0(\lambda) = -\pi \int_{\mathbb{R}^2} V(x,y)dxdy, \quad \text{for } \lambda \gg 1,$$

and

$$c_0(\lambda) = 0, \ \text{ for } \lambda \ll -1.$$

In particular, if we normalize $\xi_h(\lambda)$ by $\lim_{\lambda \to -\infty} \xi_h(\lambda) = 0$, we get

$$\xi_h(\lambda) = (2\pi h)^{-2} c_0(\lambda) + \mathcal{O}(h^{-1}).$$

Remark 4. *The results of this section can be generalized for potentials $V(x,y)$ for which there exists $\delta_1 \in \mathbb{R}$ such that supp $V \subset \{(x,y) \in \mathbb{R}^2 : x \geq \delta_1\}$ by using the techniques in [2]. For simplicity we treated the case of $V \in C_0^\infty(\mathbb{R}^2)$ to avoid the complications caused by the calculus of pseudodifferential operators.*

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