ON THE NONLINEAR WAVE EQUATION WITH TIME PERIODIC POTENTIAL

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Abstract. It is known that for some time periodic potentials $q(t,x) \geq 0$ having compact support with respect to x some solutions of the Cauchy problem for the wave equation $\partial_t^2 u - \Delta_x u + q(t,x)u = 0$ have exponentially increasing energy as $t \to \infty$. We show that if one adds a nonlinear defocusing interaction $|u|^r u, 2 \leq r < 4$, then the solution of the nonlinear wave equation exists for all $t \in \mathbb{R}$ and its energy is polynomially bounded as $t \to \infty$ for every choice of q. Moreover, we prove that the zero solution of the nonlinear wave equation is instable if the corresponding linear equation has the property mentioned above.

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1. Introduction

Our goal in this paper is to show that a defocusing nonlinear interaction may improve, in a certain sense, the long time properties of the solutions of the wave equation with a time periodic potential.

Consider the Cauchy problem with potential perturbation of the classical wave equation in the Euclidean space $\mathbb{R}^3$

$$\partial_t^2 u - \Delta_x u + q(t,x)u = 0, \quad u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x). \tag{1.1}$$

Throughout this paper $0 \leq q(t,x) \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ is periodic in time t with period $T > 0$ and has a compact support with respect to x included in $\{x \in \mathbb{R}^3 : |x| \leq \rho\}$, for some positive ρ. It is easy to show that the Cauchy problem (1.1) is globally well-posed in $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. The analysis of the long time behavior of the solution of (1.1) may be quite intricate (see e.g. [6, 1]). A slight adaptation of the arguments presented in [1] leads the following result.

Theorem 1. *There exist q and $(f_1, f_2) \in \mathcal{H}$ such that the solution of* (1.1) *satisfies :*

$$\exists\, C > 0,\ \exists\, \alpha > 0 \quad \text{such that} \quad \forall\, t \geq 0, \quad \|u(t,\cdot)\|_{H^1(\mathbb{R}^3)} \geq C\, e^{\alpha t}\,. \tag{1.2}$$

The above result has been established in [1] for the Cauchy problem with initial data in the energy space $H = H_D(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with norm

$$\|f\|_0 = \left(\|f_1\|_{H_D}^2 + \|f_2\|_{L^2}^2\right)^{1/2}, \quad f = (f_1, f_2),$$

where $H_D(\mathbb{R}^3)$ is the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|f\|_{H_D} = \|\nabla_x f\|_{L^2(\mathbb{R}^3)}$.

In fact we show that the propagator of (1.1)

$$V(T,0) : \mathcal{H} \ni (f_1(x), f_2(x)) \longrightarrow (u(T,x), u_t(T,x)) \in \mathcal{H}$$

has an eigenvalue $y, |y| > 1$ which implies (1.2).

Our purpose is to show that adding a nonlinear perturbation to (1.1) forbids the existence of solutions satisfying (1.2). Consider therefore the following Cauchy problem

$$\partial_t^2 u - \Delta_x u + q(t,x)u + |u|^r u = 0, \quad u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x)\,, \tag{1.3}$$

where $2 \le r < 4$. We have the following statement.

Theorem 2. *For any choice of q the Cauchy problem (1.3) is globally well-posed in $\mathcal{H}$. Moreover, for every $(f_1, f_2) \in \mathcal{H}$ there exists a constant $C > 0$ such that for every $t \in \mathbb{R}$, the solution of (1.3) satisfies the polynomial bound*

$$\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \le 2\Big(X(0)^{\frac{r}{r+2}} + C|t|\Big)^{\frac{r+2}{2r}},$$

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \le \|f_1\|_{L^2(\mathbb{R}^3)} + 2|t|\Big(X(0)^{\frac{r}{r+2}} + C|t|\Big)^{\frac{r+2}{2r}},$$

where

$$X(t) = \int_{\mathbb{R}^3} \big(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{2}q|u|^2 + \frac{1}{r+2}|u|^{r+2}\big)dx$$

and $C > 0$ depends only on q and r.

By global well-posedness we mean the existence, the uniqueness and the continuous dependence with respect to the data. The proof of Theorem 2 is based on the equality

$$X'(t) = \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3} (\partial_t q)|u|^2 dx \tag{1.4}$$

and the estimate

$$|X'(t)| \le C X^{1-\frac{r}{r+2}}(t).$$

It is classical to expect that the result of Theorem 1 implies the instability of the zero solution of (1.3). More precisely, we have the following instability result.

Theorem 3. *With q as in Theorem 1 the following holds true. There is $\eta > 0$ such that for every $\delta > 0$ there exists $(f_1, f_2) \in \mathcal{H}$, $\|(f_1, f_2)\|_{\mathcal{H}} < \delta$ and there exists $n = n(\delta) > 0$ such that the solution of (1.3) satisfies $\|(u(nT,\cdot), \partial_t u(nT,\cdot)\|_{\mathcal{H}} > \eta$.*

We are not aware of any nontrivial choice of $(f_1, f_2) \in \mathcal{H}$ such that the solution $u(t,x)$ of (1.3) and $u_t(t,x)$ remain uniformly bounded in $\mathcal{H}$ for all $t \ge 0$. The paper is organized as follows. In the next section, we prove Theorem 1. The third section is devoted to the proof of Theorem 2. First we obtain a local existence and uniqueness result on intervals $[s, s+\tau]$ with $\tau = c(1 + \|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma}$ with constants $c > 0$ and $\gamma > 0$ independent on f. Next we establish (1.4) for solutions

$$u(t,x) \in C([0,A], H^2_x(\mathbb{R}^3)) \cap C^1([0,A], H^1_x(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0,A], L_x^{2r+2}(\mathbb{R}^3))$$

and finally, by a local approximation in small intervals we justify (1.4) for every fixed $A > 0$ and $0 \le t \le A$. In the fourth section, we prove Theorem 3 passing to a system

$$w_{n+1} = \mathcal{F}(w_n),\ n \ge 0,$$

where $\mathcal{F} = \mathcal{U}(0,T)$ is the propagator of the nonlinear equation. In the fifth section we discuss the generalizations concerning the nonlinear equations

$$\partial_t^2 u - \Delta_x u + |u|^r u + \sum_{j=0}^{r-1} q_j(t,x)|u|^j u = 0,\ r = 2,3$$

with time-periodic functions $q_j(t+T_j, x) = q_j(t,x) \ge 0$, $j = 0, 1, r-1$ having compact support with respect to x.

2. Proof of Theorem 1

2.1. **The linear wave equation with time periodic potential.** Let $u(t, x; s)$ be the solution of the Cauchy problem

$$\partial_t^2 u - \Delta_x u + q(t, x)u = 0,\ u(s, x) = f_1(x),\ \partial_t u(s, x) = f_2(x) \tag{2.1}$$

with $f = (f_1, f_2) \in H$. Therefore the operator

$$H \ni f \to U(t, s)f = (u(t, x; s), \partial_t u(t, x; s)) \in H$$

is called the propagator (monodromy operator) of (2.1) and there exist $C > 0$ and $\alpha \geq 0$ so that

$$\|U(t, s)f\|_0 \leq Ce^{\alpha|t-s|}\|f\|_0. \tag{2.2}$$

Let $U_0(t - s)f = (u_0(t, x; s), \partial_t u_0(t, x; s))$, where u_0 solves $\partial_t^2 u_0 - \Delta_x u_0 = 0$ with initial data f for $t = s$. Then we have

$$U(t, s)f - U_0(t - s)f = -\int_s^t U_0(t - \tau)Q(\tau)U(\tau, s)f d\tau, \tag{2.3}$$

where

$$U_0(t) = \begin{pmatrix} \cos(t\sqrt{-\Delta}) & \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \\ -\sqrt{-\Delta}\sin(t\sqrt{-\Delta}) & \cos(t\sqrt{-\Delta}) \end{pmatrix}, \quad Q(t) = \begin{pmatrix} 0 & 0 \\ q(t, x) & 0 \end{pmatrix}.$$

Using the relation (2.3) and the compact support of q, allows us to obtain the estimate

$$\|U(t, s)f - U_0(t - s)f\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq C\|U(t, s)f\|_0\,.$$

Moreover the support property of q also yields

$$\text{supp}_x\,(U(t, s)f - U_0(t - s)f) \subset \{|x| \leq \rho + |t - s|\}\,.$$

Consequently $U(t, s)$ is a *compact perturbation* of the unitary operator $U_0(t - s)$.

Now consider the space $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \subset H$ with norm

$$\|f\|_1 = \left(\|f_1\|^2_{H^1(\mathbb{R}^3)} + \|f_2\|^2_{L^2(\mathbb{R}^3)}\right)^{1/2}, \quad \|f_1\|^2_{H^1(\mathbb{R}^3)} = \|\nabla_x f_1\|^2_{L^2(\mathbb{R}^3)} + \|f_1\|^2_{L^2(\mathbb{R}^3)}.$$

The map $U_0(t)$ *is not unitary* in $\mathcal{H}$. However, one easily checks that

$$\|U_0(t)f\|_1 \leq C(1 + |t|)\|f\|_1, \quad \forall\, t \in \mathbb{R},$$

with a constant $C > 0$ independent of t. Consequently, the spectral radius of the operator $U_0(T) : \mathcal{H} \to \mathcal{H}$ is not greater than 1.

By using (2.3), it is easy to show by a fixed point theorem that for small $t_0 > 0$ and $s \leq t \leq s + t_0$ we have a local solution $(v(t, x; s), \partial_t v(t, x; s)) \in \mathcal{H}$ of the Cauchy problem (2.1) with initial data $f \in \mathcal{H}$. For this solution one deduces

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left(|\partial_t v(t, x; s)|^2 + |\nabla_x v(t, x; s)|^2 + |v(t, x; s)|^2\right)dx = -2\text{Re}\int_{\mathbb{R}^3} qv\overline{\partial_t v}dx + 2\text{Re}\int_{\mathbb{R}^3} v\overline{\partial_t v}dx$$

which yields

$$\frac{d}{dt}\|(v(t, x; s), \partial_t v(t, x; s))\|_1^2 \leq C_1\|(v(t, x; s), \partial_t v(t, x; s))\|_1^2$$

with a constant $C_1 > 0$ independent of f and s. The last inequality implies an estimate

$$\|(v(t, x; s), \partial_t v(t, x; s))\|_1 \leq C_2 e^{\beta|t-s|}\|f\|_1, \quad s \leq t \leq s + t_0,\ \beta \geq 0. \tag{2.4}$$

By a standard argument this leads to a global existence of a solution of (2.1). Introduce the propagator

$$\mathcal{H} \ni f \mapsto V(t, s)f = (v(t, x; s), \partial_t v(t, x; s)) \in \mathcal{H}$$

corresponding to the Cauchy problem (1.1) with initial data $f \in \mathcal{H}$. For $V(t,s)$ we obtain an estimate similar to (2.2). As in Section 5 in [6], it is easy to see that we have the following properties

$$U(t,s) \circ U(s,r) = U(t,r),\ U(s,s) = \mathrm{Id},\ U(t+T, s+T) = U(t,s), \quad t,s,r \in \mathbb{R}.$$

The same properties hold for the propagator $V(t,s)$. In particular, $V(T,0) = V((k+1)T, kT)$, $k \in \mathbb{N}$ and $V(nT,0) = (V(T,0))^n$.

As above notice that $V(t,s) - U_0(t-s)$ is a compact operator in $\mathcal{L}(\mathcal{H})$. For $|z| \gg 1$ we have

$$(V(T,0) - zI)^{-1} = (U_0(T) - zI)^{-1} - (U_0(T) - zI)^{-1}\big(V(T,0) - U_0(T)\big)(V(T,0) - zI)^{-1},$$

hence

$$\big[I + (U_0(T) - zI)^{-1}\big(V(T,0) - U_0(T)\big)\big](V(T,0) - zI)^{-1} = (U_0(T) - zI)^{-1}.$$

Set $K(z) = I + (U_0(T) - zI)^{-1}\big(V(T,0) - U_0(T)\big)$. For $|z|$ large enough $K(z)$ is invertible. By the analytic Fredholm theorem for $|z| \geq 1 + \delta > 1$ the operator $K(z)$ is invertible outside a discreet set and the inverse $K^{-1}(z)$ is a meromorphic operator-valued function. Consequently, the operator $V(T,0) \in \mathcal{L}(\mathcal{H})$ has in the open domain $|z| > 1$ a discrete set of eigenvalues with finite multiplicities which could accumulate only to the circle $|z| = 1$.

2.2. **Extending the result of** [1] **to** $\mathcal{H}$. In [1] it was proved that there are potentials $q(t,x) \geq 0$ for which the operator $U(T,0) : H \to H$ has an eigenvalue $z, |z| > 1$. In this paper we deal with the operator $V(T,0) : \mathcal{H} \to \mathcal{H}$ and it is not clear if the eigenfunction $\psi \in H$ with eigenvalues z constructed in [1] belongs to $\mathcal{H}$.

Below we make some modifications on the argument of [1] in order to show that for the potential constructed in [1] the corresponding operator $V(T,0) : \mathcal{H} \to \mathcal{H}$ has an eigenvalue $y, |y| > 1$. For convenience we will use the notations in [1] and we recall some of them. The potential in [1] has the form $V^\epsilon(t,x) := b^\epsilon(x) + q(t)\chi^\delta(x)$ with $\epsilon > 0$, where $b^\epsilon(x) \in C_0^\infty(\mathbb{R}^3)$ is supported in $\{0 < L \leq |x| \leq L+1\}$ and equal to $1/\epsilon$ for $\{L + \epsilon \leq |x| \leq L + 1 - \epsilon\}$, $\chi^\delta(x) \geq 0$ is a smooth function with support in $|x| < L$ and equal to 1 for $|x| \leq L - \delta < L$. Finally, $q(t) \geq 0$ is a periodic smooth function with period $T > 0$. The number L is related to the interval of instability of the Hill operator associated with $q(t)$. The number $\delta > 0$ is fixed sufficiently small and the propagator $K^\delta(T)$ related to the equation

$$\partial_t^2 u - \Delta_x u + q(t)\chi^\delta(x)u = 0,\ t \geq 0,\ |x| < L$$

with Dirichlet boundary conditions on $|x| = L$ has an eigenvalue $z_1, |z_1| > 1$ with eigenfunction $\varphi \in H_0^1(|x| \leq L)$, that is $K^\delta(T)\varphi = z_1\varphi$. Let $S^\epsilon(T) : H \to H$ be the propagator corresponding to the Cauchy problem for the equation

$$\partial_t^2 u - \Delta_x u + V^\epsilon(t,x)u = 0,\ t \geq 0,\ x \in \mathbb{R}^3$$

and let $W^\epsilon(T) : \mathcal{H} \to \mathcal{H}$ be the propagator for the same problem with initial data in $\mathcal{H}$. The problem is to show that for $\epsilon > 0$ sufficiently small $W^\epsilon(T)$ has an eigenvalues $y, |y| > 1$ (here $S^\epsilon(T), W^\epsilon(T)$ correspond to our notations $U(T,0), V(T,0)$ and these operators have domains H and $\mathcal{H}$, respectively).

Extend φ as 0 outside $|x| \geq L$ and denote the new function $\varphi \in \mathcal{H}$ again by φ. Let

$$\gamma = \{z \in \mathbb{C} : |z - z_1| = \eta > 0\} \subset \{z : |z| > 1\}$$

be a circle with center z_1 such that $K^\delta(T) - zI$ is analytic on γ and z_1 is the only eigenvalue of $K^\delta(T)$ in $|z - z_1| \leq \eta$. If $W^\epsilon(T)$ has an eigenvalues on γ the problem is solved. Assume that $W^\epsilon(T)$ has no eigenvalues on γ. It is easy to see that

$$(W^\epsilon(T) - zI)^{-1}\varphi = (S^\epsilon(T) - zI)^{-1}\varphi \in \mathcal{H},\ z \in \gamma.$$

Indeed,

$$(W^\epsilon(T)-zI)^{-1}\varphi=(S^\epsilon(T)-zI)^{-1}\varphi+(S^\epsilon(T)-zI)^{-1}(S^\epsilon(T)-W^\epsilon(T))(W^\epsilon(T)-zI)^{-1}\varphi$$

and

$$(S^\epsilon(T)-W^\epsilon(T))(W^\epsilon(T)-zI)^{-1}\varphi=0.$$

Our purpose is to study

$$(\varphi,(W^\epsilon(T)-zI)^{-1}\varphi)_{\mathcal{H}}=(\varphi,(S^\epsilon(T)-zI)^{-1}\varphi)_{\mathcal{H}},$$

where $(.,,)_{\mathcal{H}}$ denotes the scalar product in $\mathcal{H}$ and $(.,.)_H$ denotes the scalar product in H. It was proved in [1] that for $z\in\gamma$ one has the weak convergence in H

$$(S^\epsilon(T)-zI)^{-1}\varphi\rightharpoonup_{\epsilon\to 0}(K^\delta(T)-zI)^{-1}\varphi,$$

so

$$(\varphi,(S^\epsilon(T)-zI)^{-1}\varphi)_H\longrightarrow(\varphi,(K^\delta(T)-zI)^{-1}\varphi)_H.$$

Here we have used the fact that $\varphi=0$ for $|x|>L$. Let $\varphi=(\varphi_1,\varphi_2)$. We claim that as $\epsilon\to 0$ we have

$$(\varphi_1,((S^\epsilon(T)-zI)^{-1}\varphi)_1)_{L^2}\longrightarrow(\varphi_1,((K^\delta(T)-zI)^{-1}\varphi)_1)_{L^2}. \tag{2.5}$$

To prove this write

$$\varphi_1=-\Delta\psi \text{ with } \psi=\Big(\frac{1}{4\pi|x|}\star\varphi_1\Big).$$

The main point is the following

Lemma 1. *We have $\psi\in H_D(\mathbb{R}^3)$.*

Proof. Since

$$|\partial_{x_j}\psi(x)|=\Big|\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x_j-y_j)\varphi_1(y)}{|x-y|^3}dy\Big|\le\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|\varphi_1(y)|}{|x-y|^2}dy,$$

we can apply the Hardy-Littlewood-Sobolev inequality. More precisely, by using Theorem 4.3 of [5] with $n=3$, $\lambda=2$, $r=2$, $p=6/5$, we obtain that

$$\|\partial_{x_j}\psi(x)\|_{L^2(\mathbb{R}^3)}\le C\|\varphi_1(x)\|_{L^{6/5}(\mathbb{R}^3)}.$$

Now using that $\varphi_1(x)$ is with compact support and the Hölder inequality, we obtain that

$$\|\varphi_1(x)\|_{L^{6/5}(\mathbb{R}^3)}\le C_1\|\varphi_1(x)\|_{L^2(\mathbb{R}^3)}.$$

This completes the proof of Lemma 1. □

Therefore

$$(-\Delta\psi,((S^\epsilon(T)-zI)^{-1}\varphi)_1)_{L^2}=\Big(\langle\nabla_x\psi,\nabla_x((S^\epsilon(T)-zI)^{-1}\varphi))_1\rangle\Big)_{L^2}$$

$$\longrightarrow_{\epsilon\to 0}\Big(\langle\nabla_x\psi,\nabla_x((K^\delta(T)-zI)^{-1}\varphi))_1\rangle\Big)_{L^2}=(-\Delta\psi,((K^\delta(T)-zI)^{-1}\varphi))_1)_{L^2}$$

which proves the claim (2.5). Consequently,

$$(\varphi,(W^\epsilon(T)-zI)^{-1}\varphi)_{\mathcal{H}}\longrightarrow(\varphi,(K^\delta(T)-zI)^{-1}\varphi)_{\mathcal{H}}. \tag{2.6}$$

Moreover, Proposition 4.2 in [1] says that with a constant $C_0>0$ we have uniformly for $z\in\gamma$ the norm estimate

$$\|(S^\epsilon(T)-zI)^{-1}\|_H\le C_0,\ \forall\epsilon\in]0,\epsilon_0].$$

Since

$$\|(S^\epsilon(T)-zI)^{-1}\varphi\|_{L^2(|x|\le L)}\le C_1\|(S^\epsilon(T)-zI)^{-1}\varphi\|_H,$$

the sequence $(\varphi, (W^\epsilon(T)-zI)^{-1}\varphi)_{\mathcal{H}}$ is bounded for $z \in \gamma$. Repeating the argument of Section 5 in [1], one deduces

$$\Big(\varphi, \frac{1}{2\pi i}\int_\gamma (W^\epsilon(T)-zI)^{-1}\varphi dz\Big)_{\mathcal{H}} \longrightarrow \Big(\varphi, \frac{1}{2\pi i}\int_\gamma (K^\delta(T)-zI)^{-1}\varphi dz\Big)_{\mathcal{H}} = \|\varphi\|^2_{\mathcal{H}} \neq 0.$$

This completes the proof that for small ϵ the operator $W^\epsilon(T)$ has an eigenvalue $y, |y| > 1$.

3. Proof of Theorem 2

3.1. **Local well-posedness.** Consider the linear problem

$$\partial_t^2 u - \Delta_x u + q(t,x)u = F,\ u(s,x) = f_1(x),\ \partial_t u(s,x) = f_2(x). \tag{3.1}$$

By using the argument in [6], one may show that the solution of (3) satisfies the same *local in time* Strichartz estimates as in the case $q = 0$. Notice that for these local Strichartz estimates we don't need a global control of the local energy and we can establish them without a condition on the cut-off resolvent $\varphi(V(T,0)-z)^{-1}\varphi$. More precisely, we have the following

Proposition 1. *For every finite $a > 0$ and $f = (f_1, f_2) \in \mathcal{H}, F \in L^1([s,s+a]; L^2(\mathbb{R}^3))$ the solution of* (3.1) *satisfies*

$$\|(u,\partial_t u)\|_{C([s,s+a];\mathcal{H})} + \|u\|_{L^p_t([s,s+a],L^q_x(\mathbb{R}^3))} \leq C(a)\big(\|(f_1,f_2)\|_{\mathcal{H}} + \|F\|_{L^1([s,s+a];L^2(\mathbb{R}^3))}\big)\,, \tag{3.2}$$

provided $\frac{1}{p}+\frac{3}{q}=\frac{1}{2}$, $p > 2$ (the constant $C(a)$ in (3.2) *depends on a, p and $q(t,x)$). Moreover, if $(f_1,f_2) \in H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ and $F \in L^1([s,s+a]; H^1(\mathbb{R}^3))$, we have*

$$\|(u,\partial_t u)\|_{C([s,s+a];H^2\times H^1)} + \|\nabla_x u\|_{L^p_t([s,s+a],L^q_x(\mathbb{R}^3))} \leq C_1(a)\big(\|(f_1,f_2)\|_{H^2\times H^1} + \|F\|_{L^1([s,s+a];H^1(\mathbb{R}^3))}\big)\,. \tag{3.3}$$

For the sake of completeness we present below the proof. The first step is to establish

Lemma 2. *Let $a > 0$, $(f_0,\ f_1) \in \mathcal{H}(\mathbb{R}^3)$ and let $F(t,x) \in L^2_t([s,s+a];\ H^1(\mathbb{R}^3))$ be supported in $\{(t,x) :\ |x| \leq R\}$. Then for every fixed $\varphi \in C_0^\infty(\mathbb{R}^3)$ the solution $u(t,x)$ of* (3.1) *satisfies the estimate*

$$\int_s^{s+a} \|(\varphi u(t,x), \varphi\partial_t u(t,x))\|^2_{\mathcal{H}(\mathbb{R}^3)}dt \leq C\Big(\|(f_0,f_1)\|_{\mathcal{H}(\mathbb{R}^3)} + \|F\|_{L^2_t([s,s+a];H^1(\mathbb{R}^3))}\Big)^2$$

with a constant $C = C(a,\varphi,R) > 0$ depending only on a, φ and R.

The proof is a trivial modification of the proof of Proposition 1 in [7] based on the estimate (2.4) and the representation

$$(u,u_t)(t,x) = U_0(t-s)(f_0,f_1) - \int_s^{s+t}\Big[V(t,\tau)Q(\tau)U_0(\tau-s)f - V(t,\tau)(0,F(\tau,x))\Big]d\tau,$$

where

$$Q(\tau) = \begin{pmatrix} 0 & 0 \\ q(\tau,x) & 0 \end{pmatrix}.$$

Next we write $u = u_0 + v$, where u_0 is the solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta)u_0 = F, \\ u_0(s,x) = f_0,\ \partial_t u_0(s,x) = f_1, \end{cases}$$

while v is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta + q)v = -qu_0, \\ v(s,x) = \partial_t v(s,x) = 0. \end{cases}$$

For u_0 we apply the Strichartz estimates for the free wave equation. On the other hand, combining Lemma 2 and the Strichartz estimate for u_0, one deduces

$$\|qv\|_{L^1([s,s+a],L^2(\mathbb{R}^3))} \le a^{1/2}\|qv\|_{L^2([s,s+a],L^2(\mathbb{R}^3))} \le Ca^{1/2}\|qu_0\|_{L^2([s,s+a],H^1(\mathbb{R}^3))}$$
$$\le C_2(a)\Big(\|(f_0,f_1)\|_{\mathcal{H}(\mathbb{R}^3)} + \|F\|_{L^1([s,s+a],L^2(\mathbb{R}^3))}\Big).$$

Since $(\partial_t^2 - \Delta)v = -qv - qu_0$, we can apply the Strichartz estimates for the free wave equation with right hand part $-(qv + qu_0)$. Taking into account the estimate for

$$\|qv + qu_0\|_{L^1([s,s+a],L^2(\mathbb{R}^3))},$$

we complete the proof of (3.2). The proof of (3.3) is similar.

A standard application of (3.2), (3.3) is the following local well-posedness result for the nonlinear wave equation

$$\partial_t^2 u - \Delta_x u + q(t,x)u + |u|^r u = 0,\ u(s,x) = f_1(x),\ \partial_t u(s,x) = f_2(x), \quad 2 \le r < 4. \tag{3.4}$$

Proposition 2. *There exist $C > 0$, $c > 0$ and $\gamma > 0$ such that for every $(f_1, f_2) \in \mathcal{H}$ there is a unique solution $(u, \partial_t u) \in C([s, s+\tau], H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ of (3.4) on $[s, s+\tau]$ with $\tau = c(1 + \|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma}$. Moreover, the solution satisfies*

$$\|(u,\partial_t u)\|_{C([s,s+\tau];\mathcal{H})} + \|u\|_{L_t^{\frac{2r+2}{r-2}}([s,s+\tau],L_x^{2r+2}(\mathbb{R}^3))} \le C\|(f_1,f_2)\|_{\mathcal{H}}\,. \tag{3.5}$$

If in addition $(f_1, f_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, then $(u, \partial_t u) \in C([s, s+\tau]; H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3))$.

Remark 1. *In the case $r = 2$ the Strichartz estimates are not needed because one may only rely on the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.*

Let us recall the main step in the proof of Proposition 2. One may construct the solutions as the limit of the sequence $(u_n)_{n\ge 0}$, where $u_0 = 0$ and u_{n+1} solves the linear problem

$$\partial_t^2 u_{n+1} - \Delta u_{n+1} + q(t,x)u_{n+1} + |u_n|^r u_n = 0,\ u(s,x) = f_1(x),\ \partial_t u(s,x) = f_2(x), \tag{3.6}$$

where $t \in [s, s+\tau]$. Set

$$\|u\|_S := \|(u,\partial_t u)\|_{C([s,s+\tau];\mathcal{H})} + \|u\|_{L_t^{\frac{2r+2}{r-2}}([s,s+\tau],L_x^{2r+2}(\mathbb{R}^3))}\,.$$

Using (3.2) for $2 < r < 4$ with

$$\frac{1}{p} = \frac{r-2}{2r+2}, \quad \frac{1}{q} = \frac{1}{2r+2}, \tag{3.7}$$

we obtain

$$\|u_{n+1}\|_S \le C\|(f_1,f_2)\|_{\mathcal{H}} + C\|u_n\|^{r+1}_{L^{r+1}([s,s+\tau];L_x^{2r+2}(\mathbb{R}^3))}\,.$$

Now using the Hölder inequality in time, we can write

$$\|u_n\|_{L^{r+1}([s,s+\tau];L_x^{2r+2}(\mathbb{R}^3))} \le \tau^{\frac{4-r}{2r+2}}\|u_n\|_{L_t^{\frac{2r+2}{r-2}}([s,s+\tau],L_x^{2r+2}(\mathbb{R}^3))} \le \tau^{\frac{4-r}{2r+2}}\|u_n\|_S\,.$$

Therefore, we arrive at the bound

$$\|u_{n+1}\|_S \le C\|(f_1,f_2)\|_{\mathcal{H}} + C\tau^{\frac{4-r}{2}}\|u_n\|_S^{r+1}\,. \tag{3.8}$$

Assume that we have the estimate

$$\|u_n\|_S \le 2C\|(f_1,f_2)\|_{\mathcal{H}}.$$

Applying (3.8), and choosing τ so that

$$\tau^{\frac{4-r}{2}}(2C)^{r+1}\|(f_1,f_2)\|^r_{\mathcal{H}} \le 1,$$

we obtain the same bound for $\|u_{n+1}\|_S$. By recurrence we conclude that

$$\|u_{n+1}\|_S \leq 2C\|(f_1, f_2)\|_{\mathcal{H}}, \quad \forall n \geq 0.$$

Next, let $w_n = u_{n+1} - u_n$ be a solution of the problem

$$\partial_t^2 w_n - \Delta w_n + q(t,x) w_n = |u_n|^r u_n - |u_{n+1}|^r u_{n+1},\ w_n(0,x) = \partial_t w_n(0,x) = 0.$$

By using the inequality

$$\Big||v|^r v - |w|^r w\Big| \leq D_r |v - w| \Big(|v|^r + |w|^r\Big),$$

with constant D_r depending only on r, we can similarly show that

$$\|u_{n+1} - u_n\|_S \leq \frac{1}{2}\|u_n - u_{n-1}\|_S$$

which implies the convergence of $(u_n)_{n\geq 0}$ with respect to the $\|\cdot\|_S$ norm.

Now assume that $(f_1, f_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and introduce the norm

$$\|u\|_{S_1} := \|(u, \partial_t u)\|_{C([s,s+\tau]; H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3))} + \|\nabla_x u\|_{L_t^{\frac{2r+2}{r-2}}([s,s+\tau], L_x^{2r+2}(\mathbb{R}^3))}.$$

Therefore the sequence $(u_n)_{n\geq 0}$ satisfies the estimate

$$\|u_{n+1}\|_{S_1} \leq C\|(f_1, f_2)\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} + C\||u_n|^r u_n\|_{L^1([s,s+a]; H^1(\mathbb{R}^3))}$$

and we have

$$\||u_n|^r u_n\|_{L^1([s,s+a]; H^1(\mathbb{R}^3))} \leq C_r \tau^{\frac{4-r}{2}} \|u_n\|_S^r \|u_n\|_{S_1}.$$

which leads to

$$\|u_{n+1}\|_{S_1} \leq C_1\|(f_1, f_2)\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} + C_1 \tau^{\frac{4-r}{2}} \|u_n\|_S^r \|u\|_{S^1}\,. \tag{3.9}$$

Indeed, we can write

$$|u_n|^r u_n = u_n^{r/2+1} \overline{u_n}^{r/2}$$

and therefore

$$\partial_{x_j}(u_n^{r/2+1}\overline{u_n}^{r/2}) = (r/2+1)\partial_{x_j} u_n u_n^{r/2} \overline{u_n}^{r/2} + r/2\ \overline{\partial_{x_j} u_n} u_n^{r/2+1} \overline{u_n}^{r/2-1}$$

yields

$$|\nabla_x(|u_n|^r u_n)| \leq C_r |\nabla_x u_n| |u_n|^r.$$

Applying the Hölder inequality, one obtains

$$\|\nabla_x(|u_n|^r u_n)|\|_{L_x^2} \leq C_1 \|\nabla_x u_n\|_{L_x^{2r+2}(\mathbb{R}^3)} \||u_n|^r\|_{L_x^{\frac{2r+2}{r}}(\mathbb{R}^3)} = C_1\|\nabla_x u_n\|_{L_x^{2r+2}(\mathbb{R}^3)} \|u_n\|^r_{L_x^{2r+2}(\mathbb{R}^3)}.$$

Increasing, if it is necessary, the constant $C > 0$ we may arrange that (3.8) and (3.9) hold with the same constant. Therefore we obtain a local solution $u(t,x) \in C([s, s+\tau], H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3))$ in the same interval $[s, s+\tau]$.

Remark 2. *We work in the complex setting, but if (f_1, f_2) is real valued, then the solution remains real valued. Indeed, if u is a solution of* (3.4) *then so is $\overline{u}$ and we may apply the uniqueness to conclude that $u = \overline{u}$.*

3.2. **Global well-posedness and polynomial bounds.** Fix $(f_1, f_2) \in \mathcal{H}$. Let u be the local solution of (3.4) obtained in Proposition 2 (with $s = 0$). First we prove the following

Lemma 3. *The solutions*

$$u(t,x) \in C([0,A], H^2_x(\mathbb{R}^3)) \cap C^1([0,A], H^1_x(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0,A], L_x^{2r+2}(\mathbb{R}^3))$$

of (3.4) *satisfy the relation*

$$\frac{d}{dt}\int_{\mathbb{R}^3}\big(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{2}q|u|^2 + \frac{1}{r+2}|u|^{r+2}\big)dx = \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3}(\partial_t q)|u|^2 dx,\ 0 \le t \le A. \quad (3.10)$$

Remark 3. *We show that* (3.10) *holds in the sense of distributions* $\mathcal{D}'(]0,A[)$. *Since the right hand side of* (3.10) *is continuous in* $]0,A[$ *the derivative of the left hand side can be taken in the classical sense.*

Proof. Let us first remark that $\int_{\mathbb{R}^3}|u|^{j+2}(t,x)dx \le \|u(t,x)\|^{j+2}_{H^1_x(\mathbb{R}^3)}$ for $0 \le j < 4$, thanks to the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{j+2}(\mathbb{R}^3)$. Moreover, from our assumption it follows that $u(t,x) \in C([0,A], L^\infty_x(\mathbb{R}^3))$ and this implies

$$|u|^r(t,x)u(t,x) \in C([0,A], L^2_x(\mathbb{R}^3)).$$

Therefore, from the equation (3.4) we deduce $\partial_t^2 u(t,x) \in C([0,A], L^2_x(\mathbb{R}^3))$.

To verify (3.10), notice that

$$\begin{aligned}\mathrm{Re}\big(\int_{\mathbb{R}^3}(\partial_t^2 u - \Delta_x u + |u|^r u)\overline{\partial_t u}dx\big) &= -\mathrm{Re}\big(\int_{\mathbb{R}^3} q(t,x)u\overline{\partial_t u}dx\big)\\ &= -\frac{1}{2}\frac{d}{dt}\big(\int_{\mathbb{R}^3} q|u|^2dx\big) + \frac{1}{2}\int_{\mathbb{R}^3}(\partial_t q)|u|^2dx\end{aligned}$$

and the integrals

$$\int_{\mathbb{R}^3}(\partial_t^2 u - \Delta_x u)\overline{\partial_t u}dx,\ \int_{\mathbb{R}^3}|u|^r u\bar{u}_t dx$$

are well defined. After an approximation with smooth functions and integration by parts we deduce

$$\mathrm{Re}\int_{\mathbb{R}^3}\Big(\partial_t^2 u - \Delta_x u\Big)\overline{\partial_t u}dx = \frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}(|\partial_t u|^2 + |\nabla_x u|^2)dx.$$

On the other hand,

$$(r/2+1)(u^{\frac{r}{2}}\bar{u}^{\frac{r}{2}+1}\partial_t u + u^{\frac{r}{2}+1}\bar{u}^{\frac{r}{2}}\partial_t \bar{u}) = \partial_t(u^{\frac{r}{2}+1})\bar{u}^{\frac{r}{2}+1} + \partial_t(\bar{u}^{\frac{r}{2}+1})u^{\frac{r}{2}+1}$$

and hence

$$\mathrm{Re}\int_{\mathbb{R}^3}|u|^r u\bar{u}_t dx = \frac{1}{r+2}\frac{d}{dt}\big(\int_{\mathbb{R}^3}|u|^{r+2}dx\big).$$

Thus (3.10) holds for $0 < t < A$ and by continuity one covers the interval $[0,A]$. □

We need the following simple lemma.

Lemma 4. *Let* $0 < \gamma < 1$ *and let* $X(t) : [0,\infty) \to [0,\infty)$ *be a derivable function such that for some* $A > 0$,

$$|X'(t)| \le CX^{1-\gamma}(t), \quad 0 \le t \le A.$$

Then

$$X(t) \le (X^\gamma(0) + C\gamma t)^{\frac{1}{\gamma}}, \quad 0 \le t \le A.$$

Proof. First assume that $X(t) > 0$ for all $0 \le t \le A$. We have

$$\Big|\frac{d}{dt}(X^\gamma(t))\Big| = \gamma\Big|X^{\gamma-1}(t)X'(t)\Big| \le C\gamma.$$

Hence

$$X^\gamma(t) = \Big|\int_0^t (X^\gamma)'(\tau)d\tau + X^\gamma(0)\Big| \le X^\gamma(0) + C\gamma t$$

and we obtain the assertion for $X(t) > 0$. In the general case, we apply the previous argument to $X(t) + \epsilon$, $\epsilon > 0$ and we let $\epsilon \to 0$. This completes the proof. □

Let $u(t,x) \in C([0,A), H_x^2(\mathbb{R}^3) \cap C^1([0,A], H_x^1(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0,A], L_x^{2r+2}(\mathbb{R}^3))$ be a solution of (3.4) and let

$$X(t) = \int_{\mathbb{R}^3} \big(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{2}q|u|^2 + \frac{1}{r+2}|u|^{r+2}\big)dx\,.$$

The support property $q(t,x) = 0$ for $|x| > \rho$ and the Hölder inequality imply

$$\Big|\int_{\mathbb{R}^3} (\partial_t q)|u|^2 dx\Big| \le C\|u(t,\cdot)\|^2_{L^2(|x|\le\rho)} \le C_1\|u(t,\cdot))\|^2_{L^{r+2}(|x|\le\rho)}.$$

Therefore

$$|X'(t)| \le C_2 X^{\frac{2}{r+2}}(t) = C_2 X^{1-\frac{r}{r+2}}(t)$$

and applying Lemma 4, we deduce

$$X(t) \le \Big(X^{\frac{r}{r+2}}(0) + \frac{C_2 r}{r+2}t\Big)^{\frac{r+2}{r}} \quad 0 \le t \le A. \tag{3.11}$$

As a consequence of (3.11) we get

$$\big(\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla_x u(t,\cdot)\|^2_{L^2(\mathbb{R}^3)}\big)^{\frac{1}{2}} \le \sqrt{2}\Big(X^{\frac{r}{r+2}}(0) + \frac{C_2 r}{r+2}t\Big)^{\frac{r+2}{2r}}$$

and therefore

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|\nabla_x u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \le 2\Big(X^{\frac{r}{r+2}}(0) + \frac{C_2 r}{r+2}t\Big)^{\frac{r+2}{2r}}.$$

On the other hand,

$$X(0) \le A_r\|(u,u_t)(0,x)\|_1^2\Big(1 + \|(u,u_t)(0,x)\|_1^r\Big)$$

with a constant A_r depending on r. Hence from (3.11) we get

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^3)} + \|\nabla_x u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \le 2\Big(X^{\frac{r}{r+2}}(0) + \frac{C_2 r}{r+2}t\Big)^{\frac{r+2}{2r}}$$

$$\le 2\Big(A_r^{\frac{r}{r+2}}\|(u,u_t)(0,x)\|_1^{\frac{2r}{r+2}}\Big[1 + \|(u,u_t)(0,x)\|_1^r\Big]^{\frac{r}{r+2}} + \frac{C_2 r}{r+2}t\Big)^{\frac{r+2}{2r}},\ 0 \le t \le A. \tag{3.12}$$

Finally, from

$$u(t,x) = u(0,x) + \int_0^t \partial_t u(\tau,x)d\tau$$

one deduces

$$\|u(t,x)\|_{L^2(\mathbb{R}^3)} \le \|u(0,x)\|_{L^2(\mathbb{R}^3)} + 2t\Big(X^{\frac{r}{r+2}}(0) + \frac{C_2 r}{r+2}t\Big)^{\frac{r+2}{2r}}.$$

This yields a polynomial bound for the solutions

$$u(t,x) \in C([0,A], H_x^2(\mathbb{R}^3)) \cap C^1([0,A], H_x^1(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0,A], L_x^{2r+2}(\mathbb{R}^3)).$$

Now we pass to the global existence of solution of (3.4). We will deal with the case $2 < r < 4$, while the case $r = 2$ can be covered by the Sobolev embedding theorem. We fix a number $a > 0$ and our purpose is to show that (3.4) has a solution for $t \in [0, a]$ with initial data $f \in \mathcal{H}$. We fix p, q by (3.7) and let the Strichartz estimate (3.2) holds in the interval $[0, a]$ with a constant $C_a > 0$. The above argument yields a local solution $u(t, x)$ with initial data $f = (f_1, f_2) \in \mathcal{H}$ for $t \in [s, s + \tau]$. Recall that $\tau = c(1 + \|f\|_{\mathcal{H}})^{-\gamma}$. Introduce the number

$$B_a := \|f\|_{\mathcal{H}} + a(B_1 + B_2 a)^{\frac{r+2}{2r}},$$

where $B_1 > 0$ and $B_2 > 0$ depend only on $\|f\|_{\mathcal{H}}$ and r. This number should be a bound of the energy of the solution $u(t, x)$ in $[0, a]$ with initial data $f \in \mathcal{H}$ if the above argument based on Lemma 3 and Lemma 4 works. However, the proof of Lemma 3 cannot be applied directly for functions $u(t, x) \in C([0, a], H_x^1(\mathbb{R}^3)) \cap C^1([0, a], L_x^2(\mathbb{R}^3))$.

Define $\tau(a) := c(1 + B_a)^{-\gamma} < 1$ with the constants $c > 0, \gamma > 0$ of Proposition 2 and observe that the local existence theorem can be applied in the interval $[s, s + \tau(a)] \subset [0, a]$ if the norm of the initial data for $t = s$ is bounded by B_a. To overcome the difficulty connected with Lemma 3 and since we did not prove in Proposition 2 the continuous dependence with respect to the initial data in $\mathcal{H}$, we need to apply an approximation argument in $[s, s + \epsilon(a)]$, where the number $0 < \epsilon(a) \le \tau(a)$ will be defined below. For simplicity we treat the case $s = 0$ below.

By the local existence let $u(t, x)$ be the solution of (3.4) in $[0, \tau(a)]$ with initial data $f = (f_1, f_2) \in \mathcal{H}$. Choose a sequence $g_n = ((g_n)_1, (g_n)_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ converging in $\mathcal{H}$ to $(f_1, f_2) \in \mathcal{H}$ as $n \to \infty$ and let $w_n(t, x)$ be the solution of the problem (3.4) in the *same interval* $[0, \tau(a)]$ with initial data g_n. Then by Proposition 2,

$$w_n(t, x) \in C([0, \tau(a)], H_x^2(\mathbb{R}^3) \cap C^1([0, \tau(a)], H_x^1(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0, \tau(a)], L_x^{2r+r}(\mathbb{R}^3)).$$

Set $v_n = w_n - u$. We claim that for $n \to \infty$ we have

$$\|(v_n, (v_n)_t)\|_{C([0,\epsilon(a)],\mathcal{H})} + \|v_n\|_{L_t^p([0,\epsilon(a)],L_x^q(\mathbb{R}^3))} \to 0$$

with $0 < \epsilon(a) \le \tau(a)$ defined below. Clearly, v_n is a solution of the equation

$$\partial_t^2 v_n - \Delta v_n + q(t, x) v_n = |u|^r u - |w_n|^r w_n.$$

Applying (3.2), one obtains

$$\begin{aligned} &\|(v_n, (v_n)_t)\|_{C([0,\epsilon(a)],\mathcal{H})} + \|v_n\|_{L_t^{\frac{2r+2}{r-2}}([0,\epsilon(a)],L_x^{2r+2}(\mathbb{R}^3))} \\ &\le C_a \|g_n - f\|_{\mathcal{H}} + C_a \||u|^r u - |w_n|^r w_n\|_{L^1([0,\epsilon(a)],L_x^2(\mathbb{R}^3))} \end{aligned} \tag{3.13}$$

and

$$\|(|u|^r u - |w_n|^r w_n)(t, .)\|_{L_x^2} \le C \|v_n(t, .)\|_{L_x^{2r+2}} \Big(\|u(t, .)\|_{L_x^{2r+2}}^r + \|w_n(t, .)\|_{L_x^{2r+2}}^r \Big).$$

Since $\frac{1}{p} + \frac{r}{p} + \Big(1 - \frac{r+1}{p}\Big) = 1$, by the generalized Hölder inequality in the integral with respect to t in (3.13) for large $n \ge n_0$ we get

$$\begin{aligned} &C_a \||u|^r u - |w_n|^r w_n\|_{L^1([0,\epsilon(a)],L_x^2(\mathbb{R}^3))} \\ &\le D_r C_a \epsilon(a)^{(1-\frac{r+1}{p})} \|v_n\|_{L^p([0,\epsilon(a)],L_x^q)} \Big(\|u\|_{L^p([0,\epsilon(a)],L_x^q)}^r + \|w_n\|_{L^p([0,\epsilon(a)],L_x^q)}^r \Big) \\ &\le 2 D_r C_a^{r+1} (\|f\|_{\mathcal{H}} + 1)^r \epsilon(a)^{(1-\frac{r+1}{p})} \|v_n\|_{L^p([0,\epsilon(a)],L_x^q)}. \end{aligned}$$

Here D_r is a constant depending only on r and we used that by Proposition 2

$$\|w_n\|_{L^{\frac{2r+2}{r-2}}([0,\epsilon(a)],L_x^{2r+2}(\mathbb{R}^3))} \le C_a \|g_n\|_{\mathcal{H}} \le C_a (\|f\|_{\mathcal{H}} + 1), \ n \ge n_0 \tag{3.14}$$

with a similar estimate for $\|u\|_{L^{\frac{2r+2}{r-2}}([0,\epsilon(a)],L_x^{2r+2}(\mathbb{R}^3))}$. Clearly, $1-\frac{r+1}{p}=2-\frac{r}{2}>0$ and we choose $0<\epsilon(a)\leq\tau(a)$, so that

$$2D_rC_a^{r+1}(B_a+1)^r\epsilon(a)^{(1-\frac{r+1}{p})}\leq\frac{1}{2}.$$

Then we may absorb the term on right hand side of (3.13) involving w_n, u and letting $n\to\infty$, we prove our claim. Moreover, for almost all $t\in[0,\epsilon(a)]$, taking into account (3.14), we have

$$\Big|\int_{\mathbb{R}^3}\Big(|u(t,x)|^{r+2}-|w_n(t,x)|^{r+2}\Big)dx\Big|$$

$$\leq D_r\|u(t,x)-w_n(t,x)\|_{L^2(\mathbb{R}^3)}\Big(\|u(t,x)\|^{r+1}_{L_x^{2r+2}(\mathbb{R}^3)}+\|w_n(t,x)\|^{r+1}_{L_x^{2r+2}(\mathbb{R}^3)}\Big)dx\longrightarrow_{n\to\infty}0.$$

Consequently, we have

$$\int_{\mathbb{R}^3}\Big(\frac{1}{2}\Big(|\partial_t w_n|^2+|\nabla_x w_n|^2+q|u|^2\Big)+\frac{1}{r+2}|w_n|^{r+2}\Big)dx$$

$$\longrightarrow_{n\to\infty}\int_{\mathbb{R}^3}\big(\frac{1}{2}(|\partial_t u|^2+|\nabla_x u|^2+q|u|^2)+\frac{1}{r+2}|u|^{r+2}\big)dx$$

in the sense of distributions $\mathcal{D}'(0,\epsilon(a))$. The equality (3.10) for $0\leq t\leq\epsilon(a)$ holds for w_n and passing to a limit in the sense of distributions, we conclude that (3.10) holds for $u(t,x)$ for $0<t<\epsilon(a)$ and hence for $0\leq t\leq\epsilon(a)$. The right hand side of (3.10) is continuous with respect to t, hence the derivative with respect to t is taken in a classical sense. Thus we are in position to apply Lemma 4 for the $u(t,x)$. Finally, we deduce (3.12) for the solution $u(t,x)$ and the norm $\|(u,u_t)(t,.)\|_{\mathcal{H}}$ for $t\in[0,\epsilon(a)]$ is bounded by B_a introduced above.

Now we pass to the second step in the interval $[\epsilon(a),2\epsilon(a)]\subset[0,a]$. As it was mentioned above, we have a bound B_a for the norm of the initial data $(u(\epsilon(a),x),u_t(\epsilon(a),x))$. By the local existence we have solution in $[\epsilon(a),2\epsilon(a)]$ and $u(t,x)$ is defined in $[0,2\epsilon(a)]$. On the other hand, we may approximate the initial data $(u(\epsilon(a),x),u_t(\epsilon(a),x))$ by functions $g_n^{(2)}\in H^2\times H^1$ and by the above argument the solution $u(t,x)$ in $[\epsilon(a),2\epsilon(a)]$ is approximated by solutions $w_n^{(2)}(t,x)$ for which (3.10) holds for $\epsilon(a)\leq t\leq 2\epsilon(a)$. Thus (3.10) is satisfied for $u(t,x)$ for $\epsilon(a)\leq t<2\epsilon(a)$ and combining this with the first step, one concludes that the same is true for $0\leq t\leq 2\epsilon(a)$. This makes possible to apply Lemma 4 for $0\leq t\leq 2\epsilon(a)$ and to deduce (3.12) with uniform constants leading to a bound by B_a. We can iterate this procedure, since $\tau(a),\epsilon(a)$ depend only on $\|f\|_{\mathcal{H}},C_a$ and r, while B_a depends on $\|f\|_{\mathcal{H}},a$ and r. The solution $u(t,x)$ will be defined globally in a interval $[0,\alpha(a)]$ with $0<a-\alpha(a)<\epsilon(a)$. Since $\alpha(a)>a-\epsilon(a)>a-1$ and a is arbitrary, we have a global solution $u(t,x)$ defined for $t\geq 0$. An application of Lemma 4 justifies the bound (3.12) for $u(t,x)$ and for all $t\geq 0$ with constants depending only on $\|f\|_{\mathcal{H}}$ and r. A similar analysis holds for negative times t.

Remark 4. *It is likely that in the case $r=2$ by using the approach of* [8] *one may obtain polynomial bounds on the higher Sobolev norms $H^\sigma(\mathbb{R}^3)\times H^{\sigma-1}(\mathbb{R}^3)$, $\sigma>1$, of the solutions of* (3.4).

3.3. **A uniform bound.** As a byproduct of the (semi-linear) global well-posedness, we have the following uniform bound on the solutions of (3.4).

Proposition 3. *Let $R>0$ and $A>0$. Then there exists a constant $C(A,R)>0$ such that for every $(f_1,f_2)\in\mathcal{H}$ such that $\|(f_1,f_2)\|_{\mathcal{H}}<R$ the solution $u(t,x)$ of* (3.4) *satisfies*

$$\|u\|_{L^{r+1}([0,A];L_x^{2r+2}(\mathbb{R}^3))}\leq C(A,R)\|(f_1,f_2)\|_{\mathcal{H}}\,. \tag{3.15}$$

Proof. Thanks to the global bounds on the solutions, we obtain that there exists $R' = R'(R, A)$ such that if $\|(f_1, f_2)\|_{\mathcal{H}} < R$, then the corresponding solutions satisfies

$$\sup_{0\leq t\leq A} \|(u(t,\cdot), \partial_t u(t,\cdot)\|_{\mathcal{H}} \leq R' .$$

Denote by $\tau = \tau(A, R') > 0$ the local existence time for initial data having $\mathcal{H}$ norm $\leq R'$, i.e. $\tau = c(1 + R')^{-\gamma}$ with the notations of Proposition 2. Next we split the interval $[0, A]$ in intervals of size τ. In every interval $[k\tau, (k + 1)\tau]$ we apply the estimate (3.2) with $F = 0$ and constant C_A independent on k. Thus we obtain a bound

$$\|u(t, x)\|_{L^{\frac{2r+2}{r-2}}([k\tau,(k+1)\tau],L_x^{2r+2}(\mathbb{R}^3))} \leq C_A^k \|(f_1, f_2)\|_{\mathcal{H}},\ 1 \leq k + 1 \leq A/\tau.$$

By using the Hölder inequality for the integral with respect to t, we obtain easily (3.15). □

4. Proof of Theorem 3

Let

$$\mathcal{H} \ni f \to \mathcal{U}(t, s)f = (v(t, x; s), v_t(t, x; s)) \in \mathcal{H}$$

be the monodromy operator corresponding to the Cauchy problem (3.4) with initial data f for $t = s$. For $\mathcal{U}(t, s)$ we have the representation

$$\mathcal{U}(t, s)f = V(t, s)f - \int_s^t V(t, \tau)Q_0\big(|\mathcal{U}(\tau, s)f|^r \mathcal{U}(\tau, s)f\big)d\tau, \tag{4.1}$$

where

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore we can write $\mathcal{U}(t + T, s + T)f$ as

$$V(t + T, s + T)f - \int_{s+T}^{t+T} V(t + T, \tau)Q_0\big(|\mathcal{U}(\tau, s + T)f|^r \mathcal{U}(\tau, s + T)f\big)d\tau$$

which in turn can be written as

$$V(t, s)f - \int_s^t V(t, \tau)Q_0\big(|\mathcal{U}(\tau + T, s + T)f|^r \mathcal{U}(\tau + T, s + T)f\big)d\tau.$$

By the uniqueness of the solution of the equation

$$\mathcal{U}(t, s)f = V(t, s)f - \int_s^t V(t, \tau)Q_0(|\mathcal{U}(\tau, s)f|^r \mathcal{U}(\tau, s)f)d\tau,$$

one deduces $\mathcal{U}(t + T, s + T) = \mathcal{U}(t, s)$. Moreover, one has the property

$$\mathcal{U}(p, r) = \mathcal{U}(p, s) \circ \mathcal{U}(s, r), \quad p, r, s \in \mathbb{R}.$$

For the solution $u(t, x; 0)$ of (3.4) (with $s = 0$) with initial data $f \in \mathcal{H}$, set

$$w_n = (u(nT, x; 0), \partial_t u(nT, x; 0)) = \mathcal{U}(nT, 0)f,\ n \in \mathbb{N}.$$

Therefore

$$w_{n+1} = \mathcal{U}((n + 1)T, 0)f = \mathcal{U}((n + 1)T, nT) \circ \mathcal{U}(nT, 0)f = \mathcal{U}(T, 0)w_n. \tag{4.2}$$

Setting $\mathcal{F} = \mathcal{U}(T, 0)$, we obtain a system

$$w_{n+1} = \mathcal{F}(w_n), \quad n \geq 0. \tag{4.3}$$

with a *nonlinear map* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$. Consider the linear map $L = V(T, 0) : \mathcal{H} \to \mathcal{H}$. Our purpose is to how that L is the Fréchet derivative of $\mathcal{F}$ at the origin in the Hilbert space $\mathcal{H}$. We use the representation

$$\mathcal{F}(h) = Lh - \int_0^T V(T, \tau) Q_0\big(|u(\tau, x; h)|^r u(\tau, x; h)\big) d\tau,$$

where $u(t, x; h)$ is the solution of (3.4) with $s = 0$ and initial data h at time 0. Using the Strichartz estimate and Proposition 3, we obtain for $\|h\|_1 \leq 1$ the bound

$$\sup_{0 \leq t \leq T} \|\mathcal{F}(h) - Lh\|_1 \leq C \|u(t, x; h)\|^{r+1}_{L^{r+1}([0,T]; L^{2r+2}_x(\mathbb{R}^3))} \leq C \|h\|_1^{r+1},$$

where $C > 0$ depends on T but is independent of h. This implies immediately that L is the Fréchet derivative of $\mathcal{F}$ at the origin.

For the exponential instability at $u = 0$ we use following definition (see [2]).

Definition 1. (i) *The equilibrium* $u = 0$ *is unstable if there exists* $\epsilon > 0$ *such that for every* $\delta > 0$ *one can find a sequence* $\{u_n\}$ *of solution of* (4.3) *such that* $0 < \|u_0\|_1 \leq \delta$ *and* $\|u_n\|_1 \geq \epsilon$ *for some* $n \in \mathbb{N}$.
(ii) *The equilibrium* $u = 0$ *is exponentially unstable at rate* $\rho > 1$ *if there exist* $\epsilon > 0$ *and* $C > 0$ *such that for every* $\delta > 0$ *one can find a sequence* $\{u_n\}$ *of solution of* (4.3) *satisfying* $0 < \|u_0\|_1 \leq \delta$ *and* $\|u_N\|_1 \geq C\rho^N \|u_0\|_1$ *for any* N *for which we have*

$$\max\{\|u_0\|_1, ..., \|u_N\|_1\} \leq \epsilon.$$

Clearly, the exponential instability implies instability. We consider the case when the spectral radius $r(L)$ of L is greater than 1. The analysis in Section 2 shows that there exist positive potentials $q(t, x) \geq 0$ for which $r(L) > 1$. We will apply the Rutman-Dalecki theorem or a more general version due to D. Henry (Theorem 5.1.5 in [4]). This theorem says that if the Fréchet derivative L of $\mathcal{F}$ at zero is such that

$$\|\mathcal{F}(u) - Lu\|_1 \leq b \|u\|_1^{1+p} \text{ whenever } \|u\|_1 \leq a \tag{4.4}$$

for some $a > 0$, $b > 0$ and $p > 0$ and if the spectral radius $r(L)$ of L satisfies $r(L) > 1$, then $\mathcal{F}$ is exponentially unstable at $u = 0$. In our case the condition (4.4) holds with $p = r$ and $a = 1$. Thus we obtain the following

Theorem 4. *Assume that the linear operator* L *has spectral radius* $r(L) > 1$. *Then* $\mathcal{F}$ *is exponentially unstable at* $u = 0$ *with rate* $r(L)$.

It remains to observe that Theorem 4 implies Theorem 3.

Remark 5. *The above argument showing nonlinear instability crucially relies on the fact that we deal with a semi-linear problem, i.e. the solution map of* (3.4) *is of class* C^1 *on* $\mathcal{H}$. *It is worth to mention that there are examples of problems which are not semi-linear (the solution map is not of class* C^1*) for which one can still get the nonlinear instability of some particular solutions (known to be linearly unstable). In such cases a "more nonlinear approach" is needed. We refer to* [3, 9] *for more details on this issue.*

5. Generalizations

We can consider more general nonlinear equations

$$\partial_t^2 u - \Delta_x u + |u|^r u + \sum_{j=0}^{r-1} q_j(t, x) |u|^j u = 0, \; r = 2, 3 \tag{5.1}$$

with smooth time-periodic functions $q_j(t+T_j,x) = q_j(t,x) \geq 0,\ j = 0,\cdots,r-1$ having compact support with respect to x. For solutions

$$u(t,x) \in C([0,\tau], H^2(\mathbb{R}^3)) \cap C^1([0,\tau], H^1(\mathbb{R}^3)) \cap L_t^{\frac{2r+2}{r-2}}([0,A], L_x^{2r+2}(\mathbb{R}^3))$$

we obtain

$$\mathrm{Re}\Big(\int_{\mathbb{R}^3}(\partial_t^2 u - \Delta_x u + |u|^r u)\bar{u}_t dx\Big) = -\mathrm{Re}(\int_{\mathbb{R}^3}\sum_{j=0}^{r-1} q_j(t,x)|u|^j u\bar{u}_t dx)$$

$$= -\frac{d}{dt}\sum_{j=0}^{r-1}\Big(\int_{\mathbb{R}^3}\frac{1}{j+2}q_j|u|^{j+2}dx\Big) + \sum_{j=0}^{r-1}\frac{1}{j+2}\int_{\mathbb{R}^3}(q_j)_t|u|^{j+2}dx$$

and

$$\frac{1}{j+2}\Big|\int_{\mathbb{R}^3}(q_j)_t|u|^{j+2}dx\Big| \leq C_j\Big(\int_{\mathbb{R}^3}|u|^{r+2}dx\Big)^{1-\frac{r-j}{r+2}},\ j = 0,\cdots,r-1.$$

Setting

$$X(t) \equiv \int_{\mathbb{R}^3}\Big(\frac{1}{2}|u_t|^2(t,x)+\frac{1}{2}|\nabla_x u|^2(t,x)+\sum_{j=0}^{r-1}\frac{1}{j+2}q_j|u|^{j+2}(t,x)+\frac{1}{r+2}|u|^{r+2}(t,x)\Big)dx, 0 \leq t \leq A,$$

one deduces

$$|X'(t)| \leq B_r\sum_{j=0}^{r-1}X(t)^{1-\frac{r-j}{r+2}} \leq B_r(1+X(t))^{1-\frac{1}{r+2}}.$$

Therefore we can apply Lemma 4 to the quantity $Y(t) = 1 + X(t)$ which implies, as before, the global existence and the polynomial bounds of the solutions of the Cauchy problem for (5.1).

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