RESONANCES FOR NON-TRAPPING TIME-PERIODIC PERTURBATIONS

JEAN-FRANÇOIS BONY AND VESSELIN PETKOV

ABSTRACT. For time-periodic perturbations of the wave equation in $\mathbb{R}_t \times \mathbb{R}_x^n$ given by a potential $q(t,x)$, we obtain an upper bound of the number of the resonances $\{z_j \in \mathbb{C} : |z_j| \geq \delta > 0\}$. We establish for $m \in N$ large enough a trace formula relating the iterations of the monodromy operator $U(mT, 0)$, $T > 0$, and the sum $\sum_j z_j^m$ of all resonances counted with their multiplicities.

1. Introduction

The purpose of this paper is to study the resonances of the wave equation with time-dependent potentials. Consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u + q(t,x)u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(s,x) = f_0(x), \ u_t(s,x) = f_1(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the potential $q(t,x) \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^n), n \geq 3$, n odd, satisfies the conditions:

(H_1) there exists $R > 0$ such that $q(t,x) = 0$ for $|x| \geq R$, $\forall t \in \mathbb{R}$,

(H_2) $q(t+T, x) = q(t,x)$, $\forall (t,x) \in \mathbb{R}^{n+1}$ with $T > 0$.

The solution of (1.1) is given by the propagator

$$U(t,s) : \ H_0 \ni (f_0, \ f_1) \longrightarrow U(t,s)(f_0, f_1) = (u(t,x), \ u_t(t,x)) \in H_0,$$

where H_0 is the energy space $H_0 = H_D(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ and $H_D(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H_D} = \Big(\int |\nabla_x f|^2 dx \Big)^{1/2}.$$

We refer to Chapter V, [13], for the properties of $U(t,s)$ and throughout the paper we will use freely the notations of [13].

Let $U_0(t) = e^{itG_0}$ be the unitary group in H_0 related to the Cauchy problem (1.1) with $q = 0$ and let P_+^ρ (resp. P_-^ρ) denote the orthogonal projection on the orthogonal complement of the spaces D_+^ρ (resp. D_-^ρ) introduced by Lax and Phillips (see [8], [13]) and defined by

$$D_\pm^\rho = \{f \in H_0 : \ U_0(t)f = 0 \text{ for } |x| \leq \pm t + \rho, \ \pm t \geq 0\}, \ \rho \geq R.$$

To define the resonances, we will use the operator

$$Z^\rho(T) = P_+^\rho U(T,0) P_-^\rho.$$

For non-trapping perturbations the spectrum of $Z^\rho(T)$ is formed by eigenvalues with finite multiplicity and for $m \in N$ large enough the operator $Z^\rho(mT)$ is compact (see [1], [5], [13]). Moreover, the eigenvalues and their multiplicity are independent of $\rho \geq R$ (see [5] and [13]). Let P denote the time-dependent operator related to the problem (1.1). We define the resonances following the approach of Lax-Phillips [8] for stationary perturbations and that of Cooper-Strauss [5] for time-periodic ones.

Definition 1. *We say that* $z \in \mathbb{C} \setminus \{0\}$ *is a resonance for* P *if* $z \in \sigma_{\rm pp}\, Z^\rho(T)$.

Notice that $e^{i\sigma T}$ is an eigenvalue of $Z^\rho(T)$ if and only if there exists an outgoing solution $u(t,x)$ of the problem (1.1) with non-vanishing initial data such that $e^{-i\sigma T}u(t,x)$ is periodic with period T. We refer to [13] for the definition of an outgoing solution (see also [5]) as well as for the proof of the above equivalence. The second definition presents a more precise description of the existence of outgoing modes with complex frequencies known in the physical literature.

We denote by Res P the set of resonances of P. In the following we write $Z(T)$, $P_\pm$ instead of $Z^\rho(T)$, $P_\pm^\rho$ if the dependence on ρ is not important and we set

$$U(T) = U(T,0),\ Z_0(T) = P_+ U_0(T) P_-.$$

In Section 3 we obtain an upper bound of the number of the resonances

$$N_\delta = \#\{z \in \mathrm{Res}\, P :\ |z| \geq \delta\} \leq C_\epsilon \delta^{-\epsilon},\ 0 < \epsilon \leq 1/2$$

which generalizes the known results for time-independent perturbations (see [11], [21], [14], [19] and the papers cited there). To our best knowledge this is the first upper bound for N_δ for time-periodic perturbations. Next, using the bound of the resonances, we obtain a trace formula involving the resonances. More precisely, given a function $g(z) = z^m h(z)$, holomorphic in a disk containing the resonances, we establish a trace formula involving $g(U(T))$ and the series

$$\sum_{z_j \in \mathrm{Res}\, P} g(z_j)$$

in the spirit of trace formulae obtained for stationary perturbations in [2], [11], [15], [22], [16] (see Section 4). In particular, for $h(z) = 1$ we have the following

Theorem 1. *Let* $\chi \in C_0^\infty(B(0,r_1);\ [0,1])$ *be such that* $\chi = 1$ *for* $|x| \leq R + T$ *and let*

$$\chi(U(T) - U_0(T)) = (U(T) - U_0(T))\chi = U(T) - U_0(T). \tag{1.2}$$

Let the projectors $P_\pm$ *and the number* $k \in \mathbb{N}$ *be fixed so that*

$$P_+ U_0(jT) P_- = 0,\ j \geq k,\ P_\pm \chi = \chi P_\pm = \chi. \tag{1.3}$$

Then for $m \geq 2k$ *large enough we have*

$$\mathrm{tr}\Big((U(kT) - U_0(kT)) U(mT - 2kT) (U(kT) - U_0(kT)) \Big) = \sum_{z_j \in \mathrm{Res}\, P} z_j^m, \tag{1.4}$$

where the summation is over all resonances counted with their multiplicities.

Remarks. 1) The equality (1.2) follows from the finite speed of propagation and the representation

$$U(T) = U_0(T) - \int_0^T U_0(T-s) Q(s) U(s,0) ds,$$

where

$$Q(s) = \begin{pmatrix} 0 & 0 \\ q(s,x) & 0 \end{pmatrix}.$$

2) It is clear that we can choose $\psi \in C_0^\infty(\mathbb{R}^n)$ with the property $\psi = 1$ for $|x| \leq r_1 + kT$ so that

$$(U(kT) - U_0(kT))(1 - \psi) = 0$$

which is a consequence of

$$\chi U_0(jT)(1 - \psi) = 0, \; j = 0, 1, \dots, k - 1$$

(see equality (4.1)). Thus in the trace formula (1.4) on the right and on the left of $U(mT - 2kT)$ we may put the cut-off function $\Psi = (U(kT) - U_0(kT))\psi$ acting as a multiplication operator.

Corollary 1. *Under the assumptions of Theorem* 1 *the existence of a sequence* $m_\nu \in \mathbb{N}$, $m_\nu \nearrow \infty$, *such that*

$$\Big|\mathrm{tr}\Big((U(kT) - U_0(kT))U(m_\nu T)(U(kT) - U_0(kT))\Big)\Big| \longrightarrow \infty \text{ as } m_\nu \nearrow \infty$$

is equivalent to

$$\mathrm{Res}\, P \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset.$$

The above result says that the existence of resonances z_j, $|z_j| > 1$, associated to solutions whose local energy blows up is connected to the behavior as $m \to \infty$ of the trace of a cut-off iteration $\Psi U(mT)\Psi$. It is clear that we can choose $b > 0$ so that the projectors $P_\pm^b$ satisfy $P_\pm^b \Psi = \Psi P_\pm^b = \Psi$. Then we obtain the property

$$\mathrm{Res}\, P \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset \Leftrightarrow \overline{\lim}_{m\to\infty} \; |\mathrm{tr}(Z^b(mT))| = +\infty.$$

The result of Corollary 1 seems quite natural for time-periodic perturbations. For example, the existence of intervals of instability for the Hill equation

$$y''(t) + p(t)y(t) + \lambda y(t) = 0 \tag{1.5}$$

with time-periodic $p(t)$ and $\lambda \in \mathbb{R}$ is determined by the trace

$$\mathrm{tr}\; M(\lambda) = y_1(T, \lambda) + y_2'(T, \lambda)$$

of the Wronskian $M(\lambda)$ given by

$$\begin{pmatrix} y(T) \\ y'(T) \end{pmatrix} = M(\lambda) \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}.$$

Here $y_1(t, \lambda)$ (resp. $y_2(t, \lambda)$) is the solution of (1.5) with $y_1(0, \lambda) = 1$, $y_1'(0, \lambda) = 0$ (resp. $y_2(0, \lambda) = 0$, $y_2'(0, \lambda) = 1$). The intervals of instability are described by the set $\{\lambda : |\mathrm{tr}\; M(\lambda)| > 2\}$ (see for instance, [7]). Moreover, if λ_0 lies in an interval of instability, there exists an eigenvalue $\mu(\lambda_0)$, $|\mu(\lambda_0)| > 1$, of $M(\lambda_0)$, and we have

$$\lim_{m\to\infty} \Big|\mathrm{tr}\; M^m(\lambda_0)\Big| = \infty.$$

This phenomenon appears for the so called parametric resonances [6] and if $\lambda = l\frac{\pi^2}{T^2}$ with suitable $l \in \mathbb{N}$ there exist unbounded solutions.

For stationary perturbations, given by a potential $V(x)$, we have always resonances and some lower bounds for the function counting the number of the resonances have been established (see [4], [3] and the references cited there). In contrast to the stationary case, for time-periodic perturbations, it is possible to construct a potential $q(t, x)$ such that the corresponding operator $P(t)$ has no resonances $z \neq 0$. In Section 5 we treat this problem.

2. Meromorphic continuation of the resolvent $(U(T)-z)^{-1}$

In the following $\mathcal{H}$ will denote the space H_0. Given a resonance $z_0 \in \operatorname{Res} P$, consider the projection

$$\pi_{z_0,\,Z} = \frac{1}{2\pi i}\int_{\gamma_0} (z - Z(T))^{-1} dz,$$

where $\gamma_0 = \{z \in \mathbb{C} : z = z_0 + \epsilon e^{i\varphi},\, 0 \le \varphi < 2\pi\}$ and $\epsilon > 0$ is sufficiently small. The space $\pi_{z_0,\,Z}(\mathcal{H})$ has a finite dimension, independent on ρ, and we define the multiplicity of z_0 as

$$m(z_0) = \operatorname{rank} \pi_{z_0,Z}(\mathcal{H}).$$

Let $C_0 > 0$ be a constant such that $\|U(T)\| \le C_0$ and let the cut-off function χ, the projectors $P_\pm$ and the integer $k \in N$ be fixed as in Theorem 1.

Introduce a number $a_0 > r_1 + kT$ and let $\mathcal{H}_{R+a_0}$ be the space of the elements of $\mathcal{H}$ which vanish for $|x| \ge R + a_0$. Next define the space $\mathcal{H}_{\rm loc}$ as the space of functions u for which $\psi u \in \mathcal{H}$ for each $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighborhood of $B(0, R + a_0)$. Then we have the following.

Proposition 1. *The operator $(U(T) - z)^{-1} : \mathcal{H}_{R+a_0} \longrightarrow \mathcal{H}_{\rm loc}$ admits a meromorphic continuation from $|z| > C_0$ to $\mathbb{C}$. The poles of this continuation coincide with the resonances* Res P *and the geometric multiplicities are the same. Moreover, for every $z_0 \in \operatorname{Res} P$ we have*

$$\pi_{z_0,Z}(\mathcal{H}) = \pi_{z_0,Z}(\mathcal{H}_{R+a_0}) = \pi_{z_0,U}(\mathcal{H}_{R+a_0}), \tag{2.1}$$

where

$$\pi_{z_0,U} = \frac{1}{2\pi i}\int_{\gamma_0} (z - U(T))^{-1} dz : \mathcal{H}_{R+a_0} \longrightarrow \mathcal{H}_{\rm loc}.$$

Remark. The above result is similar to Proposition 3.6 in [14], where the resonances for compactly supported perturbations are defined by the method of complex scaling.

Proof. For $|z| > C_0$ we have $\chi(Z(T) - z)^{-1}\chi = \chi(U(T) - z)^{-1}\chi$ and the poles of $\chi(U(T) - z)^{-1}\chi$ are included in the set Res P. To prove the inverse, notice that

$$W(T) = Z_0(T) - Z(T) = P_+(U_0(T) - U(T))P_- = \chi(U_0(T) - U(T))\chi = \chi V(T)\chi, \tag{2.2}$$

where $V(T) = U_0(T) - U(T)$. Next, we have

$$\begin{aligned}
(Z(T) - z)^{-1} &= (Z(T) - z)^{-1}(Z_0(T) - Z(T))(Z_0(T) - z)^{-1} + (Z_0(T) - z)^{-1} \\
&= (Z_0(T) - z)^{-1}(Z_0(T) - Z(T))(Z(T) - z)^{-1}(Z_0(T) - Z(T))(Z_0(T) - z)^{-1} \\
&\quad + (Z_0(T) - z)^{-1}(Z_0(T) - Z(T))(Z_0(T) - z)^{-1} + (Z_0(T) - z)^{-1} \\
&= (Z_0(T) - z)^{-1}\chi V(T)\chi\,(U(T) - z)^{-1}\chi V(T)\chi(Z_0(T) - z)^{-1} \\
&\quad + (Z_0(T) - z)^{-1}\chi V(T)\chi(Z_0(T) - z)^{-1} + (Z_0(T) - z)^{-1}.
\end{aligned} \tag{2.3}$$

The resolvent $(Z_0(T) - z)^{-1}$ is holomorphic in $\mathbb{C} \setminus \{0\}$ and (2.3) implies that the eigenvalues of $Z(T)$ are inside the poles of $\chi(U(T) - z)^{-1}\chi$. Thus the resonances coincide with the poles of the meromorphic continuation of $\chi(U(T) - z)^{-1}\chi$ and it follows immediately that the geometric multiplicities are the same.

To establish (2.1), notice that according to (2.3), we have

$$\pi_{z_0,Z} = \frac{1}{2\pi i}\int_{\gamma_0}(z - Z(T))^{-1}\chi V(T)\chi(Z_0(T)-z)^{-1}dz.$$

Given $f \in \mathcal{H}$, we write

$$\chi V(T)\chi(Z_0(T)-z)^{-1}f = \sum_{j=0}^{N_0}(z-z_0)^j\chi f_{j,0} + \mathcal{O}_f((z-z_0)^{N_0+1}),\ z\in\gamma_0$$

and we obtain for $N_0 \gg 1$

$$\pi_{z_0,Z}f = \frac{1}{2\pi i}\int_{\gamma_0}(z-Z(T))^{-1}\sum_{j=0}^{N_0}(z-z_0)^j\chi f_{j,0}dz.$$

On the other hand, as in the paper of Sjöstrand and Zworski [14], we get

$$(z-z_0)^j - (Z(T)-z_0)^j$$
$$= (z-Z(T))\Big[(z-z_0)^{j-1} + (z-z_0)^{j-2}(z-Z(T)) + \cdots + (z-Z(T))^{j-1}\Big].$$

For $j \geq 1$ we replace $(z-z_0)^j$ by $(Z(T)-z_0)^j$ and we deduce

$$\pi_{z_0,Z}f = \pi_{z_0,Z}\Big(\sum_{j=0}^{N_0}(Z(T)-z_0)^j(\chi f_{j,0})\Big).$$

Next we exploit the equality

$$Z(jT) - Z_0(jT)$$
$$= -\sum_{\nu=0}^{j-1}Z_0(\nu T)(Z_0(T)-Z(T))Z((j-\nu-1)T).$$

Observing that $Z_0(\nu T) = 0$ for $\nu \geq k$, we deduce

$$Z(jT)\chi = Z_0(jT)\chi - \sum_{\nu=0}^{k-1}Z_0(\nu T)\chi V(T)\chi Z((j-\nu-1)T)\chi.$$

This implies

$$Z(jT)\chi = P_+\Phi,\ \forall j\in\mathbb{N},$$

where $\Phi \in C_0^\infty(B(0,r_1+kT);\ [0,1])$ is such that $(1-\Phi)U_0(jT)\chi = 0$ for $0 \leq j \leq k-1$. Since

$$\pi_{z_0,Z}P_+\Phi = P_+\pi_{z_0,Z}\Phi = \pi_{z_0,Z}\Phi,$$

we conclude that

$$\pi_{z_0,Z}(\mathcal{H}) = \pi_{z_0,Z}(\Phi\mathcal{H}) \subset \pi_{z_0,Z}(\mathcal{H}_{R+a_0}).$$

Finally, if $P_-^\rho\Phi = \Phi$ we have

$$\pi_{z_0,Z}(\mathcal{H}) = \pi_{z_0,U}(\Phi\mathcal{H}) + \frac{1}{2\pi i}(1-P_+^\rho)\int_{\gamma_0}(z-U(T))^{-1}\Phi dz :\ \mathcal{H}_{R+a_0} \longrightarrow \mathcal{H}_{\mathrm{loc}}.$$

The term involving $(1-P_+^\rho)$ is independent on the choice of P_+^ρ, provided $P_-^\rho\Phi = \Phi$, and it vanishes on every compact set. This completes the proof. □

3. Upper bound of the number of resonances.

In this section, we give a upper bound of the number of resonances lying in the disk

$$\{z \in \mathbb{C} : |z| \geq \delta\}, \ \delta > 0.$$

We will prove the following.

Theorem 2. *Suppose the assumptions (H_1), (H_2) fulfilled. Then the number of the resonances $z \in \operatorname{Res} P(t)$, $|z| > 1$ is finite and for each $0 < \varepsilon \leq 1/2$ there exists a constant $C_\epsilon > 0$ such that for every $0 < \delta \leq 1$ we have*

$$\#\{z \in \operatorname{Res} P : \ |z| \geq \delta\} \leq C_\epsilon \delta^{-\epsilon}. \tag{3.1}$$

Remarks. 1. For stationary potentials this result has been obtained by Melrose [11] (see the estimate (44)).

2. The above bound is natural for independent on time perturbations. Indeed, in this case, Melrose [11], Zworski [21], Sjöstrand and Zworski [14], Vodev [19], have proved that

$$\#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : \ |\sigma| \leq r\} \leq C r^n. \tag{3.2}$$

Moreover, if P is non-trapping, Vainberg [17] in the classical case and Martinez [10] in the semi-classical framework have showed that for each $N \in \mathbb{N}$ we have

$$\#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : \ |\operatorname{Im} \sigma| \leq N \ln(|\sigma|)\} < \infty. \tag{3.3}$$

This implies

$$\begin{aligned} \#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : \ &|\operatorname{Im} \sigma| \leq r\} \\ &\leq \#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : \ N \ln(|\sigma|) \leq |\operatorname{Im} \sigma| \leq r\} + C_N \\ &\leq \#\operatorname{Res} P \cap \{\sigma \in \mathbb{C} : \ |\sigma| \leq e^{r/N}\} + C_N \leq C'_N e^{rn/N}. \end{aligned} \tag{3.4}$$

Now, fixing a $T > 0$ and setting $z = e^{i\sigma T}$, we obtain the estimate (3.1) with $\epsilon = \frac{n}{TN}$.

Proof. We will exploit the method developed by Melrose [11], [12] for perturbations independent on time (see also Zworski [21] and Vodev [19]). To prove the theorem, it is sufficient to show that there exists $N \in \mathbb{N}$ such that for each $\varepsilon > 0$, the eigenvalues of the operator $Z(NT)$ satisfy for all $0 < \delta \leq 1$ the estimate

$$\#\{z \in \mathbb{C} : \ z \in \sigma_{\rm pp}(Z(NT)), \ |z| \geq \delta\} \leq C_\epsilon \delta^{-\varepsilon}. \tag{3.5}$$

Given a compact operator S, we denote by $\mu_j(S)$, $j = 1, 2, \ldots$, the characteristic values of S which form a non-increasing sequence of the eigenvalues of $(S^*S)^{1/2}$ counted with their multiplicity. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $k \in \mathbb{N}$ be fixed as in Theorem 1 so that $Z_0(kT) = 0$. For $M \in \mathbb{N}$, we have

$$\begin{aligned} Z((2k+M)T) &= Z(kT) Z(MT) Z(kT) \\ &= \big(Z(kT) - Z_0(kT)\big) Z(MT) \big(Z(kT) - Z_0(kT)\big) \\ &= P_+ \big(U(kT) - U_0(kT)\big) U(MT) \big(U(kT) - U_0(kT)\big) P_- \\ &= P_+ \big(U(kT) - U_0(kT)\big) \chi U(MT) \chi \big(U(kT) - U_0(kT)\big) P_-. \end{aligned} \tag{3.6}$$

Since the perturbation of $P(t)$ is given by a potential, the results for the propagation of singularities imply that the operator $\chi U(MT) \chi$ is regularizing for $M \in \mathbb{N}$ large enough (see [5], [1], [13], [18]). Let $\Omega \subset\subset \mathbb{R}^{2n}$ be a open hypercube, with $\operatorname{supp} \chi \subset \Omega$, and let Δ_Ω be the Laplacian in Ω

with Dirichlet boundary condition. It is well known (see for instance, [21], [19]) that for all $m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$\mu_j\big((I-\Delta_\Omega)^{-m}\big) \leq C_m j^{-2m/n}, \ \ \forall j \in \mathbb{N}.$$

Consequently, using (3.6) and the inequalities

$$\mu_j(AB) \leq \mu_j(A)\|B\|,$$
$$\mu_j(AB) \leq \mu_j(B)\|A\|,$$

we get, for $m \in \mathbb{N}$,

$$\begin{aligned} \mu_j\big(Z((2k+M)T)\big) &\leq C\mu_j\big(\chi U(MT)\chi\big) \\ &\leq C\mu_j\big((I-\Delta_\Omega)^{-m}(I-\Delta_\Omega)^m\chi U(MT)\chi\big) \\ &\leq C\mu_j\big((I-\Delta_\Omega)^{-m}\big)\big\|(I-\Delta_\Omega)^m\chi U(MT)\chi\big\| \\ &\leq C_m j^{-2m/n} \end{aligned} \tag{3.7}$$

with a new constant $C_m > 0$.

We choose $N = 2k+M$, $2m > n$ and we order the eigenvalues

$$\lambda_1, \lambda_2, ..., \lambda_p, ...$$

of $Z(NT)$ counted with their multiplicities by decreasing modulus. Then

$$|\lambda_p|^p \leq \prod_{j=1}^p |\lambda_j| \leq \prod_{j=1}^p \mu_j(Z(NT)) \leq (C_k)^p (p!)^{-k},$$

where $k \in \mathbb{N}$ can be taken as large as we wish. Thus with a constant C_k', we get

$$|\lambda_p| \leq C_k (p!)^{-\frac{k}{p}} \leq C_k' p^{-k}.$$

Now for the eigenvalues $\lambda_1, ..., \lambda_p$ with modulus greater than $\delta > 0$ we deduce

$$p \leq C_k \delta^{-\frac{1}{k}}$$

and taking $k = \frac{1}{\epsilon}$, we complete the proof. □

4. Trace formula

In this section we prove Theorem 1. Recall that $\chi \in C_0^\infty(\mathbb{R}^n)$, the projectors $P_\pm$ and $k \in \mathbb{N}$ are fixed so that (1.2) and (1.3) hold. First notice that

$$U(kT) - U_0(kT) = \sum_{j=0}^{k-1} U(jT)(U(T)-U_0(T))U_0((k-j-1)T) \tag{4.1}$$

$$= P_-(U(kT)-U_0(kT)) = (U(kT)-U_0(kT))P_+.$$

The second and the third equalities follow from the fact that

$$(I-P_-)U(jT)\chi = \chi U_0(jT)(I-P_+) = 0, \ j = 0, ..., k-1.$$

The operator

$$P_+ U(mT - 2kT) P_-$$

is trace class for m sufficiently large and the cyclicity of the trace implies

$$\begin{aligned}
&\operatorname{tr}\Big((U(kT)-U_0(kT))U(mT-2kT)(U(kT)-U_0(kT))\Big)\\
&=\operatorname{tr}\Big(P_-(U(kT)-U_0(kT))P_+U(mT-2kT)P_-(U(kT)-U_0(kT))P_+\Big)\\
&=\operatorname{tr}\Big(P_+(U(kT)-U_0(kT))P_-P_+U(mT-2kT)P_-P_+(U(kT)-U_0(kT))P_-\Big)\\
&=\operatorname{tr}\Big(P_+U(kT)P_-P_+U(mT-2kT)P_-P_+(U(kT)P_-\Big)\\
&=\operatorname{tr}\Big(P_+U(mT)P_-\Big)=\operatorname{tr}(Z(mT)).
\end{aligned}$$

Applying Lidsii theorem for the trace of $Z(mT)$, we complete the proof since by Theorem 2 we have

$$\Big|\sum_j z_j^m\Big| \leq \sum_{p=1}^{\infty} \sum_{\frac{C}{p+1}<|z_j|\leq \frac{C}{p}} |z_j^m| \leq C_\epsilon \sum_{p=1}^{\infty}\Big(\frac{C}{p}\Big)^{m-\epsilon} \leq C_m,\ 0<\epsilon\leq 1/2,\ m\geq 2.$$

□

It is clear that Corollary 1 follows from the following

Lemma 1. *Let*

$$A_m=\sum_{|z_j|\leq 1} z_j^m,\ B_m=\sum_{|z_j|>1} z_j^m,\ m\in\mathbb{N}.$$

Then

$$|A_m|\leq C_0,\ \forall m\geq 1+\epsilon_0>1.$$

Moreover, if $\{z\in \operatorname{Res} P(t):\, |z|>1\}\neq\emptyset$*, then there exists a sequence* $m_\nu\nearrow\infty$*,* $m_\nu\in\mathbb{N}$*, such that*

$$\lim_{m_\nu\to\infty}|B_{m_\nu}|=\infty.$$

Proof. Let $m-\epsilon>1,\ \epsilon>0$. Using the estimate

$$\#\{z_j\in \operatorname{Res} P(t):\, |z_j|\geq\delta\}\leq C_\epsilon\delta^{-\epsilon},$$

we obtain

$$|A_m|\leq \sum_{k=1}^{\infty}\sum_{\frac{1}{k+1}<|z_j|\leq\frac{1}{k}}|z_j|^m\leq C_\epsilon\sum_{k=1}^{\infty}\Big(\frac{1}{k}\Big)^m\Big(\frac{1}{k+1}\Big)^{-\epsilon}\leq C'_\epsilon.$$

To deal with the sum B_m, introduce

$$\mu=\max\{|z_j|:\ \ z_j\in \operatorname{Res} P(t),\ |z_j|>1\}.$$

Since we have a finite number of resonances z_j with $|z_j|>1$, let

$$z_j=\mu e^{i\varphi_j},\ j=1,...,p,\ \varphi_\nu\neq\varphi_j\ (\mathrm{mod}\, 2\pi),\ \nu\neq j.$$

It is sufficient to show that for a suitable sequence $m_\nu\nearrow\infty$ we have

$$\lim_{m_\nu\to\infty}\Big|\sum_{j=1}^{p}c_je^{im_\nu\varphi_j}\Big|\geq\epsilon_0>0,$$

where $c_j\in\mathbb{N}$ is the multiplicity of the resonance $z_j,\ j=1,...,p$.

Put $a_j = e^{i\varphi_j}$, $j = 1, ..., p$ and assume that

$$\lim_{m\to\infty} \sum_{j=0}^{p} c_j a_j^m = 0$$

for some integers $c_j \in \mathbb{N}$, $j = 1, ..., p$. Obviously,

$$\sum_{j=0}^{p} a_j^q c_j a_j^m \longrightarrow_{m\to\infty} 0 \text{ for } q = 0, 1, \dots p-1.$$

This implies

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_p \\ \dots & \dots & \dots & \dots \\ a_1^{p-1} & a_2^{p-2} & \dots & a_p^{p-1} \end{pmatrix} \begin{pmatrix} c_1 a_1^m \\ c_2 a_2^m \\ \dots \\ c_p a_p^m \end{pmatrix} \longrightarrow 0$$

and we deduce that $(c_1 a_1^m, \dots, c_p a_p^m) \longrightarrow 0$ which is a contradiction. Thus there exists a sequence $m_\nu \nearrow \infty$ such that

$$\sum_j c_j a_j^{m_\nu} \longrightarrow \beta \neq 0 \text{ as } m_\nu \to \infty$$

and this completes the proof. □

Finally, we may establish a trace formula for the operator

$$g(U(T)) = U((m-2k)T) \sum_{j=0}^{\infty} b_j U(jT),$$

where the series $h(z) = \sum_{j=0}^{\infty} b_j z^j$ has a radius of convergence $R_0 > \|U(T)\|$ and $m \in \mathbb{N}$ is chosen so that $Z((m-2k)T)$ is a trace class. First notice that

$$\|Z((m-2k)T) \sum_{j=p}^{p+q} b_j Z(jT)\|_{\mathrm{tr}} \leq \|Z((m-2k)T)\|_{\mathrm{tr}} \sum_{j=p}^{p+q} |b_j| \|Z(T)\|^j \leq \epsilon$$

for p, $q \geq N(\epsilon)$. Since the space of trace class operators is complete in trace norm, we deduce that $g(Z(T))$ is trace class and this yields

$$\mathrm{tr}\Big(Z((m-2k)T) \sum_{j=0}^{N} b_j Z(jT)\Big) \longrightarrow \mathrm{tr}(g(Z(T)) \text{ as } N \to \infty.$$

Next, the operator

$$(U(kT) - U_0(kT))U(mT - 2kT) \sum_{j=0}^{N} b_j U(jT)(U(kT) - U_0(kT))$$

converges in the operator norm to $(U(kT) - U_0(kT))g(U(T))(U(kT) - U_0(kT))$ and

$$\mathrm{tr}\Big((U(kT) - U_0(kT))U(mT - 2kT) \sum_{j=0}^{N} b_j U(jT)(U(kT) - U_0(kT))\Big) \longrightarrow \mathrm{tr}\, g(Z(T)).$$

Applying the result of Gohberg and Krein (see Chapter 6 in [9]), we obtain the following

Theorem 3. *Let $g(z) = z^{m-2k}h(z) = z^{m-2k}\sum_{j=0}^{\infty} b_j z^j$ be a function such that the series $\sum_{j=0}^{\infty} b_j z^j$ has in $\mathbb{C}$ a radius of convergence $R_0 > \|U(T)\|$ and let m, k be chosen as in Theorem 1. Then*

$$\operatorname{tr}\Big((U(kT) - U_0(kT))g(U(T))(U(kT) - U_0(kT))\Big) = \sum_{z_j \in \operatorname{Res} P(t)} g(z_j).$$

5. Example

In this section we construct a potential $q(t, x)$ such that $Z(T) = 0$ which implies that we have no resonances $z \in \operatorname{Res} P \setminus \{0\}$. Assume that $T = t_1 + t_0,\ \ t_1 > 0,\ t_0 > 0$. We choose a potential $q(t, x)$ satisfying the assumptions (H_1), (H_2) such that

$$q(t, x) = 0 \ \text{ for } 0 < t_0 \le t \le T,\ \forall x. \tag{5.1}$$

Moreover, the support of $q(t, x)$ with respect to x is independent of t_0, t_1. We obtain

$$U(T, 0) = U(t_1 + t_0, 0) = U(T, t_0)U(t_0, 0)$$
$$= U_0(t_1)\Big[U_0(t_0) - \int_0^{t_0} U_0(t_0 - s)Q(s)U(s, 0)ds\Big].$$

Here we have used the fact that (5.1) implies $U(T, t_0) = U_0(T - t_0) = U_0(t_1)$. We fix the projectors P_+, P_-, independently of t_1, so that $P_\pm Q(s) = Q(s)$. Next we choose the time t_1 large enough so that

$$P_+U_0(t_1)P_- = 0.$$

This implies

$$Z(T) = P_+U(T, 0)P_- = P_+U(T, t_0)U(t_0, 0)P_- = P_+U_0(t_1)P_-U(t_0, 0)P_- = 0,$$

since $(I - P_-)U(t_0, 0)P_- = 0$.

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Département de Mathématiques Appliquées, Université Bordeaux I, 351, Cours de la Libération, 33405 Talence, France

E-mail address: `bony@@math.u-bordeaux.fr, petkov@@math.u-bordeaux.fr`