LOWER BOUNDS ON THE NUMBER OF SCATTERING POLES FOR SEVERAL STRICTLY CONVEX OBSTACLES

VESSELIN PETKOV

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open and connected domain with C^∞ boundary $\partial\Omega$ and bounded complement

$$K = \mathbb{R}^3 \setminus \Omega \subset \{x \in \mathbb{R}^3 : |x| \leq \rho_0\}.$$

Denote by $\nu(x)$ the exterior unit normal to $\partial\Omega$ at x pointing into Ω and consider the Neumann problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \text{ in } \mathbb{R} \times \Omega, \\ \partial_\nu u = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x). \end{cases} \tag{1.1}$$

We can associate to (1.1) a *scattering operator*

$$S(\lambda) : L^2(S^2) \longrightarrow L^2(S^2),\ \ \lambda \in \mathbb{R}$$

which admits a meromorphic continuation in $\mathbb{C}$ with poles λ_j, $\operatorname{Im}\lambda_j > 0$ (see [13]). Let Λ be the set of the scattering poles λ_j counted with their multiplicity.

Throughout this paper we assume that K has the form

$$K = \cup_{j=1}^{Q} K_j,\ \ K_i \cap K_j = \emptyset, \text{ for } i \neq j, \tag{1.2}$$

where K_j are strictly convex compact obstacles for $j = 1, \dots, Q$ and $Q \geq 3$. Moreover, we suppose that K satisfies the following condition introduced by Ikawa ([9]):

(H) The convex hull of every two connected components of K does not have common points with any other connected component of K.

Consider the reflecting rays in $\overline{\Omega}$ (see [9] and Chapter 2 in [19] for a precise definition). Under the condition (H) every periodic ray is ordinary reflecting, that is γ has no tangent segments. Given a *periodic reflecting ray* γ in $\overline{\Omega}$ with m_γ reflections, we denote by T_γ the primitive period (length) of γ, by d_γ the period of γ and by P_γ the linear Poincaré map related to γ. Setting $|\det(I - P_\gamma)| = |I - P_\gamma|$, it is easy to prove (see [18], Appendix) that there exist constants $b_1 > 0,\ b_2 > 0,\ B_0 > 0$ so that

$$B_0 e^{2b_1 d_\gamma} \leq |I - P_\gamma| \leq e^{2b_2 d_\gamma}. \tag{1.3}$$

Denote by Ξ the set of all reflecting periodic rays in $\overline{\Omega}$ and set

$$d_0 = \min\ \operatorname{dist}_{i\neq j}\ (K_i, K_j),\ \ d_1 = \max\ \operatorname{dist}_{i \neq j}(K_i, K_j).$$

For the counting function of the lengths of periodic rays there exists a constant $a_0 > 0$ such that

$$\sharp\{\gamma \in \Xi : d_\gamma \leq q\} \leq e^{a_0 q}$$

(see [9] and Chapter 2 in [19]). Finally, it is well known ([16], [22], [26]) that there exists a constant $A > 0$ so that

$$N(r) = \sharp\{\lambda_j \in \Lambda : |\lambda_j| \leq r\} \leq Ar^3. \tag{1.4}$$

It is important to note that the constants B_0, a_0, b_1, b_2 depend only on the geometry of the obstacle, while A depends on the diameter of K and the coercive estimate for the Neumann problem (1.1).

The problem of the existence of a domain $\{z \in \mathbb{C} : 0 < \operatorname{Im} z \leq \delta\}$ containing an infinite number of scattering poles of (1.1) has been studied by Ikawa [10], [11] who proved that there exists $\delta_0 > 0$ such that

$$\sharp\{\lambda \in \Lambda :\ 0 < \operatorname{Im} \lambda_j \leq \delta_0\} = \infty.$$

A similar result for hyperbolic surfaces has been obtained in [5]. On the other hand, Sjöstrand and Zworski [23] (see also [21] for the case of even dimension) obtained for the Neumann problem the lower bound

$$\sharp\{\lambda_j \in \Lambda :\ \operatorname{Im} \lambda_j \leq \omega |\ln \lambda_j|,\ |\operatorname{Re} \lambda_j| \leq r\} \geq C_\omega r,\ \omega > 0,\ r \geq r(\omega).$$

There are only few works concerning the lower bounds of the scattering poles in a domain $\{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \delta\}$ in the case when all periodic trajectories are hyperbolic, that is the Poincaré map P_γ has no eigenvalues on S^1. The case of two convex obstacles has been treated in [4]. Recently, for hyperbolic surfaces M of finite geometry Guillopé and Zworski [7] established the lower bound

$$\sharp\{\lambda \in \mathcal{R}_M :\ \operatorname{Im} \lambda < \epsilon^{-1},\ |\operatorname{Re} \lambda| \leq r\} = \Omega(r^{1-\epsilon}),\ 0 < \epsilon < 1/2, \tag{1.5}$$

where $\mathcal{R}_M$ denotes the set of scattering poles related to M and we have $f(r) = \Omega(g(r))$ if there does not exist a constant C for which $|f(r)| \leq C|g(r)|$. A more precise result for convex co-compact hyperbolic surfaces has been proved by Zworski [27] who obtained for the left hand side of (1.5) an upper bound $\mathcal{O}(r^{1+\delta_1})$, where $\delta_1 \geq 0$ is related to the Hausdorff dimension of the *trapped set* (see also [20] for similar results concerning semi-classical problems).

Our purpose in this note is to obtain a lower bound like (1.5) for the counting function

$$N_{0,\delta}(r) = \sharp\{\lambda_j \in \Lambda : 0 < \operatorname{Im} \lambda_j \leq \delta, |\operatorname{Re} \lambda_j| \leq r\}.$$

To describe our results we need to introduce some constants.

Choose θ and μ so that $2/3 < \theta < \mu < 1$. Next choose $\nu > 0$, fix $k \in \mathbb{R}$ so that $-\frac{1}{1-\theta} < k < -1$ and choose $a > k$ satisfying the inequalities $k\theta < a\theta < k+1$. Throughout the paper we assume that μ, θ, a, k, ν are fixed. Given $0 < \epsilon < 1/3$, fix a constant $\alpha = \frac{\theta}{\epsilon}(\max(0, a_0 - b_1) + b_2)$, put $\beta = (a_0/d_0 + \alpha)$, $\kappa = \theta/\beta$ and introduce

$$\delta = \frac{(3 - k + \nu)}{\kappa}.$$

The main result is our paper is the following.

Theorem 1. *Let K be an obstacle in the form* (1.2) *satisfying the condition* (H) *and let δ be chosen as above. Then we have*

$$N_{0,\delta}(r) \geq c_1 r^{\theta - \epsilon - b_2\kappa} \ln r - c_2,\ r \geq c_0, \tag{1.6}$$

where the constants $c_i > 0$, $i = 0, 1, 2$, depend on $\epsilon, \alpha, \beta, a, k, \theta, \mu, B_0$, the constant A in (1.4) *and the constant C_3 in Theorem* 2.

Remark 1. It is clear that $b_2\kappa \leq \epsilon$ and taking $1 - \theta = \epsilon$, we obtain

$$N_{0,\delta}(r) \geq c_1 r^{1-3\epsilon} \ln r - c_2,\ r \geq c_0(\epsilon).$$

The lower bound (1.6) is proved exploiting only the contribution of a suitable sequence of periodic rays with primitive periods $T_j \longrightarrow +\infty$. To obtain more precise results for the density of scattering poles in $\{z \in \mathbb{C} : 0 < \operatorname{Im} z \leq \delta\}$, we must have some information for the distribution of the lengths of the periodic rays in small intervals $L_j(\beta_0) = (T_j - e^{-\beta_0 T_j}, T_j + e^{-\beta_0 d_j})$, where $\beta_0 > 0$ is large enough. It is rather difficult to obtain some lower bounds for the number of periods in $L_j(\beta_0)$ and in Section 2 we show that under a separation condition (S) this number is bounded by $A_0 T_j$. A more complete analysis must take into account the Hausdorff dimension $H_0 = \dim_H(M_0)$ of the *non-wandering set* M_0 (see for instance [24] for the definition of M_0). Recently, for obstacles $K \subset \mathbb{R}^2$, satisfying (H), R. Kenny [14] proved that H_0 is always *positive* and, moreover, he established some lower and upper bounds for H_0. Thus, as in [7], it is natural to conjecture that for δ large enough the bounds of $N_{0,\delta}(r)$ will depend on H_0.

2. Lengths of primitive periodic rays

In this section we construct a sequence of primitive periods $T_j \to \infty$ so that $T_{j+1} - T_j \leq 1$. First let us recall the result of Theorem 1.3 in [24] (see also [17] for dimension n = 2) which yields the following asymptotic of the counting function

$$L(x) = \sharp\{T_\gamma,\ \gamma \in \Xi : T_\gamma \leq x\} = \frac{e^{hx}}{hx}\Big(1 + o(1)\Big),\ x \longrightarrow +\infty,$$

where $h > 0$ is the topological entropy of K. Fixing $y > 0$, it is very easy to obtain a lower bound of the function $L(x+y) - L(x)$ as $x \to \infty$. In fact, for $\eta = \eta(h, y) > 0$ sufficiently small and for $x \geq \max\{\frac{1}{h}, X(\eta)\}$, we have

$$\begin{aligned} L(x+y) - L(x) &\geq \frac{e^{hx}}{hx}\Big(\frac{e^{hy}}{1+hy}(1-\eta) - (1+\eta)\Big) \\ &\geq \frac{e^{hx}}{hx}\Big(\frac{h^2y^2}{2(1+hy)} - C_{h,y}\eta\Big) \geq \frac{hy^2 e^{hx}}{3x(1+hy)} \end{aligned}$$

taking $X(\eta)$ large enough. Thus, there exists a sequence of primitive periods $T_j \longrightarrow \infty$ such that

$$T_{j+1} - T_j \leq 1,\ \forall j \geq j_0.$$

Consider a sequence $q_j \in \mathbb{N},\ j \in \mathbb{N}$, so that

$$q_j - 2/3 < T_j < q_j + 2/3,\ \ \forall j.$$

Then we have

$$T_{j+1} - e^{-\beta q_{j+1}} \leq T_j - e^{-\beta q_j} + 2,\ \ j \geq j_0. \tag{2.1}$$

We may expect to improve Theorem 1 if the density of the periods in small neighbourhoods of T_j is sufficiently large (see Remark 3 in Section 3). It seems less hopeful that a such *clustering phenomenon* holds for generic obstacles. As an illustration consider the case when the following *separation condition* holds.

(S) All primitive periods can be ordered as follows

$$T_1 < T_2 < ... < T_n < ....,$$

and there exists $M \gg h$ so that

$$T_{k+1} - T_k \geq e^{-MT_k},\ \forall k \in \mathbb{N}.$$

Notice that the first assumption is generically fulfilled since for generic obstacles the primitive periods are rationally independent (see Chapter 3 in [19]). It is natural to *conjecture* that the second one is also true for generic obstacles. Assume (S) fulfilled and choose $\beta > M$. It is clear that $|mT_\gamma - T_j| \leq e^{-\beta T_j}$ implies $m \leq \frac{T_j+1}{2d_0}$. Moreover, if for two primitive periods $T_{\gamma_1}, T_{\gamma_2}$ we have

$$|lT_{\gamma_1} - mT_{\gamma_2}| \leq 2e^{-\beta T_j},\ \ m, l \in \mathbb{N},\ \ m \geq 2,\ l \geq 2,\ j \geq j_0,$$

then $l \neq m$. Thus, we can find a constant $A_0 > 0$, depending only on d_0, so that

$$\sharp\{d_\gamma \in \Xi : |d_\gamma - T_j| \leq e^{-\beta T_j}\} \leq A_0 T_j,\ \forall j \geq j_0$$

and the density of periods in exponentially small neighbourhoods of T_j grows as $A_0 T_j$. To our best knowledge, there are no examples, where the condition (S) holds. On the other hand, it is very difficult to construct examples when all primitive periods are different and (S) is not true. Finally, notice that a similar separation condition for the scattering poles plays an important role in the analysis of the resonances expansions for trapping obstacles [25].

3. Lower bound for the function $N_{0,\delta}(r)$

In this section we use freely the notations of the previous sections. Let $\rho(t) \in C_0^\infty(-1,1)$ be a function such that

$$\rho(t) \geq 0,\ \ \rho(-t) = \rho(t),\ \ \forall t \in \mathbb{R},\ \ \rho(t) = 1 \ \text{ on } [-\epsilon_0, \epsilon_0],\ \ 0 < \epsilon_0 < 1/2,$$

$$\hat{\rho}(0) = 1,\ \ \hat{\rho}(\xi) \geq 0,\ \ \forall \xi \in \mathbb{R},$$

where $\hat{\rho}(\xi)$ denotes the Fourier transform of $\rho(t)$. Introduce the function

$$\varphi_j(t) = \rho(e^{\beta q_j}(t - T_j)),$$

where T_j and q_j are defined in Section 2. The distribution $U(t) \in \mathcal{D}'(\mathbb{R})$, given by

$$U(t) = \begin{cases} \sum_{\lambda_j \in \Lambda} e^{i\lambda_j t} & \text{for } t > 0, \\ \sum_{\lambda_j \in \Lambda} e^{-i\lambda_j t} & \text{for } t < 0, \end{cases}$$

plays an important role in the analysis of the distribution of the scattering poles (see [8], [10], [11], [23]), [3], [4], [7]).

Our purpose is to obtain an estimate from below for $N_{0,\delta}(r)$ and to do this we will use an argument due to Farhy [4]. We take $\gamma_j = e^{-\beta q_j}$ and choose

$$\rho_j = \frac{3 - k + \nu}{T_j - \gamma_j} > 0,$$

where k and ν are fixed as in Section 1. Let $\widehat{\varphi U}(\lambda)$ denote the Fourier transform $\langle e^{i\lambda t}\varphi(t), U(t)\rangle$. For the counting function

$$N_\omega(r) = \sharp\{\lambda_j \in \Lambda : \operatorname{Im}\lambda_j \leq \omega|\ln\lambda_j|, |\operatorname{Re}\lambda_j| \leq r\}$$

Farhy, developing the method of Sjöstrand and Zworski [23], established the following.

Proposition 1. *([3], Proposition 2.3) Let $k \neq -1, r > 2, a > k,\ b,\ p > 3, \mu \in (0,1), \nu > 0$. Then we have the estimate*

$$N_{\rho_j}(r) \geq C \int_1^{r/2} |\widehat{\varphi_j U}(\lambda)| d\lambda - C_{a,k,\nu} \gamma_j^{-a} r^{k+1} - C_b \gamma_j^{-b} r^{4-b} - C_{p,\mu} \gamma_j^{-p} r^{3-\mu p}, \tag{3.1}$$

where the positive constants C, $C_{a,k,\nu}$, C_b, $C_{p,\mu}$ are independent on j and r.

Remark 2. A detailed analysis of the proof of the above result shows that the constants C, $C_{a,k,\nu}$, C_b, $C_{p,\mu}$ depend on the parameters a, k, ν, b, p, μ and the constant A in (1.4).

Modifying the proof of Theorem 2.4 in [4], we obtain the following.

Proposition 2. *Let T_j be the sequence introduced in Section 2 and let $b_\pm(r) = \kappa \ln r \pm B$, $\kappa = \frac{\theta}{\beta}$. Then for $r \geq R_0$ we have the estimate*

$$N_{0,\delta}(r) \geq C_0 \min_{T_j \in [b_-(r),\ b_+(r)]} \int_1^{C_1 r} |\widehat{\varphi_j U}(\lambda)| d\lambda - C_2, \tag{3.2}$$

where the positive constants B, R_0, C_0, C_1, C_2 depend on $\beta, \delta, d_0, \theta, \mu$ and the constant A in (1.4).

Proof. Clearly, $\mathrm{Re}\{z \in \mathbb{C} : \mathrm{Im}\, z = \delta,\ \omega \ln |z| = \delta\} = \pm\Big(e^{2\delta/\omega} - \delta^2\Big)^{1/2}$. Setting

$$a_j = \Big(\exp(2\delta/\rho_j) - \delta^2\Big)^{1/2},\ j \geq j_0,$$

and applying the estimate (3.1) for $N_{\rho_j}(a_j)$, we obtain

$$N_{0,\delta}(a_j) \geq N_{\rho_j}(a_j) \geq C \int_1^{a_j/2} |\widehat{\varphi_j U}(\lambda)| d\lambda$$
$$-C_{a,k,\nu} \exp(a\beta q_j) r^{k+1} - C_b \exp(b\beta q_j) r^{4-b} - C_{p,\mu} \exp(p\beta q_j) r^{3-\mu p},\ j \geq j_0.$$

Here and below j_0 denotes an integer independent on j which can change from line to line. Taking into account (2.1), it is easy to see that

$$a_{j+1}^2 \leq \exp\Big(\frac{4}{\kappa}\Big)(a_j^2 + \delta^2) - \delta^2. \tag{3.3}$$

Consequently, we can find a constant $C_\delta > 0$ so that

$$a_{j+1} \leq C_\delta\, a_j,\ \ \forall j \geq j_0.$$

Now assume that for some integer $j \geq j_0$ we have $a_j \leq r < a_{j+1} \leq C_\delta\, a_j$. It follows that with a constant $A_{\delta,k,\nu} > 0$, independent on r and j, we have

$$T_j \kappa^{-1} - A_{\delta,k,\nu} \leq \ln r \leq T_j \kappa^{-1} + A_{\delta,k,\nu},\ \ j \geq j_0.$$

Take $B = \kappa A_{\delta,k,\nu}$ in the definition of $b_\pm(r)$ and set $C_0 = C$, $C_1 = \frac{1}{2C_\delta}$. Since $\beta\kappa = \theta$, a combination of the above estimates yields

$$N_{0,\delta}(r) \geq C_0 \min_{T_j \in [b_-(r),\, b_+(r)]} \int_1^{C_1 r} |\widehat{\varphi_j U}(\lambda)| d\lambda$$
$$-C_{a,\kappa,\delta,k,\nu}\, r^{a\theta + k + 1} - C_{b,\kappa,\delta,k,\nu}\, r^{4 + b(\theta - 1)} - C_{p,\mu,\kappa,\delta,k,\nu}\, r^{3 + p(\theta - \mu)},\ j \geq j_0.$$

Next we choose b and p large enough to arrange $4 + b(\theta - 1) < 0,\ \ 3 + p(\theta - \mu) < 0$ and, according to the choice of a and k, we get $a\theta + k + 1 < 2k + 2 < 0$. This completes the proof. $\square$

To study $\widehat{\varphi_j U}(\lambda)$, we will exploit the trace formula (see [1], [15])

$$\widehat{\varphi_j U}(\lambda) = 2\mathrm{tr}_{L^2(\mathbb{R}^3)} \int_{\mathbb{R}} e^{i\lambda t} \varphi_j(t) \Big(\cos(t\sqrt{-\Delta}) \oplus 0 - \cos(t\sqrt{-\Delta_0}) \Big) dt.$$

Here Δ_0 is the Laplacian in $\mathbb{R}^3$, Δ is the Laplacian in Ω with Neumann boundary condition on $\partial\Omega$ and $\cos(t\sqrt{-\Delta}) \oplus 0$ acts as 0 on $L^2(K)$. Denote by $E_0(t,x,y)$, $E(t,x,y)$ the kernels of the operators $\cos(t\sqrt{-\Delta_0})$, $\cos(t\sqrt{-\Delta})$, respectively, and set

$$\tilde{E}(t,x,y) = \begin{cases} E(t,x,y), & (x,y) \in \Omega \times \Omega, \\ 0 \ \text{for} \ (x,y) \notin \Omega \times \Omega. \end{cases}$$

It is clear that

$$\mathrm{supp}_{x,y} \Big(\tilde{E}(t,x,y) - E_0(t,x,y) \Big) \subset \{(x,y) \in \mathbb{R}^6 : |x| \leq \rho_0 + t, \ |y| \leq \rho_0 + t\}$$

so for $t \in \mathrm{supp}\ \varphi_j(t)$ the integration with respect to x can be taken over $\Omega_j = \{x \in \mathbb{R}^n : |x| \leq \rho_0 + q_j + 1\}$ and we must study the trace

$$\int_{\Omega_j} \langle e^{i\lambda t} \varphi_j(t), \tilde{E}(t,x,x) \rangle dx. \tag{3.4}$$

Since $\tilde{E}(t,x,y)$ is even with respect to t, we may write (3.4) in the form

$$\int_{\Omega_j} \langle e^{-i\lambda t} \tilde{\varphi_j}(t), \tilde{E}(t,x,x) \rangle dx$$

with $\tilde{\varphi_j}(t) = \rho(e^{\beta q_j}(t+T_j))$ supported in $(-T_j - e^{-\beta q_j}, -T_j + e^{-\beta q_j})$. This shows that the behaviour of the reflecting rays for $-T_j \leq t \leq 0$ can be exploited for the analysis of the trace in the same way as the behaviour of the rays for $0 \leq t \leq T_j$.

The singularities of $U(t)$ are related only to the periodic ordinary reflecting rays and the distribution $U(t)|_{\mathbb{R}^+}$ has the form $U(t)|_{\mathbb{R}^+} = F(t) + V(t)$ with

$$F(t) = \sum_{\gamma \in \Xi} T_\gamma |I - P_\gamma|^{-1/2} \delta(t - d_\gamma), \ V(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+).$$

We refer to [6] and to Section 6.3 in [19] for the details of the calculation of the leading singularities involved in $F(t)$. Obviously,

$$|\widehat{\varphi_j V}(\lambda)| \leq C(j) e^{-\beta T_j}, \ j \geq j_0$$

with a function $C(j)$ depending on j.

To estimate the growth of $C(j)$ as $j \longrightarrow \infty$, we may apply the argument of Ikawa [8], [11], [12] based on the construction of asymptotic solutions for the problem (1.1) (see also [2]). Given a function $\psi \in C_0^\infty(\Omega)$, the kernel $E(t,x,y)\psi(y)$ of the operator $\cos\Big(t\sqrt{-\Delta}\Big)\psi$ admits the representation

$$E(t,x,y)\psi(y) = (2\pi)^{-3} \int_{S^2} d\omega \int_0^\infty k^2 u(t,x;k,\omega) e^{-ik\langle y,\omega \rangle} \psi(y) dk,$$

where $u(t,x;k,\omega)$ is the solution of the problem (1.1) with $f_1(x) = \Psi(x) e^{ik\langle x,\omega\rangle}, f_2(x) = 0$ and $\Psi(x) \in C_0^\infty(\Omega)$ is equal to 1 on supp ψ. In this way the analysis is reduced to the investigation of the integral

$$\int\int_{\Omega_j} e^{i\lambda t}\varphi_j(t)(t)E(t,x,x)\psi(x)dtdx$$
$$= (2\pi)^{-3}\int_{\mathbb{R}} dt\int_{\Omega_j} dx\int_{S^2} d\omega\int_0^\infty k^2 e^{i\lambda t}\varphi_j(t)u(t,x;k,\omega)\psi(x)e^{-ik\langle x,\omega\rangle}dk.$$

In the works of Ikawa [8], [9] and Burq [2] it was constructed an asymptotic solution $u^{(N)} = w_-^{(N)} + w_+^{(N)}$ of $u(t,x;k,\omega)$ with

$$w_\pm^{(N)}(t,x;k,\omega) = \sum_J \exp\Big(ik(\varphi_J(x,\omega)\mp t)\Big)\sum_{p=0}^N v_{\pm,J,\,p}(t,x;\omega)(ik)^{-p}.$$

Here $\omega \in S^2$, $k \in \mathbb{R}^+$, and

$$J = (j_1, j_2, ..., j_n),\ j_k \in \{1, ..., Q\},\ j_k \neq j_{k+1},\ k = 1, 2, .., n-1,\ |J| = n$$

are configurations related to the rays reflecting on $\partial K_{j_1}, \partial K_{j_2}, ..., \partial K_{j_n}$. The function $(u - u^{(N)})$ is a solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)(u - u^{(N)}) = k^{-N}F_N(t,x;k,\omega) \text{ in } \mathbb{R}\times\Omega, \\ \partial_\nu(u - u^{(N)}) = k^{-N}g_N(t,x;k,\omega) \text{ on } \mathbb{R}\times\partial\Omega, \\ (u - u^{(N)})(0,x;k,\omega) = \partial_t(u - u^{(N)})(0,x;k,\omega) = 0, \end{cases}$$

where F_N is related to the action of $(\partial_t^2 - \Delta_x)$ to the amplitudes with index N, while $g_N(t,x;k,\omega)$ is related to the trace on $\partial\Omega$ of the normal derivatives of the amplitudes with index N multiplied with the corresponding phase functions. The analysis of the integral involving $(u - u^{(N)})$ yields a term $\mathcal{O}(e^{-\beta T_j})$ (see for more details [8]) and we are going to study the integral

$$(2\pi)^{-3}\int_{\mathbb{R}} dt\int_{\Omega_j} dx\int_{S^2} d\omega\int_0^\infty k^2 e^{i\lambda t}\varphi_j(t)\Big(w_+^{(N)}(t,x;k,\omega) + w_-^{(N)}(t,x;k,\omega)\Big)\psi(x)e^{-ik\langle x,\omega\rangle}dk.$$

By a stationary phase argument we obtain the leading term (see [6] and Chapter 6 in [19]). To estimate the rest, a trivial modification of Proposition 2.2 in [11] yields the following.

Theorem 2. *There exists a constant $C_3 > 0$, depending only on K, and $J_1(\beta) > 0$ so that for $|\lambda| \le e^{\beta q_j}$ and $j \ge J_1(\beta)$ we have*

$$|\widehat{\varphi_j V}(\lambda)| \le C_3 \exp\Big((\frac{a_0}{d_0} - \beta)T_j\Big). \tag{3.5}$$

Notice that the number of configurations J such that $(|J| - 1)d_0 \le q_j + 1$ is bounded by $A_1\exp(a_0 T_j/d_0)$ and this explains the factor $\exp(\frac{a_0}{d_0}T_j)$ above.

Proof of Theorem 1. First, observe that for fixed r and for $T_j \ge b_-(r)$ we have

$$e^{\beta(1-\frac{\epsilon}{\theta})q_j} \ge e^{\beta(1-\frac{\epsilon}{\theta})(T_j-1)} \ge \exp\Big(\beta(1-\frac{\epsilon}{\theta})(\kappa\ln r - B - 1)\Big) = C_4 r^{\theta-\epsilon}$$

with a constant $C_4 = \exp(-\beta(1-\frac{\epsilon}{\theta})(B+1))$ independent on r. Thus for every r and for $T_j \in [b_-(r),\, b_+(r)]$ the inequality $|\lambda| \le C_4 r^{\theta-\epsilon}$ implies $|\lambda| \le e^{\beta(1-\frac{\epsilon}{\theta})q_j}$. Secondly, take r sufficiently large to arrange $C_1 r > C_4 r^{\theta-\epsilon}$. Then, according to Proposition 2, we obtain

$$N_{0,\delta}(r) \ge C_0 \min_{T_j\in[b_-(r),\, b_+(r)]}\int_1^{C_4 r^{\theta-\epsilon}} |\widehat{(\varphi_j U)}(\lambda)|d\lambda - C_2$$

$$\geq C_0 \min_{T_j \in [b_-(r),\, b_+(r)]} \int_1^{C_4 r^{\theta-\epsilon}} \Big(|\widehat{\varphi_j F}(\lambda)| - |\widehat{\varphi_j V}(\lambda)| \Big) d\lambda - C_2,\ \ r \geq R_1 \geq R_0.$$

Next we will estimate the Fourier transform $\langle e^{i\lambda t}\varphi_j(t), F(t)\rangle$, provided $|\lambda| \leq e^{\beta(1-\frac{\epsilon}{\theta})q_j}$. Clearly, there exists $J = J(\epsilon, \theta) > 0$ so that

$$\Big|\langle (e^{i\lambda t} - e^{i\lambda T_j})\varphi_j(t), F(t)\rangle\Big|$$

$$= \Big| \sum_{|d_\gamma - T_j| \leq e^{-\beta q_j}} \rho(e^{\beta q_j}(d_\gamma - T_j)) \Big(e^{i\lambda d_\gamma} - e^{i\lambda T_j}\Big) T_\gamma |I - P_\gamma|^{-1/2} \Big|$$

$$\leq C_\rho |\lambda| e^{-\beta q_j} \exp\Big((a_0 - b_1)(q_j+1)\Big)(q_j+1)$$

$$\leq C_\rho \exp(\frac{\beta\epsilon}{\theta}) \exp\Big((a_0 - b_1 - \beta\frac{\epsilon}{\theta})(q_j+1)\Big)(q_j+1) \leq d_0 e^{-b_2 T_j},\ \ |\lambda| \leq e^{\beta(1-\frac{\epsilon}{\theta})q_j},\ \ \forall j \geq J.$$

where we have used the inequality

$$a_0 - b_1 - \beta\frac{\epsilon}{\theta} \leq -\frac{\theta a_0}{\epsilon d_0} - b_2.$$

On the other hand, the fact that we have only positive terms in the sum below implies

$$< \varphi_j(t), F(t) > = \sum_{|d_\gamma - T_j| \leq e^{-\beta q_j}} T_\gamma |I - P_\gamma|^{-1/2} \rho(e^{\beta q_j}(d_\gamma - T_j)) \tag{3.6}$$

$$\geq T_j e^{-\beta T_j} + \sharp\{d_\gamma :\ |d_\gamma - T_j| \leq \epsilon_0 e^{-\beta q_j},\ d_\gamma \neq T_j\} 2 d_0 e^{-b_2(T_j+1)}.$$

Consequently, for $|\lambda| \leq e^{\beta(1-\frac{\epsilon}{\theta})q_j}$ and $j \geq J$ we obtain

$$|\langle e^{i\lambda t}\varphi_j(t), F(t)\rangle| \geq |\langle \varphi_j(t), F(t)\rangle| - \Big|\langle \Big(e^{i\lambda t} - e^{i\lambda T_j}\Big)\varphi_j(t), F(t)\rangle\Big| \geq \frac{T_j}{2} e^{-b_2 T_j}. \tag{3.7}$$

Combining the above estimate for $\widehat{\varphi_j F}(\lambda)$ with (3.5) and taking j sufficiently large, we get

$$N_{0,\delta}(r) \geq \frac{1}{2} C_0 C_4 r^{\theta-\epsilon} \Big[(\kappa \ln r - B) \exp\Big(-b_2(\kappa \ln r + B)\Big)$$

$$-2C_3 \exp\Big((a_0/d_0 - \beta)(\kappa \ln r - B)\Big)\Big] - C_2,$$

provided r large enough. Our choice of β yields $\beta - \frac{a_0}{d_0} \geq \frac{\theta}{\epsilon} b_2$ and we conclude that

$$N_{0,\delta}(r) \geq C_5 r^{\theta - \epsilon - b_2\kappa} \Big[\kappa \ln r - B - C_6 r^{(1-\frac{\theta}{\epsilon}) b_2 \kappa} \Big] - C_2$$

with

$$C_5 = \frac{1}{2} C_0 C_4 e^{-b_2 B},\ C_6 = 2C_3 \exp\Big((\beta - a_0/d_0)B\Big).$$

For r large enough we obtain the estimate (1.6) and the proof of Theorem 1 is complete. □

Remark 3. The estimate (3.6) shows that we can improve our principal result (1.6) if for a suitable sequence $T_j \to \infty$ we have a lower bound

$$\sharp\{d_\gamma :\ |d_\gamma - T_j| \leq \epsilon_0 e^{-\beta T_j}\} \geq \epsilon_1 e^{\eta_1 T_j},\ \ \epsilon_1 > 0,\ 0 < \eta_1 < h,\ \forall j \geq j_0.$$

This will add a factor $r^{\kappa\eta_1}$ in the lower bound of $N_{0,\delta}(r)$. On the other hand, as we have mentioned in Section 2, if the condition (S) holds, the above *clustering* phenomenon never appears.

Remark 4. In the case of Dirichlet problem the distribution $F(t)$ has the form

$$F(t) = \sum_{\gamma\in\Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} \delta(t - d_\gamma).$$

The change of signs in the representation of $F(t)$ leads to considerable difficulties. In this situation our argument can be applied if we can construct a sequence of periods $d_j \to \infty$ so that

$$d_{j+1} - d_j \leq d,\ \forall j \geq J, \tag{3.8}$$

$$|\langle \varphi(e^{\beta q_j}(t - d_j)), F(t)\rangle| \geq \eta_0 e^{-\eta d_j} \tag{3.9}$$

with $d > 0$, $\eta_0 > 0$, $\eta > 0$ independent on $j \geq J$. Notice that in [12] the existence of a sequence d_j, satisfying (3.9), was related to the analytic singularities of the dynamical zeta function

$$F_D(s) = \sum_{\gamma\in\Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-s d_\gamma},\ \mathrm{Re}\, s \gg 1.$$

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Département de Mathématiques Appliquées, Université Bordeaux I,
351, Cours de la Libération, 33405 Talence, FRANCE
petkov@math.u-bordeaux.fr