CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS WITH TRIPLE EFFECTIVE CHARACTERISTICS ON THE INITIAL PLANE

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Abstract. We study the Cauchy problem for effectively hyperbolic operators P with triple characteristics points lying on the initial plane $t = 0$. Under some conditions on the principal symbol of P one proves that the Cauchy problem for P in $[0, T] \times \Omega \subset \mathbb{R}^{n+1}$ is well posed for every choice of lower order terms. Our results improves those in [11] since we do not assume the condition (E) of [11] to be satisfied.

1. Introduction

In this paper we study the Cauchy problem for a differential operator

$$P(t,x,D_t,D_x) = \sum_{k+|\alpha|\le 3} c_{k,\alpha}(t,x) D_t^k D_x^\alpha, \quad D_t = -i\partial_t, \quad D_{x_j} = -i\partial_{x_j}$$

of order 3 with smooth coefficients $c_{k,\alpha}(t,x)$, $t \in \mathbb{R}$, $x \in \Omega \subset \mathbb{R}^n$, $c_{3,0} \equiv 1$. Denote by

$$p(t,x,\tau,\xi) = \sum_{k+|\alpha|=3} c_{k,\alpha}(t,x)\tau^k \xi^\alpha = \tau^3 + q_1(t,x,\xi)\tau^2 + q_2(t,x,\xi)\tau + q_3(t,x,\xi)$$

the principal symbol of P. Throughout the paper we work with symbols $s(t,x,\xi) \in S^m_{1,0}(\Omega \times \mathbb{R}^n)$ of pseudo-differential operators which depend smoothly on $t \in [0,T]$ and we use the Weyl quantization (see [3])

$$s(t,x,D)u = (\mathrm{Op}^w(s)u)(t,x) = (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} s\Big(t, \frac{x+y}{2}, \xi\Big) u(t,y)dyd\xi.$$

We will use the notation $S^m_{0,1}$ for the class of symbols (see [3]) and we abbreviate $S^m_{1,0}$ to S^m and $\mathrm{Op}^w(s)$ to $\mathrm{Op}(s)$.

With a real symbol $\varphi \in S^0_{1,0}$ one can write

$$\begin{aligned} P = (D_t - \mathrm{Op}(\varphi)\langle D\rangle)^3 + \mathrm{Op}(a)\langle D\rangle(D_t - \mathrm{Op}(\varphi)\langle D\rangle)^2 - \mathrm{Op}(b)\langle D\rangle^2(D_t - \mathrm{Op}(\varphi)\langle D\rangle) \\ + \mathrm{Op}(c)\langle D\rangle^3 - \sum_{j=0}^{2} \mathrm{Op}(b_j)\langle D\rangle^j (D_t - \mathrm{Op}(\varphi)\langle D\rangle)^{2-j} \end{aligned} \tag{1.1}$$

which is a differential operator in t. Here the symbols a, b, $c \in S^0_{1,0}$ coincide with

$$q_1\langle\xi\rangle^{-1} + 3\varphi, \quad -\big(q_2\langle\xi\rangle^{-2} + 2\varphi q_1\langle\xi\rangle^{-1} + 3\varphi^2\big), \quad q_3\langle\xi\rangle^{-3} + \varphi q_2\langle\xi\rangle^{-2} + \varphi^2\langle\xi\rangle^{-1} + \varphi^3,$$

respectively, $b_j \in S^0_{1,0}$, $j = 0, 1, 2$ (see [3]), and $\langle D\rangle$ has symbol $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

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First we assume that the principal symbol

$$p(t,x,\tau,\xi) = (\tau - \varphi\langle\xi\rangle)^3 + a\langle\xi\rangle(\tau - \varphi\langle\xi\rangle)^2 - b\langle\xi\rangle^2(\tau - \varphi\langle\xi\rangle) + c\langle\xi\rangle^3 \tag{1.2}$$

is hyperbolic, that is the roots of equation $p = 0$ with respect to τ are real for $(t,x,\xi) \in [0,T] \times \Omega \times \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is an open set. Recall that an operator is effectively hyperbolic if the fundamental matrix $F_p(z)$ of the principal symbol p has two non-vanishing eigenvalues $\pm\mu(z)$ at every critical point z of p, where $dp(z) = 0$. An effectively hyperbolic operator in $[0,T] \times \Omega$ may have triple characteristics only for $t = 0$ or $t = T$ (see [4, Lemma 8.1]). Second we assume that p has triple characteristic points only on $t = 0$ and P is *effectively hyperbolic* at every triple characteristic points $\rho = (0,x,\tau,\xi)$ which is equivalent (see [4, Lemma 8.1]) to the condition

$$\frac{\partial^2 p}{\partial t \partial \tau}(\rho) < 0.$$

Consequently, at a triple characteristic point $\rho_0 = (0,x_0,0,\xi_0)$, assuming $\varphi(0,x_0,\xi_0) = 0$, we have $b_t(0,x_0,\xi_0) > 0$. Moreover, at ρ_0 we have $a(0,x_0,\xi_0) = b(0,x_0,\xi_0) = c(0,x_0,\xi_0) = 0$.

Our purpose is to study the Cauchy problem for such P and to prove that under some conditions on p this problem is well posed for every choice of lower order terms (see [11] for the definition of well posed Cauchy problem). This property is called *strong hyperbolicity* and the effective hyperbolicity of P is a necessary condition for it (see [4, Theorem 3]). For operators having only double characteristics every effectively hyperbolic operator is strongly hyperbolic and we refer to [9] for the references and related works. The conjecture is that effectively hyperbolic operators with triple characteristic points on $t = 0$ are strongly hyperbolic (see [4], [6], [1], [11]). On the other hand, for some class of hyperbolic operators with triple characteristics the above conjecture has been proved in [6], [1], [11], but the general case is still an open problem.

In [11] the strong hyperbolicity was established under the condition (E) saying that for some $\delta > 0$ and small $t \geq 0$ we have the lower bound

$$\frac{\Delta}{\langle\xi\rangle^6} \geq \delta t \Big(\frac{\Delta_0}{\langle\xi\rangle^2}\Big)^2, \ (x,\xi) \in \Omega \times \mathbb{R}^n.$$

Here $\Delta \in S^6$ is the discriminant of the equation $p = 0$ with respect to τ, while $\Delta_0 \in S^2$ is the discriminant of the equation $\frac{\partial p}{\partial \tau} = 0$ with respect to τ. In [11] it was introduced also a weaker condition (H) saying that with some constant $\delta > 0$ and small $t \geq 0$ we have

$$\frac{\Delta}{\langle\xi\rangle^6} \geq \delta t^2 \frac{\Delta_0}{\langle\xi\rangle^2}, \ (x,\xi) \in \Omega \times \mathbb{R}^n.$$

We can consider a microlocal version of the conditions (E) and (H) assuming that the above inequalities hold for (t,x,ξ), $t \geq 0$, in a small conic neighborhood W_0 of every triple characteristic point $(0,x_0,\xi_0)$. The purpose of this paper is to study operators with triple characteristics on the plane $t = 0$ and our main results are stated in Theorem 4.1 and Corollary 4.5. They improve the results in [11] and show that we have a strong hyperbolicity for some operators for which (E) is not satisfied, but (H) holds. In particular, we cover the case of operators whose principal symbol p admits a microlocal factorization with one smooth root under the condition that there are no double characteristic points of p converging to a triple characteristic point $(0,x,0,\xi)$ (see Example 1.1).

Concerning the symbols $a(t,x,\xi)$, $b(t,x,\xi)$, $c(t,x,\xi)$, we assume the existence of $\delta_1 > 0$ such that

$$\begin{aligned} & b(t,x,\xi) \geq \delta_1 t, \\ & c = \mathcal{O}(b^2),\ \ \langle\xi\rangle^\alpha \partial_\xi^\alpha \partial_x^\beta c = \mathcal{O}(b),\ |\alpha+\beta| = 1,\ \ \langle\xi\rangle^\alpha \partial_\xi^\alpha \partial_x^\beta c = \mathcal{O}(\sqrt{b}),\ |\alpha+\beta| = 2, \\ & \partial_t c = \mathcal{O}(b),\ \ \langle\xi\rangle^\alpha \partial_\xi^\alpha \partial_x^\beta (ac) = \mathcal{O}(\sqrt{b}),\ \ |\alpha+\beta| = 3. \end{aligned} \tag{1.3}$$

It is clear that the condition (1.3) are satisfied if

$$b(t,x,\xi) \geq \delta_1 t,\ \ \langle\xi\rangle^\alpha \partial_t^\gamma \partial_\xi^\alpha \partial_x^\beta c = \mathcal{O}\big(b^{2-|\alpha+\beta|/2-|\gamma|}\big)\ \text{ for }\ |\alpha+\beta+\gamma| \leq 3,\ \ \gamma = 0,1. \tag{1.4}$$

In fact, we assume a slightly weaker microlocal conditions formulated in (3.11) and Theorem 4.1.

Below we present two examples of effectively hyperbolic operators with triple characteristics on $t = 0$ satisfying the above assumptions.

Example 1.1. Assume $c \equiv 0$. Then the symbol p becomes $p = ((\tau - \varphi\langle\xi\rangle)^2 + a\langle\xi\rangle(\tau - \varphi\langle\xi\rangle) - b\langle\xi\rangle^2)(\tau - \varphi\langle\xi\rangle)$. Let $\rho = (0, x_0, \varphi(0,x_0,\xi_0)\langle\xi_0\rangle, \xi_0)$, be a triple characteristic point. For small $t > 0$ we have $b(t,x_0,\xi_0) > 0$. If for some (y,η) sufficiently close to (x_0,ξ_0) we have $b(0,y,\eta) < 0$, then there exists $z = (t^*, x^*, \xi^*)$ with $t^* > 0$ such that $b(z) = 0$ and the equation $(\tau - \varphi\langle\xi\rangle)^2 + a\langle\xi\rangle(\tau - \varphi\langle\xi\rangle) - b\langle\xi\rangle^2 = 0$ has a root $\varphi(z)\langle\xi^*\rangle$ for z. This implies the existence of a double characteristic point $(t^*, x^*, \varphi(z)\langle\xi^*\rangle, \xi^*)$ of p. We exclude this possibility, assuming $b(0,x,\xi) \geq 0$ for (x,ξ) close to (x_0,ξ_0).

Remark 1.1. For the operator in Example 1.1, the discriminant of the equation $p = 0$ has the form $\Delta = b^2(a^2+4b)\langle\xi\rangle^6$, while $\Delta_0 = 4(a^2+3b)\langle\xi\rangle^2$. Therefore the condition (E) is reduced to

$$b^2(a^2+4b) \geq \delta t(a^2+3b)^2.$$

If $b = \mathcal{O}(t)$, this inequality yields $b^2a^2 + 4b^3 \geq \delta t a^4$ and hence $a^2 \leq \mathcal{O}(t^2)/\delta t = \mathcal{O}(t)$ which is not satisfied in any small neighborhood of a triple characteristic point $(0, x_0, \varphi(0,x_0,\xi_0)\langle\xi_0\rangle, \xi_0)$, unless $a(0,x,\xi) = 0$ for all $(0,x,\xi)$ close to the point $(0,x_0,\xi_0)$. On the other hand, the inequality

$$b^2(a^2+4b) \geq \delta t^2(a^2+3b)$$

obviously holds ($b \geq \delta_1 t$ is assumed), hence (H) is satisfied.

The Example 1.1 covers the case when the principal symbol p admits a factorization

$$p = (\tau^2 + 2d(t,x,\xi)\tau + f(t,x,\xi))(\tau - \lambda(t,x,\xi))$$

with C^∞ smooth real root $\lambda(t,x,\xi)$ and p has not double characteristic points in a neighborhood of $(0,x_0,\xi_0)$. In fact, we may write

$$p = \big((\tau-\lambda)^2 + 2(\lambda+d)(\tau-\lambda) + \lambda^2 + 2d\lambda + f\big)(\tau-\lambda)$$

and taking $\varphi = \lambda\langle\xi\rangle^{-1}$ we reduce the symbol to Example 1.1. Notice that effectively hyperbolic operators with principal symbols admitting above factorization have been studied by V. Ivrii in [6] who proved the strong hyperbolicity constructing parametrix. Here we present another proof based on energy estimates with weight t^{-N}, assuming P strictly hyperbolic for small $t > 0$.

Example 1.2. Consider the operator with principal symbol

$$p = \tau^3 - (t + \alpha(x,\xi))\langle\xi\rangle^2\tau - (t^2b_2 + tb_1 + b_0)\langle\xi\rangle^3,$$

where α, b_0, b_1, b_2 are zero order pseudo-differential operators and $\alpha \geq 0$. This class of operators has been studied in [11] under the condition (E). We write p as follows

$$p = (\tau + b_1\langle\xi\rangle)^3 - 3b_1\langle\xi\rangle(\tau + b_1\langle\xi\rangle)^2 - \big(t + \alpha - 3b_1^2\big)\langle\xi\rangle^2(\tau + b_1\langle\xi\rangle)$$

$$-\big[t^2 b_2 + b_0 - b_1\alpha + b_1^3\big]\langle\xi\rangle^3.$$

Choosing $\varphi = -b_1(t, x, \xi)$ one reduces the symbol p to the form (1.2) with $a = -3b_1$, $b = t + \alpha - 3b_1^2$, $c = -(t^2 b_2 + b_0 - b_1\alpha + b_1^3)$. If $\alpha \geq 3b_1^2$, $b_0 = b_1\alpha - b_1^3$, the condition (1.4) is satisfied, while for $\alpha = 3b_1^2$, $b_0 = b_1\alpha - b_1^3$ the condition (E) is not satisfied for b_1, unless $b_1(0, x, \xi) \equiv 0$. It is easy to see that with the above choice of b_0 and b_1, the condition (H) holds.

Notice that if $\rho = (t, x, \tau, \xi)$ with $t > 0$ is a double characteristic point for p, one has $\Delta(\rho) = 0$ and $\Delta_0(\rho) > 0$. Therefore the condition (H) is not satisfied and the analysis of this case is a difficult open problem. The proofs in this work are based on energy estimates with weight t^{-N} with $N \gg 1$ leading to estimates with big loss of regularity. This phenomenon is typical for effectively hyperbolic operators with multiple characteristics (see [4], [6], [1], [11]).

We follow the approach in [11] reducing the problem to the one for first order pseudo-differential system. In Section 2 we construct a symmetrizer S for the principal symbol of the system following a general result (see Lemma 2.1) which has independent interest. Moreover, $\det S = \frac{1}{27}\Delta$ and under our assumptions one shows that $\det S \geq \delta b^2(a^2 + 4b)$, $\delta > 0$. Therefore $\Delta \geq \varepsilon t^2(a^2 + 4b)$, $\varepsilon > 0$, and in general the condition (E) is not satisfied. This leads to difficulties in Section 3, where a more fine analysis of the matrix pseudo-differential operators is needed. As in [11] a detailed examination of the sharp Gårding inequality for matrix pseudo-differential operators with nonnegative definite symbols plays a crucial role in the analysis. In Section 4 we show that the microlocal conditions (1.3) are sufficient for the energy estimates in Theorems 4.1 and 4.2.

2. Symmetrizer

First we recall a general result concerning the existence of a symmetrizer. Let $p(\zeta) = \zeta^m + a_1\zeta^{m-1} + \cdots + a_m$ be a monic hyperbolic polynomial of degree m and let $q(\zeta) = p'(\zeta)$. Here $a_j(t, x, \xi)$ depend on (t, x, ξ) but we omit this in the notations below. Let

$$h_{p,q}(\zeta, \bar{\zeta}) = \frac{p(\zeta)q(\bar{\zeta}) - p(\bar{\zeta})q(\zeta)}{\zeta - \bar{\zeta}} = \sum_{i,j=1}^{m} h_{ij}\zeta^{i-1}\bar{\zeta}^{j-1}$$

be the Bézout form of p and q. It is well known that the matrix $H = (h_{ij})$ is nonnegative definite (see for example [5]).

Consider the Sylvester matrix A_p corresponding to $p(\zeta)$ which has the form

$$A_p = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \\ -a_m & -a_{m-1} & \cdots & -a_1 \end{pmatrix}.$$

One has the following result [10] and for the sake of completeness we present the proof.

Lemma 2.1. ([10, Lemma 2.3.1]) *H symmetrizes A_p and* $\det H = \Delta^2$ *where Δ is the difference-product of the roots of $p(\tau) = 0$.*

Proof. We first treat the case when $p(\zeta)$ is a strictly hyperbolic polynomial. Let λ_j, $j = 1, \ldots, m$ be the different roots of the equation $p(\zeta) = 0$. Write $p(\zeta) = \prod_{j=1}^{m}(\zeta - \lambda_j)$ and set

$$\sigma_{\ell,k} = \sum_{1\le j_1<\cdots<j_\ell\le m, j_p\neq k} \lambda_{j_1}\cdots\lambda_{j_\ell}.$$

Since $p'(\zeta) = \sum_{k=1}^{m}\prod_{j=1,j\neq k}^{m}(\zeta - \lambda_j) = \sum_{i=1}^{m}(-1)^{m-i}\sigma_{m-i,k}\zeta^{i-1}$ it is easy to see

$$h_{ij} = \sum_{k=1}^{m}(-1)^{i+j}\sigma_{m-i,k}\sigma_{j-1,k}.$$

Denote by R the Vandermonde's matrix having the form

$$R = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix}.$$

Since $\lambda_i \neq \lambda_j$, $i \neq j$, the matrix R is invertible and $|\det R| = |\Delta|$. It is clear that

$$A_pR = R\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}.$$

Denote by ${}^{co}R = (r_{ij})$ the cofactor matrix of R and by $\Delta(\lambda_1, \ldots, \lambda_k)$ the difference-product of $\lambda_1, \ldots, \lambda_k$. It is easily seen that r_{ij} is divisible by $\Delta_i = \Delta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)$, hence

$$r_{ij} = c_{ij}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)\Delta_i. \tag{2.1}$$

Since r_{ij} and Δ_i are alternating polynomials in $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)$ of degree $m(m-1)/2 - j + 1$ and $(m-1)(m-2)/2$ respectively, then c_{ij} is a symmetric polynomial of degree

$$m - j = m(m-1)/2 - j + 1 - (m-1)(m-2)/2.$$

Therefore c_{ij} is a polynomial in fundamental symmetric polynomials of $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)$. Noting that Δ_i is of degree $m-2$ and r_{ij} $(j \neq m)$ is of degree $m-1$ respectively with respect to λ_ℓ $(\ell \neq i)$, one concludes that c_{ij} is of degree 1 with respect to λ_ℓ $(\ell \neq i)$ which proves that

$$c_{ij} = (-1)^{i+j}\sigma_{m-j,i}. \tag{2.2}$$

Thus denoting $C = (c_{ij})$ we have ${}^tCC = (h_{ij}) = H$. In particular, this shows that the symmetric matrix H is nonnegative definite as it was mentioned above.

Set $D = \operatorname{diag}(\Delta_1, \ldots, \Delta_m)$ and note that D is invertible. Moreover it follows from (2.1) that $C = D^{-1}({}^{co}R) = (\det R)D^{-1}R^{-1}$ and hence

$$CA_pC^{-1} = D^{-1}(R^{-1}A_pR)D.$$

It is clear that CA_pC^{-1} is a diagonal matrix because both $R^{-1}A_pR$ and D are diagonal matrices. Then $CA_pC^{-1} = {}^tC^{-1}\,{}^tA_p{}^tC$ yields ${}^tCCA_p = {}^tA_p{}^tCC$ which proves that HA_p is symmetric. From $C = (\det R)D^{-1}R^{-1}$ it follows that

$$C = \operatorname{diag}\Big(\pm\prod_{k\neq 1}(\lambda_i - \lambda_k), \pm\prod_{k\neq 2}(\lambda_i - \lambda_k), \ldots, \pm\prod_{k\neq m}(\lambda_i - \lambda_k)\Big)R^{-1}$$

and hence $|\det C| = |\prod_{j=1}^m \prod_{k\neq j}^m (\lambda_k - \lambda_j)|/|\Delta| = |\Delta|$. Consequently, $\det H = \Delta^2$ and this completes the proof for strictly hyperbolic polynomial $p(\zeta)$.

Passing to the general case, introduce the polynomial

$$p_\varepsilon(\zeta) = \Big(1 + \varepsilon \frac{\partial}{\partial \zeta}\Big)^{m-1} p(\zeta),\ \varepsilon \neq 0.$$

According to [12], $p_\varepsilon(\zeta)$ is strictly hyperbolic and let $H_\varepsilon = {}^tC_\varepsilon C_\varepsilon$ be the symmetrizer for A_{p_ε} constructed above. Obviously, as $\varepsilon \to 0$, we have $A_{p_\varepsilon} \to A_p$ since the coefficients of $p_\varepsilon(\zeta)$ go to the ones of $p(\zeta)$. The roots of $p(\zeta)$ depend continuously on the coefficients and this yields $\lambda_{j,\varepsilon} \to \lambda_j$, $\lambda_{j,\varepsilon}$ being the roots of $p_\varepsilon(\zeta) = 0$. The equalities (2.2) imply $C_\varepsilon \to C$ and passing to the limit $\varepsilon \to 0$, we obtain the result. $\square$

Note that H is different from the Leray's symmetrizer ([7]) since if B is the Leray's symmetrizer, then $\det B = \Delta^{2(m-1)}$. Now consider

$$\tilde{A}_p = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_m \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Corollary 2.1. *Let $J = (\delta_{i,m+1-j})$, where δ_{ij} is the Kronecker's delta. Then $\tilde{H} = JH\,{}^tJ$ symmetrizes $\tilde{A}_p$ and* $\det \tilde{H} = \Delta^2$.

Proof. Since $\tilde{A}_p = JA_p\,{}^tJ$ and ${}^tJJ = I$ the proof is immediate. $\square$

With $U = {}^t\big((D_t - \mathrm{Op}(\varphi)\langle D\rangle)^2 u, \langle D\rangle (D_t - \mathrm{Op}(\varphi)\langle D\rangle)u, \langle D\rangle^2 u\big)$ the equation $Pu = f$ is reduced

$$D_t U = \mathrm{Op}(\varphi)\langle D\rangle U + (\mathrm{Op}(A)\langle D\rangle + \mathrm{Op}(B))U + F, \tag{2.3}$$

where $F = {}^t(f, 0, 0)$ and

$$A(t,x,\xi) = \begin{pmatrix} -a & b & -c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B(t,x,\xi) = \begin{pmatrix} b_{11} & b_{11} & b_{13} \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix},$$

where $b_{ij} \in S^0_{1,0}$.

Introduce

$$S(t,x,\xi) = \frac{1}{3}\begin{pmatrix} 3 & 2a & -b \\ 2a & 2(a^2+b) & -ab-3c \\ -b & -ab-3c & b^2-2ac \end{pmatrix}$$

which is a representation matrix (conjugated by J in Corollary 2.1) of the Bézout form of $p(\tau) = \tau^3 + a\tau^2 - b\tau + c$ and $p'(\tau)$ (see for example [5], [8]). Therefore S symmetrizes A so that

$$S(t,x,\xi)A(t,x,\xi) = \frac{1}{3}\begin{pmatrix} -a & 2b & -3c \\ 2b & ab-3c & -2ac \\ -3c & -2ac & bc \end{pmatrix}. \tag{2.4}$$

Note that when $c = 0$ one has

$$S_0(t,x,\xi) = \frac{1}{3}\begin{pmatrix} 3 & 2a & -b \\ 2a & 2(a^2+b) & -ab \\ -b & -ab & b^2 \end{pmatrix}$$

and hence

$$\det S_0(t,x,\xi) = \frac{1}{27} b^2(a^2+4b).$$

.

Lemma 2.2. *There exist $\bar{\varepsilon} > 0$ and $\delta > 0$ such that*

$$\det S \geq \delta b^2(a^2+b)$$

if $|ac| \leq \bar{\varepsilon}\, b^2$ and $|c| \leq \bar{\varepsilon}\, b^{3/2}$.

Proof. Note that

$$\det S = \det S_0 + \frac{1}{27}\big\{ -4a^3c - 18abc - 27c^2\big\}.$$

Since

$$|a^3c| \leq \bar{\varepsilon}\, b^2 a^2, \ \ |abc| \leq \bar{\varepsilon}\, b^3, \ \ |c^2| \leq \bar{\varepsilon}^2\, b^3$$

choosing $\bar{\varepsilon} = 1/50$ for instance, the assertion is clear. □

Lemma 2.3. *There exist $\bar{\varepsilon} > 0$ and $\varepsilon_1 > 0$ such that*

$$S(t,x,\xi) \gg \varepsilon_1 t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix} = \varepsilon_1 t J,$$

provided $|ac| \leq \bar{\varepsilon}\, b^2$ and $|c| \leq \bar{\varepsilon}\, b^{3/2}$.

Proof. Since

$$3S - \varepsilon_1 t J = \begin{pmatrix} 3-\varepsilon_1 t & 2a & -b \\ 2a & 2a^2+2b-\varepsilon_1 t & -ab-3c \\ -b & -ab-3c & b^2-\varepsilon_1 tb - 2ac \end{pmatrix},$$

one obtains

$$\det(3S - \varepsilon_1 tJ) = \det 3S + \varepsilon_1 \mathcal{O}\big(b^2(b+a^2)\big).$$

Indeed

$$\begin{aligned} (3-\varepsilon_1 t)(2a^2+2b-\varepsilon_1 t)(b^2-\varepsilon_1 tb-2ac) &= 3(2a^2+2b)(b^2-2ac) + \varepsilon_1 \mathcal{O}\big(tb(b+a^2)\big), \\ b^2(2a^2+2b-\varepsilon_1 t) &= b^2(2a^2+2b) + \varepsilon_1 \mathcal{O}\big(tb(b+a^2)\big), \\ 4a^2(b^2-\varepsilon_1 tb-2ac) &= 4a^2(b^2-2ac) + \varepsilon_1 \mathcal{O}\big(tba^2\big), \\ (3-\varepsilon_1 t)(ab+3c)^2 &= 3(ab+3c)^2 + \varepsilon_1 \mathcal{O}\big(tb^2\big). \end{aligned}$$

Noting $b \geq \delta_1 t$, one gets the above representation and we deduce $\det(3S - \varepsilon_1 tJ) \geq 0$ for small ε_1. In the same way one treats the principal minors of order 2. For example

$$(3-\varepsilon_1 t)(2a^2+2b-\varepsilon_1 t) - 4a^2 = 2a^2 + 6b - \varepsilon_1 t(2a^2+2b) + \varepsilon_1^2 t^2 \geq 2(a^2+b)(1-\varepsilon_1 t) \geq 0,$$

$$\begin{aligned} (3-\varepsilon_1 t)(b^2-\varepsilon_1 tb-2ac) - b^2 &= 2b^2 - 6ac - \varepsilon_1 t(b^2-2ac+3b) + \varepsilon_1^2 t^2 b \\ &\geq b^2 - 4ac - 3\varepsilon_1 tb + (b^2-2ac)(1-\varepsilon_1 t) \\ &\geq (1-4\bar{\varepsilon})b^2 - 3\varepsilon_1 tb + (1-2\bar{\varepsilon})(1-\varepsilon_1 t)b^2 \geq 0, \end{aligned}$$

$$
\begin{aligned}
(2a^2+2b-\varepsilon_1 t)(b^2-\varepsilon_1 tb-2ac)-(ab+3c)^2 &\geq a^2b^2+2b^3-10abc-9c^2-4a^3c\\
&\quad -3\varepsilon_1 tb^2+2\varepsilon_1 tac-2\varepsilon_1 tba^2\\
&\geq (1-4\bar{\varepsilon})a^2b^2+(2-10\bar{\varepsilon}-9\bar{\varepsilon}^2)b^3-(3\varepsilon_1+2\varepsilon_1\bar{\varepsilon})tb^2-2\varepsilon_1 tba^2\geq 0
\end{aligned}
$$

since all terms involving $\varepsilon_1 t$ can be compensated by $a^2b^2+2b^3$. ☐

Lemma 2.4. *Assume $\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)}=\mathcal{O}(\sqrt{b})$ for $|\alpha+\beta|=2$ and $\langle\xi\rangle^\alpha (ac)^{(\alpha)}_{(\beta)}=\mathcal{O}(\sqrt{b})$ for $|\alpha+\beta|=3$. There exists $C>0$ such that for $U\in C^\infty(\mathbb{R}_t : C_0^\infty(\mathbb{R}^n))$ we have*

$$
\mathsf{Re}(\mathrm{Op}(S)U,U)\geq \varepsilon_1 t\big(\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big)-Ct^{-1}\|\langle D\rangle^{-1}U\|^2.
$$

Proof. We will follow the argument of [11, Section 3] and we use the notation $\partial_\xi^\alpha D_x^\beta Q=Q^{(\alpha)}_{(\beta)}$. Recall that we have the representation

$$
Q_F-\mathrm{Op}(Q)=\mathrm{Op}\Big(\sum_{2\leq|\alpha+\beta|\leq 3}\psi_{\alpha,\beta}(\xi)Q^{(\alpha)}_{(\beta)}\Big)+\mathrm{Op}(R) \tag{2.5}
$$

with $R\in S^{-2}_{1/2,0}$ and real symbols $\psi_{\alpha,\beta}\in S^{(|\alpha|-|\beta|)/2}$, where Q_F is the Friedrichs part of Q (see [11, Appendix], [2]) and hence $(Q_FU,U)\geq 0$.

Notice that b is real, hence $(\mathrm{Op}(b)U_3,U_3)=\mathsf{Re}\,(\mathrm{Op}(b)U_3,U_3)$. Setting $Q=S-2\varepsilon_1 tJ$, we have

$$
\mathsf{Re}\,(\mathrm{Op}(S)U,U)=\mathsf{Re}\,(\mathrm{Op}(Q)U,U)+2\varepsilon_1 t\big(\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big),
$$

and it is enough to prove

$$
\big|\mathsf{Re}(\mathrm{Op}\Big(\sum_{2\leq|\alpha+\beta|\leq 3}\psi_{\alpha\beta}Q^{(\alpha)}_{(\beta)}\Big)U,U)\big|\leq \varepsilon_1 t\big(\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big)+C\varepsilon_1^{-1}t^{-1}\|\langle D\rangle^{-1}U\|^2. \tag{2.6}
$$

Indeed if this is true, then we have

$$
\begin{aligned}
\mathsf{Re}(\mathrm{Op}(Q)U,U) &\geq (Q_FU,U)-\varepsilon_1 t\big(\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big)\\
&\quad -C\varepsilon_1^{-1}t^{-1}\|\langle D\rangle^{-1}U\|^2-C\|\langle D\rangle^{-1}U\|^2\\
&\geq -\varepsilon_1 t\big(\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big)-C\varepsilon_1^{-1}t^{-1}\|\langle D\rangle^{-1}U\|^2.
\end{aligned}
$$

Thus we conclude the assertion.

To prove (2.6), consider $\mathsf{Re}(\mathrm{Op}(\psi_{\alpha\beta}Q^{(\alpha)}_{(\beta)})U,U)$ with $|\alpha+\beta|=2$. Setting $g=b^2-\varepsilon tb-2ac$, one has

$$
Q^{(\alpha)}_{(\beta)}=\begin{pmatrix} 0 & S^{-|\alpha|} & S^{-|\alpha|}\\ S^{-|\alpha|} & S^{-|\alpha|} & S^{-|\alpha|}\\ S^{-|\alpha|} & S^{-|\alpha|} & g^{(\alpha)}_{(\beta)}\end{pmatrix}.
$$

Here and below S^m denotes some symbol in the class S^m. This yields

$$\psi_{\alpha\beta} Q^{(\alpha)}_{(\beta)} = \begin{pmatrix} 0 & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & \psi_{\alpha\beta} g^{(\alpha)}_{(\beta)} \end{pmatrix}$$

and hence

$$|(\mathrm{Op}(\psi_{\alpha\beta} Q^{(\alpha)}_{(\beta)})U, U)| \le \varepsilon_1 t \sum_{j=1}^{2} \|U_j\|^2 + C\varepsilon_1^{-1} t^{-1} \|\langle D\rangle^{-1} U\|^2$$
$$+ |\mathsf{Re}\,\big(\mathrm{Op}\big(\psi_{\alpha\beta} g^{(\alpha)}_{(\beta)}\big)U_3, U_3\big)|.$$

Let $T = \psi_{\alpha\beta} g^{(\alpha)}_{(\beta)} \langle\xi\rangle$. Then $\psi_{\alpha\beta} g^{(\alpha)}_{(\beta)} = \mathsf{Re}\,(T\#\langle\xi\rangle^{-1}) + S^{-2}$ and

$$\mathsf{Re}\,(\mathrm{Op}(\psi_{\alpha\beta} g^{(\alpha)}_{(\beta)})U_3, U_3) \le \varepsilon_1 t \|\mathrm{Op}(T)U_3\|^2 + C\varepsilon_1^{-1} t^{-1} \|\langle D\rangle^{-1} U_3\|^2.$$

Note that $\|\mathrm{Op}(T)U_3\|^2 = (\mathrm{Op}(T\#T)U_3, U_3)$ and $T\#T = T^2 + S^{-2}$. Therefore there exists $C > 0$ such that

$$T^2 \le Cb$$

because $\langle\xi\rangle^{\alpha} c^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ and $\langle\xi\rangle^{\alpha}\Big(b(b-\varepsilon_1 t)\Big)^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ and $b \ge \delta t$. Applying the Fefferman-Phong inequality for the operator with symbol $Cb - T^2$, one proves the assertion.

For the case $|\alpha + \beta| = 3$ with $T_1 = \psi_{\alpha\beta} g^{(\alpha)}_{(\beta)} \langle\xi\rangle^{3/2}$ we have the inequality

$$T_1^2 \le Cb$$

with some $C > 0$. Indeed, $\langle\xi\rangle^{\alpha}(ac)^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ and $\langle\xi\rangle^{\alpha}\Big(b(b-\varepsilon_1 t)\Big)^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$. Repeating the above argument, we complete the proof. □

Corollary 2.2. *Let $\tilde S = S + \lambda\, t^{-1}\langle\xi\rangle^{-2} I$. Then there exists $\lambda_0 > 0$ such that for $\lambda \ge \lambda_0$ we have*

$$\mathsf{Re}(\mathrm{Op}(\tilde S)U, U) = \mathsf{Re}(\mathrm{Op}(S)U, U) + \lambda t^{-1} \|\langle D\rangle^{-1} U\|^2$$
$$\ge \varepsilon_1 t\big(\sum_{j=1}^{2} \|U_j\|^2 + (\mathrm{Op}\,(b)U_3, U_3)\big) + (\lambda/2) t^{-1} \|\langle D\rangle^{-1} U\|^2.$$

Corollary 2.3. *There exist $\delta_2 > 0$ and $\lambda_0 > 0$ such that*

$$\mathsf{Re}(\mathrm{Op}(\tilde S)U, U) \ge \delta_2 t^2 \|U\|^2 + (\lambda/2) t^{-1} \|\langle D\rangle^{-1} U\|^2, \quad \lambda \ge \lambda_0.$$

Proof. Since there exists $\delta_1 > 0$ such that $b \ge \delta_1 t$ from the Fefferman-Phong inequality for the scalar symbol $b - \delta_1 t$ one deduces

$$(\mathrm{Op}\,(b)U_3, U_3) \ge \delta_1 t \|U_3\|^2 - C\|\langle D\rangle^{-1} U_3\|^2$$

which proves the assertion thanks to Corollary 2.2. □

3. Energy estimates

Consider the energy $(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U)$, where $(\cdot,\cdot)$ is the $L^2(\mathbb{R}^n)$ inner product and $N>0$, $\gamma>0$ are positive parameters. Then one has

$$\begin{aligned}\partial_t(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U)&=-N(t^{-N-1}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U)-\gamma(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U)\\ +(t^{-N}e^{-\gamma t}\mathrm{Op}(\partial_t S)U,U)&-\lambda(N+1)t^{-N-2}e^{-\gamma t}\|\langle D\rangle^{-1}U\|^2-\lambda\gamma t^{-N-1}e^{-\gamma t}\|\langle D\rangle^{-1}U\|^2\\ -2\mathsf{Im}\,(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})(\varphi\langle D\rangle&+\mathrm{Op}(A)\langle D\rangle+\mathrm{Op}(B))U,U))-2\mathsf{Im}(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})F,U).\end{aligned}\tag{3.1}$$

Consider $S\#A\#\langle\xi\rangle-\langle\xi\rangle\#A^*\#S$. Note that

$$S\#A=SA+\sum_{|\alpha+\beta|=1}\frac{(-1)^{|\beta|}}{2i}S^{(\alpha)}_{(\beta)}A^{(\beta)}_{(\alpha)}+\sum_{|\alpha+\beta|=2}\cdots+S^{-3}.$$

Writing $S=(s_{ij})$ one has

$$\sum_{|\alpha+\beta|=2}\cdots=\sum_{|\alpha+\beta|=2}\cdots\Big(s^{(\alpha)}_{ij(\beta)}\Big)\begin{pmatrix}-a^{(\beta)}_{(\alpha)}&b^{(\beta)}_{(\alpha)}&-c^{(\beta)}_{(\alpha)}\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}S^{-2}&S^{-2}&\mathcal{O}(\sqrt{b})S^{-2}\\S^{-2}&S^{-2}&\mathcal{O}(\sqrt{b})S^{-2}\\S^{-2}&S^{-2}&\mathcal{O}(\sqrt{b})S^{-2}\end{pmatrix},$$

because $c^{(\beta)}_{(\alpha)}=\mathcal{O}(\sqrt{b})$ for $|\alpha+\beta|=2$. Then

$$(S\#A)\#\langle\xi\rangle=(SA)\#\langle\xi\rangle+\Big(\sum_{|\alpha+\beta|=1}\cdots\Big)\#\langle\xi\rangle+\begin{pmatrix}S^{-1}&S^{-1}&\mathcal{O}(\sqrt{b})S^{-1}\\S^{-1}&S^{-1}&\mathcal{O}(\sqrt{b})S^{-1}\\S^{-1}&S^{-1}&\mathcal{O}(\sqrt{b})S^{-1}\end{pmatrix}+S^{-2}.$$

Denoting the third term on the right-hand side by K_2, repeating the same arguments as before, it is easy to see

$$|((\mathrm{Op}(K_2)+\mathrm{Op}(S^{-2}))U,U)|\le C\Big(\|\langle D\rangle^{-1}U\|^2+\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\Big).\tag{3.2}$$

Now we turn to the term with $|\alpha+\beta|=1$. Note

$$S^{(\alpha)}_{(\beta)}A^{(\beta)}_{(\alpha)}=\Big(s^{(\alpha)}_{ij(\beta)}\Big)\begin{pmatrix}-a^{(\beta)}_{(\alpha)}&b^{(\beta)}_{(\alpha)}&-c^{(\beta)}_{(\alpha)}\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}S^{-1}&S^{-1}&\mathcal{O}(\sqrt{b})S^{-1}\\S^{-1}&S^{-1}&\mathcal{O}(\sqrt{b})S^{-1}\\\mathcal{O}(\sqrt{b})S^{-1}&\mathcal{O}(\sqrt{b})S^{-1}&\mathcal{O}(b)S^{-1}\end{pmatrix},$$

since $c^{(\beta)}_{(\alpha)}=\mathcal{O}(\sqrt{b})$ and $b^{(\alpha)}_{(\beta)}=\mathcal{O}(\sqrt{b})$ for $|\alpha+\beta|=1$ and hence

$$\Big(\sum_{|\alpha+\beta|=1}\cdots\Big)\#\langle\xi\rangle=\begin{pmatrix}S^0&S^0&\mathcal{O}(\sqrt{b})S^0+S^{-1}\\S^0&S^0&\mathcal{O}(\sqrt{b})S^0+S^{-1}\\\mathcal{O}(\sqrt{b})S^0+S^{-1}&\mathcal{O}(\sqrt{b})S^0+S^{-1}&\mathcal{O}(b)S^0+\mathcal{O}(\sqrt{b})S^{-1}+S^{-2}\end{pmatrix}=K_1.$$

The same arguments proves

$$|(\mathrm{Op}(K_1)U,U)|\le C\big(\|\langle D\rangle^{-1}U\|^2+\sum_{j=1}^2\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big).$$

Consider $A^*\#S$. We have the representation

$$A^*\#S = A^*S + \sum_{|\alpha+\beta|=1} \frac{(-1)^{|\beta|}}{2i}(A^*)^{(\alpha)}_{(\beta)} S^{(\beta)}_{(\alpha)} + \sum_{|\alpha+\beta|=2} \cdots + S^{-3} = A^*S + \tilde{K}.$$

Repeating similar arguments, one gets

$$|(\mathrm{Op}(\langle\xi\rangle\#\tilde{K})U, U)| \le C\big(\|\langle D\rangle^{-1}U\|^2 + \sum_{j=1}^{2}\|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3)\big).$$

Since $A^*S = SA$, taking (2.4) into account, we see

$$(SA)\#\langle\xi\rangle - \langle\xi\rangle\#(A^*S) = (SA)\#\langle\xi\rangle - \langle\xi\rangle\#(SA)$$

$$= \begin{pmatrix} S^0 & S^0 & \mathcal{O}(\sqrt{b})S^0 + S^{-1} \\ S^0 & S^0 & \mathcal{O}(\sqrt{b})S^0 + S^{-1} \\ \mathcal{O}(\sqrt{b})S^0 + S^{-1} & \mathcal{O}(\sqrt{b})S^0 + S^{-1} & \mathcal{O}(b)S^0 + \mathcal{O}(\sqrt{b})S^{-1} + S^{-2} \end{pmatrix}.$$

Summarizing the above estimates, we obtain the following

Lemma 3.5. *Assume $\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ for $|\alpha+\beta| \le 2$. There is $C > 0$ such that*

$$|(\mathrm{Op}(S\#A\#\langle\xi\rangle - \langle\xi\rangle\#A^*\#S)U, U)| \le C\big(\sum_{j=1}^{2}\|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3) + \|\langle D\rangle^{-1}U\|^2\big).$$

Consider $S\#\varphi\#\langle\xi\rangle - \langle\xi\rangle\#\varphi\#S$, where $\varphi \in S^0$ is scalar. Recall

$$S\#\varphi = \varphi S + \sum_{|\alpha+\beta|=1} \frac{(-1)^{|\beta|}}{2i} S^{(\alpha)}_{(\beta)}\varphi^{(\beta)}_{(\alpha)} + \sum_{|\alpha+\beta|=2} \cdots + S^{-3}.$$

For $|\alpha+\beta| = 2$ one has

$$S^{(\alpha)}_{(\beta)}\varphi^{(\beta)}_{(\alpha)} = \begin{pmatrix} S^{-2} & S^{-2} & S^{-2} \\ S^{-2} & S^{-2} & S^{-2} \\ S^{-2} & S^{-2} & \mathcal{O}(\sqrt{b})S^{-2} \end{pmatrix}$$

and hence

$$(S\#\varphi)\#\langle\xi\rangle = (\varphi S)\#\langle\xi\rangle + \Big(\sum_{|\alpha+\beta|=1} \cdots\Big)\#\langle\xi\rangle + \begin{pmatrix} S^{-1} & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & \mathcal{O}(\sqrt{b})S^{-1} + S^{-2} \end{pmatrix} + S^{-2}.$$

Denoting the third term on the right-hand side by K_2, we have the same estimate as (3.2). Similarly one has

$$\langle\xi\rangle\#(\varphi\#S) = \langle\xi\rangle\#(\varphi S) + \langle\xi\rangle\#\Big(\sum_{|\alpha+\beta|=1} \cdots\Big) + \begin{pmatrix} S^{-1} & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & \mathcal{O}(\sqrt{b})S^{-1} + S^{-2} \end{pmatrix} + S^{-2}$$

Consider the term with $|\alpha+\beta| = 1$ and observe that

$$S^{(\alpha)}_{(\beta)}\varphi^{(\beta)}_{(\alpha)} = \begin{pmatrix} S^{-1} & S^{-1} & \mathcal{O}(\sqrt{b})S^{-1} \\ S^{-1} & S^{-1} & \mathcal{O}(\sqrt{b})S^{-1} \\ \mathcal{O}(\sqrt{b})S^{-1} & \mathcal{O}(\sqrt{b})S^{-1} & g^{(\alpha)}_{(\beta)}\varphi^{(\beta)}_{(\alpha)} \end{pmatrix}$$

with $g = b^2 - 2ac$. Therefore

$$\langle\xi\rangle\#(S^{(\alpha)}_{(\beta)}\varphi^{(\beta)}_{(\alpha)}) = \begin{pmatrix} S^0 & S^0 & \mathcal{O}(\sqrt{b})S^0 + S^{-1} \\ S^0 & S^0 & \mathcal{O}(\sqrt{b})S^{-1} + S^{-1} \\ \mathcal{O}(\sqrt{b})S^0 + S^{-1} & \mathcal{O}(\sqrt{b})S^0 + S^{-1} & \mathcal{O}(b)S^0 + \mathcal{O}(\sqrt{b})S^{-1} + S^{-2} \end{pmatrix} \tag{3.3}$$

because $c^{(\alpha)}_{(\beta)} = \mathcal{O}(b)$ for $|\alpha+\beta| = 1$ and then

$$|(\mathrm{Op}(\langle\xi\rangle\#(S^{(\alpha)}_{(\beta)}\varphi^{(\beta)}_{(\alpha)}))U, U)| \leq C\big(\sum_{j=1}^{2}\|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3) + \|\langle D\rangle^{-1}U\|^2\big).$$

Similar arguments are applied to $|(\mathrm{Op}(\varphi^{(\alpha)}_{(\beta)}S^{(\beta)}_{(\alpha)})U, U)|$. Finally, since

$$\langle\xi\rangle\#(\varphi S) - (\varphi S)\#\langle\xi\rangle = \begin{pmatrix} S^0 & S^0 & \mathcal{O}(\sqrt{b})S^0 + S^{-1} \\ S^0 & S^0 & \mathcal{O}(\sqrt{b})S^{-1} + S^{-1} \\ \mathcal{O}(\sqrt{b})S^0 + S^{-1} & \mathcal{O}(\sqrt{b})S^0 + S^{-1} & \mathcal{O}(b)S^0 + \mathcal{O}(\sqrt{b})S^{-1} + S^{-2} \end{pmatrix},$$

we obtain

Lemma 3.6. *Assume* $\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)} = \mathcal{O}(b)$ *for* $|\alpha+\beta| = 1$ *and* $\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ *for* $|\alpha+\beta| = 2$. *Then there exists* $C > 0$ *such that*

$$|(\mathrm{Op}(S\#\varphi\#\langle\xi\rangle - \langle\xi\rangle\#\varphi\#S)U, U)| \leq C\big(\sum_{j=1}^{2}\|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3) + \|\langle D\rangle^{-1}U\|^2\big).$$

Combining Lemmas 3.5, 3.6 and Corollary 2.2, one concludes that for sufficiently large $N_1 > 0$ we have

$$\begin{aligned} &-N_1(\mathrm{Op}(\tilde{S})U, U) - 2t\mathrm{Im}\,(\mathrm{Op}(S)(\mathrm{Op}(\varphi)\langle D\rangle + \mathrm{Op}(A)\langle D\rangle)U, U) \\ &\leq (-N_1\varepsilon_1 + 2C)t\big(\sum_{j=1}^{2}\|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3)\big) + (-N_1(\lambda/2)t^{-1} + 2Ct)\|\langle D\rangle^{-1}U\|^2 \leq 0 \end{aligned} \tag{3.4}$$

Now we pass to the analysis of the term involving $\partial_t S$.

Lemma 3.7. *Assume* $\partial_t c = \mathcal{O}(b)$. *For* $\varepsilon > 0$ *sufficiently small we have*

$$S \gg \varepsilon t\partial_t S.$$

Proof. Since $\partial_t c = \mathcal{O}(b)$, one has

$$3S - \varepsilon t\partial_t S = \begin{pmatrix} 3 & 2a + \varepsilon\mathcal{O}(t) & -b + \varepsilon\mathcal{O}(t) \\ 2a + \varepsilon\mathcal{O}(t) & 2a^2 + 2b + \varepsilon\mathcal{O}(t) & -ab - 3c + \varepsilon\mathcal{O}(at) + \varepsilon\mathcal{O}(bt) \\ -b + \varepsilon\mathcal{O}(t) & -ab - 3c + \varepsilon\mathcal{O}(at) + \varepsilon\mathcal{O}(bt) & b^2 - 2ac + \varepsilon\mathcal{O}(bt) \end{pmatrix}.$$

It is not difficult to see that

$$\det(3S - \varepsilon t\,\partial_t S) = \det 3S + \varepsilon\mathcal{O}\big(b^2(b + a^2)\big)$$

because $t = \mathcal{O}(b)$. ∎

Lemma 3.8. *Assume $\partial_t c = \mathcal{O}(b)$, $\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ for $|\alpha+\beta| = 2$ and $\langle\xi\rangle^\alpha (ac)^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ for $|\alpha+\beta| = 3$. There exist $\varepsilon > 0$ and $C > 0$ such that for $U \in C^\infty(\mathbb{R}_t : C_0^\infty(\mathbb{R}^n))$ we have*

$$\mathsf{Re}(\mathrm{Op}(S - \varepsilon t\,\partial_t S)U, U) \ge -\varepsilon t\big(\sum_{j=1}^2 \|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3)\big) - Ct^{-1}\varepsilon^{-1}\|\langle D\rangle^{-1}U\|^2. \tag{3.5}$$

Proof. Denoting $Q = S - 2\varepsilon t\,\partial_t S$, it suffices to prove

$$\big|\mathsf{Re}(\mathrm{Op}\Big(\sum_{2\le|\alpha+\beta|\le 3} \psi_{\alpha\beta} Q^{(\alpha)}_{(\beta)}\Big)U, U)\big| \le \varepsilon t\big(\sum_{j=1}^2 \|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3)\big) + C\varepsilon^{-1}t^{-1}\|\langle D\rangle^{-1}U\|^2. \tag{3.6}$$

Consider $\mathsf{Re}\big(\mathrm{Op}\big(\psi_{\alpha\beta}Q^{(\alpha)}_{(\beta)}\big)U, U\big)$ with $|\alpha+\beta| = 2$. Note that

$$\psi_{\alpha\beta}Q^{(\alpha)}_{(\beta)} = \begin{pmatrix} 0 & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & S^{-1} \\ S^{-1} & S^{-1} & \psi_{\alpha\beta}\big(g^{(\alpha)}_{(\beta)} - \varepsilon t\,(\partial_t g)^{(\alpha)}_{(\beta)}\big) \end{pmatrix},$$

where $g = b^2 - 2ac$. Consequently, one deduce

$$\begin{aligned}|(\mathrm{Op}(\psi_{\alpha\beta}Q^{(\alpha)}_{(\beta)})U, U)| \le \varepsilon t \sum_{j=1}^2 \|U_j\|^2 + C\varepsilon^{-1}t^{-1}\|\langle D\rangle^{-1}U\|^2 \\ + |\mathsf{Re}\,(\mathrm{Op}(\psi_{\alpha\beta}(g^{(\alpha)}_{(\beta)} - \varepsilon t(\partial_t g)^{(\alpha)}_{(\beta)}))U_3, U_3)|.\end{aligned}$$

Setting

$$T = \psi_{\alpha\beta}\big(g^{(\alpha)}_{(\beta)} - \varepsilon t(\partial_t g)^{(\alpha)}_{(\beta)}\big)\langle\xi\rangle \in S^0,$$

we obtain $\mathsf{Re}\,\big(\psi_{\alpha\beta}\big(g^{(\alpha)}_{(\beta)} - \varepsilon t(\partial_t g)^{(\alpha)}_{(\beta)}\big)\big) = T\#\langle\xi\rangle^{-1} + S^{-2}$. Therefore

$$\mathsf{Re}\,\big(\mathrm{Op}\big(\psi_{\alpha\beta}(g^{(\alpha)}_{(\beta)} - \varepsilon t(\partial_t g)^{(\alpha)}_{(\beta)}\big)U_3, U_3) \le \varepsilon t\|\mathrm{Op}(T)U_3\|^2 + C\varepsilon^{-1}t^{-1}\|\langle D\rangle^{-1}U_3\|^2$$

Note that $\|\mathrm{Op}(T)U_3\|^2 = (\mathrm{Op}(T\#T)U_3, U_3)$ and $T\#T = T^2 + S^{-2}$. There is $C > 0$ such that

$$T^2 \le Cb$$

because $t = \mathcal{O}(b)$ and $\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ so that $Cb - T^2 \ge 0$. Then applying the Fefferman-Phong inequality, we prove the assertion. Let $|\alpha+\beta| = 3$ then with $T_1 = \big(\psi_{\alpha\beta}\big(g^{(\alpha)}_{(\beta)} - \varepsilon t(\partial_t g)^{(\alpha)}_{(\beta)}\big)\big)\#\langle\xi\rangle^{3/2}$

$$T_1^2 \le Cb$$

with some $C > 0$ since $t = \mathcal{O}(b)$ and $\langle\xi\rangle^\alpha (ac)^{(\alpha)}_{(\beta)} = \mathcal{O}(\sqrt{b})$ and the proof is similar. □

From (3.5) setting $N_2 = \varepsilon^{-1}$ and dividing by ε, one deduces

$$\mathsf{Re}(\mathrm{Op}(-N_2 S + t\partial_t S)U, U) \le t\big(\sum_{j=1}^2 \|U_j\|^2 + (\mathrm{Op}(b)U_3, U_3)\big) + Ct^{-1}\varepsilon^{-2}\|\langle D\rangle^{-1}U\|^2$$

and applying Corollary 2.2, this implies

$$\begin{aligned} &-(N_2+N_3)\mathsf{Re}(\mathrm{Op}(\tilde{S})U,U)+t\mathsf{Re}(\mathrm{Op}(\partial_t S)U,U)\\ &\le(-N_3\varepsilon_1+1)t\big(\sum_{j=1}^{2}\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big)+t^{-1}(C\varepsilon^{-2}-N_3\lambda)\|\langle D\rangle^{-1}\|^2. \end{aligned}\tag{3.7}$$

Fixing ε and N_2, we choose N_3 sufficiently large and we arrange the right hand side of the above inequality to be negative.

Next we turn to the analysis of $2\mathsf{Im}(\mathrm{Op}(\tilde{S})\mathrm{Op}(B)U,U)$. Recall that $(\mathrm{Op}(\tilde{S})U,U)\gg 0$ by Corollary 2.3. Consequently,

$$\begin{aligned} &2|(\mathrm{Op}(\tilde{S})\mathrm{Op}(B)U,U)|\le N^{-1/2}(t\mathrm{Op}(\tilde{S})\mathrm{Op}(B)U,\mathrm{Op}(B)U)+N^{1/2}(t^{-1}\mathrm{Op}(\tilde{S})U,U)\\ &=N^{-1/2}(t\mathrm{Op}(B^*)\mathrm{Op}(\tilde{S})\mathrm{Op}(B)U,U)+N^{1/2}(t^{-1}\mathrm{Op}(\tilde{S})U,U)\\ &\le N^{-1/2}(t^{-1}t^2\mathrm{Op}(B^*)\mathrm{Op}(S)\mathrm{Op}(B)U,U)+N^{1/2}(t^{-1}\mathrm{Op}(\tilde{S})U,U)+C_2\lambda N^{-1/2}\|\langle D\rangle^{-1}U\|^2. \end{aligned}\tag{3.8}$$

Lemma 3.9. *There exists $N_4>0$ depending on T and B such that for $0\le t\le T$ and any $\varepsilon>0$ there exists $D_\varepsilon>0$ such that*

$$\mathsf{Re}\big(\mathrm{Op}(N_4S-t^2B^*SB)U,U\big)\ge-\varepsilon t(\sum_{j=1}^{2}\|U_j\|^2+(cU_3,U_3))-D_\varepsilon t^{-1}\|\langle D\rangle^{-1}U\|^2.$$

Proof. Recall

$$3S-\varepsilon t^2B^*SB=\begin{pmatrix}3+\varepsilon\mathcal{O}(t^2) & 2a+\varepsilon\mathcal{O}(t^2) & -b+\varepsilon\mathcal{O}(t^2)\\ 2a+\varepsilon\mathcal{O}(t^2) & 2(a^2+b)+\varepsilon\mathcal{O}(t^2) & -ab-3c+\varepsilon\mathcal{O}(t^2)\\ -b+\varepsilon\mathcal{O}(t^2) & -ab-3c+\varepsilon\mathcal{O}(t^2) & b^2-2ac+\varepsilon\mathcal{O}(t^2)\end{pmatrix}$$

which proves $3S-\varepsilon t^2B^*SB\gg 0$ with some $\varepsilon=\varepsilon(T)>0$. To justify this, notice that the terms $\varepsilon\mathcal{O}(t^2b)$, $\varepsilon\mathcal{O}(t^2c)$, $\varepsilon\mathcal{O}(t^2a^2)$, $\varepsilon\mathcal{O}(t^4a)$ can be absorbed by $\det S$ because $b\ge\delta_1 t$. For example,

$$\varepsilon t^4|a|\le\frac{1}{2}\varepsilon(t^5+t^3a^2)\le C\varepsilon tb^2(a^2+b).$$

Choosing $\varepsilon(T)$ small enough, we obtain the result. Then the rest of the proof is just a repetition of the proof of Lemma 3.8. □

According to Lemma 3.9 and (3.8), one has

$$\begin{aligned} 2|(\mathrm{Op}(\tilde{S})\mathrm{Op}(B)U,U)|&\le 2N_4^{1/2}t^{-1}(\mathrm{Op}(\tilde{S})U,U)+\varepsilon t(\sum_{j=1}^{2}\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3))\\ &-N_4^{1/2}\lambda t^{-2}\|\langle D\rangle^{-1}U\|^2+D_\varepsilon t^{-1}\|\langle D\rangle^{-1}U\|^2+C_2\lambda N_4^{-1/2}\|\langle D\rangle^{-1}U\|^2. \end{aligned}\tag{3.9}$$

Combining the estimates (3.4), (3.7), (3.9), it follows that

$$\begin{aligned}
\partial_t \mathsf{Re}(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U) &\le -2\mathsf{Im}(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})F,U)\\
-(N-N_1-N_2-N_3-2N_4^{1/2})&t^{-N-1}e^{-\gamma t}\mathsf{Re}(\mathrm{Op}(\tilde{S})U,U)\\
+\Big[C_\varepsilon-\lambda\Big(N+1+N_4^{1/2}-\lambda C\varepsilon^{-1}\Big)\Big]&t^{-N-2}e^{-\gamma t}\|\langle D\rangle^{-1}U\|^2\\
&+\varepsilon t^{-N}e^{-\gamma t}\big(\sum_{j=1}^{2}\|U_j\|^2+(\mathrm{Op}(b)U_3,U_3)\big)\\
-(\gamma-D_\varepsilon-C_1\lambda-Ct\lambda N_4^{-1/2})&t^{-N-1}e^{-\gamma t}\|\langle D\rangle^{-1}U\|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
2|(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})F,U)| &\le 2(t^{-N+1}e^{-\gamma t}\mathrm{Op}(\tilde{S})F,F)^{1/2}(t^{-N-1}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U)^{1/2}\\
&\le (t^{-N+1}e^{-\gamma t}\mathrm{Op}(\tilde{S})F,F)+(t^{-N-1}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U).
\end{aligned}$$

Denote $N^*=N_1+N_2+N_3+2N_2^{1/2}+2$ and we choose $0<\varepsilon\le\varepsilon_1$. We fix ε and $\lambda>2C_\varepsilon$. Next we fix N_4 so that

$$N_4^{1/2}>\lambda C\varepsilon^{-1}+1.$$

Then the term with $t^{-N-2}e^{-\gamma t}\|\langle D\rangle^{-1}U\|^2$ is absorbed. Finally we choose $N>N^*$ and γ such that $\gamma-D_\varepsilon-C_1\lambda-C\lambda N_4^{-1/2}T\ge 0$. Then we have

$$\partial_t\mathsf{Re}(t^{-N}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U)\le(t^{-N+1}e^{-\gamma t}\mathrm{Op}(\tilde{S})F,F)-(N-N^*)\mathsf{Re}\,(t^{-N-1}e^{-\gamma t}\mathrm{Op}(\tilde{S})U,U). \tag{3.10}$$

Integrating (3.10) in τ from $\varepsilon>0$ to t and taking Corollary 2.3 into account, one obtains

Proposition 3.1. *Assume that*

$$\begin{aligned}
&b\ge\delta_1 t,\quad |ac|\le\bar{\varepsilon}\,b^2,\quad |c|\le\bar{\varepsilon}\,b^{3/2},\\
&\langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)}=\mathcal{O}\big(b\big)\ \textit{for}\ |\alpha+\beta|=1,\quad \langle\xi\rangle^\alpha c^{(\alpha)}_{(\beta)}=\mathcal{O}\big(\sqrt{b}\big)\ \textit{for}\ |\alpha+\beta|=2,\\
&\langle\xi\rangle^\alpha (ac)^{(\alpha)}_{(\beta)}=\mathcal{O}\big(\sqrt{b}\big),\ \ |\alpha+\beta|=3,\quad \partial_t c=\mathcal{O}\big(b\big)
\end{aligned}\tag{3.11}$$

hold globally where $\bar{\varepsilon}$ *is given in Lemmas* 2.2 *and* 2.3. *Then there exist* $\delta_2>0,\gamma_0>0,N\in\mathbb{N}$ *and* $C>0$ *such that for* $\gamma\ge\gamma_0$ *and* $0<\varepsilon\le t\le T$ *we have for any* $U\in C^\infty(\mathbb{R}_t:C_0^\infty(\mathbb{R}^n))$

$$\begin{aligned}
&\delta_2 t^{-N+2}e^{-\gamma t}\|U(t)\|^2+\delta_2(N-N^*)\int_\varepsilon^t\tau^{-N+1}e^{-\gamma\tau}\|U(\tau)\|^2d\tau\\
&\le C\varepsilon^{-N-1}e^{-\gamma\varepsilon}\|U(\varepsilon)\|^2+\int_\varepsilon^t\tau^{-N+1}e^{-\gamma\tau}(\mathrm{Op}(\tilde{S})F(\tau),F(\tau))d\tau.
\end{aligned}$$

4. Microlocal energy estimates

First we prove the following

Lemma 4.10. *Assume that* (1.3) *is satisfied in* $[0,T]\times\tilde{W}$ *where* $\tilde{W}$ *is a conic neighborhood of* (x_0,ξ_0). *Then there exist extensions* $\tilde{a}(t,x,\xi)\in S^0$, $\tilde{b}(t,x,\xi)\in S^0$ *and* $\tilde{c}(t,x,\xi)\in S^0$ *of* a, b *and* c *such that* (3.11) *holds globally.*

Proof. Assume that (1.3) is satisfied in $[0,T]\times\tilde{W}$. Choose conic neighborhoods U, V, W of (x_0,ξ_0) such that $U\Subset V\Subset W\Subset\tilde{W}$. Take $0\le\chi(x,\xi)\in S^0$, $0\le\tilde{\chi}(x,\xi)\in S^0$ such that $\chi=1$ on V and $\chi=0$ outside W and $\tilde{\chi}=0$ on U and $\tilde{\chi}=1$ outside V. Choosing W and T small one can assume that χb is small as we please in $[0,T]\times\mathbb{R}^{2n}$ because $b(0,x_0,\xi_0)=0$. We define the extensions of a, b, c by

$$\tilde{a}=\chi a,\quad \tilde{b}=\chi^2 b+M\tilde{\chi},\quad \tilde{c}=\chi^3 c$$

where $M>0$ is a positive constant which we will choose below. Note that

$$|\tilde{a}\tilde{c}|=\chi^4|ac|\le C|a|\chi^4 b^2\le\bar{\varepsilon}(\chi^2 b)^2\le\bar{\varepsilon}\,\tilde{b}^2,$$
$$|\tilde{c}|=\chi^3|c|\le C\chi^3 b^2=Cb^{1/2}(\chi^2 b)^{3/2}\le\bar{\varepsilon}\,\tilde{b}^{3/2}$$

taking $a(0,x_0,\xi_0)=0$, $b(0,x_0,\xi_0)=0$ into account and choosing W small.

If $(x,\xi)\in V$ then $\tilde{b}(t,x,\xi)=b+M\tilde{\chi}\ge\delta_1 t$ and if (x,ξ) is outside V then $\tilde{b}(t,x,\xi)=\chi^2 b+M\ge\delta_1 t$ for $[0,T]\times\mathbb{R}^{2n}$ choosing M so that $M\ge\delta_1 T$. Thus we have

$$\tilde{b}(t,x,\xi)\ge\delta_1 t\quad (t,x,\xi)\in[0,T]\times\mathbb{R}^{2n}.$$

We turn to estimate derivatives of $\tilde{c}$ and $\tilde{a}\tilde{c}$. For $|\alpha+\beta|=1$ it is clear that

$$\langle\xi\rangle^{|\alpha|}\big|\tilde{c}^{(\alpha)}_{(\beta)}\big|=\langle\xi\rangle^{|\alpha|}\big|(\chi^3 c)^{(\alpha)}_{(\beta)}\big|\le C(\chi^2 b^2+\chi^3 b)\le C_1\chi^2 b\le C_1\tilde{b}.$$

Similarly for $|\alpha+\beta|=2$ one sees

$$\langle\xi\rangle^{|\alpha|}\big|(\chi^3 c)^{(\alpha)}_{(\beta)}\big|\le C(\chi b^2+\chi^2 b+\chi^3\sqrt{b})\le C_1\chi\sqrt{b}=C_1(\chi^2 b)^{1/2}\le C_1\tilde{b}^{1/2}.$$

For $|\alpha+\beta|=3$, taking $\langle\xi\rangle^{\alpha}(ac)^{(\alpha)}_{(\beta)}=\mathcal{O}(\sqrt{b})$ into account, one has

$$\langle\xi\rangle^{|\alpha|}\big|(\tilde{a}\tilde{c})^{(\alpha)}_{(\beta)}\big|=\langle\xi\rangle^{|\alpha|}\big|(\chi^4 ac)^{(\alpha)}_{(\beta)}\big|$$
$$\le C(\chi b^2+\chi^2 b+\chi^3\sqrt{b}+\chi^4\sqrt{b})\le C_1\chi\sqrt{b}\le C_1\tilde{b}^{1/2}.$$

Since $|\partial_t\tilde{c}|=|\chi^3\partial_t c|\le C\chi^3 b\le C\tilde{b}$ is obvious the proof is complete. □

Remark 4.2. In the proof of Lemma 4.10 replacing $\tilde{b}$ by $\chi^2 b+M\tilde{\chi}+M'\chi_0(\xi)$ where $\chi_0(\xi)\in C_0^\infty(\mathbb{R}^n)$ which is 1 near $\xi=0$ and $M'>0$ is a suitable positive constant it suffices to assume that (1.3) is satisfied in $[0,T]\times\tilde{W}$ for $|\xi|\ge 1$.

Let $V\Subset V_1\Subset\Omega$ and $u\in C^\infty(\mathbb{R}_t: C_0^\infty(V))$ Let $\{\chi_\alpha\}$ be a finite partition of unity with $\chi_\alpha(x,\xi)\in S^0$ so that

$$\sum_\alpha\chi_\alpha^2(x,\xi)=\chi^2(x),$$

where $\chi(x)=1$ on $\overline{V}$ and $\operatorname{supp}\chi\subset V_1$. We can suppose that $\operatorname{supp}\chi_\alpha\subset V_1$. We repeat the argument in [11, Section 4], studying a system

$$D_t U_\alpha=(\mathrm{Op}(\varphi)\langle D\rangle+\mathrm{Op}(A)\langle D\rangle+\mathrm{Op}(B))U_\alpha+F_\alpha$$

with $U_\alpha={}^t\big((D_t-\mathrm{Op}(\varphi)\langle D\rangle)^2\chi_\alpha u,\langle D\rangle(D_t-\mathrm{Op}(\varphi)\langle D\rangle)\chi_\alpha u,\langle D\rangle^2\chi_\alpha u\big)$. One extends the coefficients a, b, c and φ outside the support of χ_α and one can assume that (3.11) are satisfied globally. Thus we obtain the following

Theorem 4.1. *Let $Y \Subset \Omega$. Assume that for every point $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ there exist a conic neighborhood $W \subset T^*\Omega \setminus \{0\}$ and $T(x_0, \xi_0) > 0$ such that the estimates* (3.11) *are satisfied for $0 \le t \le T(x_0, \xi_0)$ and $(x, \xi) \in W$. Then there exist $c > 0$, $T_0 > 0$, $\gamma_0 > 0, C > 0$ and $N \in \mathbb{N}$ such that for $\gamma \ge \gamma_0$, $0 < \varepsilon < t \le T_0$ we have for any $U \in C^\infty(\mathbb{R}_t : C_0^\infty(Y))$*

$$\begin{aligned} &c\,t^{-N+2}e^{-\gamma t}\|U(t)\|^2 + c\int_\varepsilon^t \tau^{-N+1}e^{-\gamma\tau}\|U(\tau)\|^2 d\tau \\ &\le C\varepsilon^{-N-1}e^{-\gamma\varepsilon}\|U(\varepsilon)\|^2 + C\int_\varepsilon^t \tau^{-N+1}e^{-\gamma\tau}\|f(\tau)\|^2 d\tau. \end{aligned} \tag{4.1}$$

Corollary 4.4. *Let $Y \Subset \Omega$. Assume that for every point $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ there exist a conic neighborhood $W \subset T^*\Omega \setminus \{0\}$ and $T(x_0, \xi_0) > 0$ such that the estimates* (1.3) *are satisfied for $0 \le t \le T(x_0, \xi_0)$ and $(x, \xi) \in W$. Then the same assertion as in Theorem* 4.1 *holds.*

The same argument can be applied for the adjoint operator P^*. With

$$V = {}^t\big((D_t - \mathrm{Op}(\varphi)\langle D\rangle)^2 v, \langle D\rangle(D_t - \mathrm{Op}(\varphi)\langle D\rangle)v, \langle D\rangle^2 v\big)$$

the equation $P^*v = g$ is reduced to

$$D_t V = \mathrm{Op}(\varphi)\langle D\rangle V + (\mathrm{Op}(A)\langle D\rangle + \mathrm{Op}(\tilde{B}))V + G, \tag{4.2}$$

with $G = {}^t(g, 0, 0)$. Here the principal symbol is the same, while the lower order terms change. To study the Cauchy problem for P^* in $0 < t < T$ with initial data on $t = T$ one considers

$$\begin{aligned} &-\partial_t(t^N e^{\gamma t}\mathrm{Op}(\tilde{S})V, V) = -N(t^{N-1}e^{\gamma t}\mathrm{Op}(\tilde{S})V, V) - \gamma(t^N e^{\gamma t}\mathrm{Op}(\tilde{S})V, V) \\ &-(t^N e^{\gamma t}\mathrm{Op}(\partial_t S)V, V) - \lambda(N-1)t^{N-2}e^{\gamma t}\|\langle D\rangle^{-1}U\|^2 - \lambda\gamma t^{N-1}e^{\gamma t}\|\langle D\rangle^{-1}U\|^2 \\ &+2\mathsf{Im}\,(t^N e^{\gamma t}(\mathrm{Op}(\tilde{S})(\mathrm{Op}(\varphi)\langle D\rangle + \mathrm{Op}(A)\langle D\rangle + \mathrm{Op}(\tilde{B}))V, V)) + 2\mathsf{Im}(t^N e^{\gamma t}\mathrm{Op}(\tilde{S})G, V). \end{aligned} \tag{4.3}$$

Repeating the argument of Section 3, one obtains the following

Theorem 4.2. *Let $Y \Subset \Omega$. Assume that for every point $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$ there exist a conic neighborhood $W \subset T^*\Omega \setminus \{0\}$ and $T(x_0, \xi_0) > 0$ such that the estimates* (3.11) *are satisfied for $0 \le t \le T(x_0, \xi_0)$ and $(x, \xi) \in W$. Then there exist $c > 0$, $T_0 > 0$, $\gamma_0 > 0, C > 0$ and $N \in \mathbb{N}$ such that for $\gamma \ge \gamma_0$, $0 < \varepsilon < t \le T_0$ we have for any $V \in C^\infty(\mathbb{R}_t : C_0^\infty(Y))$*

$$\begin{aligned} &c\,t^{N+2}e^{\gamma t}\|V(t)\|^2 + c\int_t^{T_0} \tau^{N+1}e^{\gamma\tau}\|V(\tau)\|^2 d\tau \\ &\le CT_0^{N-1}e^{\gamma T_0}\|V(T_0)\|^2 + C\int_t^{T_0} \tau^{N+1}e^{\gamma\tau}\|g(\tau)\|^2 d\tau. \end{aligned} \tag{4.4}$$

Following the argument in [11], we may absorb the weight τ^{-N} and obtain energy estimates with a loss of derivatives. For the sake of completeness we recall this argument. Consider $Pu = f$ for $u \in C^\infty(\mathbb{R}_t : C_0^\infty(\mathbb{R}^n))$. Assume $u(\varepsilon, x) = u_t(\varepsilon, x) = u_{tt}(\varepsilon, x) = 0$. Differentiating $Pu = f$ with respect to t, we determine the functions $D_t^j u(\varepsilon, x) = u_j(x) \in C_0^\infty(\mathbb{R}^n)$ and set

$$u_M(t, x) = \sum_{j=0}^M \frac{1}{j!}u_j(x)(i(t-\varepsilon))^j,\ 0 < \varepsilon \le t \le T_0.$$

Therefore $w = u - u_M \in C^\infty(\mathbb{R}_t : C_0^\infty(\mathbb{R}^n))$ satisfies $Pw = f_M$ with

$$D_t^j f_M(\varepsilon, x) = 0,\ j = 0, 1, \dots, M-3, \quad D_t^j w(\varepsilon, x) = 0,\ j = 0, 1, \dots, M.$$

Consequently, from Theorem 4.1 one deduce the existence of $N \in \mathbb{N}$ and $C > 0$ such that for $\varepsilon > 0$, and a solution $u \in C^\infty([\varepsilon, T_0] \times C_0^\infty(Y))$ to the equation $Pu = f$ with

$$u(\varepsilon, x) = u_t(\varepsilon, x) = u_{tt}(\varepsilon, x) = 0$$

we have

$$\sum_{j+|\alpha|\le 2} \int_\varepsilon^t \|\partial_t^j \partial_x^\alpha u(s,x)\| ds \le C \int_\varepsilon^t \| \sum_{j+|\alpha|\le N} \partial_t^j \partial_x^\alpha Pu(s,x)\| ds, \tag{4.5}$$

where C is independent of ε. We can obtain a similar estimates for higher order derivatives.

Note that under the assumptions of Theorem 4.1 the symbol p is strictly hyperbolic for $0 < t \le T_0$ with some $T_0 > 0$. Indeed the fact that p is strictly hyperbolic for $0 < t \le T_0$, is equivalent to $\Delta > 0$ for $0 < t \le T_0$, Δ being the discriminant of the equation $p = 0$ with respect to τ. On the other hand, $\Delta = 27 \det S$ (see also Corollary 2.1) and $\det S > 0$ for $t > 0$ by Lemma 2.2. Therefore applying the estimate (4.5) and repeating the argument in [3, Theorem 23.4.5] one can find $Z \Subset \Omega$ and $T^* > 0$ such that for $f \in C_0^\infty([0, T_0] \times \Omega)$ there exists $u \in C_0^\infty([0, T_0] \times \Omega)$ satisfying $Pu = f$ in $[0, T^*] \times Z$. The local uniqueness of the solution of the Cauchy problem for P can be obtained taking into account Theorem 4.2 for the adjoint operator P^* and using the argument of [3, Theorem 23.4.5]. We leave the details to the reader.

Finally, we deduce

Corollary 4.5. *Under the assumptions of Theorem 4.1 the Cauchy problem for P is C^∞ well posed in $[0, T^*] \times Z$ for all lower order terms.*

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