DYNAMICAL ZETA FUNCTION FOR SEVERAL STRICTLY CONVEX OBSTACLES

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Abstract. The behavior of the dynamical zeta function $Z_D(s)$ related to several strictly convex disjoint obstacles is similar to that of the inverse $Q(s) = \frac{1}{\zeta(s)}$ of the Riemann zeta function $\zeta(s)$. Let $\Pi(s)$ be the series obtained from $Z_D(s)$ summing only over primitive periodic rays. In this paper we examine the analytic singularities of $Z_D(s)$ and $\Pi(s)$ close to the line $\Re s = s_2$, where s_2 is the abscissa of absolute convergence of the series obtained by the second iterations of the primitive periodic rays. We show that at least one of the functions $Z_D(s)$, $\Pi(s)$ has a singularity at $s = s_2$.

KEYWORDS: dynamical zeta function, periodic rays

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open and connected domain with C^∞ boundary $\partial\Omega$ having the form $\Omega = \mathbb{R}^n \setminus K$, where

$$K = \cup_{j=1}^{Q} K_j, \;\; K_i \cap K_j = \emptyset, \text{ for } i \neq j \tag{1.1}$$

and K_j are strictly convex compact obstacles for $j = 1, \dots, Q$, $Q \geq 3$. Throughout this paper we suppose that K satisfies the following condition introduced by Ikawa ([6]):

(H) The convex hull of every two connected components of K does not have common points with any other connected component of K.

Consider the reflecting rays in $\overline{\Omega}$ (see [6] and Chapter 2 in [19] for a precise definition). Under the condition (H) every periodic ray is ordinary reflecting, that is γ has no tangent segments. Given a *periodic reflecting ray* γ in $\overline{\Omega}$ with m_γ reflections, we denote by T_γ the primitive period (length) of γ, by $d_\gamma = lT_\gamma$, $l \in \mathbb{N}$, the period of γ and by P_γ the linear Poincaré map related to γ. Setting $|\det(I - P_\gamma)| = |I - P_\gamma|$, it is easy to prove (see [18], Appendix) that there exist constants $b_1 > 0$, $b_2 > 0$, $B_0 > 0$ so that

$$B_0 e^{2b_1 d_\gamma} \leq |I - P_\gamma| \leq e^{2b_2 d_\gamma}. \tag{1.2}$$

Denote by Ξ the set of all reflecting periodic rays in $\overline{\Omega}$ and set

$$d_0 = \min \;\text{dist}_{i\neq j}\; (K_i, K_j), \quad D_0 = \max \;\text{dist}_{i\neq j}(K_i, K_j).$$

For the counting function of the lengths of periodic rays there exists a constant $a_0 > 0$ such that

$$\sharp\{\gamma \in \Xi : d_\gamma \leq q\} \leq e^{a_0 q} \tag{1.3}$$

(see [6], [22] and Chapter 2 in [19]). In this note we examine the dynamical zeta function

$$Z_D(s) = \sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma},\ s \in \mathbb{C}, \tag{1.4}$$

where the summation is over all periodic rays $\gamma \in \Xi$. This zeta function is related to the trace formula for the unitary group associated to the Dirichlet problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \ \text{ in } \ \mathbb{R} \times \Omega, \\ u = 0 \ \text{ on } \mathbb{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x). \end{cases} \tag{1.5}$$

The form of $Z_D(s)$ is obtained by the Laplace transformation of the distribution

$$\sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} \delta(t - d_\gamma) \tag{1.6}$$

which in turn is the sum of the principal singularities of $u(t) \in \mathcal{D}'(\mathbb{R}^+)$ given by

$$u(t) = \sum_{\lambda_j} e^{it\lambda_j},\ t > 0.$$

Here $\lambda_j \in \mathbb{C}$ are the poles of the scattering matrix $S(z)$ related to the problem (1.5) and the summation is over all poles counted with their multiplicities. We refer to [7], [8], [18], [21] for a more detailed description of this link and to [1], [28], [13], [5], [19], [21] for the trace formulas leading to (1.6).

Following a result of Ikawa [7], [8], the existence of an analytic singularity of $Z_D(s)$ implies the existence of $\delta > 0$ such that there are infinite number of poles $\{z_j\}_{j\in\mathbb{N}}$ of the scattering matrix $S(z)$ satisfying

$$0 < \Im z_j \leq \delta,\ \forall j \in \mathbb{N}$$

and the last property is known as the modified Lax-Phillips conjecture. Another motivation for the analysis of $Z_D(s)$ is the folklore conjecture that the singularities of $Z_D(s)$ should determine approximatively the scattering poles.

By using (1.2) and (1.3), it is easy to see that there exists $s_1 \in \mathbb{R}$ called abscissa of absolute convergence such that for $\Re s > s_1$ the series (1.4) is absolutely convergent. Despite many works in physical literature and the numerical analysis results concerning n-disk problems (see for example [3], [29], [12], [28] and the references cited there), to our best knowledge in the general case the problem of the existence of at *least one singularity* of $Z_D(s)$ is still open. The existence of an analytic non-real singularity has been proved by Ikawa [9] in the case when K is the union of several balls with radius $r \leq r_0$, provided $r_0 > 0$ sufficiently small. Recently, Stoyanov [25] generalized the result of Ikawa for several obstacles satisfying some geometrical conditions and having diameters less than r_0. It was proved in [18] that $Z_D(s)$ has no singularities on the line $\Re s = s_1$. In fact we have a stronger result and following the recent works of Stoyanov (see [23], [26]), we know that there exists $\delta_0 > 0$ such that $Z_D(s)$ is analytic for $\Re s > s_1 - \delta_0$ (see also [10] for the special case $s_1 > 0$.) This means that $Z_D(s)$ is analytic in a domain around $\Re s = s_1$ and this phenomenon of cancellations is typical for dynamical zeta functions (see [4], [16], [23], [24], [26]). On the other hand, since $Z_D(s)$ is a Dirichlet series with real coefficients changing their signs, the situation is very similar to that for the inverse $Q(s) = \frac{1}{\zeta(s)}$ of the classical Riemann zeta function $\zeta(s)$. It is well known that $Q(s)$ is analytic on the line $\Re s = 1$ and $Q(s)$ has non-real singularities on the *critical line* $\Re s = 1/2$. Moreover, we have the representation

$$\log \zeta(s) = \sum_{m=1}^{\infty} \sum_{p \in \mathbf{P}} \frac{1}{m} \frac{1}{p^{ms}}, \ \ \Re s > 1, \tag{1.7}$$

where $\mathbf{P}$ denotes the set of prime numbers. Consequently, the analytic behavior of $\log \zeta(s)$ for $1/2 < \Re s \leq 1$ is characterized by the continuation of the function

$$\pi(s) = \sum_{p \in \mathbf{P}} \frac{1}{p^s}, \ \Re s > 1$$

and the critical line $\Re s = 1/2$ is related to $m = 2$ in the representation (1.7).

Denote by $\mathcal{P}$ the set of all primitive periodic rays. In this note we examine the analytic singularities of $Z_D(s)$ close to the line $\Re s = s_2$, where $s_2 < s_1$ is the abscissa of the absolute convergence of the series $\Pi_2(s)$ obtained from $Z_D(s)$ when we sum only over the rays 2γ, $\gamma \in \mathcal{P}$, that is over the second iteration of primitive rays (see Section 4 for a precise definition). We show that the line $\Re s = s_2$ plays a role in the investigation of the singularities of $Z_D(s)$. Similarly to $\pi(s)$, introduce the function

$$\Pi(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sT_\gamma}, \ \Re s > s_1,$$

where the summation is over the primitive rays $\gamma \in \mathcal{P}$. Next let $h_\Pi < s_1$ be the abscissa of holomorphy of $\Pi(s)$ given by

$$h_\Pi = \inf\{t \in \mathbb{R} : \ \Pi(s) \ \text{is analytic for} \ \Re s > t\}.$$

Our main result is the following

Theorem 1. *At least one of the functions $Z_D(s), \Pi(s)$ has a singularity at $s = s_2$ and the difference $Z_D(s) - \Pi(s)$ is analytic for*

$$s \in \{z \in \mathbb{C} : \ \Re z > s_2\}.$$

Moreover, if $s_2 \neq h_\Pi$, then $Z_D(s)$ has a singularity at z with $\Re z > \max\{s_2, h_\Pi\} - \epsilon_1$, where $\epsilon_1 > 0$ is sufficiently small.

Following the same way, we may show that if we consider the series obtained by summing over all iterations of the primitive rays of order $(2m - 1)$, the corresponding function will be singular at $s = s_{2m}$ if $Z_D(s)$ is analytic at $s = s_{2m}$. Here s_k is the abscissa of absolute convergence of the series obtained by summing over all iterations of order $k \geq 2$ and we show that $s_1 - h_t < s_k < s_{k-1}$, $h_t > 0$ being the topological entropy of the billiard flow (see Proposition 2). Thus if $Z_D(s)$ is analytic for $\Re s > s_1 - h_t$, for any fixed $M \geq 2$ one obtains a singularity of the sum of series related to the iterations $m \leq M$. This corollary yields some information for the numerical analysis since in the numerical experiences one treats series with finite number iterations.

The existence of a singularity z_0 of $\Pi(s)$ such that $\Re z_0 > s_2 - \epsilon_0$, $\epsilon_0 > 0$, $\Im z_0 \neq 0$, is an interesting open problem, but it seems that the difficulty of this problem could be compared with that of the existence or the absence of singularities of $\pi(s)$ for $1/2 < \Re s < 1$. If fact, the dynamics of the periodic orbits is chaotic and the random change of signs of the coefficients in (1.4) plays some essential role. We *conjecture* that in general $Z_D(s)$ is not singular at s_2 and Theorem 1 shows that in this case $\Pi(s)$ must be singular at s_2. It is expected that there exist non-real singularities z of $\Pi(s)$ with $\Re z$ arbitrary close to line of holomorphy $\Re s = h_\Pi$ of $\Pi(s)$. This will lead to singularities of $Z_D(s)$. In fact we have two possibilities:

$$(a)\ s_2 \neq h_\Pi,\ (b)\ s_2 = h_\Pi.$$

Our analysis in Section 4 implies that in the case (a) the function $Z_D(s)$ must be singular either at $s = s_2$ $(s_2 > h_\Pi)$ or at a point z close to the line $\Re s = h_\Pi$ $(s_2 < h_\Pi)$ and we obtain a solution of the modified Lax-Phillips conjecture (see [8], [9], [25]). In the case (b) we have a phenomenon similar to the famous Riemann conjecture for $\zeta(s)$ and the maximal domain $\Re s > t$, where $\Pi(s)$ is analytic, is determined by the line $\Re s = s_2$. Finally, it is not clear if the singularities found in [9] and [25] lie in the domain $\Re s > s_2$ and we will discuss this problem in Section 4.

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2. Symbolic dynamics

We will write $Z_D(s)$ as a Selberg zeta function using the argument of Section 5, [18]. First assume $n = 3$ and let $\lambda_{\gamma,i},\ i = 1, 2,\ |\lambda_{\gamma,i}| > 1$, be the eigenvalues of the Poincaré map P_γ of the ray $\gamma \in \mathcal{P}$. Set

$$\delta_\gamma = -\frac{1}{2}\log(\lambda_{\gamma,1}\lambda_{\gamma,2}),\ \nu_\gamma = -\log\lambda_{\gamma,1},\ \mu_\gamma = -\log\lambda_{\gamma,2}.$$

The product $\lambda_{\gamma,1}\lambda_{\gamma,2}$ and the sum $\lambda_{\gamma,1} + \lambda_{\gamma,2}$ are positive and $\delta_\gamma < 0$. Given $\gamma \in \mathcal{P}$, introduce

$$r_\gamma = \begin{cases} 0 & \text{if } m_\gamma = 2k, \\ 1 & \text{if } m_\gamma = 2k+1. \end{cases}$$

Then for $\Re s \gg s_1$ we have

$$Z_D(s) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma\in\mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma+p\mu_\gamma)}. \tag{2.1}$$

We refer to [18] for the details of the proof of this representation. For $n = 2$ we have a simpler formula since there is only one eigenvalue $\lambda_\gamma > 1$ and we get

$$Z_D(s) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma\in\mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma)}, \tag{2.2}$$

where $\delta_\gamma = -\frac{1}{2}\log\lambda_\gamma,\ \nu_\gamma = 2\delta_\gamma$. Consider the leading term of $Z_D(s)$ obtained for $k = p = 0$ (resp. $k = 0$ for $n = 2$) and having the form

$$Z(s) = -\frac{d}{ds}Z_0(s),\ \ Z_0(s) = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{\gamma\in\mathcal{P}}(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma)}.$$

We will write $Z_0(s)$ by using a symbolic model. Let us recall some notations concerning the symbolic dynamics. Given a $Q \times Q$ matrix $A(i, j)_{i,j=1,\dots,Q}$ such that

$$A(i, j) = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

introduce the spaces

$$\Sigma_A = \{\xi = \{\xi_i\}_{i=-\infty}^{\infty} :\ \xi_i \in \{1, \dots, Q\},\ A(\xi_i, \xi_{i+1}) = 1\},$$

$$\Sigma_A^+ = \{\xi = (\xi_0, \xi_1, \dots) :\ A(\xi_i, \xi_{i+1}) = 1,\ \forall i \geq 0\}.$$

Let σ_A be the shift on Σ_A, Σ_A^+ given, respectively, by

$$(\sigma_A \xi)_i = \xi_{i+1},\ \forall i \in \mathbb{Z},\ \ (\sigma_A \xi)_i = \xi_{i+1},\ \forall i \geq 0.$$

For every $\xi \in \Sigma_A$ there exists a unique ray $\gamma(\xi)$ with successive reflection points on

$$..., \partial K_{j-1}, \partial K_j, \partial K_{j+1}, ...$$

(see [6], [19]). Let $P_j(\xi)$ be the j^{th} reflection point of $\gamma(\xi)$ and let $f(\xi) = \|P_0(\xi) - P_1(\xi)\|$. If $\gamma = \gamma(\xi) \in \mathcal{P}$ has m reflections and primitive period T_γ, then

$$T_\gamma = f(\xi) + f(\sigma_A \xi) + \cdots + f(\sigma_A^{m-1} \xi) = S_m f(\xi).$$

Also (see [8], [9]), there exists a function $g(\xi)$ such that

$$\delta_\gamma = g(\xi) + g(\sigma_A \xi) + \cdots + g(\sigma_A^{m-1} \xi) = S_m g(\xi).$$

For $\Re s$ large we may write $Z_0(s)$ as follows

$$Z_0(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-sf(\xi) + g(\xi))}.$$

Given a continuous function $F(\xi) \in C(\Sigma_A)$, introduce

$$\mathrm{var}_n F = \sup_{\xi, \eta \in \Sigma_A} \{|F(\xi) - F(\eta)| :\ \xi_i = \eta_i \text{ for } |i| \leq n\}$$

and for $0 < \theta < 1$ consider the norms

$$|F|_\theta = \sup_n \frac{\mathrm{var}_n F}{\theta^n},\ \|F\|_\infty = \sup_{\xi \in \Sigma_A} |F(\xi)|,\ \|F\|_\theta = \|F\|_\infty + |F|_\theta.$$

Let $\mathcal{F}_\theta(\Sigma_A) \subset C(\Sigma_A)$, $\mathcal{F}_\theta(\Sigma_A^+) \subset C(\Sigma_A^+)$ be Banach spaces with norm $\|.\|_\theta$. It follows from the exponential instability of the billiard ball map that with some constant $0 < \theta < 1$, depending on the geometry of K, we have $f(\xi),\ g(\xi) \in \mathcal{F}_\theta(\Sigma_A)$ (see for more details [8], [9], [18], [23], [25]). We introduce the suspended flow σ^f over the space

$$\Sigma_A^f = \{(\xi, t) :\ \xi \in \Sigma_A,\ 0 \leq t \leq f(\xi)\}$$

with the identification $(\xi, f(\xi)) \sim (\sigma_a(\xi), 0)$ (see [17]) and notice that the topological entropy $h_t > 0$ of the suspended flow σ^f over Σ_A^f is given by

$$h_t = \sup_{\mu \in \mathcal{M}} \frac{h_\mu(\sigma_A)}{\int_{\Sigma_A} f d\mu}.$$

Finally, recall that the pressure $P(F)$ of a function $F \in C(\Sigma_A)$ is given by

$$P(F) = \sup_{m \in \mathcal{M}} \Big(h_\mu(\sigma_A) + \int_{\Sigma_A} F d\mu \Big),$$

where $h_\mu(\sigma_A)$ is the measure entropy of σ_A and the sup is taken over the set $\mathcal{M}$ of all probabilistic measures on Σ_A invariant with respect to σ_A.

3. Summation over the iterated periodic rays

It is well known (see [17]) that for every function $\varphi(\xi) \in \mathcal{F}_\theta(\Sigma_A)$ there exists $h,\ \psi \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A)$ so that
$$\varphi(\xi) = h(\xi) + \psi(\sigma_A(\xi)) - \psi(\xi)$$
and the function $h(\xi)$ depends only on the coordinates $(\xi_0, \xi_1, \dots)$. In this case we will write $\varphi \sim h$. Obviously, if $F \sim \tilde{F}$, we have $P(F) = P(\tilde{F})$. Passing to functions $f \sim \tilde{f},\ g \sim \tilde{g}$, we get
$$Z_0(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-s\tilde{f}(\xi) + \tilde{g}(\xi))}.$$

The function
$$\mathbb{R} \ni s \longrightarrow P(-skf + kg)$$
is strictly decreasing and given an integer $k \geq 1$ we may introduce the number $s_k \in \mathbb{R}$ determined uniquely by the equality
$$P(-s_k kf + kg) = 0.$$
It follows easily from the results in [17] that s_k is the abscissa of absolute convergence of the series
$$P_k(s) = \frac{1}{k} \sum_{\gamma \in \mathcal{P}} (-1)^{km_\gamma} e^{-ksT_\gamma + k\delta_\gamma}.$$
Indeed, s_k is the abscissa of absolute convergence of the series
$$G_k(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-skf(\xi) + kg(\xi))}.$$
On the other hand, for $\Re s > s_k$ we have
$$G_k(s) = \sum_{\gamma \in \mathcal{P}} e^{-skT_\gamma + k\delta_\gamma} + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{m(-skT_\gamma + k\delta_\gamma)}$$
and as in [17], Chapter 6 and [18], Section 4, we deduce that the series
$$\sum_{m=2}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{m(-skT_\gamma + k\delta_\gamma)}$$
is absolutely convergent for $\Re s \geq s_k - \epsilon$ for some small $\epsilon > 0$. Next we will prove the following

Lemma 1. *For all $k \geq 1$ we have $s_{k+1} < s_k$.*

Proof. The pressure of the function $-s_k kf + kg$ is zero, so we may find a function $h \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A^+)$ so that $h \sim -s_k kf + kg,\ P(h) = 0$ and we may choose h (see for more details [17]) so that
$$\sum_{\sigma_A \eta = \xi} e^{h(\eta)} = 1,\ \forall \xi \in \Sigma_A^+.$$
This implies $h(\eta) \leq \alpha_k < 0$ for all $\eta \in \Sigma_A^+$ and
$$k \int_{\Sigma_A} (-s_k f + g) d\mu \leq \alpha_k$$

for each $\mu \in \mathcal{M}$. It is clear that

$$h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g)d\mu$$
$$\leq \sup_{\mu \in \mathcal{M}} \Big[h_\mu(\sigma) + \int_{\Sigma_A} (-s_k kf + kg)d\mu\Big] + \frac{\alpha_k}{k} = \frac{\alpha_k}{k} < 0,\ \forall \mu \in \mathcal{M}.$$

This implies

$$P\Big(-s_k(k+1)f + (k+1)g\Big) = \sup_{\mu \in \mathcal{M}} \Big[h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g)d\mu\Big] \leq \frac{\alpha_k}{k}.$$

On the other hand, $P\Big(-s_{k+1}(k+1)f + (k+1)g\Big) = 0$ and since the function

$$\mathbb{R} \ni s \longrightarrow P\Big(-s(k+1)f + (k+1)g\Big)$$

is strictly decreasing, we get $s_{k+1} < s_k$.

To study the convergence of the series over the iterated rays we need the following

Proposition 1. *For every $k \geq 1$ there exists $\epsilon_o(k) > 0$, depending on k, such that the series*

$$\sum_{m=k+1}^{\infty} P_m(s) = \sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma + \delta_\gamma)}$$

is absolutely convergent for $\Re s \geq s_k - \epsilon_o(k)$.

Proof. As in the proof of Lemma 1, we choose h so that $h \sim -ks_k f + kg$, $h(\eta) < 0$, for all $\eta \in \Sigma_A^+$. First, assume that $s_k < 0$. We choose $\epsilon = \epsilon(k) > 0$ small enough in order to arrange the inequality

$$\sup_{\eta \in \Sigma_A^+} h(\eta) = \alpha_k \leq (k+1)k\epsilon s_k \|f\|_\infty.$$

Let $\eta \in \Sigma_A^+$ correspond to a primitive periodic ray $\gamma \in \mathcal{P}$ with m reflections as it has been explained in Section 2. We obtain

$$S_m(-ks_k f + kg)(\eta) = -ks_k T_\gamma + k\delta_\gamma.$$

On the other hand, it is clear that $T_\gamma \leq m\|f\|_\infty$ and we get

$$S_m h(\eta) \leq m(k+1)k\epsilon s_k \|f\|_\infty \leq (k+1)k\epsilon s_k T_\gamma.$$

From the equality $S_m(-ks_k f + kg)(\eta) = S_m h(\eta)$, we deduce

$$-s_k T_\gamma + \delta_\gamma \leq (k+1)\epsilon s_k T_\gamma,\ \forall \gamma \in \mathcal{P}.$$

Now let $0 \leq u \leq \frac{\epsilon}{k+1}$. Then

$$-s_k(1+u)T_\gamma + \delta_\gamma \leq (k+1)\epsilon s_k T_\gamma - s_k u T_\gamma$$
$$\leq \Big((k+1)\epsilon - \frac{\epsilon}{k+1}\Big) s_k T_\gamma \leq \epsilon s_k T_\gamma$$

and we get the lower bound

$$1 > 1 - e^{-s_k(1+u)T_\gamma + \delta_\gamma} \geq 1 - e^{\epsilon s_k T_\gamma} \geq 1 - e^{2s_k \epsilon d_0} = \frac{1}{C_{\epsilon,k}} > 0.$$

Thus for $0 \le u \le \frac{\epsilon}{k+1}$ the series

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1+u)T_\gamma+\delta_\gamma)} = \frac{e^{(k+1)(-s_k(1+u)T_\gamma+\delta_\gamma)}}{1-e^{-s_k(1+u)T_\gamma+\delta_\gamma}} \le C_{\epsilon,k}e^{(k+1)(-s_k(1+u)T_\gamma+\delta_\gamma)} \tag{3.1}$$

is convergent.

Next we obtain

$$-(k+1)s_k(1+u)T_\gamma+(k+1)\delta_\gamma$$
$$\le -s_k kT_\gamma + k\delta_\gamma + (k+1)\epsilon s_k T_\gamma - (k+1)us_k T_\gamma \le -s_k(1-\epsilon)kT_\gamma + k\delta_\gamma.$$

Since s_k is the abscissa of absolute convergence of the series of k iterated rays, we deduce

$$\sum_{\gamma\in\mathcal{P}} e^{-s_k(1-\epsilon)kT_\gamma+k\delta_\gamma} < \infty.$$

Thus we conclude that

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(-s_k(1+u)T_\gamma+\delta_\gamma)} < \infty$$

and the series

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma+\delta_\gamma)}$$

is absolutely convergent for $\Re s \ge s_k - \frac{\epsilon}{k+1}$. Setting $\epsilon_o(k) = \frac{\epsilon}{k+1}$, we obtain the result in this case.

Passing to the case $s_k > 0$, choose $\epsilon = \epsilon(k) > 0$ to arrange the inequalities

$$\sup_{\eta\in\Sigma_A^+} h(\eta) \le -(k+1)k\epsilon s_k \|f\|_\infty,$$

$$-s_kT_\gamma + \delta_\gamma \le -(k+1)\epsilon s_k T_\gamma,\ \forall T_\gamma \in \mathcal{P}.$$

For $0 \le u \le \frac{\epsilon}{k+1}$ we deduce

$$-s_k(1-u)T_\gamma + \delta_\gamma \le -(k+1)\epsilon s_k T_\gamma + s_k u T_\gamma \le -\epsilon s_k T_\gamma$$

which yields

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1-u)T_\gamma+\delta_\gamma)} \le C_{\epsilon,k}e^{(k+1)(-s_k(1-u)T_\gamma+\delta_\gamma)}. \tag{3.2}$$

On the other hand,

$$-(k+1)s_k(1-u)T_\gamma + (k+1)\delta_\gamma \le -s_k(1+\epsilon)kT_\gamma + k\delta_\gamma$$

and this leads to

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(-s_k(1-u)T_\gamma+\delta_\gamma)} < \infty\,.$$

Finally, in the case $s_k = 0$, we arrange

$$\sup_{\eta\in\Sigma_A^+} h(\eta) \le -(k+1)k\epsilon\|f\|_\infty,$$

$$\delta_\gamma \le -(k+1)\epsilon T_\gamma,\ \forall T_\gamma \in \mathcal{P}.$$

Repeating the above argument, we establish for $0 \leq u \leq \frac{\epsilon}{k+1}$ the convergence of the series

$$\sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} e^{m(uT_\gamma + \delta_\gamma)} < \infty ,$$

and this completes the proof.

To compare s_k and s_1, consider the measure $\nu \in \mathcal{M}$ for which we have

$$P(-s_1 f + g) = h_\nu(\sigma_A) + \int_{\Sigma_A} (-s_1 f + g) d\nu = 0.$$

This measure is called equilibrium state of $-s_1 f + g$ (see [17]). Then we obtain

$$P\Big(-k(s_1 - \frac{k-1}{k} h_t) f + kg\Big) \geq h_\nu(\sigma_A) + k \int_{\Sigma_A} (-s_1 f + g) d\nu + (k-1) h_t \int_{\Sigma_A} f d\nu$$

$$= (k-1)\Big[h_t \int_{\Sigma_A} f d\nu - h_\nu(\sigma_A)\Big] \geq 0.$$

Comparing this with $P(-k s_k f + kg) = 0$, we deduce

$$s_k \geq s_1 - \frac{k-1}{k} h_t. \tag{3.3}$$

Thus we have proved the following

Proposition 2. *The sequence s_k is convergent and* $\lim_{k \to \infty} s_k \geq s_1 - h_t$.

It is interesting to note that the abscissa c_0 of simple convergence of the Dirichlet series $Z_0(s)$ satisfies the estimate

$$c_0 \geq s_1 - h_t,$$

but it is difficult to compare c_0 with s_k.

4. Singularities on the line $\Re s = s_2$

Consider the Dirichlet series

$$P_2(s) = \frac{1}{2} \sum_{\gamma \in \mathcal{P}} e^{-2sT_\gamma + 2\delta_\gamma}$$

with positive coefficients. According to a classical result, this series has an analytic singularity at $s = s_2$. On the other hand, Proposition 1 implies that the sum over all iterated rays $k\gamma, \gamma \in \mathcal{P}, k \geq 3$, given by

$$\sum_{k=3}^{\infty} P_k(s),$$

is analytic for $\Re s \geq s_2 - \epsilon_o(2)$ for some $\epsilon_o(2) > 0$. It is clear that the singularities of $Z_0(s)$ for $\Re s > s_2$ are related to those of the series obtained by summing only over the primitive rays

$$P_1(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{r_\gamma} e^{-sT_\gamma + \delta_\gamma}.$$

Let h_p be the abscissa of *holomorphy* of the Dirichlet series $P_1(s)$. More precisely, h_p is the *minimal* real number t such that $P_1(s)$ is analytic for $\Re s > t$. We have three possibilities:

$$(i)\ h_p > s_2,\quad (ii)\ h_p = s_2,\quad (iii)\ h_p < s_2.$$

In the case (i), the function $P_1(s)$, and hence $Z_0(s)$, has either a singularity on the line $\Re s = h_p$ or there exists a sequence of singularities z_j with $\Re z_j \to h_p$, $|\Im z_j| \to \infty$. In the case (iii), the function $P_2(s)$ produces a singularity of $Z_0(s)$ at $s = s_2$. In the case (ii) we must examine the singularities of the sum $P_1(s) + P_2(s)$. Of course, if $P_1(s)$ is analytic at $s = s_2$, we have the same situation as in the case (iii). Thus a cancellation of the singularities of $P_1(s) + P_2(s)$ at the point s_2 is possible only if $P_1(s)$ is singular at $s = s_2$. Thus we have the following

Theorem 2. *At least one of the functions $Z_0(s)$, $P_1(s)$ has a singularity at $s = s_2$. Moreover, the difference $Z_0(s) - P_1(s)$ is analytic for*

$$s \in \{z \in \mathbb{C} :\ \Re z > s_2\}.$$

We may compare the functions $Z_0(s)$ and $Z_D(s)$. As it was shown in [8], [18], [25] there exists $\mu_1 > 0$ such that $Z_D(s) - Z_0(s)$ is analytic for $\Re s > s_1 - \mu_1$. The number μ_1 depends on the geometry of obstacles (see Appendix in [18] and [25]). In some cases we may show that $s_2 > s_1 - \mu_1$. For example this is true if $n = 2$ and $s_2 < 0$. Nevertheless, it is more natural to deal with the function $\Pi(s)$ introduced in Section 1. As above, let h_Π be the abscissa of the holomorphy of the Dirichlet series $\Pi(s)$ introduced in Section 1. We consider again three cases:

$$(i)\ h_\Pi > s_2,\quad (ii)\ h_\Pi = s_2,\quad (iii)\ h_\Pi < s_2.$$

For $m \geq 2$ and $n = 3$ the analysis of the series

$$\Pi_m(s) = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{1}{m} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)},\ \Re s > s_1$$

is completely similar to that of $P_m(s)$. In fact the abscissa of absolute convergence of $\Pi_m(s)$ coincides with that of $P_m(s)$ and we may apply Proposition 1 for the series

$$\sum_{m=j+1}^{\infty} \Pi_m(s) = \sum_{m=j+1}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{1}{m} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)}$$

assuming $j \geq 1$. The case $n = 2$ is treated in a similar way and repeating the argument of the proof of Theorem 2, we obtain Theorem 1.

Following the same way, we may consider the function

$$\mathbf{\Pi_3}(s) = \Pi(s) + \Pi_2(s) + \Pi_3(s) = \sum_{\gamma \in \Xi_3} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma},\ \Re s > s_1,$$

where the summation is over all rays $\gamma \in \Xi_3 \subset \Xi$ which are either primitive or they are obtained by two or three iterations of primitive periodic rays. Then at least one of the functions $Z_D(s)$, $\mathbf{\Pi_3}(s)$ has a singularity at $s = s_4$ and it is possible to iterate this argument.

Let us mention that from our results it is not clear if the analytic singularity z of $\Pi(s)$ or $Z_D(s)$ given by Theorem 1 is a pole. In fact, it is known that the function $Z_0(s)$ is meromorphic for

$$\Re s \geq s_1 - \frac{|\log \theta|}{2\|f\|_\infty},$$

$0 < \theta < 1$ being the constant introduced in Section 2. On the other hand, we have $s_2 \geq h_t/2$ and s_2 lies in the above domain if $h_t \|f\|_\infty \leq |\log \theta|$. It is expected that $Z_0(s)$ and $Z_D(s)$ are meromorphic in a more larger domain or in the whole complex plan. For $n = 2$ some results in this direction are obtained by Morita [15].

It is interesting to mention that for all $k \in \mathbb{N}$ we have

$$s_k > b_0 = \sup_{\gamma \in \mathcal{P}} \frac{\delta_\gamma}{T_\gamma}. \tag{4.1}$$

In [18] it was established that $b_0 < 0$, so we need to check (4.1) only for $s_k < 0$. In this case the argument of the proof of Proposition 1 shows that

$$-s_k T_\gamma + \delta_\gamma \leq \epsilon_k T_\gamma,\ \forall \gamma \in \mathcal{P}$$

with some $\epsilon_k < 0$ and we obtain (4.1). The number b_0 has been introduced in [18] and it is related to the sequence of poles

$$s_{m,\gamma} = \frac{\delta_\gamma}{T_\gamma} + \frac{2m\pi}{T_\gamma}\mathbf{i},\ \ r_{m,\gamma} = \frac{\delta_\gamma}{T_\gamma} + \frac{(2m+1)\pi}{T_\gamma}\mathbf{i},\ m \in \mathbb{Z}$$

obtained from the series formed by all iteration of a *fixed* periodic primitive ray γ.

For several strictly convex small obstacles Ikawa [9] and Stoyanov [25] established the existence of a non-real singularity

$$z_0 = \alpha + \mathbf{i}\frac{\pi}{d_1},\ \ \alpha \in \mathbb{R}$$

of $Z_D(s)$ with d_1 sufficiently close to D_0. Following the analysis in Section 7, [25], we conclude that $s_1 - b_K \leq \alpha < s_1$ with

$$b_K \geq \frac{1}{D_0} \ln\Big(1 + \frac{\kappa_{\min}}{\nu_0} D_0\Big).$$

Here $\kappa_{\min} > 0$ is the minimal normal curvature of ∂K and $\nu_0 > 0$ is a constant depending on d_0, the diameter of K and

$$\chi_0 = \min\{\text{dist}\Big(K_j,\ \text{convex hull}\ (K_i \cup K_l)\Big) :\ j \neq i, i \neq l, l \neq j\} > 0.$$

For obstacles having sufficiently small diameters, we may arrange the inequality $b_K \geq h_t$. Indeed, it is sufficient to have

$$h_\mu(\sigma_A) \leq \frac{d_0}{D_0} \ln\Big(1 + \frac{\kappa_{\min}}{\nu_0} D_0\Big) \leq b_K \int_{\Sigma_A} f d\mu$$

for every σ_A invariant measure $\mu \in \mathcal{M}$. If the diameters of the obstacles are sufficiently small, then $\kappa_{\min}$ is large enough, while $\frac{d_0}{D_0}$ and χ_0 remain bounded from below. Thus in this case we have

$$\sup_{\mu \in \mathcal{M}} h_\mu(\sigma_A) \leq \frac{d_0}{D_0} \ln\Big(1 + \frac{\kappa_{\min}}{\nu_0} D_0\Big)$$

which implies $b_K \geq h_t$. Combining this with (3.3), we obtain immediately

$$s_1 - b_K \leq s_1 - h_t < s_k,\ \forall k \in \mathbb{N}.$$

Consequently, the line $\Re s = s_k$ lies in the domain where we have complex singularities and this agrees with the conjecture that we must have complex singularities of $Z_D(s)$ close to the line $\Re s = h_\Pi$ or close to the line $\Re s = s_2$.

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