RUELLE TRANSFER OPERATORS WITH TWO COMPLEX PARAMETERS AND APPLICATIONS

Vesselin Petkov

Université de Bordeaux, Institut de Mathématiques de Bordeaux
351, Cours de la Libération, 33405 Talence, France
Email: petkov@math.u-bordeaux1.fr

Luchezar Stoyanov

University of Western Australia, School of Mathematics and Statistics
Perth, WA 6009, Australia
Email: luchezar.stoyanov@uwa.edu.au

Abstract. For a C^2 Axiom A flow $\phi_t : M \longrightarrow M$ on a Riemannian manifold M and a basic set Λ for ϕ_t we consider the Ruelle transfer operator $L_{f-s\tau+zg}$, where f and g are real-valued Hölder functions on Λ, τ is the roof function and $s, z \in \mathbb{C}$ are complex parameters. Under some assumptions about ϕ_t we establish estimates for the iterations of this Ruelle operator in the spirit of the estimates for operators with one complex parameter (see [4], [21], [22]). Two cases are covered: (i) for arbitrary Hölder f, g when $|\operatorname{Im} z| \leq B|\operatorname{Im} s|^\mu$ for some constants $B > 0$, $0 < \mu < 1$ ($\mu = 1$ for Lipschitz f, g), (ii) for Lipschitz f, g when $|\operatorname{Im} s| \leq B_1|\operatorname{Im} z|$ for some constant $B > 0$. Applying these estimates, we obtain a non zero analytic extension of the zeta function $\zeta(s, z)$ for $P_f - \epsilon < \operatorname{Re}(s) < P_f$ and $|z|$ small enough with simple pole at $s = s(z)$. Two other applications are considered as well: the first concerns the Hannay-Ozorio de Almeida sum formula, while the second deals with the asymptotic of the counting function $\pi_F(T)$ for weighted primitive periods of the flow ϕ_t.

1. Introduction

Let M be a C^2 complete (not necessarily compact) Riemannian manifold, $\phi_t : M \longrightarrow M$ ($t \in \mathbb{R}$) a C^2 flow on M and let $\varphi_t : M \longrightarrow M$ be a C^2 weak mixing Axiom A flow ([2], [11]). Let Λ be a *basic set* for ϕ_t, i.e. Λ is a compact locally maximal invariant subset of M and ϕ_t is hyperbolic and transitive on Λ.

Given a Hölder continuous function $F : \Lambda \longrightarrow \mathbb{R}$ and a primitive periodic orbit γ of ϕ_t, denote by $\lambda(\gamma)$ the *least period* of γ. The *weighted period* of γ is defined by $\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))dt$, where $x_\gamma \in \gamma$. The weighted version of the dynamical zeta function (see Sect. 9 in [11]) is given by

$$\zeta_\phi(s, F) := \prod_\gamma \Big(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\Big)^{-1}.$$

For $F = 0$ we obtain the classical Ruelle dynamical zeta function.

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It is well known (see for instance Chapter 6 in [11]) that the analysis of the dynamical zeta function can be reduced to that of a Dirichlet series by using a symbolic coding of Λ given by a fixed Markov family $\{R_i\}_{i=1}^k$. For our analysis it is convenient to consider a Markov family of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$ (see section 2 for the notation and more details). Let $\mathcal{P} : R = \cup_{i=1}^k R_i \longrightarrow R$ be the related Poincaré map, let $\tau(x) > 0$ be the first return time function on R, and let $\sigma : U = \cup_{i=1}^k U_i \longrightarrow U$ be the *shift map* given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. The flow ϕ_t on Λ is naturally related to the suspension flow σ_t^τ on the suspension space R^τ (see section 2 for details). There exists a natural semi-conjugacy projection $\pi(x, t) : R^\tau \longrightarrow \Lambda$ which is one-to-one on a residual set (see [2]). Then following the results in [2], [3], a closed σ-orbit $\{x, \sigma x, ..., \sigma^{n-1}x\}$ is projected to a closed orbit γ in Λ with a least period $\lambda(\gamma) = \tau^n(x) := \tau(x) + \tau(\sigma(x)) + ... + \tau(\sigma^{n-1}(x))$.

Passing to the symbolic model (see [2], Chapter 6 in [11]), the analysis of $\zeta_\varphi(s, F)$ is reduced to that of the Dirichlet series

$$\eta(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x)}$$

with a Hölder continuous function $f(x) = \int_0^{\tau(x)} F(\pi(x, t))dt : R \longrightarrow \mathbb{R}$. To deal with certain problems (see Chapter 9 in [11] and [17]) it is necessary to study a more general series

$$\eta_g(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} g^n(x) e^{f^n(x) - s\tau^n(x)}$$

with a Hölder continuous function $G : \Lambda \longrightarrow \mathbb{R}$ and $g(x) = \int_0^{\tau(x)} G(\pi(x, t))dt : R \longrightarrow \mathbb{R}$. For this purpose it is convenient to examine the zeta function

$$\zeta(s, z) := \prod_{\gamma} \Big(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma) + z\lambda_G(\gamma)}\Big)^{-1} = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x) + zg^n(x)}\Big) \tag{1.1}$$

depending on **two complex variables** $s, z \in \mathbb{C}$. Formally, we have

$$\eta_g(s) = \frac{\partial \log \zeta(s, z)}{\partial z}\Big|_{z=0}.$$

The analysis of the series in (1.1) is based on the investigation of the iterations of the Ruelle operator

$$L_{f - s\tau + zg} v(x) = \sum_{\sigma y = x} e^{f(y) - s\tau(y) + zg(y)} v(y), \; s, z \in \mathbb{C},$$

since

$$L^n_{f - s\tau + zg} v(x) = \sum_{\sigma^n y = x} e^{f^n(y) - s\tau^n(y) + zg^n(y)} v(y), \; n \in \mathbb{N}.$$

The precise definition of the Ruelle operator acting on spaces of Hölder functions is given in section 4. Thus, the strategy for the proof of the analytic continuation of the dynamical zeta function comprises two majors steps:

(I) Prove that suitable "contraction" estimates for the iterations of the Ruelle operator $L^n_{f - s\tau + zg}$ imply the *convergence by packets* of the Dirichlet series which yields an analytic continuation of the corresponding zeta function.

(II) Establish suitable "contraction" estimates for the iterations.

This strategy has been used for zeta functions depending on one complex parameter and related spectral estimates, called Dolgopyat estimates, have been proved in many cases ([4], [20], [21], [22]) under some conditions on ϕ_t. The most general case of such estimates known so far for Ruelle operators with one complex parameter is that described by the Standing Assumptions in section 4 below (see [21], [22]).

In this paper we study both problems (I) and (II) for zeta functions and Ruelle operators depending on two parameters $s, z \in \mathbb{C}$. These problems are motivated by particular important applications in mind, however we believe they are also of an independent interest.

1.1. **Results.** Under some hypothesis on the flow ϕ_t (see section 4 for our standing assumptions) we prove spectral estimates for the iterations of Ruelle operator $L^n_{f-s\tau+zg}$ with **two complex parameters** $s, z \in \mathbb{C}$. These estimates are in the spirit of those obtained in [4], [20], [21], [22] for Ruelle operators with **one complex parameter** $s \in \mathbb{C}$. It should be emphasized that the transition from one to two complex parameters is highly non-trivial, and so far there have been no results of this kind in the literature. In particular, in the treatment of this case completely new difficulties appear when $|\operatorname{Im} s| \to \infty$ and $|\operatorname{Im} z| \to \infty$.

In what follows, first in Theorem 5 we prove spectral estimates in the case of arbitrary Hölder continuous functions f, g when there exist constants $B > 0$ and $0 < \mu < 1$ such that $|\operatorname{Im} z| \leq B|\operatorname{Im} s|^{\mu}$ and $|\operatorname{Im} s| \geq b_0 > 0$. When f, g are Lipschitz one can take $\mu = 1$. This covers completely the case when $|z|$ is bounded and the estimates have the same form as those for operators with one complex parameter. Moreover, these estimates are sufficient for the applications in [11] and [18] when $|z|$ runs in a small neighbourhood of 0 (see sections 7 and 8). Notice that in the special case of a geodesic flow on a surface with negative curvature in the proof of Lemma 3.5 in [18] it was mentioned that one can obtain a non-vanishing extension of $\zeta(s, z)$ for sufficiently small $|z|$. However no proof of this result was given, and indeed one needs some of the results in this paper to obtain this – in particular, the generalisation of Ruelle's lemma to the case of two complex parameters (see section 3) and the estimates of the corresponding Ruelle operator established in sections 5 and 6 below. In fact, in section 6 we deal with the more difficult situation when f, g are Lipschitz and there exists a constant $B_1 > 0$ such that $|\operatorname{Im} s| \leq B_1 |\operatorname{Im} z|$ (see Theorem 6).

To study the analytic continuation of $\zeta(s, z)$ for $P_f - \eta_0 < \operatorname{Re} s < P_f$, we need a generalisation of Ruelle's lemma mentioned above which yields a link between the convergence by packets of a Dirichlet series like (1.3) below and $\log \zeta(s, z)$ and the estimates of the iterations of the corresponding Ruelle operator. The reader may consult [24] for a precise result in this direction completing some points the previous works ([19], [16], [9]), treating this question. For our needs in this paper we prove in section 3 an analogue of this lemma for Dirichlet series with two complex parameters following the approach in [24]. Combining Theorem 4 with the estimates in Theorem 5 (b), we obtain the following

Theorem 1. *Assume the standing assumptions in section 4 fulfilled for a basic set Λ. Then for any Hölder continuous functions $F, G : \Lambda \longrightarrow \mathbb{R}$ there exists $\eta_0 > 0$ such that the function $\zeta(s, z)$ admits a non vanishing analytic continuation for*

$$(s, z) \in \{(s, z) \in \mathbb{C}^2 : \; P_f - \eta_0 \leq \operatorname{Re} s, \; s \neq s(z), \; |z| \leq \eta_0\}$$

with a simple pole at $s(z)$. The pole $s(z)$ is determined as the root of the equation $Pr(f - s\tau + zg) = 0$ with respect to s for $|z| \leq \eta_0$.

Applying the results of sections 5 and 6, we study also the analytic continuation of $\zeta(s, \mathbf{i}w)$ for $P_f - \eta_0 < \operatorname{Re} s$ and $w \in \mathbb{R}$, $|w| \geq \eta_0$, in the case when $F, G : \Lambda \longrightarrow \mathbb{R}$ are Lipschitz functions (see Theorem 7). Here both complex parameters s, w may go to infinity, the analysis of this case is more complicated and we study the situation when $z = \mathbf{i}w$. Our investigation was motivated by the necessity to have an analytic continuation of the zeta functions appearing in the arguments in [23], [7]. This analytic continuation combined with the arguments in [23] opens some perspectives for investigations on sharp large deviations for Anosov flows with exponentially shrinking intervals in the spirit of [12]. Some other applications are also possible, in particular we expect to obtain the result of Theorem 7 for arbitrary Hölder functions $F, G : \Lambda \longrightarrow \mathbb{R}$, which for now is an open problem.

Our first application concerns the so called Hannay-Ozorio de Almeida sum formula (see [5], [10], [18]). Let $\phi_t : M \longrightarrow M$ be the geodesic flow on the unit-tangent bundle over a compact negatively curved surface M. In [18] it was proved that there exists $\epsilon > 0$ such that if $(\delta(T))^{-1} = \mathcal{O}(e^{\epsilon T})$, then for every Hölder continuous function $G : M \longrightarrow \mathbb{R}$, we have

$$\lim_{T\to+\infty} \frac{1}{\delta(T)} \sum_{T-\frac{\delta(T)}{2} \leq \lambda(\gamma) \leq T+\frac{\delta(T)}{2}} \lambda_G(\gamma) e^{-\lambda^u(\gamma)} = \int_M G d\mu, \tag{1.2}$$

where γ runs over the set of primitive periodic orbits of the flow in M, $\lambda^u(\gamma) = \lambda_E(\gamma)$ with $E(x) = \lim_{t\to 0} \frac{1}{t} \log |\mathrm{Jac}\,(D\phi_t|_{E^u(x)})|$, while μ is the unique ϕ_t-invariant probability measure which is absolutely continuous with respect to the volume measure on M. The measure μ is called SRB (Sinai-Ruelle-Bowen) measure (see [3]). Notice that in the above case the Anosov flow ϕ_t is weak mixing and M is an attractor. Applying Theorem 1 and the arguments in [18], we prove the following

Theorem 2. *Let Λ be an attractor, that is there exists an open neighborhood V of Λ such that $\Lambda = \cap_{t\geq 0}\phi_t(V)$. Assume the standing assumptions of section 4 fulfilled for the basic set Λ. Then there exists $\epsilon > 0$ such that if $(\delta(T))^{-1} = \mathcal{O}(e^{\epsilon T})$, then for every Hölder function $G : \Lambda \longrightarrow \mathbb{R}$ the formula (1.2) holds with the SRB measure μ for ϕ_t.*

Our second application concerns the counting function

$$\pi_F(T) = \sum_{\lambda(\gamma)\leq T} e^{\lambda_F(\gamma)},$$

where γ is a primitive period orbit for $\phi_t : \Lambda \longrightarrow \Lambda$, $\lambda(\gamma)$ is the least period and $\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))dt$, $x_\gamma \in \gamma$. For $F = 0$ we obtain the counting function $\pi_0(T) = \#\{\gamma : \lambda(\gamma) \leq T\}$. These counting functions have been studied in many works (see [16] for references concerning $\pi_0(T)$ and [11], [15] for the function $\pi_F(T)$). The study of $\pi_F(T)$ is based on the analytic continuation of the function

$$\zeta_F(s) = \prod_\gamma \Big(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\Big)^{-1},\ s \in \mathbb{C},$$

which is just the function $\zeta(s, 0)$ defined above. We prove the following

Theorem 3. *Let Λ be a basic set and let $F : \Lambda \longrightarrow \mathbb{R}$ be a Hölder function. Assume the standing assumptions of section 4 fulfilled for Λ. Then there exists $\epsilon > 0$ such that*

$$\pi_F(T) = li(e^{Pr(F)T})(1 + \mathcal{O}(e^{-\epsilon T})),\ T \to \infty,$$

where $li(x) := \int_2^x \frac{1}{\log y} dy \sim \frac{x}{\log x}$, $x \to +\infty$.

In the case when $\phi_t : T^1(M) \longrightarrow T^1(M)$ is the geodesic flow on the unit tangent bundle $T^1(M)$ of a compact C^2 manifold M with negative section curvatures which are $\frac{1}{4}$-pinching the above result has been established in [15]. It follows from [21] and [22] that the special case of a geodesic flow in [15] is covered by Theorem 3.

The proof of Theorem 5 for Hölder functions f and $g \equiv 0$ implies some new result even for the Ruelle operator with one complex parameter under the standing assumptions. For example, we have to study quite precisely the approximations of f by smooth functions and estimate the Lipschitz constants of the corresponding eigenfunctions related to maximal eigenvalues. This particular result is given in Lemma 4 and appears to be of an independent interest.

The results of our work for contact Anosov flows satisfying some pinching conditions, called in section 4 simplifying assumptions, have been announced in [13]. Here we treat a more general case and present detailed proofs of the results.

1.2. **Examples.** Here we describe several examples that provide specific applications of the results in this paper.

Example 1. If $G = 0$ we obtain the classical Ruelle *dynamical zeta function*

$$\zeta_\phi(s) = \prod_\gamma \Big(1 - e^{-s\lambda(\gamma)}\Big)^{-1}.$$

Then $Pr(0) = h$, where $h > 0$ is the topological entropy of ϕ_t and $\zeta_\phi(s)$ is absolutely convergent for $\operatorname{Re} s > h$ (see Chapter 6 in [11]).

Example 2. Consider the expansion function $E : \Lambda \longrightarrow \mathbb{R}$ defined by

$$E(x) := \lim_{t \to 0} \frac{1}{t} \log |\text{Jac}\,(D\phi_t|_{E^u(x)})|.$$

Introduce the function $\lambda^u(\gamma) = \lambda_E(\gamma)$ and we define $f : R \longrightarrow \mathbb{R}$ by

$$f(x) = -\int_0^{\tau(x)} E(\pi(x,t))dt.$$

Then we have $-\lambda^u(\gamma) = f^n(x)$, f is Hölder continuous and $Pr(f) = 0$ (see [3]). Consequently, the series

$$\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x)} \tag{1.3}$$

is absolutely convergent for $\operatorname{Re} s > 0$ and nowhere zero and analytic for $\operatorname{Re} s \geq 0$ except for a simple pole at $\operatorname{Re} s = 0$ (see Theorem 9.2 in [11]). The roof functions $\tau(x)$ is constant on stable leaves of rectangles R_i of the Markov family, so we can assume that $\tau(x)$ depends only on $x \in U$. By a standard argument (see [11]) we can replace f by a Hölder function $\hat{f}(x)$ which depends only on $x \in U$ so that $f \sim \hat{f}$. Thus the series (1.3) can be written by functions $\hat{f}$, τ depending on only $x \in U$. We keep the notation f below assuming that f depends only on $x \in U$. The analysis of the analytic continuation of (1.3) is based on spectral estimates for the iterations of the Ruelle operator

$$L_{f - s\tau} v(x) = \sum_{\sigma y = x} e^{f(y) - s\tau(y)} v(y),\ v \in C^\alpha(U),\ s \in \mathbb{C}.$$

(see [4], [16], [21], [22], [24] for more details).

Example 3. Let f, τ be real-valued Hölder functions and let $P_f > 0$ be the unique real number such that $Pr(f - P_f\tau) = 0$. Let $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$, where $G : \Lambda \longrightarrow \mathbb{R}$ is a Hölder function. Then if the suspended flow σ_t^τ is weak-mixing, the function (1.1) is nowhere zero analytic function for $\mathrm{Re}\, s > P_f$ and z in a neighborhood of 0 (depending on s) with nowhere zero analytic extension to $\mathrm{Re}\, s = P_f$ $(s \neq P_f)$ for small $|z|$. This statement is just Theorem 6.4 in [11]. To examine the analytic continuation of $\zeta(s,z)$ for $P_f - \eta_0 \leq \mathrm{Re}\, s$ and small $|z|$, it is necessary to have some spectral estimates for the iterations of the Ruelle operator

$$L_{f-s\tau+zg}v(x) = \sum_{\sigma y = x} e^{f(y)-s\tau(y)+zg(y)}v(y),\ v \in C^\alpha(U),\ s \in \mathbb{C}, z \in \mathbb{C}. \tag{1.4}$$

The analytic continuation of $\zeta(s,z)$ for small $|z|$ and that of $\eta_g(s)$ play a crucial role in the argument in [18] concerning the Hannay-Ozorio de Almeida sum formula for the geodesic flow on compact negatively curved surfaces. We deal with the same question for Axiom A flows on basic sets in section 8.

Example 4. In [7] for Anosov flows the authors examine the spectral properties of the Ruelle operator (1.4) with $f = 0$ and $z = \mathbf{i}w$, $w \in \mathbb{R}$ and the analyticity of the corresponding L-function $L(s,z)$. The properties of the Ruelle operator

$$L^n_{f-(P_f+a+\mathbf{i}b)\tau+\mathbf{i}w},\ w \in \mathbb{R},\ n \in \mathbb{N},$$

are also rather important in the paper [23] dealing with the large deviations for Anosov flows. Here as above $P_f \in \mathbb{R}$ is such that $Pr(f - P_f\tau) = 0$. However, it is important to note that in [7] and [23] the analysis of the Ruelle operators covers mainly the domain $\mathrm{Re}\, s \geq P_f$ and there are no results treating the spectral properties for $P_f - \eta_0 \leq \mathrm{Re}\, s < P_f$ and $z = \mathbf{i}w$, $w \in \mathbb{R}$. To our best knowledge the analytic continuation of the function $\zeta(s,z)$ for these values of s and z has not been investigated in the literature so far which makes it quite difficult to obtain sharper results.

2. Preliminaries

As in section 1, let $\phi_t : M \longrightarrow M$ be a C^2 Axiom A flow on a Riemannian manifold M, and let Λ be a basic set for ϕ_t. The restriction of the flow on Λ is a hyperbolic flow [11]. For any $x \in M$ let $W^s_\epsilon(x), W^u_\epsilon(x)$ be the local stable and unstable manifolds through x, respectively (see [2], [6], [11]). When M is compact and M itself is a basic set, ϕ_t is called an *Anosov flow*. It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x,y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x,y] \in \Lambda$ (cf. [6]). That is, there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\phi_t([x,y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x,y) = t$, defines the so called *temporal distance function*.

We will use the set-up and some arguments from [21]. As in [21], fix a *(pseudo-) Markov family* $\mathcal{R} = \{R_i\}_{i=1}^k$ of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x,y] : x \in U_i, y \in S_i\}$. Set $R = \cup_{i=1}^k R_i$, $U = \cup_{i=1}^k U_i$. Consider the *Poincaré map* $\mathcal{P} : R \longrightarrow R$, defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in R$. The function τ is the so called *first return time* associated with $\mathcal{R}$. Let $\sigma : U \longrightarrow U$ be the *shift map* given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. Let $\widehat{U}$ be the set of those points $x \in U$ such that $\mathcal{P}^m(x)$ is not a boundary point of a rectangle for any integer m. In a similar way define $\widehat{R}$. Clearly in general τ is not continuous on U, however under the assumption that the holonomy maps are Lipschitz (see section 4) τ is *essentially Lipschitz* on U in the sense that there exists a constant $L > 0$ such that if $x, y \in U_i \cap \sigma^{-1}(U_j)$ for some i, j, then $|\tau(x) - \tau(y)| \leq L\, d(x,y)$. The same applies to $\sigma : U \longrightarrow U$.

The hyperbolicity of the flow on Λ implies the existence of constants $c_0 \in (0,1]$ and $\gamma_1 > \gamma_0 > 1$ such that

$$c_0\gamma_0^m \, d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma_1^m}{c_0} d(u_1, u_2) \tag{2.1}$$

whenever $\sigma^j(u_1)$ and $\sigma^j(u_2)$ belong to the same U_{i_j} for all $j = 0, 1 \dots, m$.

Define a $k \times k$ matrix $A = \{A(i,j)\}_{i,j=1}^k$ by

$$A(i,j) = \begin{cases} 1 \text{ if } \mathcal{P}(\mathrm{Int}\, R_i) \cap \mathrm{Int}\, R_j \neq \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

It is possible to construct a Markov family $\mathcal{R}$ so that A is irreducible and aperiodic (see [2]).

Consider the suspension space $R^\tau = \{(x,t) \in R \times \mathbb{R} : 0 \leq t \leq \tau(x)\}/\sim$, where by $\sim$ we identify the points $(x, \tau(x))$ and $(\sigma x, 0)$. The corresponding suspension flow is defined by $\sigma_t^\tau(x,s) = (x, s+t)$ on R^τ taking into account the identification $\sim$. For a Hölder continuous function f on R, the *topological pressure* $\Pr(f)$ with respect to σ is defined as

$$\Pr(f) = \sup_{m \in \mathcal{M}_\sigma} \{h(\sigma, m) + \int f dm\},$$

where $\mathcal{M}_\sigma$ denotes the space of all σ-invariant Borel probability measures and $h(\sigma, m)$ is the entropy of σ with respect to m. We say that f and g are *cohomologous* and we denote this by $f \sim g$ if there exists a continuous function w such that $f = g + w \circ \sigma - w$. For a function v on R one defines

$$v^n(x) := v(x) + v(\sigma(x)) + ... + v(\sigma^{n-1}(x)).$$

3. Ruelle's lemma with two complex parameters

Let $B(\widehat{U})$ be the *space of bounded functions* $q : \widehat{U} \longrightarrow \mathbb{C}$ with its standard norm $\|q\|_0 = \sup_{x \in \widehat{U}} |g(x)|$. Given a function $q \in B(\widehat{U})$, the *Ruelle transfer operator* $L_q : B(\widehat{U}) \longrightarrow B(\widehat{U})$ is defined by $(L_q h)(u) = \sum_{\sigma(v)=u} e^{q(v)} h(v)$. If $q \in B(\widehat{U})$ is Lipschitz on $\widehat{U}$ with respect to the Riemann metric, then L_q preserves the space $C^{\mathrm{Lip}}(\widehat{U})$ of *Lipschitz functions* $q : \widehat{U} \longrightarrow \mathbb{C}$. Similarly, if q is ν-Hölder for some $\nu > 0$, the operator L_q preserves the *space* $C^\nu(\widehat{U})$ *of* ν*-Hölder functions* on $\widehat{U}$. In this section we assume that g, τ and f are real-valued $\nu-$Hölder continuous functions on $\hat{U}$. Then we can extend these functions as Hölder continuous on U.

We define the Ruelle operator $L_{g-sr+zf} : C^\nu(\hat{U}) \longrightarrow C^\nu(\hat{U})$ by

$$L_{f-s\tau+zg} v(x) = \sum_{\sigma y = x} e^{f(y) - s\tau(y) + zg(y)} v(y), \; s, z \in \mathbb{C}.$$

Next, for $\nu > 0$ define the ν-norm on a set $B \subset U$ by

$$|w|_\nu = \sup\Big\{\frac{|w(x) - w(y)|}{d(x,y)^\nu} : x, y \in B \cap U_i, \; i = 1, ..., k, \; x \neq y\Big\}.$$

Let $\|w\|_\nu = \|w\|_\infty + |w|_\nu$, and denote by $\|.\|_\nu$ be the corresponding norm for operators. Let $\chi_i(x)$ be the characteristic function of U_i.

Introduce the sum

$$Z_n(f - sr + zg) := \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x) + zg^n(x)}.$$

Our purpose is to prove the following statement which can be considered as Ruelle's lemma with two complex parameters.

Theorem 4. *For every Markov leaf U_i fix an arbitrary point $x_i \in U_i$. Then for every $\epsilon > 0$ and sufficiently small $a_0 > 0, c_0 > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\begin{aligned} &\Big| Z_n(f - s\tau + zg) - \sum_{i=1}^{k} L^n_{f-s\tau+zg}\chi_i(x_i)\Big| \\ \leq\ & C_\epsilon(1+|s|)(1+|z|)\sum_{m=2}^{n} \|L^{n-m}_{f-s\tau+zg}\|_\nu \gamma_0^{-m\nu} e^{m(\epsilon + Pr(f-a\tau+cg))},\ \forall n \in \mathbb{N} \end{aligned} \tag{3.1}$$

for $s = a + \mathbf{i}b,\ z = c + \mathbf{i}w,\ |a| \leq a_0, |c| \leq c_0$.

The proof of this theorem follows that of Theorem 3.1 in [24] with some modifications. We have to take into account the presence of a second complex parameter z. Given a string $\alpha = (\alpha_0, ..., \alpha_{n-1})$ of symbols α_j taking the values in $\{1, ..., k\}$, we say that α is an admissible word if $A(\alpha_j, \alpha_{j+1}) = 1$ for all $0 \leq j \leq n-1$. Set $|\alpha| = n$ and define the cylinder of length n in the leaf U_{α_0} by

$$U_\alpha = U_{\alpha_0} \cap \sigma^{-1}U_{\alpha_1} \cap ... \cap \sigma^{-(n-1)}U_{\alpha_{n-1}}.$$

Each U_i is a cylinder of length 1. Next we introduce some other words (see [24]). Given a word $\alpha = (\alpha_0, ..., \alpha_{n-1})$ and $i = 1, ..., k$, if $A(\alpha_{n-1}, i) = 1$ and $A(i, \alpha_0) = 1$, we define

$$\alpha i = (\alpha_0, ..., \alpha_{n-1}, i),\ i\alpha = (i, \alpha_0, ..., \alpha_{n-1}),\ \bar{\alpha} = (\alpha_0, ..., \alpha_{n-2}).$$

We have the following

Lemma 1. *Let w be a ν-Hölder real-valued function. Let x and y be on the same cylinder U_α with $|\alpha| = m$. Then there exists a constant $B > 0$ depending only on w, ν and the constants c_0 and γ_0 in (2.1) such that*

$$|w^m(x) - w^m(y)| \leq B(d(\sigma^{m-1}x, \sigma^{m-1}y))^\nu.$$

The proof is a repetition of that of Lemma 2.5 in [24] and we leave the details to the reader.

Proposition 1. *Let $m \geq 1$ and let w be a function which is ν-Hölder continuous on all cylinder of length $m+1$. Then for the transfer operator $L_{f-s\tau+zg}$ we have*

$$L_{f-s\tau+zg} := \oplus_{|\alpha|=m+1} C^\nu(U_\alpha) \ni w \longrightarrow L_{f-s\tau+zg}w \in \oplus_{|\alpha|=m} C^\nu(U_\alpha).$$

Proof. Let w be ν-Hölder on $U_{i\alpha}$ for all i such that $A(i, \alpha_0) = 1$. Let $x, y \in \operatorname{Int} U_\alpha$ and let $|U| = \max_{i=1,,,k} \operatorname{diam}(U_i)$. Then

$$\begin{aligned} & |L_{f-s\tau+zg}w(x) - L_{f-s\tau+zg}w(y)| \\ =\ & \Big| \sum_{A(i,\alpha_0)=1} e^{f(ix) - s\tau(ix) + zg(ix)} w(ix) - \sum_{A(i,\alpha_0)=1} e^{f(iy) - s\tau(iy) + zg(iy)} w(iy)\Big| \\ \leq\ & \sum_{A(i,\alpha_0)=1} |e^{-s\tau(iy)}| \Big(|e^{s\tau(iy) - s\tau(ix)} - 1||e^{f(iy)+zg(iy)}w(ix)| + |e^{f(iy)+zg(iy)}w(iy) - e^{f(ix)+zg(ix)}w(ix)|\Big) \\ \leq\ & e^{a_0|\tau|_\infty} \sum_{A(i,\alpha_0)=1} \Big(|s||\tau|_\beta e^{a_0|\tau|_\nu |U|^\nu} e^{|f|_\infty + c_0|g|_\infty}|w|_\infty + |e^{f(iy)+zg(iy)}w(iy) - e^{f(ix)+zg(ix)}w(ix)|\Big). \end{aligned}$$

Repeating this argument, we get

$$\sum_{A(i,\alpha_0)=1} |e^{f(iy)+zg(iy)}w(iy) - e^{f(ix)+zg(ix)}w(ix)|$$
$$\leq \quad e^{c_0|g|_\infty} \sum_{A(i,\alpha_0)=1} \Big(|z||g|_\nu e^{c_0|g|_\nu |U|^\nu} e^{|f|_\infty} |w|_\infty + |e^{f(iy)}w(iy) - e^{f(ix)}w(ix)| \Big),$$

and we conclude that

$$|L_{f-s\tau+zg}w(x) - L_{f-s\tau+zg}w(y)| \leq C|w|_\nu d(x,y)^\nu. \quad \square$$

Now, as in [24], we will choose in every cylinder U_α a point $x_\alpha \in U_\alpha$. For the reader's convenience we recall the choice of x_α.
(1) If U_α has an n-periodic point, then we take $x_\alpha \in U_\alpha$ so that $\sigma^n x_\alpha = x_\alpha$.
(2) If U_α has no n-periodic point and $n > 1$ we choose $x_\alpha \in U_\alpha$ arbitrary so that $x_\alpha \notin \sigma(U_{\alpha_{n-1}})$.
(3) if $|\alpha| = n = 1$, then we take $x_\alpha = x_i$, where $i = \alpha_0$ and $x_i \in U_i$ is one of the points fixed in Theorem 4.

Let χ_α be the characteristic function of U_α. Then Lemma 3.4 and Lemma 3.5 in [24] are applied without any change and we get

$$Z_n(f - s\tau + zg) = \sum_{|\alpha|=n} (L^n_{f-s\tau+zg}\chi_\alpha)(x_\alpha).$$

Proposition 2. *We have*

$$Z_n(f - s\tau + zg) - \sum_{i=1}^{k} L^n_{f-s\tau+zg}\chi_i(x_i)$$
$$= \sum_{m=2}^{n} \Big(\sum_{|\alpha|=m} L^n_{f-s\tau+zg}\chi_\alpha(x_\alpha) - \sum_{|\beta|=m-1} L^n_{f-s\tau+zg}\chi_\beta(x_\beta) \Big). \tag{3.2}$$

The proof is elementary by using the fact that

$$\sum_{i=1}^{k} (L^n_{f-s\tau+zg}\chi_{U_i})(x_i) = \sum_{|\alpha|=1} (L^n_{f-s\tau+zg}\chi_\alpha)(x_\alpha).$$

Now we repeat the argument in [24] without any change and conclude that

$$\sum_{|\beta|=m-1} L^n_{f-s\tau+zg}\chi_\beta(x_\beta) = \sum_{|\alpha|=m} L^n_{f-s\tau+zg}\chi_\alpha(x_{\bar{\alpha}}).$$

Thus, the proof of (3.1) is reduced to an estimate of the difference

$$L^n_{f-s\tau+zg}\chi_\alpha(x_\alpha) - L^n_{f-s\tau+zg}\chi_\alpha(x_{\bar{\alpha}}).$$

Observe that x_α and $x_{\bar{\alpha}}$ are on the same cylinder $U_{\bar{\alpha}}$. According to Proposition 1, the function $L^n_{f-s\tau+zg}\chi_\alpha$ is ν-Hölder continuous on $U_{\bar{\alpha}}$. Consequently, for every $n \geq 2$ we obtain

$$|L^n_{f-s\tau+zg}\chi_\alpha(x_\alpha) - L^n_{f-s\tau+zg}\chi_\alpha(x_{\bar{\alpha}})| \leq \|L^n_{f-s\tau+zg}\chi_\alpha\|_\nu d(x_\alpha, x_{\bar{\alpha}})^\nu,$$

where $\|.\|_\mu$ denotes the operator norm derived from the ν-Hölder norm. Going back to (3.2), we deduce

$$\Big|Z_n(f - s\tau + zg) - \sum_{i=1}^{k} L^n_{f-s\tau+zg}\chi_i(x_i)\Big|$$
$$\leq \sum_{m=2}^{n}\sum_{|\alpha|=m} \|L^{n-m}_{f-s\tau+zg}\|_\nu \|L^m_{f-s\tau+zg}\chi_\alpha\|_\nu d(x_\alpha, x_{\bar{\alpha}}). \tag{3.3}$$

This it makes possible to apply (2.1) and to conclude that

$$d(x_\alpha, x_{\bar{\alpha}}) \leq C^\nu \gamma_0^{-\nu(m-2)} d(\sigma^{m-2}x_\alpha, \sigma^{m-2}x_{\bar{\alpha}})^\nu \leq C_2\gamma_0^{-m\nu}.$$

To finish the proof we have to estimate the term $\|L^m_{g-sr+zf}\chi_\beta\|_\nu$. Given a word α of length $n > 1$ and $x \in \sigma(U_{\alpha_{n-1}}) \cap \operatorname{Int} U_i$ for any i with $A(\alpha_{n-1}, i) = 1$, we define $\sigma_\alpha^{-1}(x)$ to be the unique point y such that $\sigma^n(y) = x$ and $y \in U_\alpha$. For a symbol i we define $ix = \sigma_i^{-1}(x)$.

First we have

Lemma 2.

$$(L^m_{f-s\tau+zg}\chi_\beta)(x) = \begin{cases} e^{(f-s\tau+zg)^m(\sigma_\beta^{-1}x)}, \text{ if } x \in \sigma(U_{\beta_{m-1}}), \\ 0, \text{ otherwise.} \end{cases}$$

The proof is a repetition of that of Lemma 3.7 in [24] and it is based on the definition of σ_α^{-1} above and the fact that

$$(L^m_{f-s\tau+zg}\chi_\beta)(x) = \sum_{\sigma^m y = x} e^{f^m - s\tau^m + zg^m}(y)\chi_\beta(y).$$

For every admissible word β with $|\beta| = m$, we fix a point $y_\beta \in \sigma(U_{\beta_{m-1}})$ which will be chosen as in [24]. Define $z_\beta = \sigma_\beta^{-1}(y_\beta)$.

Lemma 3. *There exist constants $B_0 > 0, B_1 > 0, B_2 > 0$ such that we have the estimate*

$$\|L^m_{f-s\tau+zg}(\chi_\beta)\|_\nu \leq B_0\Big(e^{a_0|U|^\nu B_1} + B_1|s|e^{a_0|U|^\nu(1+\gamma_0^{-\nu})B_1}\Big)$$
$$\times\Big(e^{c_0|U|^\nu B_2} + B_2|z|e^{c_0|U|^\nu(1+\gamma_0^{-\nu})B_2}\Big)e^{(f^m - a\tau^m + cg^m)(z_\beta)}.$$

Proof. We will follow the proof of Lemma 3.8 in [24]. Let x and y be in the same Markov leaf. If $y \notin \sigma(U_{\beta_{m-1}})$, then $|L^m_{f-s\tau+zg}(\chi_\beta)(x)| = |L^m_{f-s\tau+zg}(\chi_\beta)(x) - L^m_{f-s\tau+zg}(\chi_\beta)(y)| = 0$. In the case when $x \notin \sigma(U_{\beta_{m-1}})$, we repeat the same argument. So we will consider the case when both x and y are in $\sigma(U_{\beta_{m-1}})$.

We have

$$|L^m_{f-s\tau+zg}(\chi_\beta)(x)| = |e^{(f^m - (a+ib)\tau^m + (c+id)g^m)(\sigma_\beta^{-1}x)}|$$
$$\leq \exp\Big((f^m - a\tau^m + cg^m)(\sigma_\beta^{-1}x) - (f^m - a\tau^m + cg^m)(\sigma_\beta^{-1}y)\Big)e^{(f^m - a\tau^m + cg^m)(z_\beta)}.$$

On the other hand, applying Lemma 1 with $w = \tau$, we get

$$|\tau^m(\sigma_\beta^{-1}x) - \tau^m(\sigma_\beta^{-1}y)| \leq B_1(d(\sigma^{m-1}\sigma_\beta^{-1}x, \sigma^{m-1}\sigma_\beta^{-1}y))^\nu \leq B_1|U|^\nu.$$

The same argument works for the terms involving f^m and g^m, applying Lemma 1 with $w = f, g$, respectively. Thus we obtain

$$|L^m_{f-s\tau+zg}(\chi_\beta)(x)| \leq e^{(C_0 + a_0B_1 + c_0B_2)|U|^\mu} e^{(f^m - a\tau^m + cg^m)(z_\beta)}.$$

and this implies an estimate for $|L^m_{f-s\tau+zg}(\chi_\beta)|_\infty$. Next,

$$|L^m_{f-s\tau+zg}(\chi_\beta)(x) - L^m_{f-s\tau+zg}(\chi_\beta)(y)|$$
$$\leq |e^{f^m(\sigma_\beta^{-1}(x))-f^m(\sigma_\beta^{-1}(y))} - 1||e^{f^m(\sigma_\beta^{-1}(y))}||e^{-s\tau^m(\sigma_\beta^{-1}(x))+s\tau^m(\sigma_\beta^{-1}(y))} - 1||e^{-s\tau^m(\sigma_\beta^{-1}(y))}|$$
$$\times|e^{zg^m(\sigma_\beta^{-1}(x))-zg^m(\sigma_\beta^{-1}(y))} - 1||e^{zg^m(\sigma_\beta^{-1}(y))}|.$$

As in [24], we have

$$|e^{-sr^m(\sigma_\beta^{-1}(x))+sr^m(\sigma_\beta^{-1}(y))} - 1||e^{-sr^m(\sigma_\beta^{-1}(y))}| \leq B_1\gamma_0^\nu|s|e^{a_0B_1(1+\gamma_0^{-\nu})|U|^\nu}e^{-ar^m(z_\beta)}d(x,y)^\nu.$$

For the product involving zg^m we have the same estimate with $B_2, |z|, c_0$ and c in the place of $B_1, |s|, a_0$ and a. A similar estimate holds for the term containing f^m with a constant B_3 in the place of B_1. Taking the product of these estimates we obtain a bound for $|L^m_{f-s\tau+zg}(\chi_\beta)(x) - L^m_{f-s\tau+zg}(\chi_\beta)(y)|$, this implies the desired estimate for the μ-Hölder norm of $L_{f-ms\tau+zg}(\chi_\beta)$. This completes the proof. □

Now the proof of Theorem 4 is reduced to the estimate of $\sum_{|\beta|=m} e^{(f^m-a\tau^m+cg^m)(z_\beta)}$. Introduce the real-valued function $h = f - a\tau + cg$. Then we have to estimate $\sum_{|\beta|=m} e^{h^m(z_\beta)}$. For this purpose we repeat the argument on pages 232-234 in [24] and deduce with some constant $d_0 > 0$ depending only on the matrix A and every $\epsilon > 0$ the bound

$$\sum_{|\beta|=m} e^{h^m(z_\beta)} \leq e^{d_0|h|_\infty}B_\epsilon e^{(m+d_0)(\epsilon+\Pr(h))}.$$

Combing this with the previous estimates, we get (3.1) which completes the proof of Theorem 4. □

4. Ruelle operators – definitions and assumptions

For a contact Anosov flows ϕ_t with Lipschitz local stable holonomy maps it is proved in section 6 in [21] that the following *local non-integrability condition* holds:

(LNIC): *There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and $\theta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\hat{z} \in \Lambda \cap W^u_\epsilon(z_0)$ and any tangent vector $\eta \in E^u(\hat{z})$ to Λ at $\hat{z}$ with $\|\eta\| = 1$ there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z})$, $\tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z})$ with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta = \delta(\tilde{z}, \tilde{y}_1, \tilde{y}_2) > 0$ and $\epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon]$ such that*

$$|\Delta(\exp^u_z(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp^u_z(v), \pi_{\tilde{y}_2}(z))| \geq \delta\,\|v\|$$

for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$ and $v \in E^u(z; \epsilon')$ with $\exp^u_z(v) \in \Lambda$ and $\langle \frac{v}{\|v\|}, \eta_z\rangle \geq \theta_0$, where η_z is the parallel translate of η along the geodesic in $W^u_{\epsilon_0}(z_0)$ from $\hat{z}$ to z.

For any $x \in \Lambda$, $T > 0$ and $\delta \in (0, \epsilon]$ set

$$B^u_T(x, \delta) = \{y \in W^u_\epsilon(x) : d(\phi_t(x), \phi_t(y)) \leq \delta\ ,\ 0 \leq t \leq T\}.$$

We will say that ϕ_t has a *regular distortion along unstable manifolds* over the basic set Λ if there exists a constant $\epsilon_0 > 0$ with the following properties:

(a) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $R = R(\delta, \epsilon) > 0$ such that

$$\mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon)) \leq R\,\mathrm{diam}(\Lambda \cap B^u_T(z, \delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(b) For any $\epsilon \in (0, \epsilon_0]$ and any $\rho \in (0,1)$ there exists $\delta \in (0, \epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\mathrm{diam}(\Lambda \cap B^u_T(z, \delta)) \leq \rho \, \mathrm{diam}(\Lambda \cap B^u_T(z, \epsilon))$.

A large class of flows on basic sets having regular distortion along unstable manifolds is described in [22].

In this paper we work under the following **Standing Assumptions:**

(A) ϕ_t has Lipschitz local holonomy maps over Λ,

(B) the local non-integrability condition (LNIC) holds for ϕ_t on Λ,

(C) ϕ_t has a regular distortion along unstable manifolds over the basic set Λ.

A rather large class of examples satisfying the above conditions is provided by imposing the following *pinching condition*:

(P): *There exist constants $C > 0$ and $\beta \geq \alpha > 0$ such that for every $x \in M$ we have*

$$\frac{1}{C} \, e^{\alpha_x \, t} \, \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C \, e^{\beta_x \, t} \, \|u\| \quad , \quad u \in E^u(x) \ , t > 0$$

for some constants $\alpha_x, \beta_x > 0$ with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in M$.

We should note that (P) holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so called $\frac{1}{4}$-pinching condition. (P) always holds when $\dim(M) = 3$.

Simplifying Assumptions: ϕ_t is a C^2 contact Anosov flow satisfying the condition (P).

As shown in [22] the pinching condition (P) implies that ϕ_t has Lipschitz local holonomy maps and regular distortion along unstable manifolds. Combining this with Proposition 6.1 in [21], shows that the Simplifying Assumptions imply the Standing Assumptions.

As in section 2 consider a **fixed Markov family** $\mathcal{R} = \{R_i\}_{i=1}^k$ for the flow ϕ_t on Λ consisting of rectangles $R_i = [U_i, S_i]$ and let $U = \cup_{i=1}^k U_i$. The Standing Assumptions imply the existence of constants $c_0 \in (0,1]$ and $\gamma_1 > \gamma_0 > 1$ such that (2.1) hold.

In what follows we will assume that f **and** g **are fixed real-valued functions in** $C^\alpha(\widehat{U})$ **for some fixed** $\alpha > 0$. Let $P = P_f$ be the unique real number so that $\Pr(f - P\,\tau) = 0$, where $\Pr(h)$ is the *topological pressure* of h with respect to the shift map σ defined in Section 2. Given $t \in \mathbb{R}$ with $t \geq 1$, following [4], denote by f_t the *average of f over balls in U of radius* $1/t$. To be more precise, first one has to fix an arbitrary extension $f \in C^\alpha(V)$ (with the same Hölder constant), where V is an open neighborhood of U in M, and then take the averages in question. Then $f_t \in C^\infty(V)$, so its restriction to U is Lipschitz (with respect to the Riemann metric) and:

(a) $\|f - f_t\|_\infty \leq |f|_\alpha / t^\alpha$;

(b) $\mathrm{Lip}(f_t) \leq \mathrm{Const}\, \|f\|_\infty t$;

(c) For any $\beta \in (0, \alpha)$ we have $|f - f_t|_\beta \leq 2\,|f|_\alpha / t^{\alpha - \beta}$.

In the special case $f \in C^{\mathrm{Lip}}(U)$ we set $f_t = f$ for all $t \geq 1$. Similarly for g. Let $\lambda_0 > 0$ be the largest eigenvalue of $L_{f - P\tau}$, and let $\hat{\nu}_0$ be the (unique) probability measure on U with $L^*_{f-P\tau} \hat{\nu}_0 = \hat{\nu}_0$. Fix a corresponding (positive) eigenfunction $h_0 \in \hat{C}^\alpha(U)$ such that $\int_U h_0 \, d\hat{\nu}_0 = 1$. Then $d\nu_0 = h_0 \, d\hat{\nu}_0$ defines a *σ-invariant probability measure* ν_0 on U. Setting

$$f_0 = f - P\,\tau + \ln h_0(u) - \ln h_0(\sigma(u)),$$

we have $L^*_{f^{(0)}} \nu_0 = \nu_0$, i.e. $\displaystyle\int_U L_{f^{(0)}} H \, d\nu_0 = \int_U H \, d\nu_0$ for any $H \in C(U)$, and $L_{f_0} 1 = 1$.

Given real numbers a and t (with $|a| + \frac{1}{|t|}$ small), denote by λ_{at} the *largest eigenvalue* of $L_{f_t-(P+a)\tau}$ on $C^{\mathrm{Lip}}(U)$ and by h_{at} the corresponding (positive) eigenfunction such that $\int_U h_{at}\, d\nu_{at} = 1$, where ν_{at} is the unique probability measure on U with $L^*_{f_t-(P+a)\tau}\nu_{at} = \nu_{at}$.

As is well-known the shift map $\sigma : \widehat{U} \longrightarrow \widehat{U}$ is naturally isomorphic to an one-sided subshift of finite type. Given $\theta \in (0,1)$, a natural metric associated by this isomorphism is defined (for $x \neq y$) by $d_\theta(x,y) = \theta^m$, where m is the largest integer such that x, y belong to the same cylinder of length m. There exist $\theta = \theta(\alpha) \in (0,1)$ and $\beta \in (0,\alpha)$ such that $(d(x,y))^\alpha \leq \text{Const}\; d_\theta(x,y)$ and $d_\theta(x,y) \leq \text{Const}\; (d(x,y))^\beta$ for all $x, y \in \widehat{U}$. One can then apply the Ruelle-Perron-Frobenius theorem to the sub-shift of fine type and deduce that $h_{at} \in C^\beta(\widehat{U})$. However this is not enough for our purposes – in Lemma 4 below we get a bit more.

Consider an arbitrary $\beta \in (0,\alpha)$. It follows from properties (a) and (c) above that there exists a constant $C_0 > 0$, depending on f and α but independent of β, such that

$$\|[f_t - (P+a)\tau] - (f - P\tau)\|_\beta \leq C_0\, [|a| + 1/t^{\alpha-\beta}] \tag{4.1}$$

for all $|a| \leq 1$ and $t \geq 1$. Since $\Pr(f - P\tau) = 0$, it follows from the analyticity of pressure and the eigenfunction projection corresponding to the *maximal eigenvalue* $\lambda_{at} = e^{\Pr(f_t-(P+a)\tau)}$ of the Ruelle operator $L_{f_t-(P+a)\tau}$ on $C^\beta(U)$ (cf. e.g. Ch. 3 in [11]) that there exists a constant $a_0 > 0$ such that, taking $C_0 > 0$ sufficiently large, we have

$$|\Pr(f_t - (P+a)\tau)| \leq C_0\, \left(|a| + \frac{1}{t^{\alpha-\beta}}\right) \quad , \quad \|h_{at} - h_0\|_\beta \leq C_0\, \left(|a| + \frac{1}{t^{\alpha-\beta}}\right) \tag{4.2}$$

for $|a| \leq a_0$ and $1/t \leq a_0$. We may assume $C_0 > 0$ and $a_0 > 0$ are taken so that $1/C_0 \leq \lambda_{at} \leq C_0$, $\|f_t\|_\infty \leq C_0$ and $1/C_0 \leq h_{at}(u) \leq C_0$ for all $u \in U$ and all $|a|, 1/t \leq a_0$.

Given real numbers a and t with $|a|, 1/t \leq a_0$ consider the functions

$$f_{at} = f_t - (P+a)\tau + \ln h_{at} - \ln(h_{at}\circ\sigma) - \ln\lambda_{at}$$

and the operators

$$\mathcal{L}_{abt} = L_{f_{at} - \mathbf{i}\, b\tau} : C(U) \longrightarrow C(U) \quad , \quad \mathcal{M}_{at} = L_{f_{at}} : C(U) \longrightarrow C(U).$$

One checks that $\mathcal{M}_{at}\, 1 = 1$.

Taking the constant $C_0 > 0$ sufficiently large, we may assume that

$$\|f_{at} - f_0\|_\beta \leq C_0\, \left[|a| + \frac{1}{t^{\alpha-\beta}}\right] \quad , \quad |a|, 1/t \leq a_0. \tag{4.3}$$

We will now prove a simple uniform estimate for $\mathrm{Lip}(h_{at})$. With respect to the usual metrics on symbol spaces this a consequence of general facts (see e.g. Sect. 1.7 in [1] or Ch. 3 in [11]), however here we need it with respect to the Riemann metric.

The proof of the following lemma is given in the Appendix.

Lemma 4. *Taking the constant $a_0 > 0$ sufficiently small, there exists a constant $T' > 0$ such that for all $a, t \in \mathbb{R}$ with $|a| \leq a_0$ and $t \geq 1/a_0$ we have $h_{at} \in C^{\mathrm{Lip}}(\widehat{U})$ and* $\mathrm{Lip}(h_{at}) \leq T't$.

It follows from the above that, assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $T > 0$ (depending on $|f|_\alpha$ and a_0) such that

$$\|f_{at}\|_\infty \leq T \quad , \quad \|g_t\|_\infty \leq T \quad , \quad \mathrm{Lip}(h_{at}) \leq T\, t \quad , \quad \mathrm{Lip}(f_{at}) \leq T\, t \tag{4.4}$$

for $|a|, 1/t \leq a_0$. We will also assume that $T \geq \max\{\, \|\tau\|_0\,, \mathrm{Lip}(\tau_{|\widehat{U}})\,\}$. From now on **we will assume that** a_0**,** C_0**,** T**,** $1 < \gamma_0 < \gamma_1$ **are fixed constants** with (2.1) and (4.1) – (4.4).

5. Ruelle operators depending on two parameters – the case when b is the leading parameter

Throughout this section we work under the Standing Assumptions made in section 4 and with fixed real-valued functions $f, g \in C^\alpha(\widehat{U})$ as in section 4. Throughout $0 < \beta < \alpha$ are fixed numbers.

We will study Ruelle operators of the form $L_{f-(P_f+a+\mathbf{i}b)\tau+zg}$, where $z = c + \mathbf{i}w$, $a, b, c, w \in \mathbb{R}$, and $|a|, |c| \leq a_0$ for some constant $a_0 > 0$. Such operators will be approximates by operators of the form

$$\mathcal{L}_{abtz} = L_{f_{at} - \mathbf{i}\, b\tau + zg_t} : C^\alpha(\widehat{U}) \longrightarrow C^\alpha(\widehat{U}).$$

In fact, since $f_{at} - \mathbf{i}b\tau + zg_t$ is Lipschitz, the operators $\mathcal{L}_{abtz}$ preserves each of the spaces $C^{\alpha'}(\widehat{U})$ for $0 < \alpha' \leq 1$ including the space $C^{\mathrm{Lip}}(\widehat{U})$ of Lipschitz functions $h : \widehat{U} \longrightarrow \mathbb{C}$. For such h we will denote by $\mathrm{Lip}(h)$ the Lipschitz constant of h. Let $\|h\|_0$ denote the *standard* sup *norm* of h on $\widehat{U}$. For $|b| \geq 1$, as in [4], consider the norm $\|.\|_{\mathrm{Lip},b}$ on $C^{\mathrm{Lip}}(\widehat{U})$ defined by $\|h\|_{\mathrm{Lip},b} = \|h\|_0 + \frac{\mathrm{Lip}(h)}{|b|}$, and also the norm $\|h\|_{\beta,b} = \|h\|_\infty + \frac{|h|_\beta}{|b|}$ on $C^\beta(U)$.

Our aim in this section is to prove the following

Theorem 5. *Let $\phi_t : M \longrightarrow M$ satisfy the Standing Assumptions over the basic set Λ, and let $0 < \beta < \alpha$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t over Λ as in section 2. Then for any real-valued functions $f, g \in C^\alpha(\widehat{U})$ we have:*

(a) *For any constants $\epsilon > 0$, $B > 0$ and $\nu \in (0, 1)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$, $A_0 > 0$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f_{at} - \mathbf{i}b\tau + (c+\mathbf{i}w)g_t} h\|_{\mathrm{Lip},b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\mathrm{Lip},b}$$

for all $h \in C^{\mathrm{Lip}}(\widehat{U})$, all integers $m \geq 1$ and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$, $1 \leq t \leq \frac{1}{A_0} \log |b|^\nu$ and $|w| \leq B\, |b|^\nu$.

(b) *For any constants $\epsilon > 0$, $B > 0$, $\nu \in (0, 1)$ and $\beta \in (0, \alpha)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f-(P_f+a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\beta,b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\beta,b}$$

for all $h \in C^\beta(\widehat{U})$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B\, |b|^\nu$.

(c) *If $f, g \in C^{\mathrm{Lip}}(\widehat{U})$, then for any constants $\epsilon > 0$, $B > 0$ and $\beta \in (0, \alpha)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f-(P_f+a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\mathrm{Lip},b} \leq C\, \rho^m\, |b|^\epsilon\, \|h\|_{\mathrm{Lip},b}$$

for all $h \in C^\beta(\widehat{U})$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B\, |b|$.

We will first prove part (a) of the above theorem and then derive part (b) by a simple approximation procedure. To prove part (a) we will use the main steps in section 5 in [21] with necessary modifications. The proof of part (c) is just a much simpler version of the proof of (b).

Define a *new metric* D on $\widehat{U}$ by

$$D(x, y) = \min\{\mathrm{diam}(\mathcal{C}) : x, y \in \mathcal{C}\,,\ \mathcal{C} \text{ a cylinder contained in } U_i\}$$

if $x, y \in U_i$ for some $i = 1, \ldots, k$, and $D(x, y) = 1$ otherwise. Rescaling the metric on M if necessary, we will assume that $\mathrm{diam}(U_i) < 1$ for all i. As shown in [20], D is a metric on $\widehat{U}$ with $d(x, y) \leq D(x, y)$ for $x, y \in \widehat{U}_i$ for some i, and for any cylinder $\mathcal{C}$ in U the characteristic function $\chi_{\widehat{\mathcal{C}}}$ of $\widehat{\mathcal{C}}$ on $\widehat{U}$ is Lipschitz with respect to D and $\mathrm{Lip}_D(\chi_{\widehat{\mathcal{C}}}) \leq 1/\mathrm{diam}(\mathcal{C})$.

We will denote by $C_D^{\mathrm{Lip}}(\widehat{U})$ the *space of all Lipschitz functions* $h : \widehat{U} \longrightarrow \mathbb{C}$ *with respect to the metric* D on $\widehat{U}$ and by $\mathrm{Lip}_D(h)$ the *Lipschitz constant* of h with respect to D.

Given $A > 0$, denote by $K_A(\widehat{U})$ *the set of all functions* $h \in C_D^{\mathrm{Lip}}(\widehat{U})$ such that $h > 0$ and $\frac{|h(u)-h(u')|}{h(u')} \leq A\, D(u, u')$ for all $u, u' \in \widehat{U}$ that belong to the same $\widehat{U}_i$ for some $i = 1, \ldots, k$. Notice that $h \in K_A(\widehat{U})$ implies $|\ln h(u) - \ln h(v)| \leq A\ D(u, v)$ and therefore $e^{-A\ D(u,v)} \leq \frac{h(u)}{h(v)} \leq e^{A\ D(u,v)}$ for all $u, v \in \widehat{U}_i$, $i = 1, \ldots, k$.

We begin with a lemma of Lasota-Yorke type, which necessarily has a more complicated form due to the more complex situation considered. It involves the operators $\mathcal{L}_{abtz}$, and also operators of the form

$$\mathcal{M}_{atc} = L_{f_{at}+cg_t} : C^\alpha(\widehat{U}) \longrightarrow C^\alpha(\widehat{U}).$$

Fix arbitrary constants $\nu \in (0, 1)$ **and** $\hat{\gamma}$ **with** $1 < \hat{\gamma} < \gamma_0$.

Lemma 5. *Assuming* $a_0 > 0$ *is chosen sufficiently small, there exists a constant* $A_0 > 0$ *such that for all* $a, c, t \in \mathbb{R}$ *with* $|a|, |c| \leq a_0$ *and* $t \geq 1$ *the following hold:*

(a) If $H \in K_E(\widehat{U})$ *for some* $E > 0$*, then*

$$\frac{|(\mathcal{M}_{atc}^m H)(u) - (\mathcal{M}_{atc}^m H)(u')|}{(\mathcal{M}_{atc}^m H)(u')} \leq A_0 \left[\frac{E}{\hat{\gamma}^m} + e^{A_0 t}\, t\right] D(u, u')$$

for all $m \geq 1$ *and all* $u, u' \in U_i$, $i = 1, \ldots, k$.

(b) If the functions h *and* H *on* $\widehat{U}$ *and* $E > 0$ *are such that* $H > 0$ *on* $\widehat{U}$ *and* $|h(v) - h(v')| \leq E\, H(v')\, D(v, v')$ *for any* $v, v' \in \widehat{U}_i$, $i = 1, \ldots, k$, *then for any integer* $m \geq 1$ *and any* $b, w, t \in \mathbb{R}$ *with* $|b|, t, |w| \geq 1$, *for* $z = c + \mathbf{i}w$ *we have*

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_0 \left(\frac{E}{\hat{\gamma}^m}(\mathcal{M}_{atc}^m H)(u') + (|b| + e^{A_0 t} t + t|w|)(\mathcal{M}_{atc}^m |h|)(u')\right) D(u, u')$$

whenever $u, u' \in \widehat{U}_i$ *for some* $i = 1, \ldots, k$. *In particular, if*

$$t \leq \frac{\log |b|^\nu}{A_0} \quad , \quad t \leq B|b|^{1-\nu} \quad , \qquad |w| \leq B|b|^\nu \tag{5.1}$$

for some constant $B > 0$*, then*

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_1 \left(\frac{E}{\hat{\gamma}^m}(\mathcal{M}_{atc}^m H)(u') + |b|(\mathcal{M}_{atc}^m |h|)(u')\right) D(u, u').$$

for some constant $A_1 > 0$.

A proof of this lemma is given in the Appendix.

From now on we will assume that a_0**,** η_0 **and** A_0 **are fixed with the properties in Lemma 5 above and** $a, b, c, w, t \in \mathbb{R}$ **are such that** $|a| \leq a_0$**,** $c \leq \eta_0$**,** $|b|, t, |w| \geq 1$ **and (5.1) hold. As before, set** $z = c + \mathbf{i}d$**.**

We will use the entire set-up and notation from section 4 in [21]. In what follows we recall the main part of it.

Following section 4 in [21], **fix an arbitrary point** $z_0 \in \Lambda$ **and constants** $\epsilon_0 > 0$ **and** $\theta_0 \in (0,1)$ **with the properties described in** (LNIC). Assume that $z_0 \in \mathrm{Int}_\Lambda(U_1)$, $U_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$ and $S_1 \subset \Lambda \cap W^s_{\epsilon_0}(z_0)$. Fix an arbitrary constant θ_1 such that

$$0 < \theta_0 < \theta_1 < 1 .$$

Next, fix an arbitrary orthonormal basis $e_1, \ldots, e_n$ in $E^u(z_0)$ and a C^1 parametrization $r(s) = \exp^u_{z_0}(s)$, $s \in V_0'$, of a small neighborhood W_0 of z_0 in $W^u_{\epsilon_0}(z_0)$ such that V_0' is a convex compact neighborhood of 0 in $\mathbb{R}^n \approx \mathrm{span}(e_1, \ldots, e_n) = E^u(z_0)$. Then $r(0) = z_0$ and $\frac{\partial}{\partial s_i} r(s)_{|s=0} = e_i$ for all $i = 1, \ldots, n$. Set $U_0' = W_0 \cap \Lambda$. Shrinking W_0 (and therefore V_0' as well) if necessary, we may assume that $\overline{U_0'} \subset \mathrm{Int}_\Lambda(U_1)$ and $\left| \left\langle \frac{\partial r}{\partial s_i}(s), \frac{\partial r}{\partial s_j}(s) \right\rangle - \delta_{ij} \right|$ is uniformly small for all $i,j = 1, \ldots, n$ and $s \in V_0'$, so that

$$\frac{1}{2}\langle \xi, \eta\rangle \leq \langle\, dr(s)\cdot \xi \;,\; dr(s)\cdot \eta \,\rangle \leq 2\,\langle \xi, \eta\rangle \quad , \quad \xi, \eta \in E^u(z_0) \;,\; s \in V_0' \;,$$

and $\frac{1}{2}\, \|s - s'\| \leq d(r(s), r(s')) \leq 2\, \|s - s'\|$, $s, s' \in V_0'$.

Definitions ([21]): (a) For a cylinder $\mathcal{C} \subset U_0'$ and a unit vector $\xi \in E^u(z_0)$ we will say that a *separation by a ξ-plane occurs* in $\mathcal{C}$ if there exist $u, v \in \mathcal{C}$ with $d(u,v) \geq \frac{1}{2}\mathrm{diam}(\mathcal{C})$ such that $\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|} \;,\; \xi \right\rangle \geq \theta_1$.

Let $\mathcal{S}_\xi$ be the *family of all cylinders* $\mathcal{C}$ contained in U_0' such that a separation by an ξ-plane occurs in $\mathcal{C}$.

(b) Given an open subset V of U_0' which is a finite union of open cylinders and $\delta > 0$, let $\mathcal{C}_1, \ldots, \mathcal{C}_p$ ($p = p(\delta) \geq 1$) be the family of maximal closed cylinders in $\overline{V}$ with $\mathrm{diam}(\mathcal{C}_j) \leq \delta$. For any unit vector $\xi \in E^u(z_0)$ set $M^{(\delta)}_\xi(V) = \cup\{\mathcal{C}_j : \mathcal{C}_j \in \mathcal{S}_\xi \;,\; 1 \leq j \leq p\}$.

In what follows we will construct, amongst other things, a sequence of unit vectors $\xi_1, \xi_2, \ldots, \xi_{j_0} \in E^u(z_0)$. For each $\ell = 1, \ldots, j_0$ set $B_\ell = \{\eta \in \mathbf{S}^{n-1} : \langle \eta, \xi_\ell\rangle \geq \theta_0\}$. For $t \in \mathbb{R}$ and $s \in E^u(z_0)$ set $I_{\eta,t}\, g(s) = \frac{g(s + t\,\eta) - g(s)}{t}$, $t \neq 0$ (*increment* of g in the direction of η).

Lemma 6. ([21]) *There exist integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, a sequence of unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of U_0' which is a finite union of open cylinders of length n_1 such that setting $\mathcal{U} = \sigma^{n_1}(U_0)$ we have:*

(a) *For any integer $N \geq N_0$ there exist Lipschitz maps $v_1^{(\ell)}, v_2^{(\ell)} : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$) such that $\sigma^N(v_i^{(\ell)}(x)) = x$ for all $x \in \mathcal{U}$ and $v_i^{(\ell)}(\mathcal{U})$ is a finite union of open cylinders of length N ($i = 1, 2$; $\ell = 1, 2, \ldots, \ell_0$).*

(b) *There exists a constant $\hat{\delta} > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ with $s + h\,\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h} \left(\tau^N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \tau^N(v_1^{(\ell)}(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{\hat{\delta}}{2} \, .$$

(c) *We have $\overline{v_i^{(\ell)}(U)} \bigcap \overline{v_{i'}^{(\ell')}(U)} = \emptyset$ whenever $(i,\ell) \neq (i', \ell')$.*

(d) *For any open cylinder V in U_0 there exists a constant $\delta' = \delta'(V) > 0$ such that*

$$V \subset M^{(\delta)}_{\eta_1}(V) \cup M^{(\delta)}_{\eta_2}(V) \cup \ldots \cup M^{(\delta)}_{\eta_{\ell_0}}(V)$$

for all $\delta \in (0, \delta']$.

Fix U_0 and $\mathcal{U}$ with the properties described in Lemma 1; then $\overline{\mathcal{U}} = U$.

Set $\hat{\delta} = \min_{1\leq \ell\leq \ell_0} \hat{\delta}_j$, $n_0 = \max_{1\leq \ell\leq \ell_0} m_\ell$, and fix an arbitrary point $\hat{z}_0 \in U_0^{(\ell_0)} \cap \widehat{U}$.

Fix integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^u(z_0)$ and a non-empty open subset U_0 of W_0 with the properties described in Lemma 6. By the choice of U_0, $\sigma^{n_1} : U_0 \longrightarrow \mathcal{U}$ is one-to-one and has an inverse map $\psi : \mathcal{U} \longrightarrow U_0$, which is Lipschitz.

Set $E = \max\left\{4A_0\ ,\ \frac{2A_0 T}{\gamma-1}\right\}$, where $A_0 \geq 1$ is the constant from Lemma 5.4, and **fix an integer** $N \geq N_0$ such that

$$\gamma^N \geq \max\left\{ 6A_0\ ,\ \frac{200\,\gamma_1^{n_1}\,A_0}{c_0^2}\ ,\ \frac{512\,\gamma^{n_1}\,E}{c_0\,\hat{\delta}\,\rho} \right\}.$$

Then fix maps $v_i^{(\ell)} : U \longrightarrow U$ ($\ell = 1, \ldots, \ell_0$, $i = 1, 2$) with the properties (a), (b), (c) and (d) in Lemma 6. In particular, (c) gives

$$\overline{v_i^{(\ell)}(U)} \cap \overline{v_{i'}^{(\ell')}(U)} = \emptyset \quad , \quad (i,\ell) \neq (i',\ell').$$

Since U_0 is a finite union of open cylinders, it follows from Lemma 6(d) that there exist a constant $\delta' = \delta'(U_0) > 0$ such that

$$M_{\eta_1}^{(\delta)}(U_0) \cup \ldots \cup M_{\eta_{\ell_0}}^{(\delta)}(U_0) \supset U_0 \quad , \quad \delta \in (0, \delta'].$$

Fix δ' with this property. Set

$$\epsilon_1 = \min\left\{ \frac{1}{32C_0}\ ,\ c_1\ ,\ \frac{1}{4E}\ ,\ \frac{1}{\hat{\delta}\,\rho^{p_0+2}}\ ,\ \frac{c_0 r_0}{\gamma_1^{n_1}}\ ,\ \frac{c_0^2(\gamma-1)}{16T\gamma_1^{n_1}} \right\},$$

and let $b \in \mathbb{R}$ be such that $|b| \geq 1$ and

$$\frac{\epsilon_1}{|b|} \leq \delta'.$$

Let $\mathcal{C}_m$ $(1 \leq m \leq p)$ be the family of *maximal closed cylinders* contained in $\overline{U_0}$ with $\mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|b|}$ such that $U_0 \subset \cup_{j=m}^p \mathcal{C}_m$ and $\overline{U_0} = \cup_{m=1}^p \mathcal{C}_m$. As in [21],

$$\rho\,\frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq m \leq p\,. \tag{5.2}$$

Fix an integer $q_0 \geq 1$ such that

$$\theta_0 < \theta_1 - 32\,\rho^{q_0-1}.$$

Next, let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all closed cylinders contained in $\overline{U_0}$ that are *subcylinders of co-length* $p_0\,q_0$ of some $\mathcal{C}_m$ $(1 \leq m \leq p)$. Then $\overline{U_0} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q$. Moreover,

$$\rho^{p_0\,q_0+1} \cdot \frac{\epsilon_1}{|b|} \leq \mathrm{diam}(\mathcal{D}_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|b|} \quad , \quad 1 \leq j \leq q.$$

Given $j = 1, \ldots, q$, $\ell = 1, \ldots, \ell_0$ and $i = 1, 2$, set $\widehat{\mathcal{D}}_j = \mathcal{D}_j \cap \widehat{U}$, $Z_j = \overline{\sigma^{n_1}(\widehat{\mathcal{D}}_j)}$, $\widehat{Z}_j = Z_j \cap \widehat{U}$, $X_{i,j}^{(\ell)} = \overline{v_i^{(\ell)}(\widehat{Z}_j)}$, and $\widehat{X}_{i,j}^{(\ell)} = X_{i,j}^{(\ell)} \cap \widehat{U}$. It then follows that $\mathcal{D}_j = \psi(Z_j)$, and $U = \cup_{j=1}^q Z_j$. Moreover, $\sigma^{N-n_1}(v_i^{(\ell)}(x)) = \psi(x)$ for all $x \in \mathcal{U}$, and all $X_{i,j}^{(\ell)}$ are cylinders such that $X_{i,j}^{(\ell)} \cap X_{i',j'}^{(\ell')} = \emptyset$ whenever $(i,j,\ell) \neq (i',j',\ell')$, and

$$\mathrm{diam}(X_{i,j}^{(\ell)}) \geq \frac{c_0\,\rho^{p_0\,q_0+1}}{\gamma_1^N} \cdot \frac{\epsilon_1}{|b|}$$

for all $i = 1,2$, $j = 1,\dots,q$ and $\ell = 1,\dots,\ell_0$. The *characteristic function* $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \longrightarrow [0,1]$ of $\widehat{X}_{i,j}^{(\ell)}$ belongs to $C_D^{\rm Lip}(\widehat{U})$ and $\mathrm{Lip}_D(X_{i,j}^{(\ell)}) \leq 1/\mathrm{diam}(X_{i,j}^{(\ell)})$.

Let J be a *subset of the set* $\Xi = \{\ (i,j,\ell)\ :\ 1 \leq i \leq 2\ ,\ 1 \leq j \leq q\ ,\ 1 \leq \ell \leq \ell_0\ \}$. Set

$$\mu_0 = \mu_0(N) = \min \left\{ \frac{1}{4}\ ,\ \frac{c_0\, \rho^{p_0 q_0 + 2}\, \epsilon_1}{4\, \gamma_1^N}\ ,\ \frac{1}{4\, e^{2TN}}\ \sin^2\left(\frac{\hat{\delta}\, \rho\, \epsilon_1}{256}\right) \right\},$$

and define the function $\omega = \omega_J : \widehat{U} \longrightarrow [0,1]$ by $\omega = 1 - \mu_0 \displaystyle\sum_{(i,j,\ell)\in J} \omega_{i,j}^{(\ell)}$. Clearly $\omega \in C_D^{\rm Lip}(\widehat{U})$ and $1 - \mu \leq \omega(u) \leq 1$ for any $u \in \widehat{U}$. Moreover,

$$\mathrm{Lip}_D(\omega) \leq \Gamma = \frac{2\mu\, \gamma_1^N}{c_0\, \rho^{p_0 q_0 + 2}} \cdot \frac{|b|}{\epsilon_1}.$$

Next, define *the contraction operator* $\mathcal{N} = \mathcal{N}_J(a,b,t,c) : C_D^{\rm Lip}(\widehat{U}) \longrightarrow C_D^{\rm Lip}(\widehat{U})$ by

$$(\mathcal{N} h) = \mathcal{M}_{atc}^N(\omega_J \cdot h).$$

Using Lemma 5 above, the proof of the following lemma is the same as that of Lemma 5.6 in [21].

Lemma 7. *Under the above conditions for N and μ the following hold :*

(a) $\mathcal{N} h \in K_{E|b|}(\widehat{U})$ for any $h \in K_{E|b|}(\widehat{U})$;

(b) If $h \in C_D^{\rm Lip}(\widehat{U})$ and $H \in K_{E|b|}(\widehat{U})$ are such that $|h| \leq H$ in $\widehat{U}$ and $|h(v) - h(v')| \leq E|b| H(v')\, D(v,v')$ for any $v, v' \in U_j$, $j = 1,\dots,k$, then for any $i = 1,\dots,k$ and any $u, u' \in \widehat{U}_i$ we have

$$|(\mathcal{L}_{abtz}^N h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E|b| (\mathcal{N} H)(u')\, D(u,u').$$

Definition. A subset J of Ξ will be called *dense* if for any $m = 1,\dots,p$ there exists $(i,j,\ell) \in J$ such that $\mathcal{D}_j \subset \mathcal{C}_m$.

Denote by $\mathsf{J} = \mathsf{J}(a,b)$ the *set of all dense subsets* J of Ξ.

Although the operator $\mathcal{N}$ here is different, the proof of the following lemma is very similar to that of Lemma 5.8 in [21].

Lemma 8. *Given the number N, there exist $\rho_2 = \rho_2(N) \in (0,1)$ and $a_0 = a_0(N) > 0$ such that $\displaystyle\int_{\widehat{U}} (\mathcal{N}_J H)^2 d\nu \leq \rho_2 \int_{\widehat{U}} H^2 d\nu$ whenever $|a|, |c| \leq a_0$, $t \geq 1/a_0$, J is dense and $H \in K_{E|b|}(\widehat{U})$.*

In what follows we assume that $h, H \in C_D^{\rm Lip}(\widehat{U})$ are such that

$$H \in K_{E|b|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \widehat{U}\ , \tag{5.3}$$

and

$$|h(u) - h(u')| \leq E|b| H(u')\, D(u,u') \quad \text{whenever } u, u' \in \widehat{U}_i\ ,\ i = 1,\dots,k\ . \tag{5.4}$$

Let again $z = c + \mathbf{i}w$. Define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbb{C}$ ($\ell = 1,\dots,j_0$, $i = 1,2$) by

$$\chi_\ell^{(1)}(u) = \frac{\left| e^{(f_{at}^N - \mathbf{i}b\tau^N + z g_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{at}^N - \mathbf{i}b\tau^N + z g_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{(1-\mu) e^{f_{at}^N(v_1^{(\ell)}(u)) + c g_t^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f_{at}^N(v_2^{(\ell)}(u)) + c g_t^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left| e^{(f_{at}^N - \mathbf{i}b\tau^N + zg_t^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f_{at}^N - \mathbf{i}b\tau^N + zg_t^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{e^{f_{at}^N(v_1^{(\ell)}(u)) + cg_t^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1-\mu) e^{f_{at}^N(v_2^{(\ell)}(u)) + cg_t^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

and set $\gamma_\ell(u) = b\,[\tau^N(v_2^{(\ell)}(u)) - \tau^N(v_1^{(\ell)}(u))]$, $u \in \widehat{U}$.

Definitions. We will say that the cylinders $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are *adjacent* if they are subcylinders of the same $\mathcal{C}_m$ for some m. If $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ for some m and for some $\ell = 1, \ldots, \ell_0$ there exist $u \in \mathcal{D}_j$ and $v \in \mathcal{D}_{j'}$ such that $d(u, v) \geq \frac{1}{2}\operatorname{diam}(\mathcal{C}_m)$ and $\left\langle \frac{r^{-1}(v) - r^{-1}(u)}{\|r^{-1}(v) - r^{-1}(u)\|}\ ,\ \eta_\ell \right\rangle \geq \theta_1$, we will say that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are *η_ℓ-separable in $\mathcal{C}_m$*.

As a consequence of Lemma 6(b) one gets the following.

Lemma 9. (Lemma 5.9 in [21]) *Let $j, j' \in \{1, 2, \ldots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$ for some $m = 1, \ldots, p$ and $\ell = 1, \ldots, \ell_0$. Then $|\gamma_\ell(u) - \gamma_\ell(u')| \geq c_2\epsilon_1$ for all $u \in \widehat{Z}_j$ and $u' \in \widehat{Z}_{j'}$, where $c_2 = \dfrac{\hat{\delta}\,\rho}{16}$.*

The following lemma is the analogue of Lemma 5.10 in [21] and represents the main step in proving Theorem 1.

Lemma 10. *Assume $|b| \geq b_0$ for some sufficiently large $b_0 > 0$, $|a|, |c| \leq a_0$, and let* (5.1) *hold. Then for any $j = 1, \ldots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \ldots, q\}$ and $\ell \in \{1, \ldots, \ell_0\}$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$.*

To prove this we need the following lemma which coincides with Lemma 14 in [4] and its proof is almost the same.

Lemma 11. *If h and H satisfy* (5.3)-(5.4)*, then for any $j = 1, \ldots, q$, $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have:*

(a) $\dfrac{1}{2} \leq \dfrac{H(v_i^{(\ell)}(u'))}{H(v_i^{(\ell)}(u''))} \leq 2$ *for all $u', u'' \in \widehat{Z}_j$;*

(b) Either for all $u \in \widehat{Z}_j$ we have $|h(v_i^{(\ell)}(u))| \leq \frac{3}{4} H(v_i^{(\ell)}(u))$, or $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$ for all $u \in \widehat{Z}_j$.

Sketch of proof of Lemma 10. We use a modification of the proof of Lemma 5.10 in [21].

Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $\mathcal{D}_j \subset \mathcal{C}_m$. As in [21] we find $j', j'' = 1, \ldots, q$ such that $\mathcal{D}_{j'}, \mathcal{D}_{j''} \subset \mathcal{C}_m$ and $\mathcal{D}_{j'}$ and $\mathcal{D}_{j''}$ are η_ℓ-separable in $\mathcal{C}_m$.

Fix ℓ, j' and j'' with the above properties, and set $\widehat{Z} = \widehat{Z}_j \cup \widehat{Z}_{j'} \cup \widehat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \widehat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, i.e. $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4}\, H(v_i^{(\ell)}(u))$, $u \in \widehat{Z}$.

Since $\psi(\widehat{Z}) = \widehat{\mathcal{D}}_j \cup \widehat{\mathcal{D}}_{j'} \cup \widehat{\mathcal{D}}_{j''} \subset \mathcal{C}_m$, given $u, u' \in \widehat{Z}$ we have $\sigma^{N-n_1}(v_i^{(\ell)}(u)), \sigma^{N-n_1}(v_i^{(\ell)}(u')) \in \mathcal{C}_m$. Moreover, $\mathcal{C}' = v_i^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m))$ is a cylinder with $\operatorname{diam}(\mathcal{C}') \leq \frac{\epsilon_1}{c_0\,\gamma^{N-n_1}\,|b|}$. Thus, the estimate (9.3) in the Appendix below implies

$$|g_t^N(v_i^{(\ell)}(u)) - g_t^N(v_i^{(\ell)}(u'))| \leq \frac{C_1 t \epsilon_1}{c_0\,\gamma^{N-n_1}\,|b|}.$$

Using the above assumption, (5.1), (5.2) and (4.4), and assuming e.g.

$$e^{cg_t^N(v_i^{(\ell)}(u))}|h(v_i^{(\ell)}(u))| \geq e^{cg_t^N(v_i^{(\ell)}(u))}|h(v_i^{(\ell)}(u'))|,$$

we get[1]

$$\begin{aligned}
&\frac{|e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u)) - e^{zg_t^N(v_i^{(\ell)}(u'))}h(v_i^{(\ell)}(u'))|}{\min\{|e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u))|, |e^{zg_t^N(v_i^{(\ell)}(u'))}h(v_i^{(\ell)}(u'))|\}} \\
= \; & \frac{|e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u)) - e^{zg_t^N(v_i^{(\ell)}(u'))}h(v_i^{(\ell)}(u'))|}{e^{cg_t^N(v_i^{(\ell)}(u'))}|h(v_i^{(\ell)}(u'))|} \\
\leq \; & \frac{|e^{zg_t^N(v_i^{(\ell)}(u))} - e^{zg_t^N(v_i^{(\ell)}(u'))}|}{e^{cg_t^N(v_i^{(\ell)}(u'))}} + \frac{e^{cg_t^N(v_i^{(\ell)}(u))}|h(v_i^{(\ell)}(u)) - h(v_i^{(\ell)}(u'))|}{e^{cg_t^N(v_i^{(\ell)}(u'))}|h(v_i^{(\ell)}(u'))|} \\
\leq \; & \frac{|e^{zg_t^N(v_i^{(\ell)}(u))} - e^{zg_t^N(v_i^{(\ell)}(u'))}|}{e^{cg_t^N(v_i^{(\ell)}(u'))}} + \frac{e^{c(g_t^N(v_i^{(\ell)}(u'))-g_t^N(v_i^{(\ell)}(u')))}\, E|b|H(v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|} D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) \\
\leq \; & \frac{|e^{cg_t^N(v_i^{(\ell)}(u))} - e^{cg_t^N(v_i^{(\ell)}(u'))}|}{e^{cg_t^N(v_i^{(\ell)}(u'))}} + |e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u))} - e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u'))}| + 4E|b|e^{2a_0NT}\operatorname{diam}(\mathcal{C}') \\
\leq \; & (e^{C_1t}C_1t + |w|C_1t)\, D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + 4E|b|e^{2Na_0T}\,\frac{\gamma^{n_1}\epsilon_1}{c_0\gamma^N} \\
\leq \; & \frac{(B+A_0)\gamma^{n_1}\epsilon_1}{c_0\gamma^N} + \frac{4E\gamma^{n_1}\epsilon_1}{c_0(e^{-2a_0T}\gamma)^N} < \frac{\pi}{12}
\end{aligned}$$

assuming $a_0 > 0$ is chosen sufficiently small and N sufficiently large. So, the angle between the complex numbers

$$e^{zg_t^N(v_i^{(\ell)}(u)}h(v_i^{(\ell)}(u)) \quad \text{and } e^{zg_t^N(v_i^{(\ell)}(u')}h(v_i^{(\ell)}(u'))$$

(regarded as vectors in $\mathbb{R}^2$) is $< \pi/6$. In particular, for each $i = 1, 2$ we can choose a real continuous function $\theta_i(u)$, $u \in \widehat{Z}$, with values in $[0, \pi/6]$ and a constant λ_i such that

$$e^{zg_t^N(v_i^{(\ell)}(u))}h(v_i^{(\ell)}(u)) = e^{\mathbf{i}(\lambda_i+\theta_i(u))}e^{cg_t^N(v_i^{(\ell)}(u))}|h(v_i^{(\ell)}(u))|$$

for all $u \in \widehat{Z}$. Fix an arbitrary $u_0 \in \widehat{Z}$ and set $\lambda = \gamma_\ell(u_0)$. Replacing e.g λ_2 by $\lambda_2 + 2m\pi$ for some integer m, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \leq \pi$. Using the above, $\theta \leq 2\sin\theta$ for $\theta \in [0, \pi/6]$, and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2\sin|\theta_i(u) - \theta_i(u')| < \frac{c_2\epsilon_1}{8}$.

The difference between the arguments of the complex numbers

$$e^{\mathbf{i}\,b\,\tau^N(v_1^{(\ell)}(u))}e^{zg_t^N(v_1^{(\ell)}(u)}h(v_1^{(\ell)}(u)) \quad \text{and} \quad e^{\mathbf{i}\,b\,\tau^N(v_2^{(\ell)}(u))}e^{zg_t^N(v_2^{(\ell)}(u)}h(v_2^{(\ell)}(u))$$

is given by the function

$$\Gamma^{(\ell)}(u) = [b\,\tau^N(v_2^{(\ell)}(u)) + \theta_2(u) + \lambda_2] - [b\,\tau^N(v_1^{(\ell)}(u)) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1) + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u))\,.$$

Given $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$, since $\widehat{\mathcal{D}}_{j'}$ and $\widehat{\mathcal{D}}_{j''}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$, it follows from Lemma 9 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{c_2\epsilon_1}{2}\,.$$

[1] Using some estimates as in the proof of Lemma 5(b) in the Appendix below and $\|cg_t^N\|_0 \leq a_0NT$ by (4.4).

Thus, $|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq \frac{c_2}{2}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$. Hence either $|\Gamma^{(\ell)}(u')| \geq \frac{c_2}{4}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ or $|\Gamma^{(\ell)}(u'')| \geq \frac{c_2}{4}\epsilon_1$ for all $u'' \in \widehat{Z}_{j''}$.

Assume for example that $|\Gamma^{(\ell)}(u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in \widehat{Z}_{j'}$. Since $\widehat{Z} \subset \sigma^{n_1}(\mathcal{C}_m)$, as in [21] we have for any $u \in \widehat{Z}$ we get $|\Gamma_\ell(u)| < \frac{3\pi}{2}$. Thus, $\frac{c_2}{4}\epsilon_1 \leq |\Gamma^{(\ell)}(u)| < \frac{3\pi}{2}$ for all $u \in \widehat{Z}_{j'}$. Now as in [4] (see also [21]) one shows that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$. $\square$

Parts (a) and (b) of the following lemma can be proved in the same way as the corresponding parts of Lemma 5.3 in [21], while part (c) follows from Lemma 5(b).

Lemma 12. *There exist a positive integer N and constants $\hat{\rho} = \hat{\rho}(N) \in (0,1)$, $a_0 = a_0(N) > 0$, $b_0 = b_0(N) > 0$ and $E \geq 1$ such that for every $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ such that (5.1) hold, there exists a finite family $\{\mathcal{N}_J\}_{J \in \mathsf{J}}$ of operators*

$$\mathcal{N}_J = \mathcal{N}_J(a,b,t,c) : C_D^{\text{Lip}}(\widehat{U}) \longrightarrow C_D^{\text{Lip}}(\widehat{U}),$$

where $\mathsf{J} = \mathsf{J}(a,b,t,c)$, with the following properties:

(a) The operators $\mathcal{N}_J$ preserve the cone $K_{E|b|}(\widehat{U})$;

(b) For all $H \in K_{E|b|}(\widehat{U})$ and $J \in \mathsf{J}$ we have $\int_{\widehat{U}} (\mathcal{N}_J H)^2 \, d\nu_0 \leq \hat{\rho} \int_{\widehat{U}} H^2 \, d\nu_0$.

(c) If $h, H \in C_D^{\text{Lip}}(\widehat{U})$ are such that $H \in K_{E|b|}(\widehat{U})$, $|h(u)| \leq H(u)$ for all $u \in \widehat{U}$ and $|h(u) - h(u')| \leq E|b|H(u')\, D(u,u')$ whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$, then there exists $J \in \mathsf{J}$ such that $|\mathcal{L}_{abw}^N h(u)| \leq (\mathcal{N}_J H)(u)$ for all $u \in \widehat{U}$ and for $z = c + \mathbf{i}w$ we have

$$|(\mathcal{L}_{abtz}^N h)(u) - (\mathcal{L}_{abtz}^N h)(u')| \leq E|b|(\mathcal{N}_J H)(u')\, D(u,u')$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$.

Proof of Theorem 5(a). Using an argument from [4] one derives from Lemma 12 that there exist a positive integer N and constants $\hat{\rho} \in (0,1)$ and $a_0 > 0$, $b_0 \geq 1$, $A_0 > 0$ such that for any $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ for which (5.1) hold, and for any $h \in C^{\text{Lip}}(\widehat{U})$ with $\|h\|_{\text{Lip},b} \leq 1$ we have

$$\int_U |\mathcal{L}_{abtz}^{Nm} h|^2 \, d\nu_0 \leq \hat{\rho}^m \quad , \quad m \geq 0. \tag{5.5}$$

Then the estimate claimed in Theorem 5(a) follows as in [4] (see also the proof of Corollary 3.3(a) in [20]). $\square$

The proof of Theorem 5(b) can be derived using an approximation procedure as in [4] – see the Appendix below for some details.

6. Spectral estimates when w is the leading parameter

Here we try to repeat the arguments from the previous section however changing the roles of the parameters b and w. We continue to use the assumptions made at the beginning of section 5, however now we suppose that $f \in C^{\text{Lip}}(\widehat{U})$. We will consider the case

$$|b| \leq B\,|w| \tag{6.1}$$

for an arbitrarily large (but fixed) constant $B > 0$.

Assume that $G : \Lambda \longrightarrow \mathbb{R}$ is a Lipschitz functions which is constant on stable leaves of $B_i = \{\phi_t(x) : x \in R_i, 0 \leq t \leq \tau(x)\}$ for each rectangle R_i of the Markov family and $A = \min_{x\in\Lambda} G(x) > 0$. Set

$$L = \mathrm{Lip}(G) \quad , \quad D = \mathrm{diam}(\Lambda)\,,$$

where without loss of generality we may assume that $D \geq 1$. We will also assume that

$$L \leq \hat{\mu}\, A \quad , \text{where} \quad \hat{\mu} = \frac{c_0\,\hat{\delta}}{128\, C_0\, C_1\, D}. \tag{6.2}$$

The function

$$g(x) = \int_0^{\tau(x)} G(\phi_t(x))\, dt \quad , \quad x \in R,$$

is constant on stable leaves of R, so it can be regarded as a function on U. Clearly $g \in C^{\mathrm{Lip}}(\widehat{U})$.

Remark. Notice that if we replace G by $G + d$ for some constant $d > 0$, then

$$g'(x) = \int_0^{\tau(x)} (G(\phi_t(x)) + d)\, dt = g(x) + d\,\tau(x),$$

so

$$\mathcal{L}_{f_a - \mathbf{i}\, b\tau + \mathbf{i}\, wg} = \mathcal{L}_{f_a - \mathbf{i}\, b\tau + \mathbf{i}\, w(g' - d\tau)} = \mathcal{L}_{f_a - \mathbf{i}\,(b+dw)\tau - \mathbf{i}\, wg'}\,.$$

Choose and fix $d > 0$ so that $\frac{\mathrm{Lip}(G)}{G_0 + d} \leq \hat{\mu}$. Then for $G' = G + d$ and $g' = g + d\tau$ we have $\frac{\mathrm{Lip}(G')}{\min G'} \leq \hat{\mu}$, and the operator $\mathcal{L}_{f_a - \mathbf{i}\, b\tau + \mathbf{i}\, wg} = \mathcal{L}_{f_a - \mathbf{i}\, b'\tau + \mathbf{i}\, wg'}$, where $b' = b + dw$. Thus, without loss of generality we may assume that $\frac{\mathrm{Lip}(G)}{\min G} \leq \hat{\mu}$, which is equivalent to (6.2). As in [12], this will imply a non-integrability property for g (see Lemma 10 below). In other words, dealing with an initial function G one has to first change it to arrange (6.2), and then with the new parameters b and w that appear in front of $\mathbf{i}\tau$ and $\mathbf{i}g$ consider the cases $|w| \leq B|b|$ (as in Theorem 5(c)) and $|b| \leq B|w|$, which is considered in this section.

As in section 5, we will use the set-up and some arguments from [21]. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t over Λ as in section 2.

Here we prove the following analogue of Theorem 5(c).

Theorem 6. *Let $\phi_t : M \longrightarrow M$ be a C^2 flow satisfying the Standing Assumptions over the basic set Λ. Assume in addition that* (6.2) *holds. Then for any real-valued functions $f, g \in C^{\mathrm{Lip}}(\widehat{U})$, any constants $\epsilon > 0$ and $B > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $w_0 \geq 1$ and $C = C(B,\epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\| L^m_{f - (P_f + a + \mathbf{i}b)\tau + (c + \mathbf{i}w)g} h\|_{\mathrm{Lip},b} \leq C\, \rho^m\, |b|^{\epsilon}\, \|h\|_{\mathrm{Lip},b} \tag{6.3}$$

for all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|w| \geq w_0$ and $|b| \leq B\,|w|$.

Recall the definitions of $\lambda_0 > 0$, $\hat{\nu}_0$, h_0, f_0 from section 4; now we have $h_0, f_0 \in C^{\mathrm{Lip}}(\widehat{U})$. Fix a small $a_0 > 0$. Given a real number a with $|a| \leq a_0$, denote by λ_a the *largest eigenvalue* of $L_{f-(P+a)\tau}$ on $C^{\mathrm{Lip}}(U)$ and by h_a the corresponding (positive) eigenfunction such that $\int_U h_a\, d\nu_a = 1$, where ν_a is the unique probability measure on U with $L^*_{f-(P+a)\tau}\nu_a = \nu_a$. Given real numbers a, b, c, w with $|a|, |c| \leq a_0$ consider the function

$$\tilde{f}_a = f - (P + a)\tau + \ln h_a - \ln(h_a \circ \sigma) - \ln \lambda_a$$

and the operators

$$\mathcal{L}_{abz} = L_{\tilde{f}_a - \mathbf{i}\, b\,\tau + zg} : C(U) \longrightarrow C(U) \quad , \quad \tilde{\mathcal{M}}_{ac} = L_{\tilde{f}_a + cg} : C(U) \longrightarrow C(U),$$

where $z = c + \mathbf{i}w$. Notice that $L_{\tilde{f}_a} 1 = 1$.

Taking the constant $C_0 > 0$ sufficiently large, we may assume that

$$\mathrm{Lip}(\tilde{f}_a - f_0) \leq C_0 |a| \quad , \quad , \|\tilde{f}_a - f_0\|_0 \leq C_0\, |a| \quad , \quad |a| \leq a_0. \tag{6.4}$$

Thus, ssuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $T > 0$ (depending on f and a_0) such that

$$\|\tilde{f}_a\|_\infty \leq T \quad , \quad \mathrm{Lip}(h_a) \leq T \quad , \quad \mathrm{Lip}(\tilde{f}_a) \leq T \tag{6.5}$$

for $|a| \leq a_0$. As before, we will assume that $T \geq \max\{\, \|\tau\|_0\,, \mathrm{Lip}(\tau_{|\widehat{U}})\,\}$, and also that $\mathrm{Lip}(g) \leq T$ and $\|g\|_0 \leq T$.

Essentially in what follows we will repeat (a simplified version of) the proof of Theorem 5, so we will use the set-up in section 5 – see the text after Lemma 6, up to and including the definition of ϵ_1.

Let $a, b, c, w \in \mathbb{R}$ be so that $|a|, |c| \leq a_0$, $|w| \geq w_0$, where w_0 is a sufficiently large constant defined as b_0 in section 5, and $|b| \leq B|w|$. Set $z = c + \mathbf{i}w$.

Let $\mathcal{C}_m$ $(1 \leq m \leq p)$ be the family of *maximal closed cylinders* contained in $\overline{U_0}$ with $\mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|w|}$ such that $U_0 \subset \cup_{j=m}^p \mathcal{C}_m$ and $\overline{U_0} = \cup_{m=1}^p \mathcal{C}_m$. As before we have

$$\rho\, \frac{\epsilon_1}{|w|} \leq \mathrm{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|w|} \quad , \quad 1 \leq m \leq p.$$

Fix an integer $q_0 \geq 1$ as in Sect. 5, and let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all closed cylinders contained in $\overline{U_0}$ that are *subcylinders of co-length* $p_0\, q_0$ of some $\mathcal{C}_m$ $(1 \leq m \leq p)$. Then $\overline{U_0} = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q$ and

$$\rho^{p_0\, q_0 + 1} \cdot \frac{\epsilon_1}{|w|} \leq \mathrm{diam}(\mathcal{D}_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|w|} \quad , \quad 1 \leq j \leq q.$$

Next, define the cylinders $Z_j = \overline{\sigma^{n_1}(\widehat{\mathcal{D}}_j)}$ and $X_{i,j}^{(\ell)} = \overline{v_i^{(\ell)}(\widehat{Z}_j)}$ as in section 5, and consider the characteristic functions $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \longrightarrow [0,1]$. Let J be a *subset of the set* $\Xi = \Xi(a,w) = \{\ (i,j,\ell)\ :\ 1 \leq i \leq 2\ ,\ 1 \leq j \leq q\ ,\ 1 \leq \ell \leq \ell_0\ \}$. Define $\mu_0 > 0$ as in section 4 and $\omega = \omega_J : \widehat{U} \longrightarrow [0,1]$ by $\omega = 1 - \mu_0 \displaystyle\sum_{(i,j,\ell) \in J} \omega_{i,j}^{(\ell)}$. Finally define $\mathcal{N} = \mathcal{N}_J(a,b,c) : C_D^{\mathrm{Lip}}(\widehat{U}) \longrightarrow C_D^{\mathrm{Lip}}(\widehat{U})$ by $(\mathcal{N}h) = \tilde{\mathcal{M}}_{ac}^N(\omega_J \cdot h)$.

Then we have the following analogue of Lemma 5.

Lemma 13. *Assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $A_0 > 0$ such that for all $a, c \in \mathbb{R}$ with $|a|, |c| \leq a_0$ the following hold:*

(a) If $H \in K_E(\widehat{U})$ for some $E > 0$, then

$$\frac{|(\tilde{\mathcal{M}}_{ac}^m H)(u) - (\tilde{\mathcal{M}}_{ac}^m H)(u')|}{(\tilde{\mathcal{M}}_{ac}^m H)(u')} \leq A_0 \left[\frac{E}{\gamma_0^m} + 1\right] D(u,u')$$

for all $m \geq 1$ and all $u, u' \in U_i$, $i = 1, \ldots, k$.

(b) If the functions h and H on $\widehat{U}$ and $E > 0$ are such that $H > 0$ on $\widehat{U}$ and $|h(v) - h(v')| \leq E\, H(v')\, D(v, v')$ for any $v, v' \in \widehat{U}_i$, $i = 1, \dots, k$, then for any integer $m \geq 1$ and any $b, w \in \mathbb{R}$ with $|b|, |w| \geq 1$, for $z = c + \mathbf{i}w$ we have

$$|(\mathcal{L}_{abw}^N h)(u) - (\mathcal{L}_{abw}^N h)(u')| \leq E|w|(\mathcal{N} H)(u')\, D(u, u').$$

whenever $u, u' \in \widehat{U}_i$ for some $i = 1, \dots, k$.

The proof is a simplified version of that of Lemma 5 and we omit it.

Next, changing appropriately the definition of a dense subset J of Ξ, Lemma 8 holds again replacing $K_{E|b|}(\widehat{U})$ by $K_{E|w|}(\widehat{U})$.

Assume that $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ are such that

$$H \in K_{E|w|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \widehat{U}, \tag{6.6}$$

and

$$|h(u) - h(u')| \leq E|w| H(u')\, D(u, u') \quad \text{whenever } u, u' \in \widehat{U}_i\ ,\ i = 1, \dots, k. \tag{6.7}$$

Define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbb{C}$ by

$$\chi_\ell^{(1)}(u) = \frac{\left| e^{(\tilde{f}_a^N - \mathbf{i} b\tau^N + zg^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(\tilde{f}_a^N - \mathbf{i} b\tau^N + zg^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{(1-\mu) e^{\tilde{f}_a^N(v_1^{(\ell)}(u)) + cg^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{\tilde{f}_a^N(v_2^{(\ell)}(u)) + cg^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left| e^{(\tilde{f}_a^N - \mathbf{i} b\tau^N + zg^N)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(\tilde{f}_a^N - \mathbf{i} b\tau^N + zg^N)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{e^{\tilde{f}_a^N(v_1^{(\ell)}(u)) + cg^N(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1-\mu) e^{\tilde{f}_a^N(v_2^{(\ell)}(u)) + cg^N(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

and set $\gamma_\ell(u) = w\,[\tau_N(v_2^{(\ell)}(u)) - \tau_N(v_1^{(\ell)}(u))]$, $u \in \widehat{U}$. The crucial step in this section is to prove the following analogue of Lemma 9:

Lemma 14. *Let $j, j' \in \{1, 2, \dots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$ for some $m = 1, \dots, p$ and $\ell = 1, \dots, \ell_0$. Then $|\gamma_\ell(u) - \gamma_\ell(u')| \geq c_3 \epsilon_1$ for all $u \in \widehat{Z}_j$ and $u' \in \widehat{Z}_{j'}$, where $c_3 = \dfrac{A\hat{\delta}\,\rho}{32}$.*

To prove the above we need the following.

Lemma 15. (Lemma 6 in [12]) *Assume that* (6.2) *holds. Under the assumptions and notation in Lemma 1, for all $\ell = 1, \dots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ so that $s + h\,\eta \in r^{-1}(U_0 \cap \Lambda)$ we have*

$$\left[I_{\eta,h} \left(g^N(v_2^{(\ell)}(\tilde{r}(\cdot))) - g^N(v_1^{(\ell)}(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{A\hat{\delta}}{4}\,.$$

Proof of Lemma 14. This just a repetition of the proof of Lemma 5.9 in [21], where instead of using Lemma 6(b) we use the above Lemma 14. We omit the details. □

Next, we need to prove the analogue of Lemma 10.

Lemma 16. *Assume $|w| \geq w_0$ for some sufficiently large $w_0 > 0$ and let $|b| \leq B|w|$. Then for any $j = 1, \dots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \dots, q\}$ and $\ell \in \{1, \dots, \ell_0\}$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$.*

Sketch of proof of Lemma 16. We will use Lemma 11 which holds again with (5.3)-(5.4) replaced by (6.6)-(6.7).

Given $j = 1, \dots, q$, let $m = 1, \dots, p$ be such that $\mathcal{D}_j \subset \mathcal{C}_m$. As in [21] we find $j', j'' = 1, \dots, q$ such that $\mathcal{D}_{j'}, \mathcal{D}_{j''} \subset \mathcal{C}_m$ and $\mathcal{D}_{j'}$ and $\mathcal{D}_{j''}$ are η_ℓ-separable in $\mathcal{C}_m$.

Fix ℓ, j' and j'' with the above properties, and set $\widehat{Z} = \widehat{Z}_j \cup \widehat{Z}_{j'} \cup \widehat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \widehat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 11(b) holds for $\widehat{Z}_t$, ℓ and i, i.e. $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4}\, H(v_i^{(\ell)}(u))$, $u \in \widehat{Z}$.

Again we have $\psi(\widehat{Z}) = \widehat{\mathcal{D}}_j \cup \widehat{\mathcal{D}}_{j'} \cup \widehat{\mathcal{D}}_{j''} \subset \mathcal{C}_m$, and $\mathcal{C}' = v_i^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m))$ is a cylinder with $\mathrm{diam}(\mathcal{C}') \leq \frac{\epsilon_1}{c_0\, \gamma^{N-n_1}\, |w|}$. Thus, assuming e.g. $|h(v_i^{(\ell)}(u))| \geq |h(v_i^{(\ell)}(u'))|$, we get

$$
\begin{aligned}
& \frac{|e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u)}h(v_i^{(\ell)}(u)) - e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u')}h(v_i^{(\ell)}(u'))|}{\min\{|h(v_i^{(\ell)}(u))|, |h(v_i^{(\ell)}(u'))|\}} \\
\leq\ & |e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u)} - e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u')}| + \frac{E|w|\, H(v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|} D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) \\
\leq\ & |b| C_1\, D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + 4E|w|\, D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) \\
\leq\ & (B|w|\, C_1 + 4E|w|)\, \mathrm{diam}(\mathcal{C}') \leq \frac{(BC_1 + 4E)\epsilon_1}{\gamma_1^{N-n_1}} < \frac{\pi}{12}
\end{aligned}
$$

assuming N is chosen sufficiently large. So, the angle between the complex numbers

$$e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u)}h(v_i^{(\ell)}(u)) \quad \text{and } e^{\mathbf{i} b\tau_N(v_i^{(\ell)}(u')}h(v_i^{(\ell)}(u'))$$

(regarded as vectors in $\mathbb{R}^2$) is $< \pi/6$. In particular, for each $i = 1, 2$ we can choose a real continuous function $\theta_i(u)$, $u \in \widehat{Z}$, with values in $[0, \pi/6]$ and a constant λ_i such that $h(v_i^{(\ell)}(u)) = e^{\mathbf{i}(\lambda_i + \theta_i(u))}|h(v_i^{(\ell)}(u))|$ for all $u \in \widehat{Z}$. Fix an arbitrary $u_0 \in \widehat{Z}$ and set $\lambda = \gamma_\ell(u_0)$. Replacing e.g λ_2 by $\lambda_2 + 2m\pi$ for some integer m, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \leq \pi$. Using the above, $\theta \leq 2\sin\theta$ for $\theta \in [0, \pi/6]$, and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2\sin|\theta_i(u) - \theta_i(u')| < \frac{c_2\epsilon_1}{8}$.

The difference between the arguments of the complex numbers

$$e^{\mathbf{i}\, b\, \tau_N(v_1^{(\ell)}(u))}e^{\mathbf{i} w g_N(v_1^{(\ell)}(u)}h(v_1^{(\ell)}(u)) \quad \text{and} \quad e^{\mathbf{i}\, b\, \tau_N(v_2^{(\ell)}(u))}e^{\mathbf{i} w g_N(v_2^{(\ell)}(u)}h(v_2^{(\ell)}(u))$$

is given by the function

$$\Gamma^{(\ell)}(u) = [w\, g_N(v_2^{(\ell)}(u)) + \theta_2(u) + \lambda_2] - [w\, g_N(v_1^{(\ell)}(u)) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1) + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u)) \;.$$

Given $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$, since $\widehat{\mathcal{D}}_{j'}$ and $\widehat{\mathcal{D}}_{j''}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$, it follows from Lemma 9 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{c_3\epsilon_1}{2} \;.$$

Thus, $|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq \frac{c_3}{2}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$. Hence either $|\Gamma^{(\ell)}(u')| \geq \frac{c_3}{4}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ or $|\Gamma^{(\ell)}(u'')| \geq \frac{c_3}{4}\epsilon_1$ for all $u'' \in \widehat{Z}_{j''}$.

Assume for example that $|\Gamma^{(\ell)}(u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in \widehat{Z}_{j'}$. Since $\widehat{Z} \subset \sigma^{n_1}(\mathcal{C}_m)$, as in [21] we have for any $u \in \widehat{Z}$ we get $|\Gamma_\ell(u)| < \frac{3\pi}{2}$. Thus, $\frac{c_2}{4}\epsilon_1 \leq |\Gamma^{(\ell)}(u)| < \frac{3\pi}{2}$ for all $u \in \widehat{Z}_{j'}$. Now as in [4] (see also [21]) one shows that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$. □

Proof of Theorem 6. This is now the same as the proof of Theorem 5(a). □

7. Analytic continuation of the function $\zeta(s, z)$

Consider the function $\zeta(s, z)$ introduced in section 1. Recall that $s = a + \mathbf{i}b, z = c + \mathbf{i}w$ with real $a, b, c, w \in \mathbb{R}$. First, we assume that f and g are functions in $C^\alpha(\Lambda)$ with some $0 < \alpha < 1$. Passing to the symbolic model defined by the Markov family $\mathcal{R}$ we obtain functions[2] in $C^\alpha(R)$ which we denote again by f and g. We assume that $Pr(f - P_f\tau) = 0$ and we set $s = P_f + a + \mathbf{i}b$. The functions f, g depend on $x \in R$. A second reduction is to replace f and g by functions $\hat{f}$, $\hat{g} \in C^{\alpha/2}(U)$ depending only on $x \in U$ so that $f = \hat{f} + h_1 - h_1 \circ \sigma$, $g = \hat{g} + h_2 - h_2 \circ \sigma$ (see Proposition 1.2 in [11]). Since for periodic points with $\sigma^n x = x$ we have $f^n(x) = \hat{f}^n(x)$, $g^n(x) = \hat{g}^n(x)$, we obtain the representation

$$\zeta(s, z) = \exp\Big(\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{\hat{f}^n(x) - (P_f + a + \mathbf{i}b)\tau^n(x) + (c + \mathbf{i}w)\hat{g}^n(x)}\Big).$$

In this section we will prove under the standing assumptions that there exists $\epsilon > 0$ and $\epsilon_0 > 0$ such that the function $\zeta(s, z)$ has a non-vanishing zero analytic continuation for $-\epsilon \leq a \leq 0$ and $|z| \leq \epsilon_0$ with a simple pole at $s = s(z)$, $s(0) = P_f$. Here $s(z)$ is determined from the equation $Pr(f - s\tau + zg) = 0$. For simplicity of the notation we denote below $\hat{f}$ and $\hat{g}$ again by f, g.

First consider the case $0 < \delta \leq |b| \leq b_0$. Since our standing assumptions imply that the flow ϕ_t is weak mixing, Theorem 6.4 in [11] says that for every fixed b lying in the compact interval $[\delta, b_0]$ there exists $\epsilon(b) > 0$ so that the function $\zeta(s, z)$ is analytic for $|s - P_f + \mathbf{i}b| \leq \epsilon(b)$, $|z| \leq \epsilon(b)$. This implies that there exists $\eta_0 = \eta_0(\delta, b_0) > 0$ such that $\zeta(s, z)$ is analytic for $P_f - \eta_0 \leq \operatorname{Re} s \leq P_f + \eta_0, \delta \leq |\operatorname{Im} s| \leq b_0$, $|z| \leq \eta_0$. Decreasing $\delta > 0$ and η_0, if it is necessary, we apply once more Theorem 6.4 in [11], to conclude that $\zeta(s, z)(1 - e^{Pr(f - s\tau + zg)})$ is analytic for

$$s \in \{s \in \mathbb{C} : |\operatorname{Re} s - P_f| \leq \eta, |\operatorname{Im} s| \leq \delta\}$$

and $|z| \leq \eta_0$. Consequently, the singularities of $\zeta(s, z)$ are given by (s, z) for which we have $Pr(f - s\tau + zg) = 0$ and, solving this equation, we get $s = s(z)$ with $s(0) = P_f$. It is clear that we have a simple pole at $s(z)$ since $\frac{d}{ds}Pr(f - s\tau + zg) \neq 0$ for $|z|$ small enough.

Now we pass to the case when $|\operatorname{Im} s| = |b| \geq b_0 > 0$, $|z| \leq \eta_0$. Then we fix a $\beta \in (0, \alpha/2)$ and we get with $0 < \mu < 1$ the inequality $|\operatorname{Im} b| \geq B_0 |z|^\mu$ with $B_0 = \frac{b_0}{\eta_0^\mu}$. Thus we are in position to apply the estimates of Theorem 5(b) saying that for every $\epsilon > 0$ there exist $0 < \rho < 1$ and $C_\epsilon > 0$ so that

$$\|L^m_{f - (P_f + a + \mathbf{i}b)\tau + zg}\|_{\beta, b} \leq C_\epsilon \rho^m |b|^\epsilon, \ \forall m \in \mathbb{N} \tag{7.1}$$

[2]In fact, one has to define first f and g as functions in $C^\alpha(\hat{R})$ and then extend them as α-Hölder functions on R. In the same way one should proceed with Hölder functions on U.

for $|a| \leq a_0, |b| \geq b_0,\ |z| \leq \eta_0$. Next we apply Theorem 4 with functions $f, g \in C^\beta(U)$. For $|\operatorname{Re} s - P_f| \leq \eta_0,\ |\operatorname{Im} s| \geq b_0$ and $|z| \leq \eta_0$ we deduce

$$|Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)| \leq \sum_{i=1}^{k} |L^n_{f-(P_f+\mathbf{i}a+b)\tau+zg}(\chi_i)(x_i)|$$

$$+C(1+|b|)\sum_{m=2}^{n} \|L^m_{f-(P_f+a+\mathbf{i}b)\tau+zg}\|_\beta \gamma_0^{-m\beta} e^{mPr(f-(P_f+a)\tau+(\operatorname{Re} z)g)}$$

$$\leq \quad k\|L^n_{f-(P_f+a+\mathbf{i}b)\tau+zg}\|_\beta + C_\epsilon(1+|b|)|b|^\epsilon \sum_{m=2}^{n} \rho^{n-m}\gamma_0^{-m\beta} e^{m(\epsilon+Pr(f-(P_f+a)\tau+cg))}.$$

Taking η_0 and ϵ small, we arrange

$$\gamma_0^{-\beta} e^{\epsilon+Pr(f-(P_f+a)\tau+cg)} \leq \gamma_2 < 1$$

for $|a| \leq \eta_0,\ |c| \leq \eta_0$, since $Pr(f - P_f\tau) = 0$ and $\gamma_0^{-\nu} < 1$. Next increasing $0 < \rho < 1$, if it is necessary, we get $\frac{\gamma_2}{\rho} < 1$. Thus the sum above will be bounded by

$$C_\epsilon(1+|b|)|b|^\epsilon \rho^n \sum_{m=2}^{\infty} \Big(\frac{\gamma_2}{\rho}\Big)^m \leq C'_\epsilon |b|^{1+\epsilon}\rho^n$$

for $|a| \leq \eta_0,\ |z| \leq \eta_0$. The analysis of the term $\|L^n_{f-(P_f+a+\mathbf{i}b)+zg}\|_\beta$ follows the same argument and it is simpler. Finally, we get

$$|Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)| \leq B_\epsilon |b|^{1+\epsilon}\rho^n,\ \forall n \in \mathbb{N}$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n} Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)$$

is absolutely convergent for $|a| \leq \eta_0, |b| \geq b_0, |z| \leq \eta_0$. This implies the analytic continuation of $\zeta(s, z)$ for $|\operatorname{Re} s - P_f| \leq \eta_0, |\operatorname{Im} s| \geq b_0,\ |z| \leq \eta_0$, thus completing the proof of Theorem 1.

To obtain a representation of the function $\eta_g(s) = \frac{\partial \log \zeta(s,z)}{\partial z}\big|_{z=0}$ for s sufficiently close to P_f, notice that for such values of s we have

$$\eta_g(s) = -\frac{\partial \log(1 - e^{Pr(f-s\tau+zg)})}{\partial z}\Big|_{z=0} + A_0(s) = \frac{1}{s - P_f}\frac{\int g dm}{\int \tau dm} + A_1(s) = \frac{\int G d\mu_F}{s - P_f} + A_1(s),$$

where m is the equilibrium state of $f - P_f\tau$, μ_F is the equilibrium state of F and $A_0(s)$ and $A_1(s)$ are analytic in a neighborhood of P_f (see Chapter 6 in [11]). More precisely, μ_F is a σ_t^τ invariant probability measure on R^τ such that

$$Pr(F) = h(\sigma_1^\tau, \mu_F) + \int F(\pi(x,t))d\mu_F,$$

where $h(\sigma_1^\tau, \mu_F)$ is the metric entropy of σ_1^τ with respect to μ_F (see Chapter 6 in [11]).

Taking η_0 small enough, for $|z| \leq \eta_0$, $|\operatorname{Re} s - P_f| \leq \eta_0$ and $|\operatorname{Im} s| \geq \eta_0$ from the estimates for $Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)$ above, we deduce

$$|\log \zeta(s,z)| \leq C_\epsilon \max\Big(1, |\operatorname{Im} s|^{1+\epsilon}\Big).$$

To estimate $\eta_g(s)$, as in [17], we apply the Cauchy theorem for the derivative

$$\frac{\partial}{\partial z}\log\zeta(s,z)\big|_{z=0} = \frac{1}{2\pi\mathbf{i}\delta}\int_{|\xi|=\delta}\frac{\log\zeta(s,\xi)}{\xi^2}d\xi = \mathcal{O}(|\operatorname{Im} s|^{1+\epsilon}),\ |\operatorname{Im} s| \geq 1.$$

with $\delta > 0$ sufficiently small. Thus we obtain a $\mathcal{O}\Big(\max\Big(1, |\operatorname{Im} s|^{1+\epsilon}\Big)\Big)$ bound for the function

$$A(s) = \eta_g(s) - \frac{1}{s - P_f}\int G d\mu_F$$

which is analytic for $|\operatorname{Re} s - P_f| \leq \eta_0$. Decreasing η_0 and applying Phragmén-Lindelöf theorem, by a standard argument we obtain a bound $\mathcal{O}\Big(\max\Big(1, |\operatorname{Im} s|^{\alpha}\Big)\Big)$ with $0 < \alpha < 1$. Consequently, we have the following

Proposition 3. *Under the assumptions of Theorem 1 there exist $\eta_0 > 0$ and $0 < \alpha < 1$ such that for $\operatorname{Re} s > P_f - \eta_0$ we have*

$$\eta_g(s) = \frac{1}{s - P_f}\int G d\mu_F + A(s) \tag{7.2}$$

with an analytic function $A(s)$ satisfying the estimate

$$|A(s)| \leq C\max\Big(1, |\operatorname{Im} s|^{\alpha}\Big). \tag{7.3}$$

Next define $\mathcal{F}^\tau(\mathbb{C}) := \{F : R^\tau \longrightarrow \mathbb{C}\}$ and $\mathcal{F}^\tau(\mathbb{R}) := \{F : R^\tau \longrightarrow \mathbb{R}\}$ the spaces of complex-valued (real-valued) functions which are continuous. If $G \in \mathcal{F}^\tau(\mathbb{C})$ is Lipschitz continuous and if the standing assumptions for Λ are satisfied, the function

$$g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$$

is Lipschitz continuous on R. Moreover, if the representative of G in the suspension space R^τ is constant on stable leaves, the function $g(x)$ depends only on $x \in U$. Now we introduce two definitions of independence.

Definition 1. *Two functions $f_1, f_2 : U \to \mathbb{R}$ are called $\sigma-$ independent if whenever there are constants $t_1, t_2 \in R$ such that $t_1 f_1 + t_2 f_2$ is co homologous to a function in $C(U : 2\pi\mathbb{Z})$, we have $t_1 = t_2 = 0$.*

For a function $G \in \mathcal{F}^\tau(\mathbb{R})$ consider the skew product flow S_t^G on $\mathbb{S}^1 \times R^\tau$ by

$$S_t^G(e^{2\pi\mathbf{i}\alpha}, y) = \Big(e^{2\pi\mathbf{i}(\alpha + G^t(y))}, \sigma_t^\tau(y)\Big).$$

Definition 2 ([8])**.** *Let $G \in \mathcal{F}^\tau(\mathbb{R})$. Then G and σ_t^τ are flow independent if the following condition is satisfied. If $t_0, t_1 \in \mathbb{R}$ are constants such that the skew product flow S_t^H with $H = t_0 + t_1 G$ is not topologically mixing, then $t_0 = t_1 = 0$.*

Notice that if G and σ_t^τ are flow independent, then the flow σ_t^τ is topologically weak mixing and the function G is not co homologous to a constant function. On the other hand, if G and σ_t^τ are flow independent, then $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$ and τ are $\sigma-$ independent.
Below we assume that g and τ are $\sigma-$ independent and we suppose that F, G is a Lipschitz functions Λ having representative in R^τ which are constant on stable leaves. Thus we obtain functions f, g which are in $C^{\mathrm{Lip}}(\widehat{U})$. We will now obtain an analytic continuation of $\zeta(s,z)$ for $P_f - \eta_0 < \operatorname{Re} s < P_f$

and $z = \mathbf{i}w$. Set $r(s, w) = f - (P_f + a + \mathbf{i}b)\tau + \mathbf{i}wg$. We choose $M > 0$ large enough so that we can apply Theorem 6 for $|w| \geq M$. We consider two cases.

Case 1. $\eta_0 \leq |w| \leq M$. We consider two sub cases: 1a) $|\operatorname{Im} s| \leq M_1$, $1b)$ $|\operatorname{Im} s| \geq M_1$. Here $M_1 > 0$ is chosen large enough so that Theorem 5 (b) holds with $|\operatorname{Im} s| \geq M_1$.

Let $|\operatorname{Im} s| \leq M_1$. Assume first that $\operatorname{Im} r(s_0, w_0)$ is cohomologous to $c + 2\pi Q$ with an integer-valued function $Q \in C(U; \mathbb{Z})$ and a constant $c \in [0, 2\pi)$. If $c = 0$, since g and τ are $\sigma-$ independent, from the fact that $b\tau + wg$ is co homologous to a function in $C(U; 2\pi\mathbb{Z})$, we deduce $b = w = 0$ which is impossible because $b = \operatorname{Im} s \neq 0$. Thus we have $c \neq 0$. Consequently, the operator $L_{f - s_0\tau + \mathbf{i}wg}$ has an eigenvalue $e^{\mathbf{i}c}$. Then there exists a neighborhood $U_1 \subset \mathbb{C} \times \mathbb{R}$ of (s_0, w_0) such that for $(s, w) \in U_1$ we have $Pr(r(s, w)) \neq 0$ and for $(s, w) \in U_2$ we have an analytic extension of $\log \zeta(s, w)$ given by

$$\log \zeta(s, w) = \frac{K_1(s, w)}{1 - e^{Pr(r(s,w))}} + J_1(s, w)$$

with functions $K_1(s, w), J_1(s, w)$ analytic with respect to s for $(s, w) \in U_1$. Second, let $\operatorname{Im} r(s_0, w_0)$ be not cohomologous to $c + 2\pi Q$. Then the spectral radius of $L_{f - s_0\tau + \mathbf{i}wg}$ is strictly less than 1 and this will be the case for (s, w) is a small neighborhood $U_2 \subset \mathbb{C} \times \mathbb{R}$ of (s_0, w_0). Applying Theorem 4, this implies easily that $\log \zeta(s, \mathbf{i}w)$ has an analytic continuation with respect to s.

Passing to the case 1b), we observe that $|\operatorname{Im} s| \geq \frac{M_1}{\eta_0}|w|$. Then, we apply Theorem 5(c) combined with Theorem 4 to obtain an analytic continuation of $\log \zeta(s, \mathbf{i}w)$. Moreover, our argument works for $z = c + \mathbf{i}w$ with $|c| \leq \eta_0$ and $\eta_0 \leq |w| \leq M$ and we obtain an analytic continuation of $\log \zeta(s, z)$ for $P_f - \eta_0 \leq \operatorname{Re} s < P_f, |c| \leq \eta_0,\ \eta_0 \leq |w| \leq M$.

Case 2. $|w| \geq M$. We consider two-sub cases: 2a) $|\operatorname{Im} s| \geq B|w|$, 2b) $|\operatorname{Im} s| \leq B|w|$, $B = \frac{M_1}{M}$. If we have 2a), we apply Theorem 5(c). In the case 2b) we use the argument of section 6 replacing $g(x)$ by $g'(x) = g(x) + d\tau(x)$, where the constant $d > 0$ is chosen so that for the function $G' = G + d$ we have $\frac{\operatorname{Lip} G'}{\min G'} \leq \hat{\mu}$, where $\hat{\mu} > 0$ is the constant introduced in section 5. Next we write

$$L_{f - (P_f + a + \mathbf{i}b)\tau + \mathbf{i}wg} = L_{f - (P_f + a + \mathbf{i}(b + dw)\tau + \mathbf{i}wg'}.$$

For the Ruelle operator involving g' we can apply Theorem 6 since $|b + dw| \leq (B + d)|w|$, $|w| \geq M$ and g is a Lipschitz function. An application of Theorem 4 implies the analytic continuation of $\log \zeta(s, \mathbf{i}w)$ for $P_f - \eta_0 \leq \operatorname{Re} s \leq P_f$ and $|w| \geq M$. From the above analysis we deduce the following

Theorem 7. *Assume the standing assumptions fulfilled for the basic set Λ. Let $F, G : \Lambda \longrightarrow \mathbb{R}$ be Lipschitz functions having representatives in R^τ which are constant on stable leaves. Assume that g and τ are σ-independent. Then there exists $\eta_0 > 0$ such that $\zeta(s, \mathbf{i}w)$ admits a non zero analytic continuation with respect to s for $P_f - \eta_0 \leq \operatorname{Re} s$, $w \in \mathbb{R}$ and $|w| \geq \eta_0$.*

8. Applications

8.1. **Hannay-Ozorio de Almeida sum formula.** The proof of (1.2) in [18] is based on the analytic continuation of the Dirichlet series

$$\eta(s) = \sum_{\gamma} \sum_{m=1}^{\infty} \lambda_G(\gamma) e^{m(-\lambda^u(\gamma) - (s-1)\lambda(\gamma))},\ s \in \mathbb{C}$$

for $1-\eta_0 \leq \operatorname{Re} s < 1$. For this purpose the authors examine the analytic continuation of the symbolic function $\eta_g(s)$ with $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$ defined in section 1 and they use the fact that the difference $\eta(s) - \eta_g(s)$ is analytic in a region $\operatorname{Re} s > 1 - \epsilon'$, $\epsilon' > 0$. Next for the geodesic flow on surfaces with negative curvature they establish Proposition 3 with $P_f = 1$. Since M is an attractor, the equilibrium state of the function $-E(x)$ is just the SRB measure μ of ϕ_t (see [3]) and the residuum in (7.2) becomes $\int G d\mu$.

For the proof of Proposition 3 in [18] the authors exploit the link between the analytic continuation of $\zeta(s,z)$ and the spectral estimates of the Ruelle operator obtained by Dolgopyat [4]. However, in [18] Ruelle's lemma in [16] was used whose proof is rather sketchy and contains some steps which are not done in detail (see [24] for more information and comments concerning these steps and the gaps in their proofs). On the other hand, the estimates of Dolgopyat [4] are established only for Ruelle operators with one complex parameter, and to take into account the second parameter z some complementary analysis is necessary.

We should mention that [24] contains a correct and complete proof of Ruelle's lemma in the case of one complex parameter and a Hölder function $\tau(x)$. A version of this lemma with two complex parameters is given in section 3 above. Next, in Theorem 5 the spectral estimates for the Ruelle operator with two complex parameters are established for Axiom A flows on a basic set Λ of arbitrary dimension under the standing assumptions. If Λ is an attractor, according to [3], the equilibrium state of $-E(x)$ coincides with the SRB measure μ of ϕ_t. Thus we can apply Proposition 3 to obtain a representation of $\eta_g(s)$ with residue $\int G d\mu$. Using (7.2) and repeating the argument of section 4 in [18], we obtain Theorem 2.

8.2. **Asymptotic of the counting function for period orbits.** As we mentioned in section 1, the analysis of $\pi_F(T)$ is based on the analytic continuation of the function $\zeta(s,0)$ defined in section 1. From the arguments in section 7 with $z = 0$ and the proof of Proposition 3 we get the following

Proposition 4. *Under the standing assumptions in section 4 there exists $\eta_0 > 0$ such that $\frac{\zeta_F'(s)}{\zeta_F(s)}$ admits an analytic continuation for $Pr(F) - \eta_0 \leq \operatorname{Re} s$ with a simple pole at $s = Pr(F)$ with residue 1. Moreover, there exists $0 < \alpha < 1$ such that for $|\operatorname{Im} s| \geq 1$ we have*

$$\Big|\frac{\zeta_F'(s)}{\zeta_F(s)}\Big| \leq C|\operatorname{Im} s|^{\alpha}. \tag{8.1}$$

To obtain an asymptotic of $\pi_F(T)$, we examine the functions

$$\Psi(T) = \sum_{e^{nPr(F)\lambda(\gamma)} \leq T} \lambda(\gamma) e^{Pr(F)\lambda(\gamma)},\ \Psi_1(T) = \int_0^T \Psi(y)dy.$$

By a standard argument (see [16] and [15]) we obtain the representation

$$\psi_1(T) = \frac{T^2}{2} + \int_{\operatorname{Re} s = (1-\eta_0)Pr(F)} \Big(-\frac{\zeta_F'(s)}{\zeta_F(s)}\Big) \frac{T^s}{s(s+1)} ds = \frac{T^2}{2} + \mathcal{O}(T^{1+\alpha}),$$

where in the second equality the estimate (8.1) is used. This implies an asymptotic for $\Psi(T)$ and repeating the argument in [16], [15], one obtains Theorem 3.

9. Appendix: Proofs of some lemmas

Proof of Lemma 4. Denote by $\mathcal{F}_\theta(\widehat{U})$ the space of all functions $h : \widehat{U} \longrightarrow \mathbb{R}$ that are Lipschitz with respect to d_θ. Let $g \in C^{\rm Lip}(\widehat{U})$, and let $\theta = \theta_\alpha \in (0,1)$ be as in section 4. Then $g \in \mathcal{F}_\theta(\widehat{U})$. Let $\lambda > 0$ be the maximal positive eigenvalue of L_g on $\mathcal{F}_\theta(\widehat{U})$ and let $h > 0$ be a corresponding normalized eigenfunction. By the Ruelle-Perron-Frobenius theorem, we have that $\frac{1}{\lambda^m} L_g^m 1$ converges uniformly to h. We will show that there exists a constant $C > 0$ such that $\frac{1}{\lambda^m}\mathrm{Lip}(L_g^m 1) \leq C$ for all m; this would then imply immediately that $h \in C^{\rm Lip}(\widehat{U})$ and $\mathrm{Lip}(h) \leq C$.

Take an arbitrary constant $K > 0$ such that $1/K \leq h(x) \leq K$ for all $x \in \widehat{U}$. Given $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$ and an integer $m \geq 1$ for any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. By (2.1) we have

$$|g_m(v) - g_m(v')| \leq \sum_{j=0}^{m-1} |g(\sigma^j(v)) - g(\sigma^j(v'))| \leq \mathrm{Lip}(g) \sum_{j=0}^{m-1} \frac{d(u,u')}{c_0 \gamma^m} \leq C' \,\mathrm{Lip}(g)\, d(u,u')$$

for some constant $C' > 0$. Thus,

$$\begin{aligned}
|(L_g^m 1)(u) - (L_g^m 1)(u')| &\leq \sum_{\sigma^m(v)=u} \left| e^{g_m(v)} - e^{g_m(v')} \right| = \sum_{\sigma^m(v)=u} e^{g_m(v)} \left| e^{g_m(v) - g_m(v')} - 1 \right| \\
&\leq e^{C' \,\mathrm{Lip}(g)} \sum_{\sigma^m(v)=u} e^{g_m(v)} \left| g_m(v) - g_m(v') \right| \\
&\leq e^{C' \,\mathrm{Lip}(g)}\, C' \mathrm{Lip}(g)\, d(u,u') \sum_{\sigma^m(v)=u} e^{g_m(v)} \\
&\leq e^{C' \,\mathrm{Lip}(g)}\, C' \mathrm{Lip}(g)\, d(u,u') \sum_{\sigma^m(v)=u} e^{g_m(v)}\, K h(v) \\
&= e^{C' \,\mathrm{Lip}(g)}\, C' K \mathrm{Lip}(g)\, d(u,u')\, (L_g^m h)(u) \\
&= e^{C' \,\mathrm{Lip}(g)}\, C' K \mathrm{Lip}(g)\, d(u,u')\, \lambda^m h(u) \\
&\leq \lambda^m\, C' K^2 e^{C' \,\mathrm{Lip}(g)}\, \mathrm{Lip}(g)\, d(u,u').
\end{aligned}$$

Thus, for every integer m the function $\frac{1}{\lambda^m} L_g^m 1 \in C^{\rm Lip}(\widehat{U})$ and $\frac{1}{\lambda^m}\mathrm{Lip}(L_g^m 1) \leq C' K^2 e^{C' \,\mathrm{Lip}(g)}\, \mathrm{Lip}(g)$. As mentioned above this proves that the eigenfunction $h \in C^{\rm Lip}(\widehat{U})$.

Using this with $g = f_t$ proves that $h_{at} \in C^{\rm Lip}(\widehat{U})$ for all $|a| \leq a_0$ and $t \geq 1/a_0$. However the above estimate for $\mathrm{Lip}(h_{at})$ would be of the form $\leq C\, e^{C\,t}\, t$ for some constant $C > 0$, which is not good enough.

We will now show that, taking $a_0 > 0$ sufficiently small, we have $\mathrm{Lip}(h_{at}) \leq Ct$ for some constant $C > 0$ independent of a and t.

Using (4.2) and choosing $a_0 > 0$ sufficiently small, we have $\lambda_{at}\gamma > \hat{\gamma}$ for all $|a| \leq a_0$ and $t > 1/a_0$. Fix an integer $m_0 \geq 1$ so large that $\frac{C_0^2}{c_0 \hat{\gamma}^m} < \frac{1}{2}$ for $m \geq m_0$. There exists a constant $d_0 > 0$ depending on m_0 such that for any u, u' belonging to the same U_i but not to the same cylinder of length m_0 we have $d(u,u') \geq d_0$. For such u, u' we have

$$\frac{|h_{at}(u) - h_{at}(u')|}{d(u,u')} \leq \frac{2\|h_{at}\|_0}{d_0} \leq \frac{2C_0}{d_0}.$$

So, to estimate $\mathrm{Lip}(h_{at})$ it is enough to consider pairs u, u' that belong to the same cylinder of length m_0.

Fix for a moment a, t with $|a| \le a_0$ and $t \ge 1/a_0$. Set $L = \sup_{u \neq u'} \frac{|h_{at}(u) - h_{at}(u')|}{d(u,u')}$, where the supremum is taken over all pairs $u \neq u'$ that belong to the same cylinder of length m_0. If $L < \mathrm{Lip}(h_{at})$, then the above implies $\mathrm{Lip}(h_{at}) \le \frac{2C_0}{d_0} \le \frac{2C_0}{d_0}\, t$.

Assume that $L = \mathrm{Lip}(h_{at})$. Then there exist u, u' belonging to the same cylinder of length m_0 such that

$$\frac{3L}{4} < \frac{|h_{at}(u) - h_{at}(u')|}{d(u,u')}. \tag{9.1}$$

Fix such a pair u, u'. Let $m \ge m_0$ be an integer. For any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. By (2.1), $d(\sigma^j(v), \sigma^j(v')) \le \frac{1}{c_0\, \gamma^{m-j}}\, d(u,u')$ for all $j = 0, 1, \ldots, m-1$, so

$$|f_t^m(v) - f_t^m(v')| \le \sum_{j=0}^{m-1} |f_t(\sigma^j(v)) - f_t(\sigma^j(v'))| \le \mathrm{Const}\mathrm{Lip}(f_t)\, d(u,u') \le \mathrm{Const}\, t\, d(u,u').$$

At the same time, by property (i), $\|f_t\|_0 \le T''$ for some constant $T'' > 0$, so

$$|f_t^m(v) - f_t^m(v'(v))| \le 2m\|f_t\|_0 \le 2mT''.$$

Similarly,

$$|(P+a)\tau^m(v) - (P+a)\tau^m(v')| \le \mathrm{Const}\, d(u,u') \le T'',$$

assuming $T'' > 0$ is chosen sufficiently large. Thus,

$$\begin{aligned} &\left| e^{(f_t - (P+a)\tau)^m(v') - (f_t-(P+a)\tau)^m(v)} - 1 \right| \\ \le\ & e^{3mT''} \left| (f_t - (P+a)\tau)^m(v) - (f_t - (P+a)\tau)^m(v') \right| \le e^{3mT''}\, \mathrm{Const}\, t\, d(u,u'). \end{aligned}$$

Using $L^m_{f_t - (P+a)\tau} h_{at} = \lambda_{at}^m h_{at}$, we obtain

$$\begin{aligned} &\lambda_{at}^m\, |h_{at}(u) - h_{at}(u')| = \left| \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)}\, h_{at}(v) - \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v'(v))}\, h_{at}(v') \right| \\ \le\ & \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)}\, |h_{at}(v) - h_{at}(v')| + \|h_{at}\|_0 \sum_{\sigma^m v = u} \left| e^{(f_t - (P+a)\tau)^m(v)} - e^{(f_t - (P+a)\tau)^m(v')} \right| \\ \le\ & \frac{\mathrm{Lip}(h_{at})\, d(u,u')}{c_0 \gamma^m} \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)} \\ & + C_0 \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)} \left| 1 - e^{(f_t - (P+a)\tau)^m(v') - (f_t - (P+a)\tau)^m(v)} \right| \\ \le\ & \frac{L\, d(u,u')}{c_0 \gamma^m} \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)} + C_0 e^{3mT''}\, \mathrm{Const}\, t\, d(u,u') \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)} \\ \le\ & \left(\frac{L}{c_0 \gamma^m} + C_0 e^{3mT''}\, \mathrm{Const}\, t \right) d(u,u') \sum_{\sigma^m v = u} e^{(f_t - (P+a)\tau)^m(v)}\, C_0 h_{at}(v) \\ =\ & \left(\frac{L}{c_0 \gamma^m} + C_0 e^{3mT''}\, \mathrm{Const}\, t \right) d(u,u')\, C_0 \lambda_{at}^m h_{at}(u) \le \left(\frac{L}{c_0 \gamma^m} + C_0 e^{3mT''}\, \mathrm{Const}\, t \right) d(u,u')\, C_0^2 \lambda_{at}^m. \end{aligned}$$

This, (9.1) and the choice of m_0 imply $\frac{3L}{4} < \frac{LC_0^2}{c_0\gamma^m} + C_0^3 e^{3mT''}\,\mathrm{Const}\, t \leq \frac{L}{2} + C_0^3 e^{3mT''}\,\mathrm{Const}\, t$. This is true for all $m \geq m_0$. In particular for $m = m_0$ we get $\frac{L}{4} < C_0^3 e^{3m_0T''}\,\mathrm{Const}\, t$, and so $\mathrm{Lip}(h_{at}) = L \leq \mathrm{Const}\ t$. ■

Proof of Lemma 5. (a) Let $u, u' \in \widehat{U}_i$ for some $i = 1, \dots, k$ and let $m \geq 1$ be an integer. For any $v \in \widehat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \widehat{U}$ in the cylinder of length m containing v such that $\sigma^m(v') = u'$. Then

$$|f_{at}^m(v) - f_{at}^m(v')| \leq \sum_{j=0}^{m-1} |f_{at}(\sigma^j(v)) - f_{at}(\sigma^j(v'))| \leq \frac{Tt}{c_0\,(\gamma-1)}\, d(u,u') \leq C_1\, t\, D(u,u') \tag{9.2}$$

for some constant $C_1 > 0$. Similarly,

$$|g_t^m(v) - g_t^m(v')| \leq C_1\, t\, D(u,u'). \tag{9.3}$$

Also notice that if $D(u,u') = \mathrm{diam}(\mathcal{C}')$ for some cylinder $\mathcal{C}' = C[i_{m+1}, \dots, i_p]$,then $v, v'(v) \in \mathcal{C}'' = C[i_0, i_1, \dots, i_p]$ for some cylinder $\mathcal{C}''$ with $\sigma^m(\mathcal{C}'') = \mathcal{C}'$, so

$$D(v,v') \leq \mathrm{diam}(\mathcal{C}'') \leq \frac{1}{c_0\,\gamma^m}\,\mathrm{diam}(\mathcal{C}') = \frac{D(u,u')}{c_0\,\gamma^m}.$$

Using the above, $\mathrm{diam}(U_i) \leq 1$, the definition of $\mathcal{M}_{atc}$, we get

$$\begin{aligned}
\frac{|(\mathcal{M}_{atc}^m H)(u) - (\mathcal{M}_{atc}^m H)(u')|}{\mathcal{M}_{atc}^m H(u')} &= \frac{\left|\sum_{\sigma^m v = u} e^{f_{at}^m(v) + cg_t^m(v)}\, H(v) - \sum_{\sigma^m v = u} e^{f_{at}^m(v') + cg_t^m(v')}\, H(v')\right|}{\mathcal{M}_{atc}^m H(u')} \\
\leq\ & \frac{\left|\sum_{\sigma^m v = u} e^{f_{at}^m(v) + cg_t^m(v)}\, (H(v) - H(v'))\right|}{\mathcal{M}_{atc}^m H(u')} + \frac{\sum_{\sigma^m v = u} \left|e^{f_{at}^m(v) + cg_t^m(v)} - e^{f_{at}^m(v') + cg_t^m(v')}\right|\, H(v')}{\mathcal{M}_{atc}^m H(u')} \\
\leq\ & \frac{\sum_{\sigma^m v = u} e^{f_{at}^m(v) + cg_t^m(v)}\, E\, H(v')\, D(v,v')}{\mathcal{M}_{atc}^m H(u')} \\
&+ \frac{\sum_{\sigma^m v = u} \left|e^{[f_{at}^m(v) + cg_t^m(v)] - [f_{at}^m(v') + cg_t^m(v')]} - 1\right|\, e^{f_{at}^m(v') + cg_t^m(v')}\, H(v')}{\mathcal{M}_{atc}^m H(u')}.
\end{aligned}$$

Using (9.2) and (9.3) and assuming $\eta_0 \leq 1$, one obtains

$$|f_{at}^m(v) + cg_t^m(v)] - [f_{at}^m(v') + cg_t^m(v')| \leq 2C_1 t\, D(u,u') \leq 2C_1 t, \tag{9.4}$$

and therefore $\left|e^{[f_{at}^m(v) + cg_t^m(v)] - [f_{at}^m(v') + cg_t^m(v')]} - 1\right| \leq e^{2C_1 t} 2C_1 t\, D(u,u')$. However (9.4) is not good enough to estimate the first term in the right-hand-side above. Instead we use (4.3) and (4.4) to get

$$\begin{aligned}
& |f_{at}^m(v) + cg_t^m(v)] - [f_{at}^m(v') + cg_t^m(v')| \\
\leq\ & |f_t^m(v) - f_t^m(v)| + |P-a|\,|\tau^m(v) - \tau^m(v')| + |(h_{at}(v) - h_{at}(u)) - (h_{at}(v') - h_{at}(u')| \\
& + a_0|g_t^m(v) - g_t^m(v')| \\
\leq\ & 2m\|f_t - f_0\|_0 + |f_0^m(v) - f_0^m(v')| + \mathrm{Const}\, D(u,u') + 4C_0 + 2ma_0\|g_t - g\|_0 \\
\leq\ & \mathrm{Const}\, D(u,u') + C_2 m a_0 \leq C_2 + C_2 m\, a_0 \qquad (9.5)
\end{aligned}$$

for some constant $C_2 > 0$. We will now assume that $a_0 > 0$ is chosen so small that $e^{C_2 a_0} < \gamma/\hat{\gamma}$. Then

$$\begin{aligned}
& \frac{|(\mathcal{M}_{atc}^m H)(u) - (\mathcal{M}_{atc}^m H)(u')|}{\mathcal{M}_{atc}^m H(u')} \\
\leq\ & \frac{E\, D(u,u')}{c_0 \gamma^m} \frac{\sum_{\sigma^m v = u} e^{[f_{at}^m(v) + cg_t^m(v)] - [f_{at}^m(v') + cg_t^m(v')]} e^{f_{at}^m(v') + cg_t^m(v')}\, H(v')}{\mathcal{M}_{atc}^m H(u')} \\
& + e^{2C_1 t} \frac{\sum_{\sigma^m v = u} 2C_1 t\, e^{f_{at}^m(v'(v))}\, H(v'(v))}{\mathcal{M}_{atc}^m H(u')} \\
\leq\ & e^{C_2}\, e^{C_2 m a_0} \frac{E\, D(u,u')}{c_0 \gamma^m} + 2C_1 t e^{2C_1 t}\, D(u,u') \leq A_0 \left[\frac{E}{\hat{\gamma}^m} + e^{A_0 t}\, t\right] D(u,u'),
\end{aligned}$$

for some constant $A_0 > 0$ independent of a, c, t, m and E.

(b) Let $m \geq 1$ be an integer and $u, u' \in \widehat{U}_i$ for some $i = 1, \ldots, k$. Using the notation $v' = v'(v)$ and the constant $C_2 > 0$ from part (a) above, where $\sigma^m v = u$ and $\sigma^m v' = u'$, and some of the estimates from the proof of part (a), we get

$$\begin{aligned}
& |\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \\
=\ & \left| \sum_{\sigma^m v = u} \left(e^{f_{at}^m(v) + cg_t^m(v) - \mathbf{i} b \tau^m(v) + \mathbf{i} w g_t^m(v)}\, h(v) - e^{f_{at}^m(v') + cg_t^m(v') - \mathbf{i} b \tau^m(v') + \mathbf{i} w g_t^m(v')}\, h(v') \right) \right| \\
\leq\ & \left| \sum_{\sigma^m v = u} e^{f_{at}^m(v) + cg_t^m(v) - \mathbf{i} b \tau^m(v) + \mathbf{i} w g_t^m(v)}\, [h(v) - h(v')] \right| \\
& + \sum_{\sigma^m v = u} \left| e^{f_{at}^m(v) + cg_t^m(v)} - e^{f_{at}^m(v') + cg_t^m(v')} \right| |h(v')| \\
& + \sum_{\sigma^m v = u} \left| e^{-\mathbf{i} b \tau^m(v) + \mathbf{i} w g_t^m(v)} - e^{-\mathbf{i} b \tau^m(v') - \mathbf{i} w g_t^m(v')} \right| e^{f_{at}^m(v') + cg_t^m(v')} |h(v')| \\
\leq\ & \sum_{\sigma^m v = u} e^{f_{at}^m(v) + cg_t^m(v)}\, |h(v) - h(v')| \\
+\ & \sum_{\sigma^m v = u} \left| e^{[f_{at}^m(v) + cg_t^m(v)] - [f_{at}^m(v') + cg_t^m(v')]} - 1 \right| e^{f_{at}^m(v') + cg_t^m(v')}\, |h(v')| \\
& \sum_{\sigma^m v = u} \left(|b|\, |\tau^m(v) - \tau^m(v')| + |w|\, |g_t^m(v) - g_t^m(v')| \right) e^{f_{at}^m(v') + cg_t^m(v')} |h(v')|
\end{aligned}$$

Using the constants $C_1, C_2 > 0$ from the proof of part (a), (9.5) and $e^{C_2 a_0} < \gamma/\hat{\gamma}$ we get

$$\begin{aligned}\sum_{\sigma^m v = u} e^{f^m_{at}(v)+cg^m_t(v)}\, |h(v) - h(v')| &\leq e^{C_2}\, e^{C_2 m a_0} \frac{E\, D(u,u')}{c_0 \gamma^m} \sum_{\sigma^m v=u} e^{f^m_{at}(v')+cg^m_t(v')}\, H(v') \\ &\leq \frac{e^{C_2} E}{c_0 \hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u')\, D(u,u').\end{aligned}$$

This, (9.3) and (9.5) imply

$$\begin{aligned}&|\mathcal{L}^m_{abtz} h(u) - \mathcal{L}^m_{abtz} h(u')| \\ \leq\ & \frac{e^{C_2} E}{c_0 \hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u')\, D(u,u') + e^{2C_1 t} 2C_1 t\, D(u,u')\, (\mathcal{M}^m_{atc}|h|)(u') + (\mathrm{Const}\, |b| + |w| C_1\, t)\, D(u,u')\end{aligned}$$

Thus, taking the constant $A_0 > 0$ sufficiently large we get

$$|(\mathcal{L}^N_{abtz} h)(u) - (\mathcal{L}^N_{abtz} h)(u')| \leq A_0 \left(\frac{E}{\hat{\gamma}^m} (\mathcal{M}^m_{atc} H)(u') + (|b| + e^{A_0 t} t + t|w|)(\mathcal{M}^m_{atc}|h|)(u') \right) D(u,u'),$$

which proves the assertion. ■

As in [4] and [21] we need the following lemma whose proof is omitted here, since it is very similar to the proof of Lemma 5 given above.

Lemma 17. *Let $\beta \in (0,\alpha)$. There exists a constants $A'_0 > 0$ such that for all $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c|, 1/|b|, 1/t \leq a_0$ such that (5.1) hold, and all positive integers m and all $h \in C^\beta(U)$ we have*

$$|\mathcal{L}^m_{abtz} h(u) - \mathcal{L}^m_{abtz} h(u')| \leq A'_0 \left[\frac{|h|_\beta}{\hat{\gamma}^{m\beta}} + |b|\, (\mathcal{M}^m_{atc}|h|)(u') \right] (d(u,u'))^\beta$$

for all $u, u' \in U_i$.

We will derive Theorem 5(b) from Theorem 5(a), proved in section 5, and Lemma 17 above.

Proof of Theorem 5(b). We essentially repeat the proofs of Corollaries 2 and 3 in [4] (cf. also section 3 in [20]).

Let $\epsilon > 0$, $B > 0$ and $\beta \in (0,\alpha)$. Take $\hat{\rho} \in (0,1)$, $a_0 > 0$, $b_0 > 0$, $A_0 > 0$ and N as in Theorem 2(a). We will assume that $\hat{\rho} \geq \frac{1}{\gamma_0}$. Let $a, b, c, w \in \mathbb{R}$ be such that $|a|, |c| \leq a_0$ and $|b| \geq b_0$. Let $t > 0$ be such that $1/t^{\alpha-\beta} \leq a_0$. Assume that (5.1) hold and set $z = c + \mathbf{i}w$.

First, as in [4] (see also section 3 in [20]) one derives from Theorem 5(a) and Lemma 17 (approximating functions $h \in C^\beta(\widehat{U})$ by Lipschitz functions as in section 4) that there exist constants $C_3 > 0$ and $\rho_1 \in (0,1)$ such that

$$\|\mathcal{L}^n_{abtz} h\|_{\beta,b} \leq C_3 |b|^\epsilon \rho_1^n \quad , \quad n \geq 0, \tag{9.6}$$

for all $h \in C^\beta(\widehat{U})$.

Next, given $h \in C^\beta(\widehat{U})$, we have $\mathcal{L}^n_{abtz}(h/h_{at}) = \frac{1}{\lambda^n_{at} h_{at}} L_{f_t - (P+a+\mathbf{i}b)\tau + z g_t} h$, so by (9.6) we get

$$\begin{aligned}\|L^n_{f_t-(P+a+\mathbf{i}b)\tau+zg_t} h\|_{\beta,b} &\leq \lambda^n_{at} \|h_{at}\, \mathcal{L}^n_{abtz}(h/h_{at})\|_{\beta,b} \\ &\leq \mathrm{Const} (\lambda_{at}\rho_1)^n |b|^\epsilon\, \|h/h_{at}\|_{\beta,b} \leq \mathrm{Const}\, \rho_2^n\, |b|^\epsilon\, \|h\|_{\beta,b}\ ,\end{aligned}$$

where $\lambda_{at}\rho_1 \leq e^{2C_0 a_0} \rho_2 = \rho_2 < 1$, provided $a_0 > 0$ is small enough.

We will now approximate $L_{f-(P+a+\mathbf{i}b)\tau+zg}$ by $L_{f_t-(P+a+\mathbf{i}b)\tau+g_t}$ in two steps. First, using the above it follows that

$$\begin{aligned}\|L^n_{f-(P+a+\mathbf{i}b)\tau+cg+\mathbf{i}wg_t} h\|_{\beta,b} &= \left\|L^n_{f_t-(P+a+\mathbf{i}b)\tau+zg_t}\left(e^{(f^n-f^n_t)+c(g^n-g^n_t)}h\right)\right\|_{\beta,b}\\ &\leq \text{Const}\,\rho_2^n\,|b|^\epsilon\,\left\|e^{(f^n-f^n_t)+c(g^n-g^n_t)}h\right\|_{\beta,b}.\end{aligned}$$

Choosing the constant $C_4 > 0$ appropriately, $\|f - f_t\|_0 \leq C_4\, a_0$ and $|f - f_t|_\beta \leq C_4/t^{\alpha-\beta} \leq C_4 a_0$, so $\|f^n - f^n_t\|_0 \leq n\,\|f - f_t|_0 \leq C_4 n a_0$, and similarly $|f^n - f^n_t|_\beta \leq C_4 n a_0$. Similar estimates hold for $g^n - g^n_t$. Thus, $\|e^{(f^n-f^n_t)+c(g^n-g^n_t)}h\|_0 \leq e^{C_4 n a_0}\|h\|_0$, and

$$\begin{aligned}|e^{(f^n-f^n_t)+c(g^n-g^n_t)}h|_\beta &\leq \|e^{(f^n-f^n_t)+c(g^n-g^n_t)}\|_0\,|h|_\beta + |e^{(f^n-f^n_t)+c(g^n-g^n_t)}|_\beta\,\|h\|_\infty\\ &\leq e^{C_4 n a_0}|h|_\beta + e^{C_4 n a_0}\,|(f^n - f^n_t) + c(g^n - g^n_t)|_\beta\,\|h\|_\infty\\ &\leq C_5'\,n\,e^{C_4 n a_0}\,\|h\|_\beta.\end{aligned}$$

Combining this with the previous estimate gives

$$\|e^{(f^n-f^n_t)+c(g^n-g^n_t)}h\|_{\beta,b} \leq C_5''\,n\,e^{C_4 n a_0}\,\|h\|_\beta,$$

so

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau+cg+\mathbf{i}wg_t} h\|_{\beta,b} \leq C_5\,\rho_2^n\,|b|^\epsilon\,n\,e^{C_4 n a_0}\,\|h\|_{\beta,b}.$$

Taking $a_0 > 0$ sufficiently small, we may assume that $\rho_2\,e^{C_4 a_0} < 1$. Now take an arbitrary ρ_3 with $\rho_2\,e^{C_4 a_0} < \rho_3 < 1$. Then we can take the constant $C_6 > 0$ so large that $n\,\rho_2^n\,e^{C_4 n a_0} \leq C_6 \rho_3^n$ for all integers $n \geq 1$. This gives

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau+cg+\mathbf{i}wg_t} h\|_{\beta,b} \leq C_6\,\rho_3^n\,|b|^\epsilon\,\|h\|_{\beta,b} \quad , \quad n \geq 0.$$

Using the latter we can write

$$\begin{aligned}\|L^n_{f-(P+a+\mathbf{i}b)\tau+zg} h\|_{\beta,b} &= \left\|L^n_{f-(P+a+\mathbf{i}b)\tau+cg+\mathbf{i}wg_t}\left(e^{\mathbf{i}w(g^n-g^n_t)}h\right)\right\|_{\beta,b}\\ &\leq C_6\,\rho_3^n\,|b|^\epsilon\,\left\|e^{\mathbf{i}w(g^n-g^n_t)}h\right\|_{\beta,b}.\end{aligned}$$

However, $\|e^{\mathbf{i}w(g^n-g^n_t)}h\|_0 = \|h\|_0$, $|g - g_t|_\beta \leq C_4/t^{\alpha-\beta} \leq C_4 a_0 \leq 1$ (assuming $a_0 > 0$ is sufficiently small), and by (5.1), $|w| \leq B|b|^\mu \leq B|b|$, so

$$\begin{aligned}|e^{\mathbf{i}w(g^n-g^n_t)}h|_\beta &\leq \|e^{\mathbf{i}w(g^n-g^n_t)}\|_0\,|h|_\beta + |e^{\mathbf{i}w(g^n-g^n_t)}|_\beta\,\|h\|_\infty\\ &\leq |h|_\beta + |w|\,|g^n - g^n_t|_\beta\,\|h\|_\infty \leq \|h\|_\beta + Bn|b|\|h\|_\infty.\end{aligned}$$

Thus,

$$\|e^{\mathbf{i}w(g^n-g^n_t)}h\|_{\beta,b} = \|e^{\mathbf{i}w(g^n-g^n_t)}h\|_0 + \frac{1}{|b|}|e^{\mathbf{i}w(g^n-g^n_t)}h|_\beta \leq 2Bn\|h\|_{\beta,b},$$

and therefore $\|L^n_{f-(P+a+\mathbf{i}b)\tau+zg} h\|_{\beta,b} \leq C_7\,\rho_3^n\,|b|^\epsilon\,n\,\|h\|_{\beta,b}$. Now taking an arbitrary ρ with $\rho_3 < \rho < 1$ and taking the constant $C_8 > C_7$ sufficiently large, we get

$$\|L^n_{f-(P+a+\mathbf{i}b)\tau+zg} h\|_{\beta,b} \leq C_8\,\rho^n\,|b|^\epsilon\,\|h\|_{\beta,b}$$

for all integers $n \geq 0$. ■

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